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Distributed tracking of graph parameters in anonymous networks with time-varying topology

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Abstract—In this paper, we propose a distributed protocol for multi-agent systems to estimate and track changes to the diameter, and radius of a network with time-varying topology, as well as the eccentricity of each agent within it. The main strengths of the proposed protocol are its finite-time convergence and robustness to re-initialization, i.e., if there are changes in the network topology or in the agents' states during the protocol execution then it does not need to be re-initialized to converge to the correct estimation at the steady-state. The expected estimation error of the protocol can be traded-off by increasing the size of locally exchanged messages. We provide a theoretical characterization of the expected steady-state error and some numerical simulations.

I. INTRODUCTION

In networks of collaborating agents, the properties of their communication network topology are crucial. An effective model of the network is drawn by graph theory: each agent is modeled as a node and the interaction between two agents is modeled as an edge, thus constituting a graph. The properties of the graph modeling the communication network highly influences the behavior and the performance of almost any distributed algorithm executed by the network. Examples of distributed algorithms which estimate the spectrum and other graph properties such as controllability, observability, and the Fiedler vector can be found in [1]–[3].

Among the many metrics that have been proposed to characterize the influence of the agents in the network [4], [5], one of particular significance is the *eccentricity* of a node, which is defined as the maximum distance to any other node in the graph. This metric allows to easily define two important features of the graph, namely the *diameter* and *radius* of the network, which are formally equivalent to the maximum and minimum eccentricities among all nodes, respectively. Possible applications of such metrics are straightforward, such as selection of agents for maximizing the spread of influence in social networks [6], optimal coordination of cellular networks [7], efficiency maintenance in wireless networks [8], implementation of a stopping criterion in distributed algorithms [9], and many others.

In the literature, much effort has been spent going beyond centralized approaches [10] by focusing on the design of parallel [11] and distributed computation of these graph param-

eters, by means of flooding techniques [12], [13] and resorting to max-consensus protocols [14], [15]. The latter approach is particularly interesting since it allows to compute the desired parameters without the need of disclosing the identity of the agents within the networks, a framework known as *anonymous networks* [16]. However, these algorithms inherit the main drawback of max-consensus protocols: they cannot be applied to networks with time-varying topology. In fact, if a change of the topology occurs then a variation of the quantity of interest will arise, which in turn entails the need of re-initializing the algorithm in the whole network, an unsuitable feature for large scale networks.

The main contribution of this paper the first distributed protocol for anonymous multi-agent networks which estimates and tracks the eccentricity of each node, along with the diameter and radius of the graph modeling the time-varying topology of the network. Our protocol exploits only anonymous local interactions among agents and is robust to re-initialization, thus it is suitable to be implemented in large time-varying networks. We characterize the steady-state estimation error and the algorithm convergence time.

The main novelty with respect to the current literature is the ability of the protocol to track changes in the graph parameters without the need to be re-initialized in the network. In fact, in the current literature there are no other protocols available to solve the same problem, and thus we can not provide a comparison with existing approaches.

This paper is organized as follows. In Section III we formulate the graph parameters tracking problem along with some working assumptions, then we present the proposed protocol and discuss it in plain words. In Section IV we characterize the steady-state estimation error. Numerical simulations are presented in Section V and concluding remarks are given in Section VI.

II. NOTATION AND PRELIMINARIES

We denote with \mathbb{R} and \mathbb{N} the sets of real and natural numbers respectively. Moreover, we denote with \mathbb{R}_+ and \mathbb{N}_+ their restriction to strictly positive numbers.

A. Multi-Agent systems

A multi-agent system (MAS) consists of a network of agents modeled as dynamical systems interacting among each other. The network is modeled by an undirected graph $\mathcal{G}(k) = (V, E(k))$ which represents the pattern of interactions among the agents at time $k \in \mathbb{N}$: $V \subseteq \mathbb{N}$ is the set of *nodes* modeling the agents, and $E(k) \subseteq V \times V$ is the set of *edges* modeling interactions at time k between them. The

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total number of nodes in the network is constant and equal to $n = |V|$, where $|\cdot|$ denotes the cardinality of a set.

A path between two nodes i and j in a graph is a sequence of consecutive edges $\pi_{ij} = (i, p), (p, q), \dots, (r, s), (s, j)$ where each successive edge shares a node with its predecessor. An undirected graph $\mathcal{G}(k)$ is said to be *connected* if there exists a path π_{ij} between any pair of nodes $i, j \in V$. The *distance* between two nodes $i, j \in V$ at time k is denoted by $\text{dist}_{ij}(k)$ and it is defined as the length (number of edges) of the shortest path between nodes i and j .

The *eccentricity* $e_i(k)$ of node $i \in V$ at time k is defined as the maximal distance from i of any other node,

$$e_i(k) = \max_{j \in V} \text{dist}_{ij}(k).$$

The *diameter* $d(k)$ of graph $\mathcal{G}(k)$ at time k is defined as the maximal eccentricity among the nodes,

$$d(k) = \max_{i \in V} e_i(k).$$

The *radius* $r(k)$ of graph $\mathcal{G}(k)$ at time k is defined as the minimal eccentricity among the nodes,

$$r(k) = \min_{i \in V} e_i(k).$$

Agents i and j are said to be *neighbors* at time k if there exists an edge between i and j , i.e., $(i, j) \in E(k)$ or equivalently $(j, i) \in E(k)$. At any time k , the set of neighbors of the i -th agent is denoted by $\mathcal{N}_i(k) = \{j \in V : (i, j) \in E(k)\}$: it represents the agents in the graph sharing a point-to-point communication channel with agent i and interacting with it at time k . For sake of simplicity, we denote $\mathcal{N}_i^\circ(k) = \mathcal{N}_i(k) \cup \{i\}$. Similarly, the set of h -hops neighbors at time k is denoted by $\mathcal{N}_i^h(k)$ and it comprises the set of agents j which share a path π_{ij} between i and j and $\text{dist}_{ij} \equiv h$. Furthermore, $\mathcal{N}^h(k) = \{i \in V : e_i(k) = h\}$ denotes the set of all nodes with eccentricity equal to h .

B. Static and dynamic max-consensus protocols

Consider a MAS wherein each agent $i \in V$ with state $s_i(k) \in \mathbb{R}$ has access to the reference signal $v_i(k) \in \mathbb{R}$.

The max-consensus problem consists of the design of a local interaction rule enabling the agents' state to converge to the maximum among the reference signals. If the reference signals are assumed to be constant over time, i.e., $v(k) = v(0)$ for all $k \in \mathbb{N}_+$, the problem is solved by the *max-consensus protocol*

$$s_i(k) = \max_{j \in \mathcal{N}_i^\circ(k)} \{s_j(k-1)\}, \quad s_i(0) = v_i(0), \quad (1)$$

which has been proved to converge in finite-time and with zero error [17]–[20]. On the other hand, if the reference signals are assumed to be time-varying, the problem can be approximately solved by the *dynamic max-consensus protocol* presented by Deplano et al. [21], [22],

$$s_i(k) = \max_{j \in \mathcal{N}_i^\circ(k-1)} \{s_j(k-1) - \alpha, v_i(k)\}. \quad (2)$$

where $\alpha \geq 0$ is a design parameter, which has been proved to converge in finite-time and with bounded error.

III. DISTRIBUTED TRACKING OF GRAPH PARAMETERS

A. Problem statement

We consider a network of n agents that synchronously gather state information from their neighbors and update their state at discrete instants of time, indexed by the positive integer $k \in \mathbb{N}_+$. At each iteration k the pattern of communication among the agents may change but the set of agents does not change (see Remark 1). Thus, the network of agents can be effectively represented by a graph $\mathcal{G}(k) = (V, E(k))$ where V is the time-invariant set of nodes, representing the $n = |V|$ agents, and $E(k) \subseteq \{V \times V\}$ is the time-varying set of edges, representing the interactions among agents.

Assumption 1 *There exists a minimum dwell time $\Upsilon \in \mathbb{N}_+$ between two consecutive changes of the graph $\mathcal{G}(k)$.*

The problem addressed in this paper is the on of dynamically tracking the diameter $d(k)$, the radius $r(k)$ of the network, as well as the eccentricities $e_i(k)$ of the agents, which are time-varying parameters. We propose a local interaction rule that we call the Eccentricity-Diameter-Radius (EDR) Protocol, to distributedly solve this problem within the framework of anonymous networks, i.e., networks where the identity of the agents must be kept hidden.

Remark 1 *The protocol proposed in this paper can also be employed if the number of nodes is time-varying. However, in this case the analysis become more subtle and several issues arise, among them is the design of a proper join/leave logic for the agents that avoids the degradation of the performance. We postpone such analysis to further publication.*

B. Proposed estimation protocol

The proposed estimation methodology is detailed in the EDR Protocol, which envisages three operations/steps described next. In the remainder of this section, we use the notation $a \rightarrow b$ to denote that a tracks the value b .

1) *Initialization (lines 1 – 6)*: The i -th agent selects $L \in \mathbb{N}$ random numbers $u_{i\ell}(0) \in [0, 1]$ with $\ell = 1, \dots, L$ with uniform distribution and initialize its state variables¹ $x_{i\ell}, y_{i\ell}, z_{i\ell} \in \mathbb{R}$ according to

$$x_{i\ell}(0) = u_{i\ell}(0), \quad y_{i\ell}(0) = u_{i\ell}(0), \quad z_{i\ell}(0) = 0$$

2) *Execution - Distances tracking (lines 8–16)*: This step makes use of an ingenious combination of static and dynamic max-consensus protocols in eq. (1)-(2). The following explanation has to be intended for each $\ell = 1, \dots, L$.

Each variable $x_{i\ell} \in \mathbb{R}$ runs the max-consensus protocol in eq. (1) over the set $[u_{1\ell}, \dots, u_{n\ell}]$ and thus it tracks the maximum of the set with zero error,

$$x_{i\ell} \rightarrow \max_{i \in V} u_{i\ell} = u_{j_\ell^*},$$

where

$$j_\ell^* = \operatorname{argmax}_{j \in V} u_{j\ell}. \quad (3)$$

¹Note that each variable $x_{i\ell}, y_{i\ell}, z_{i\ell}$, with $i \in V$ and $\ell = 1, \dots, L$, plays the same role of the state variable s_i of Section II-B.

Assuming no quantization of the real numbers, the maximum number $u_{j_\ell^*}$ is ensured to be unique with probability one by the continuity of the distribution. Thus, the update law in line 12 of EDR Protocol ensures that the signals $u_{i\ell}$ will eventually be set to $-\infty$, unless $i \equiv j^*$.

Each variable $y_{i\ell} \in \mathbb{R}$ runs the dynamic max-consensus protocol in eq. (2) over the set of numbers $[u_{1\ell}, \dots, u_{n\ell}]$ and thus it tracks the maximum $u_{j_\ell^*}$ up to an error, which is characterized in Lemma 2 in Section IV,

$$y_{i\ell} \rightarrow u_{j_\ell^*} - \alpha \cdot \text{dist}_{ij_\ell^*}.$$

The agent i can now infer its distance $\text{dist}_{ij_\ell^*}$ to the node j_ℓ^* attaining the maximum value by

$$\varepsilon_{i\ell} = \frac{|x_{i\ell} - y_{i\ell}|}{\alpha} \rightarrow \text{dist}_{ij_\ell^*}.$$

Each variable $z_{i\ell} \in \mathbb{R}$ runs the dynamic max-consensus protocol in eq. (2) over the set of reference signals $[\varepsilon_{1\ell}, \dots, \varepsilon_{n\ell}]$, whose maximum tracks the maximum distance from agent j_ℓ^* to all other agents in the network, in fact,

$$\max_{i \in V} \varepsilon_{i\ell} \rightarrow \max_{i \in V} \text{dist}_{ij_\ell^*} = \text{dist}_{i_\ell^* j_\ell^*} \quad (4)$$

with

$$i_\ell^* = \underset{i \in V}{\text{argmax}} \text{dist}_{ij_\ell^*}. \quad (5)$$

Therefore, variable $z_{i\ell}$ tracks the value in eq. (4) up to an error, which is characterized in Lemma 2 in Section IV,

$$z_{i\ell} \rightarrow \text{dist}_{i_\ell^* j_\ell^*} - \alpha \cdot \text{dist}_{i_\ell^* i}$$

In other words, the variable $z_{i\ell}$ tracks the maximum distance from any node in the network and the node attaining the maximum value, up to an error proportional to its distance to the node attaining such maximum distance.

3) *Execution - Parameters inference (lines 17–20)*: Agent i considers as an estimation of its eccentricity the maximum distance between itself and any agent attaining a maximum,

$$\hat{e}_i = \max_{\ell=1, \dots, L} \varepsilon_{i\ell}(k) \rightarrow \max_{\ell=1, \dots, L} \text{dist}_{ij_\ell^*},$$

Then, by assuming the parameter $\alpha \in \mathbb{R}_+$ to be small enough, it considers as an estimation of the network diameter the maximum distance of any agent in the network to all agents attaining a maximum,

$$\hat{d}_i = \max_{\ell=1, \dots, L} \lceil z_{i\ell} \rceil \rightarrow \max_{\ell=1, \dots, L} \text{dist}_{i_\ell^* j_\ell^*}.$$

Finally, it considers as an estimation of the network radius the minimum distance of any agent in the network to all agents attaining a maximum,

$$\hat{r}_i = \min_{\ell=1, \dots, L} \lceil z_{i\ell} \rceil \rightarrow \min_{\ell=1, \dots, L} \text{dist}_{i_\ell^* j_\ell^*}.$$

IV. CONVERGENCE ANALYSIS OF THE EDR PROTOCOL

Before providing the characterization of the expected estimation error provided by the EDR Protocol, we need two preliminary lemmas. The first lemma concerns the specific steady-state reached by a network running the dynamic max-consensus protocol in eq. (2) with constant reference signals.

EDR Protocol : Distributed tracking of Eccentricities, Diameter, and Radius (EDR) in networks with time-varying topology

(Input): Tuning parameters $\alpha \in \mathbb{R}_+$ and $L \in \mathbb{N}$

(Output): $\hat{e}_i(k), \hat{d}_i(k), \hat{r}_i(k) \in \mathbb{R}$ for $i \in V$.

1 **(Initialization)**: for $\ell = 1, \dots, L$ each node i does

2 Select numbers with uniform distribution

3 $u_{i\ell}(0) \sim \mathcal{U}(0, 1)$

4 Initialize state variables according to

5 $x_{i\ell}(0) = u_{i\ell}(0), y_{i\ell}(0) = u_{i\ell}(0), z_{i\ell}(0) = 0$

6 Send $x_{i\ell}(0), y_{i\ell}(0), z_{i\ell}(0)$ to its neighbors

7 **(Execution)**: for $k = 1, 2, 3, \dots$ each node i does

8 for $\ell = 1, \dots, L$ it does

9 Gather $x_{j\ell}(k-1), y_{j\ell}(k-1), z_{j\ell}(k-1)$,
from its neighbors $j \in \mathcal{N}_i(k-1)$

10 Update its state variables according to

11 $x_{i\ell}(k) = \max_{j \in \mathcal{N}_i^\circ(k-1)} \{x_{j\ell}(k-1)\}$

12 $u_{i\ell}(k) = \begin{cases} -\infty & \text{if } x_{i\ell}(k) > x_{i\ell}(k-1) \\ u_{i\ell}(k-1) & \text{otherwise} \end{cases}$

13 $y_{i\ell}(k) = \max_{j \in \mathcal{N}_i^\circ(k-1)} \{y_{j\ell}(k-1) - \alpha, u_{i\ell}(k)\}$

14 $\varepsilon_{i\ell}(k) = \frac{|x_{i\ell}(k) - y_{i\ell}(k)|}{\alpha}$

15 $z_{i\ell}(k) = \max_{j \in \mathcal{N}_i^\circ(k-1)} \{z_{j\ell}(k-1) - \alpha, \varepsilon_{i\ell}(k)\}$

16 Send $x_{i\ell}(k), y_{i\ell}(k), z_{i\ell}(k)$ to its neighbors
 $j \in \mathcal{N}_j(k)$

17 Estimate the graph parameters according to

18 $\hat{e}_i(k) = \max_{\ell=1, \dots, L} \varepsilon_{i\ell}(k)$

19 $\hat{d}_i(k) = \max_{\ell=1, \dots, L} \lceil z_{i\ell}(k) \rceil$

20 $\hat{r}_i(k) = \min_{\ell=1, \dots, L} \lceil z_{i\ell}(k) \rceil$

Lemma 1 Consider a network of n agents, each of which has access to a constant $v_i \in \mathbb{R}$ and updates its state $s_i(k)$ according to the dynamic max-consensus protocol in eq. (2) and let $k_0 \in \mathbb{N}$ be a generic instant of time.

If graph \mathcal{G} is time-invariant, then there exists a convergence time $T_c \in \mathbb{N}$ such that

$$T_c \leq d + \max \left\{ d, \max_{j \in V} \frac{s_j(k_0)}{\alpha} - \max_{j \in V} \frac{v_j}{\alpha} \right\} \quad (6)$$

so that each agent reaches an equilibrium state for $k \geq k_0 + T_c$ such that

$$s_i(k) \geq v_{j^*}(k) - \alpha \cdot \text{dist}_{ij^*}, \quad (7)$$

where

$$j^* = \underset{j \in V}{\text{argmax}} v_j. \quad (8)$$

Moreover, if $v_{j^*} - v_i > \alpha \cdot \text{dist}_{ij^*}$ holds for all $i \neq j^*$, then the inequality holds strictly.

Proof: Let $k_0 \in \mathbb{N}$ be a generic instant of time. By [22],[21, Theorem 1], there exists a time

$$T^* \leq \max \left\{ d, \max_{j \in V} \frac{s_j(k_0)}{\alpha} - \max_{j \in V} \frac{v_j}{\alpha} \right\} \quad (9)$$

such that agent j^* as in eq. (8) satisfies

$$s_{j^*}(k) = v_{j^*}, \quad k \geq k_0 + T_c. \quad (10)$$

and $s_{j^*}(k)$ is the maximum among all agents at time k ,

$$s_{j^*}(k) \geq s_j(k), \quad \forall j \in V. \quad (11)$$

At time $k^* = k_0 + T^*$ we define the set of one-hop neighbors of agent j^* , formally

$$\mathcal{V}_1 = \{i \in V : (i, j^*) \in E\}.$$

The dynamic max-consensus protocol in eq. (2) at $k^* + 1$ for the agents belonging to this set reduces to

$$s_i(k^* + 1) = \max\{v_{j^*} - \alpha, v_i\}, \quad \forall i \in \mathcal{V}_1$$

since they have state as in eq. (11) and agent j^* as a neighbor with state as in eq. (10). By induction, define the sets

$$\mathcal{V}_\ell = \left\{ i \in V : (i, j) \in E, j \in \bigcup_{s=0}^{\ell-1} \mathcal{V}_s \right\}, \quad \ell = 1, 2, \dots$$

It can be noticed that the parameter ℓ denotes the distance dist_{ij^*} of agent $i \in \mathcal{V}_\ell \setminus \mathcal{V}_{\ell-1}$ to agent j^* . Since the longest shortest path between two nodes in a connected graph is at most equal to its diameter d , then it holds that $\mathcal{V}_d \equiv V$ and thus for $k \geq k^* + d = k_0 + T^* + d$ it holds

$$s_i(k) = \max\{v_{j^*} - \alpha \cdot \text{dist}_{ij^*}, v_i\}, \quad \forall i \in V.$$

This proves that: the convergence time $T_c = T^* + d$ is upper bounded as in eq. (6) due to eq. (9); the steady-state reached by the network is lower bounded as in eq. (7) since the maximum of two values is greater or equal than one of them. Moreover, if $v_i < v_{j^*} - \alpha \cdot \text{dist}_{ij^*}$ for all $i \neq j^*$ then it exactly holds $s_i(k) = v_{j^*}(k) - \alpha \cdot \text{dist}_{ij^*}(k)$. ■

The second lemma concerns the specific steady-state reached by a network running the EDR Protocol. We recall that for each ℓ , the agents run two instances of the dynamic max-consensus protocol, respectively at variables $y_{i\ell}$ and $z_{i\ell}$ for $i \in \mathcal{V}$, and we denote by $T_c(y, \ell)$ and $T_c(z, \ell)$ their convergence time. More precisely, given a generic initial instant of time $k_0 \in \mathbb{N}$, the convergence times are bounded as in eq. (6) in Lemma 1,

$$T_c(y, \ell) \leq d + \max \left\{ d, \max_{j \in V} \frac{y_{j\ell}(k_0)}{\alpha} - \max_{j \in V} \frac{u_{j\ell}(k_0)}{\alpha} \right\} \quad (12)$$

$$T_c(z, \ell) \leq d + \max \left\{ d, \max_{j \in V} \frac{z_{j\ell}(k_0)}{\alpha} - \max_{j \in V} \frac{\varepsilon_{j\ell}(k_0)}{\alpha} \right\} \quad (13)$$

Lemma 2 Consider a network that implements the EDR Protocol under Assumption 1 and let $k_0 \in \mathbb{N}$ be a generic instant of time. If the tuning parameter $\alpha \in \mathbb{R}_+$ satisfies

$$\alpha < 1/d, \quad (14)$$

and if the dwell time Υ is greater than the convergence time T_c of all dynamic max-consensus protocols, i.e.,

$$\Upsilon \geq \max_{\ell=1, \dots, L} \{T_c(y, \ell), T_c(z, \ell)\} = T'_c, \quad (15)$$

with $T_c(y, \ell)$ and $T_c(z, \ell)$ given in eq. (12)-(13), respectively, then for $k \in [k_0 + T_c, k_0 + \Upsilon]$ it holds

$$\hat{e}_i(k) = \max_{\ell=1, \dots, L} \text{dist}_{ij_\ell^*}(k_0), \quad (16)$$

$$\hat{d}_i(k) = \max_{\ell=1, \dots, L} \text{dist}_{i_\ell^* j_\ell^*}(k_0), \quad (17)$$

$$\hat{r}_i(k) = \min_{\ell=1, \dots, L} \text{dist}_{i_\ell^* j_\ell^*}(k_0). \quad (18)$$

Proof: For each $\ell \in [0, L]$, each node $i \in V$ runs at variable $x_{i\ell}$ the popular (static) max-consensus protocol in eq. (1), and thus is a standard result (cfr. [17]) that

$$x_{i\ell}(k) = u_{j_\ell^* \ell}(k), \quad k \in [k_0 + d, k_0 + \Upsilon],$$

with j_ℓ^* as in eq. (3). For each $\ell \in [0, L]$, each node $i \in V$ runs at variable $y_{i\ell}$ the dynamic max-consensus protocol in eq. (2) with reference signals $u_{i\ell}$. By the update rule of $u_{i\ell}$ is straightforward to conclude that for $k \geq k_0 + d$ all numbers are eventually set to $-\infty$, unless the unique maximum², i.e.,

$$u_{j_\ell^* \ell}(k) = u_{j_\ell^* \ell}(k_0), \quad u_{i\ell}(k) = -\infty, \quad \forall i \neq j_\ell^*.$$

Since the distance $u_{j_\ell^* \ell}(k) - u_{i\ell}(k)$ is infinite, then Lemma 1 holds strictly and for $k \in [k_0 + T_c(y, \ell), k_0 + \Upsilon]$ it follows

$$y_{i\ell}(k) = u_{j_\ell^* \ell}(k) - \alpha \cdot \text{dist}_{ij_\ell^*}(k) = x_{i\ell}(k) - \alpha \cdot \text{dist}_{ij_\ell^*}(k)$$

Therefore, the estimation of the eccentricity converges to

$$\begin{aligned} \hat{e}_i(k) &= \max_{\ell=1, \dots, L} \varepsilon_{i\ell}(k) = \max_{\ell=1, \dots, L} \frac{|y_{i\ell}(k) - x_{i\ell}(k)|}{\alpha} \\ &= \max_{\ell=1, \dots, L} \text{dist}_{ij_\ell^*}(k), \quad \forall k \in [k_0 + T_c(y, \ell), k_0 + \Upsilon], \end{aligned}$$

thus proving the veracity of eq. (16). For each $\ell \in [0, L]$, each node $i \in V$ runs at variable $z_{i\ell}$ the dynamic max-consensus protocol in eq. (2). The variable $z_{i\ell}$ tracks the value

$$\max_{i \in V} \varepsilon_{i\ell}(k) = \max_{i \in V} \text{dist}_{ij_\ell^*}(k) = \text{dist}_{i_\ell^* j_\ell^*}(k)$$

with i_ℓ^* as in eq. (5). Thus, by Lemma 1, for $k \in [k_0 + \max\{T_c(y, \ell), T_c(z, \ell)\}, k_0 + \Upsilon]$ it holds

$$z_{i\ell}(k) \geq \text{dist}_{i_\ell^* j_\ell^*}(k) - \alpha \cdot \text{dist}_{ij_\ell^*}(k).$$

By eq. (14), it holds $[-\alpha \cdot \text{dist}_{ij_\ell^*}(k_0)] = 0$ and $\lceil z_{i\ell}(k) \rceil = \text{dist}_{i_\ell^* j_\ell^*}(k)$. Thus, the diameter estimation converges to

$$\hat{d}_i(k) = \max_{\ell=1, \dots, L} \lceil z_{i\ell}(k) \rceil = \max_{\ell=1, \dots, L} \text{dist}_{i_\ell^* j_\ell^*}(k)$$

and the radius estimation converges to

$$\hat{r}_i(k) = \min_{\ell=1, \dots, L} \lceil z_{i\ell}(k) \rceil = \min_{\ell=1, \dots, L} \text{dist}_{i_\ell^* j_\ell^*}(k),$$

thus proving eq. (17)-(18). The above considerations are true if $\Upsilon \geq \max\{T_c(y, \ell), T_c(z, \ell)\}$, completing the proof. ■

²Assuming no quantization of the real numbers, the maximum is unique with probability one, by the continuity of the distribution.

Remark 2 We point out that the previous Lemma 2 implicitly proves the robustness to re-initialization of the EDR Protocol. In fact, each time a change in the network occurs, the network is able to reach a new steady-state characterized in eqs. 16-18 without the need of being re-initialized.

Next, we derive the steady-state expected estimation error of the eccentricities, diameter, and radius between any two consecutive changes of the network topology. We consider time windows $[k_0, k_0 + \Upsilon]$ such that $\Upsilon \geq T'_c$ where $k_0 \in \mathbb{N}$ is a generic initial time at which the network changes, T'_c is the convergence time of the protocol as in eq. (15) and Υ is the dwell time before a new change may occur. The parameters of interest are constant over these time windows and this allows providing a precise characterization of the expected errors in the next theorem.

Theorem 1 Consider the scenario of Lemma 2. Then, for all $k \in [k_0 + T'_c, k_0 + \Upsilon]$ the expected estimation errors of the EDR protocol are:

$$\mathbb{E}[e_i(k) - \hat{e}_i(k)] = \sum_{\varepsilon=1}^{e_i} \left(1 - \frac{\sum_{h=\varepsilon}^{e_i} |\mathcal{N}_i^h(k)|}{|V|} \right)^L \quad (19)$$

$$\mathbb{E}[d(k) - \hat{d}_i(k)] = \sum_{\delta=r+1}^d \left(1 - \frac{\sum_{h=\delta}^d |\mathcal{N}^h(k)|}{|V|} \right)^L \quad (20)$$

$$\mathbb{E}[\hat{r}_i(k) - r(k)] = \sum_{\rho=r}^{d-1} \left(1 - \frac{\sum_{h=\rho}^{\rho} |\mathcal{N}^h(k)|}{|V|} \right)^L \quad (21)$$

Sketch of the Proof: We provide next a sketch of the proof for the expected error on the eccentricities. First of all, one needs to compute the probability that the estimated eccentricity greater or equal than a certain threshold $\varepsilon \in \{1, \dots, e_i\}$, which is given by

$$\mathbb{P}[\hat{e}_i \geq \varepsilon] = 1 - \left(1 - \frac{|\mathcal{N}_i^{\varepsilon}|}{|V|} \right)^L.$$

The expected error value can be calculated by multiplying each of the possible outcomes by the probability each outcome will occur and then summing all of those values,

$$\mathbb{E}[e_i - \hat{e}_i] = e_i - \mathbb{E}[\hat{e}_i] = e_i - \sum_{\varepsilon=1}^{e_i} \varepsilon \cdot \mathbb{P}[\hat{e}_i = \varepsilon].$$

The probability of the event $\hat{e}_i = \varepsilon$ can be computed by computing the probability of event $\hat{e}_i \geq \varepsilon$ and then

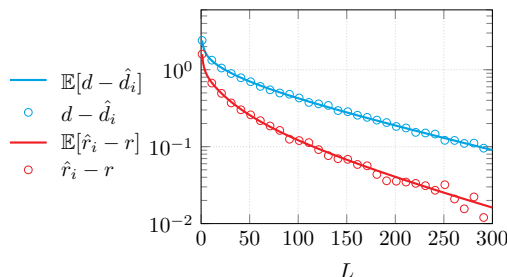


Fig. 1. Expected error and actual error for increasing values of L in a random network of $n = 300$ nodes.

subtracting the probability of event $\hat{e}_i \geq \varepsilon + 1$. This is possible since the events $\hat{e}_i = \varepsilon$ for any $\varepsilon \in \{1, \dots, e_i\}$ are independent, and thus one can write

$$\mathbb{E}[e_i - \hat{e}_i] = e_i - \sum_{\varepsilon=1}^{e_i} \varepsilon \cdot (\mathbb{P}[\hat{e}_i \geq \varepsilon] - \mathbb{P}[\hat{e}_i \geq \varepsilon + 1]),$$

from which eq. (19) follows, as well as the other proofs. ■

The above characterization shows that the quality of the estimates heavily depends not only on the specific topology of the network, but also on the choice of the parameter L . In particular, for $L \rightarrow \infty$ the expected errors go to zero, while the decaying of each term is geometrical in L .

Since the parameter L also determines the number of selected numbers and therefore the number of agents' state variables which have to be exchanged among neighbours, the design of the parameter L trades-off memory burden and communication complexity with the estimation accuracy. If alternative approaches exploiting flooding techniques usually require a number of exchanged messages which is equal to the number of nodes n in the network [12], [13], our approach allows to choose a smaller number $L \leq n$ of exchanged messages, at the expense of a worse estimation accuracy. In large-scale networks this is a useful fact, since if n is too large flooding techniques become infeasible, while our approach can still be employed.

V. SIMULATION RESULTS

In the first simulation, we consider 10^3 random graphs with $n = 300$ nodes, diameter $d = 9$ and radius $r = 5$. For each graph, and for all choices of the parameter $L = 1, \dots, n$ we compute the expected errors on the diameter, and on the radius according to eq. (20)-(21). Then, we also compute the average error made by executing the EDR Protocol for 10 times. The results of the simulation, which are given in Fig. 1, corroborates the theoretical results in the sense that the average actual error made by the proposed protocol is exactly the expected error given in Theorem 1.

As discussed in Section IV, the expected errors decay exponentially with L . In particular, while each term in the summation of eq. (20)-(21) decays exponentially, the convergence rate is determined by the largest addend in the summation, which by definition has always magnitude strictly less than one. It is also interesting to notice that it

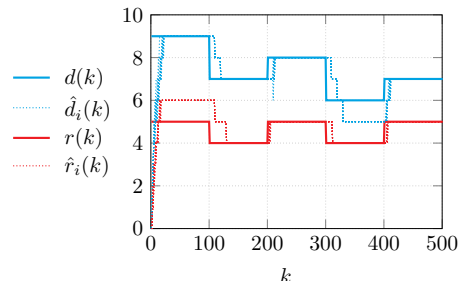


Fig. 2. Dynamic tracking of diameter, and radius in a random network of $|V| = 300$ nodes.

is possible to identify a threshold value, which in our case is $L^* \approx 50$, after which the decay is almost exponential. A pragmatic choice of the parameter L could be around such pivot point $L^* \approx 50$ in the decaying curve, which ensures that both expected errors are lesser or equal than 1.

As a second simulation, we consider a random graph with the same number of $n = |V| = 300$ nodes but with time-varying topology, thus allowing fluctuations in the values of the radius and diameter. Changes in the network occur every $\Upsilon = 100$ iterations and the agents execute the EDR Protocol without the need of being re-initialized.

The estimations $\hat{d}_i(k)$, $\hat{r}_i(k)$ with the choice of $\alpha = 0.1$ and $L = L^* = 50$, the value suggested by the above analysis, are plotted in Fig. 2. Every time a change in the topology occurs, both the diameter and radius estimation show a transient behavior leading all agents to a new steady-state, thus validating the use of the proposed protocol in networks with time-varying topology, theoretically characterized in terms of steady-state in Lemma 2 and in terms of expected error in Theorem 1. It is worth mentioning that due to the rounding up operation implemented by the EDR Protocol at lines 19-20, one could mistake the transient behavior of the estimation for a systematic delay, which is not the case. Moreover, it can be noticed that at the end of each time window of length $\Upsilon = 100$ the estimation error is not greater than 1, in accordance to the expected error analysis.

VI. CONCLUSION

This paper solves the problem of distributed tracking of important graph parameters in networks with time-varying topology, namely the eccentricities of the nodes, the diameter, and radius. The proposed approach consists of a distributed estimation protocol that exploits static and dynamic max-consensus protocols as subroutines. Its main advantage is the possibility to be implemented on networks with time-varying topology without the need of being re-initialized after a change of the graph parameters. Moreover, the proposed estimation protocol is developed within the framework of anonymous networks, wherein agents' identity must not be disclosed. As future work we aim to characterize and extend the current approach to open multi-agent networks, i.e., networks where agents can join or leave the network.

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