Annals of Physics 460 (2024) 169576

Contents lists available at ScienceDirect

Annals of Physics

journal homepage: www.elsevier.com/locate/aop

Monatomic gas as a singular limit of relativistic theory of 15 moments with non-linear contribution of microscopic energy of molecular internal mode

Takashi Arima [a,b,c], Maria Cristina Carrisi [d,*]

[a] Department of Engineering for Innovation, National Institute of Technology, Tomakomai College, Tomakomai, Japan
[b] Department of Mathematics, University of Bologna, Bologna, Italy
[c] Alma Mater Research Center on Applied Mathematics (AM^2), University of Bologna, Bologna, Italy
[d] Department of Mathematics and Informatics, University of Cagliari, Cagliari, Italy

A R T I C L E I N F O

Keywords:
Relativistic extended thermodynamics
Rarefied polyatomic gas
Causal theories
Relativistic fluids
Monatomic limit

A B S T R A C T

Recently a new relativistic model of polyatomic gases has been proposed, by Arima-Carrisi-Pennisi-Ruggeri (2022), in the context of Rational Extended Thermodynamics. It is based on a hierarchy of 15 moments of the Boltzmann–Chernikov equation that appropriately takes into account the non-linear contribution of the microscopic total energy of the molecule (the sum of the rest energy and of the energy of the molecular internal modes). In this paper, in the singular limit, under initial conditions compatible with monatomic gases, we prove that the 15-moments model for polyatomic gases leads, to the well-known 14-moments model of monatomic gases.

1. Introduction

The dynamics of relativistic gases are integral to understanding a wide range of physical phenomena, from astrophysical events to high-energy particle collisions. Theoretical frameworks for describing these dynamics have been proposed using causal thermodynamic theories or kinetic theories. A pivotal development in this field is Rational Extended Thermodynamics (RET) [1,2], which has undergone significant advancements in recent years. RET serves as a bridge between the kinetic description of gases and the macroscopic fluid-dynamical description. Importantly, this theory provides a hyperbolic system of differential field equations, and, in the coarse-graining limit, converges to the well-established Eckart theory (or the Navier–Stokes–Fourier theory in the classical non-relativistic framework).

The theoretical foundation of the RET model for relativistic gases was formulated by Liu, Müller and Ruggeri [3]. In the paper, the theory of field equations for 14 fields ($\mathrm{RET}_{14}^{R,M}$, with the superscript R signifying a theory for relativistic gases and M denoting a theory for monatomic gases) was derived in closed form through a phenomenological approach using the following universal principles: the objectivity principle, the entropy principle, and the principle of causality and stability. The 14 fields are the number density, four-velocity, energy density, dynamic pressure (nonequilibrium part of pressure), shear stress and heat flux. This model is supported by the moment equations based on the Boltzmann–Chernikov relativistic equation [4–6] and the closure is determined by using the maximum entropy principle (MEP) [7] (see, for example, [1]). The MEP was introduced first for moments by Kogan [8] and later in [9,10] in the classical framework. In paper [11], it is proved that, in the limit to classical gases, $\mathrm{RET}_{14}^{R,M}$ converges to the classical theory with 14 moments for rarefied monatomic gases [12] ($\mathrm{RET}_{14}^{C,M}$ where the superscript C represents a theory for classical gases).

* Corresponding author.
E-mail addresses: arima@tomakomai-ct.ac.jp, takashi.arima@unibo.it (T. Arima), mariacri.carrisi@unica.it (M.C. Carrisi).

https://doi.org/10.1016/j.aop.2023.169576
Received 7 July 2023; Accepted 13 December 2023
Available online 16 December 2023
0003-4916/

For many years, RET has been applied to monatomic gases, and several attempts to extend its validity to polyatomic gases have been made. A new sophisticated model in the classical framework has been constructed for the description of polyatomic gases with 14 variables (RET^{C}_{14}) based on the binary hierarchy structures of the field equations [13] that differ from the single hierarchy structure for monatomic gases [1]. From the microscopic viewpoint, this model takes into account the internal structure such as molecular rotational and vibrational modes which manifest macroscopically as bulk viscosity and dynamic pressure that underlies the origin of the bulk viscosity. The RET^{C}_{14} model has, as a singular limit to monatomic gases [14], the RET theory with 13 variables without the dynamic pressure ($\mathrm{RET}^{C,M}_{13}$) which is equivalent to the well-known Grad's 13 moments system [15]. This new model leads to satisfactory results in many fields like the study of wave propagation such as sound waves [16], shock waves [17], heat conduction [18], nozzle flow [19] and oscillating gas bubbles [20], or the study of biological problems [21], and can be hopefully applied in many other fields that involve highly non-equilibrium processes like astrophysics or nuclear physics. It is also worth mentioning that RET^{C}_{14} includes the simplest nonequilibrium hyperbolic system, namely, the 6-moments model (RET^{C}_{6}) [22] as a *principal subsystem* in the sense of the definition of paper [23]. Although the RET^{C}_{6} model ignores the shear stress and heat flux, it is the simplest model able to describe the nonequilibrium process due to the dynamic pressure [24].

In the relativistic regime, in 2017, Pennisi and Ruggeri [25] first constructed a relativistic theory of polyatomic gases in the context of RET (see also [26,27]). They achieved this by incorporating the microscopic internal structure into the Boltzmann–Chernikov equation through the distribution function, and as a result, accounted for the dynamic pressure. The physical background for considering relativistic polyatomic gases lies in scenarios with temperatures high enough for relativistic effects to be significant, yet not disruptive to the formation and stability of polyatomic molecules. Therefore, the RET model is crucial for understanding relativistic gas dynamics, such as those observed in heavy-ion collisions. Additionally, when dealing with large length-scale gas dynamics such as in cosmological problems [28,29], the model accommodates small length-scale structure manifesting as non-zero bulk viscosity.

In spite of the successful development of the relativistic RET model of polyatomic gases, an issue remains that the definition of moments incorporates quantities that lack clear physical interpretation. A recent proposal by Arima, Carrisi, Pennisi and Ruggeri [30] has addressed this issue based on a model of moments that incorporates physically appropriate moments with respect to microscopic energy [31,32]. As its particular case, the model predicts the 15 fields equations (RET^{R}_{15}) with an additional scalar nonequilibrium variable which plays a role in characterizing the polyatomic gases although it is absent in the previous study [25]. In [30], by using the technique of MEP, it was proved that the closure gives a symmetric hyperbolic system; the authors have been able to express all the tensors appearing in the system in terms of 15 independent variables, near the equilibrium state. Moreover, the authors determined, in [26], the expression of the production tensor by adopting a variant of the BGK model appropriate for polyatomic gases [31] and, by using Maxwellian iteration [33], they obtained explicit values of the phenomenological coefficients (heat conductivity, shear viscosity and bulk viscosity) [27,30]. It is noteworthy to mention the principal subsystem of RET^{R}_{15}. There exist two principal subsystems: the 14-moment model (RET^{R}_{14}), which differs from the system proposed in [25], and the 6-moment model (RET^{R}_{6}). It is also confirmed in [30] that, in the classical limit, RET^{R}_{15} converges to the corresponding model with 15 moments (RET^{C}_{15}) [34] which includes RET^{C}_{14} as a principal subsystem. The classical limit of RET^{R}_{14} also converges to RET^{C}_{14}, and the one of RET^{R}_{6} converges to RET^{C}_{6}.

Similar to the classical case, in the relativistic case, there are structural differences between the theories of monatomic and polyatomic gases. The points that until now have not been completely clarified in the new relativistic RET theory are the limiting process from polyatomic to monatomic gases and which polyatomic models correspond to which monatomic models in their limits. The aim of this paper is to elucidate these points. Through this analysis, as shown in Fig. 1, it is possible to clarify the interrelationships among the models derived from RET^{R}_{15} categorizing gases into four types depending on whether they are in the relativistic or classical regime, and whether the gases under consideration are polyatomic or monatomic.

Specifically, this paper seeks to demonstrate how the additional scalar field of RET^{R}_{15} vanishes for compatible initial data in the limit of monatomic gases and to prove the convergence of RET^{R}_{15} to $\mathrm{RET}^{R,M}_{14}$, as already shown in Fig. 1. Furthermore, it aims to prove that $\mathrm{RET}^{R,M}_{14}$ and the monatomic Euler theory emerge as the monatomic limits of RET^{R}_{14} and RET^{R}_{6}. The limits of RET^{R}_{15} and RET^{R}_{6} are singular in the sense that the system for rarefied polyatomic gases with 15 and 6 independent variables, respectively, converge to the system with only 14 and 5 independent variables for monatomic gases.

The organization of the present paper is as follows. In Section 2, we introduce the moment equations for monatomic and polyatomic gases, which are discussed above, particularly focusing on the difference of the microscopic energy included as internal variable in the definition of the moments. In Section 3, we provide a brief but necessary introduction to the relativistic model for polyatomic gases. Then, in Section 4, we calculate the monatomic limit of the polyatomic model, particularly investigating how the system of field equations transforms under a condition compatible with monatomic gases. We remark that, in the present paper, we focus on considering polytropic polyatomic gases. In Section 5, we also examine the monatomic limit of two principal subsystems of RET^{R}_{15}.

2. Relativistic moment equations of polyatomic gases

As discussed in the Introduction, the definitions of moments for monatomic and polyatomic gases differ. In this section, we will explicitly present these definitions.

The first tentative of the RET model for relativistic monatomic gases, proposed by Liu, Müller and Ruggeri [3], i.e., $\mathrm{RET}^{R,M}_{14}$, consists in the following set of balance equations

$$\partial_\alpha A^\alpha = 0, \qquad \partial_\alpha A^{\alpha\beta} = 0, \qquad \partial_\alpha A^{\alpha\beta\gamma} = I^{\beta\gamma}, \qquad \Big(\alpha,\beta,\gamma = 0,1,2,3\Big). \tag{1}$$

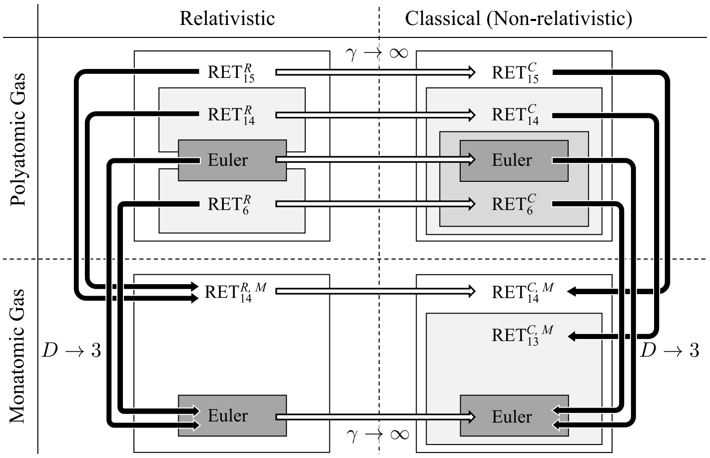

Fig. 1. Relation among hyperbolic theories in relativistic or classical regimes for monatomic or polyatomic gases. The subscript in RET indicates the number of variables, while the superscript indicates whether the theory is relativistic (R) or classical (C), and/or monatomic gases (M). The hollow arrow represents the correspondence of theories in the non-relativistic limit ($\gamma \to \infty$ where γ is the relativistic coldness defined in (11)) proved in [30] for polyatomic gases and in [11] for monatomic gases. The filled arrow represents the monatomic limit when $D \to 3$ where D is the degrees of freedom of a molecule introduced in (25). The limit in the relativistic regime is proved in the present paper and the one in the classical regime is proved in [30] for RET_{15}^{C} and in [14] for RET_{14}^{C} and RET_{6}^{C}. The nested structure indicates principal subsystems. For example, the (relativistic) Euler theory for polyatomic gases [35] is included as a principal subsystem of both RET_{6}^{R} and RET_{14}^{R}.

The tensors $A^{\alpha} \equiv V^{\alpha}$ and $A^{\alpha\beta} \equiv T^{\alpha\beta}$ represent, respectively, the particle number vector and the energy–momentum tensor, $\partial_\alpha = \partial/\partial x^\alpha$ where x^α are the space–time coordinates. $A^{\alpha\beta\gamma}$ is a symmetric third-order tensor of fluxes whose explicit expression is shown below. The right-hand sides are the production terms which are zero in the first two equations because they represent the conservation laws of the particle number and of the energy–momentum.

Such equations, as usual in RET, have been justified by the moment equations associated with the Boltzmann–Chernikov equation: $p^\alpha \partial_\alpha f = Q$. The distribution function f depends on (x^α, p^β), where p^α is the four-momentum and Q is the collisional term. The moments

$$A^{\alpha\alpha_1\cdots\alpha_n} = \frac{c}{m^{n-1}} \int_{\mathbb{R}^3} f\, p^\alpha p^{\alpha_1} \cdots p^{\alpha_n}\, d\boldsymbol{P}, \qquad I^{\alpha_1\cdots\alpha_n} = \frac{c}{m^{n-1}} \int_{\mathbb{R}^3} Q\, p^{\alpha_1} \cdots p^{\alpha_n}\, d\boldsymbol{P}, \tag{2}$$

satisfy the infinite hierarchy of balance laws

$$\partial_\alpha A^{\alpha\alpha_1\cdots\alpha_n} = I^{\alpha_1\cdots\alpha_n} \quad \text{with} \quad n = 0, 1, \dots, \tag{3}$$

where c denotes the light speed, m is the particle mass in the rest frame and $d\boldsymbol{P} = \frac{dp^1\, dp^2\, dp^3}{p^0}$.

System (1) is a particular case of (3), truncated at $n = 2$. Because of the definition (2) of the moments, the following trace conditions hold

$$A^{\alpha\beta}{}_{\beta} = A^{\alpha\beta\gamma} g_{\beta\gamma} = c^2 V^\alpha, \qquad I^{\beta}{}_{\beta} = I^{\beta\gamma} g_{\beta\gamma} = 0, \tag{4}$$

where $g^{\alpha\beta} = \mathrm{diag}(1, -1, -1, -1)$ is the metric tensor. Therefore, the scalar equation $\partial_\alpha A^{\alpha\beta}{}_{\beta} = 0$ in (1) is not independent from the other equations and the system (1) has 14 independent equations for 14 variables. The closed system of $\text{RET}_{14}^{R,M}$ was obtained in [3].

The tentative model of the relativistic polyatomic gases by Pennisi and Ruggeri [25] considers the following moment equations for 14 fields:

$$\partial_\alpha A^\alpha = 0, \qquad \partial_\alpha A^{\alpha\beta} = 0, \qquad \partial_\alpha A^{\alpha\langle\beta\gamma\rangle} = I^{\langle\beta\gamma\rangle}, \tag{5}$$

where $\langle\cdots\rangle$ denotes the traceless part of a tensor. In this theory, the moments are defined in a different way by taking into account the contribution of the energy due to the internal structure of a molecule. By analogy with the non-relativistic kinetic model of polyatomic gases, that is, the Borgnakke–Larsen model [36,37], the distribution function of polyatomic gases $f(x^\alpha, p^\beta, \mathcal{I})$ also depends on the microscopic internal energy $\mathcal{I}$ as an extra variable, and the moments of the distribution function are defined as follows:

$$\begin{aligned} A^{\alpha\alpha_1\cdots\alpha_n} &= \frac{1}{m^n c} \int_{\mathbb{R}^3} \int_0^{+\infty} f\, p^\alpha p^{\alpha_1} \cdots p^{\alpha_n} \left(mc^2 + n\mathcal{I}\right) \phi(\mathcal{I})\, d\mathcal{I}\, d\boldsymbol{P}, \\ I^{\alpha_1\cdots\alpha_n} &= \frac{1}{m^n c} \int_{\mathbb{R}^3} \int_0^{+\infty} Q\, p^{\alpha_1} \cdots p^{\alpha_n} \left(mc^2 + n\mathcal{I}\right) \phi(\mathcal{I})\, d\mathcal{I}\, d\boldsymbol{P}, \end{aligned} \tag{6}$$

where $\phi(\mathcal{I})$ is the state density of the internal mode, and the collision integral Q takes into account the influence of the internal degrees of freedom through the collisional cross-section. With this definition, the moments do not satisfy the trace conditions (4) which characterize the model for monatomic gases. Although the definitions of moments of RET theories for monatomic gases and for polyatomic gases are different due to the existence of the microscopic energy $\mathcal{I}$, it has been shown in [38] that the system (5) with (6) contains the system of RET$^{R,M}_{14}$ in the monatomic limit.

The refined model of relativistic polyatomic gases, proposed by Arima, Carrisi, Pennisi and Ruggeri, is RET$^{R}_{15}$ [30]. In this model, instead of the moments (6) with an unphysical element in their definition, i.e., the quantity $n\mathcal{I}$ of which n was necessary to recover the RET$^{C}_{14}$ model in the classical limit, a more physical definition of moments is considered

$$
\begin{aligned}
A^{\alpha\alpha_1\cdots\alpha_n} &= \left(\frac{1}{mc}\right)^{2n-1}\int_{\mathbb{R}^3}\int_0^{+\infty} f\, p^\alpha p^{\alpha_1}\cdots p^{\alpha_n}\left(mc^2+\mathcal{I}\right)^n \phi(\mathcal{I})\, d\mathcal{I}\, d\boldsymbol{P}\,,\\
I^{\alpha_1\cdots\alpha_n} &= \left(\frac{1}{mc}\right)^{2n-1}\int_{\mathbb{R}^3}\int_0^{+\infty} Q\, p^{\alpha_1}\cdots p^{\alpha_n}\left(mc^2+\mathcal{I}\right)^n \phi(\mathcal{I})\, d\mathcal{I}\, d\boldsymbol{P},
\end{aligned}
\tag{7}
$$

which preserves the microscopic full energy (the sum of the rest frame energy and the energy of internal modes). This definition of moments includes the non-linear contribution of the energy of internal mode by increasing tensorial order n whereas the expressions (6) only take into account up to first-order terms with respect to $\mathcal{I}$ in the binomial expansion of (7). The moments (7) satisfy the system (3). The system with $n = 0, 1, 2$, i.e., the system (1) with the third-order tensor including the trace part, which differs from the case represented by (5), incorporates 15 independent fields because the trace conditions are not satisfied also in this case. The result of the closure for this RET$^{R}_{15}$ model is shown in the subsequent section.

3. Brief summary of relativistic RET of polyatomic gases

Before discussing the monatomic limit of RET theory, a brief survey on the RET$^{R}_{15}$ model based on (1) with (7) is exhibited.

As usual in RET, the system of balance equations that constitutes the model has more variables than equations, so it is necessary to close it by choosing which fields are the independent variables and expressing the remaining tensors in terms of them. For RET$^{R}_{15}$, in paper [30], the 15 independent quantities have been identified with the traditional 14 physical variables:

- $\rho = nm$ (mass density, with the particle number density n),
- T (absolute temperature),
- U^α (4-velocity in the Eckart frame),
- Π (dynamic pressure),
- $q^\alpha = -h^\alpha_\mu U_\nu T^{\mu\nu}$ (heat flux), where $h^{\alpha\beta} = \frac{U^\alpha U^\beta}{c^2} - g^{\alpha\beta}$ is the projector tensor,
- $t^{\langle\alpha\beta\rangle_3} = T^{\mu\nu}\left(h^\alpha_\mu h^\beta_\nu - \frac{1}{3}h^{\alpha\beta}h_{\mu\nu}\right)$ (deviatoric shear viscous stress tensor),

and an additional scalar field:

- $\Delta = \frac{4}{c^2}U_\alpha U_\beta U_\gamma\left(A^{\alpha\beta\gamma} - A^{\alpha\beta\gamma}|_E\right)$, where the index E represents the value of the tensor calculated at thermodynamic equilibrium.

The above quantities are subjected to the following constraints:

$$
U^\alpha U_\alpha = c^2, \qquad q^\alpha U_\alpha = 0, \qquad t^{\langle\alpha\beta\rangle_3}U_\alpha = 0, \qquad t^{\langle\alpha}{}_{\alpha\rangle_3} = 0.
$$

With these choices and by using the MEP technique, the expression of the tensors appearing in system (1) has been determined in terms of the 15 independent variables [30], as follows

$$
V^\alpha = \rho U^\alpha\,, \qquad T^{\alpha\beta} = \frac{e}{c^2}U^\alpha U^\beta + (p+\Pi)\,h^{\alpha\beta} + \frac{1}{c^2}(U^\alpha q^\beta + U^\beta q^\alpha) + t^{\langle\alpha\beta\rangle_3}\,,
$$

$$
\begin{aligned}
A^{\alpha\beta\gamma} = &\left(\rho\,\theta_{0,2} + \frac{1}{4c^4}\Delta\right)U^\alpha U^\beta U^\gamma + \left(\rho\, c^2\,\theta_{1,2} - \frac{3}{4c^2}\frac{N^\Delta}{D_4}\Delta - 3\frac{N^\Pi}{D_4}\Pi\right)h^{(\alpha\beta}U^{\gamma)} + \frac{3}{c^2}\frac{N_3}{D_3}q^{(\alpha}U^\beta U^{\gamma)} + \frac{3}{5}\frac{N_{31}}{D_3}h^{(\alpha\beta}q^{\gamma)}\\
&+ 3C_5 t^{(\langle\alpha\beta\rangle_3}U^{\gamma)}\,,
\end{aligned}
\tag{8}
$$

where the pressure p and the energy e appearing in the expression of $T^{\alpha\beta}$ are functions of (ρ, T)

$$
p = \frac{k_B}{m}\rho T\,, \qquad e = \rho c^2\,\omega(\gamma) \quad \text{with} \quad \omega(\gamma) = \theta_{0,1} \tag{9}
$$

being k_B the Boltzmann constant. The coefficients $\theta_{i,j}$ including $\theta_{0,1}$, $\theta_{0,2}$ and $\theta_{1,2}$ are given by

$$
\theta_{k,j} = \frac{1}{2k+1}\binom{j+1}{2k}\frac{\int_0^{+\infty} J^*_{2k+2,j+1-2k}\left(1+\frac{\mathcal{I}}{mc^2}\right)^j \phi(\mathcal{I})\, d\mathcal{I}}{\int_0^{+\infty} J^*_{2,1}\,\phi(\mathcal{I})\, d\mathcal{I}}\,, \tag{10}
$$

and are dimensionless functions depending only on

$$
\gamma = \frac{m\,c^2}{k_B T}. \tag{11}
$$

Here, we have introduced the following quantities:

$$\gamma^* = \gamma\left(1 + \frac{\mathcal{I}}{m\,c^2}\right), \qquad J^*_{m,n} = J_{m,n}(\gamma^*), \qquad J_{m,n}(\gamma) = \int_0^{+\infty} e^{-\gamma\cosh s}\sinh^m s\cosh^n s\, d\, s\,.$$

Regarding $J_{m,n}$, we have the following recurrence relation [3,25]:

$$J_{m+2,n}(\gamma) = J_{m,n+2}(\gamma) - J_{m,n}(\gamma)\,, \tag{12}$$

valid also for $J^*_{m,n}$. The other coefficients present in the expression (8) of $A^{\alpha\beta\gamma}$, whose explicit forms are needed in the proof of the monatomic limit in the subsequent section, are defined only by using $\theta_{i,j}$ in [30] as follows:

$$D_4 = \begin{vmatrix} \theta_{0,0} & \theta_{0,1} & \theta_{0,2} & \frac{1}{3}\theta_{1,2} \\ \theta_{0,1} & \theta_{0,2} & \theta_{0,3} & \frac{1}{6}\,\theta_{1,3} \\ \theta_{0,2} & \theta_{0,3} & \theta_{0,4} & \frac{1}{10}\,\theta_{1,4} \\ \theta_{1,1} & \frac{1}{3}\,\theta_{1,2} & \frac{1}{6}\,\theta_{1,3} & \frac{5}{9}\,\theta_{2,3} \end{vmatrix} \qquad N^{\Pi} = -\begin{vmatrix} \theta_{0,0} & \theta_{0,1} & \theta_{0,2} & \frac{1}{3}\theta_{1,2} \\ \theta_{0,1} & \theta_{0,2} & \theta_{0,3} & \frac{1}{6}\,\theta_{1,3} \\ \theta_{0,2} & \theta_{0,3} & \theta_{0,4} & \frac{1}{10}\,\theta_{1,4} \\ \frac{1}{3}\,\theta_{1,2} & \frac{1}{6}\,\theta_{1,3} & \frac{1}{10}\,\theta_{1,4} & \frac{1}{9}\,\theta_{2,4} \end{vmatrix},$$

$$N^{\Delta} = \begin{vmatrix} \theta_{0,0} & \theta_{0,1} & \theta_{0,2} & \frac{1}{3}\theta_{1,2} \\ \theta_{0,1} & \theta_{0,2} & \theta_{0,3} & \frac{1}{6}\,\theta_{1,3} \\ \theta_{1,1} & \frac{1}{3}\,\theta_{1,2} & \frac{1}{6}\,\theta_{1,3} & \frac{5}{9}\,\theta_{2,3} \\ \frac{1}{3}\,\theta_{1,2} & \frac{1}{6}\,\theta_{1,3} & \frac{1}{10}\,\theta_{1,4} & \frac{1}{9}\,\theta_{2,4} \end{vmatrix}, \tag{13}$$

$$D_3 = \begin{vmatrix} \theta_{1,1} & \theta_{1,2} \\ \theta_{1,2} & \frac{3}{2}\,\theta_{1,3} \end{vmatrix}, \qquad N_3 = \frac{1}{2}\begin{vmatrix} \theta_{1,1} & \theta_{1,2} \\ \theta_{1,3} & \frac{9}{5}\,\theta_{1,4} \end{vmatrix}, \qquad N_{31} = \begin{vmatrix} \theta_{1,1} & \theta_{1,2} \\ 5\,\theta_{2,3} & 3\,\theta_{2,4} \end{vmatrix} \qquad C_5 = \frac{1}{5}\,\frac{\theta_{2,4}}{\theta_{2,3}}\,. \tag{14}$$

By adopting a variant of the BGK model with a relaxation time τ appropriate for polyatomic gases [31], in paper [30], the following explicit expression of the production term was also obtained:

$$I^{\beta\gamma} = B_1^{\Delta}\,\Delta\,U^{\beta}U^{\gamma} + (B_2^{\Delta}\,\Delta + B_2^{\Pi}\,\Pi)h^{\beta\gamma} + B^q\,U^{(\beta}\,q^{\gamma)} + B^t\,t^{\langle\beta\gamma\rangle_3}, \tag{15}$$

with

$$B_1^{\Delta} = -\frac{1}{4c^4\tau}, \quad B_2^{\Delta} = \frac{1}{4c^2\tau}\,\frac{N^{\Delta}}{D_4}, \quad B_2^{\Pi} = \frac{1}{\tau}\,\frac{N^{\Pi}}{D_4} \quad B^q = \frac{1}{c^2\tau}\left(\frac{\theta_{1,3}}{\theta_{1,2}} - 2\frac{N_3}{D_3}\right), \quad B^t = -\frac{1}{\tau}\,C_5\,. \tag{16}$$

With these fields, we can write explicitly the closed system of differential equations that models the thermodynamics of a relativistic polyatomic gas. Let us introduce the relativistic material derivative of a generic tensor $\psi^{\alpha\alpha_1\cdots\alpha_n}$, i.e., the derivative with respect to the proper time $\bar{\tau}$ along the path of the particle, denoted by a dot on a quantity, as follows:

$$\dot{\psi}^{\alpha\alpha_1\cdots\alpha_n} = \frac{d\psi^{\alpha\alpha_1\cdots\alpha_n}}{d\bar{\tau}} = \frac{d\psi^{\alpha\alpha_1\cdots\alpha_n}}{dt}\,\frac{dt}{d\bar{\tau}} = \Gamma(\partial_t\psi^{\alpha\alpha_1\cdots\alpha_n} + v^j\partial_j\psi^{\alpha\alpha_1\cdots\alpha_n}) = U^{\beta}\partial_{\beta}\psi^{\alpha\alpha_1\cdots\alpha_n}, \tag{17}$$

where Γ is the Lorentz factor, $U^{\alpha} = \frac{dx^{\alpha}}{d\bar{\tau}} \equiv (\Gamma c, \Gamma v^j)$ and v^j is the velocity. Since we can observe that any balance laws can be written with the material derivative as follows:

$$I^{\alpha_1\cdots\alpha_n} = \partial_{\alpha}\,A^{\alpha\alpha_1\cdots\alpha_n} = g_{\alpha}^{\beta}\,\partial_{\beta}\,A^{\alpha\alpha_1\cdots\alpha_n} = \left(-h_{\alpha}^{\beta} + \frac{U^{\beta}U_{\alpha}}{c^2}\right)\partial_{\beta}\,A^{\alpha\alpha_1\cdots\alpha_n} = \frac{U_{\alpha}}{c^2}\,\dot{A}^{\alpha\alpha_1\cdots\alpha_n} - h_{\alpha}^{\beta}\,\partial_{\beta}\,A^{\alpha\alpha_1\cdots\alpha_n}\,, \tag{18}$$

the closed set of equations for 15 fields; $\{\rho, U_\delta, T, \Pi, t_{\langle\alpha\beta\rangle_3}, q_\delta, \Delta\}$ are evaluated in [30][1]:

$$
\begin{aligned}
&\dot\rho + \rho\,\partial_\alpha U^\alpha = 0\,,\\
&-\frac{e+p+\Pi}{c^2}\,\dot U^\delta + \frac{1}{c^2}\,h^\delta_\beta\,\dot q^\beta + \frac{1}{c^2}\,t^{\langle\alpha\delta\rangle_3}\,\dot U_\alpha - h^{\delta\mu}\,\partial_\mu(p+\Pi) - \frac{1}{c^2}\,q^\mu\,\partial_\mu U^\delta\\
&\qquad - \frac{1}{c^2}\,q^\delta\,\partial_\alpha U^\alpha - h^\delta_\beta\,h^\mu_\alpha\,\partial_\mu\,t^{\langle\alpha\beta\rangle_3} = 0\,,\\
&\dot e + 2\,\frac{U_\alpha}{c^2}\,\dot q^\alpha + (e+p+\Pi)\,\partial_\alpha U^\alpha - h^\mu_\alpha\,\partial_\mu q^\alpha - t^{\langle\alpha\beta\rangle_3}\,\partial_\alpha U_\beta = 0\,,\\
&-\left(\rho c^2\theta_{1,2} - \frac{3}{4\,c^2}\,\frac{N^\Delta}{D_4}\,\Delta - 3\frac{N^\Pi}{D_4}\,\Pi\right)^{\bullet} + \frac{1}{c^2}\left(2\frac{N_3}{D_3} + \frac{N_{31}}{D_3}\right) q_\gamma\,\dot U^\gamma\\
&\qquad + 5\Big(-\frac{1}{3}\rho c^2\theta_{1,2} + \frac{1}{4\,c^2}\,\frac{N^\Delta}{D_4}\,\Delta + \frac{N^\Pi}{D_4}\,\Pi\Big)\,\partial_\alpha U^\alpha - q^\mu\partial_\mu\left(\frac{N_{31}}{D_3}\right)\\
&\qquad + \frac{N_{31}}{D_3}\,h^\mu_\alpha\,\partial_\mu\,q^\alpha + 2C_5\,t^{\langle\mu}{}_{\gamma\rangle_3}\,\partial_\mu\,U^\gamma = -\frac{3}{\tau}\left(\frac{1}{4c^2}\,\frac{N^\Delta}{D_4}\,\Delta + \frac{N^\Pi}{D_4}\,\Pi\right),\\
&C_5\,h_{\gamma\langle\delta}\,h_{\beta\rangle_3\theta}\dot t^{\langle\gamma\theta\rangle_3} + t_{\langle\delta\beta\rangle_3}\,\dot C_5 + \frac{2}{c^2}\left(\frac{N_3}{D_3} + \frac{1}{5}\,\frac{N_{31}}{D_3}\right) q_{\langle\delta}\,\dot U_{\beta\rangle_3}\\
&\qquad + 2\left(-\frac{1}{3}\,\rho\,c^2\theta_{1,2} + \frac{1}{4\,c^2}\,\frac{N^\Delta}{D_4}\,\Delta + \frac{N^\pi}{D_4}\,\pi\right) h_{\gamma\langle\delta}\,h^\mu{}_{\beta\rangle_3}\,\partial_\mu\,U^\gamma\\
&\qquad + \frac{2}{5}\left(q_{\langle\delta}h^\mu{}_{\beta\rangle_3}\right)\partial_\mu\left(\frac{N_{31}}{D_3}\right) - \frac{2}{5}\,\frac{N_{31}}{D_3}\left(h_{\gamma\langle\delta}h^\mu{}_{\beta\rangle_3}\,\partial_\mu\,q^\gamma\right) +\\
&\qquad + C_5\left[t_{\langle\delta\beta\rangle_3}\,\partial_\alpha\,U^\alpha + 2\,t^{\langle\mu\gamma\rangle_3}\,h_{\gamma<\beta}\,h_{\delta>_3\nu}\,\partial_\mu\,U^\nu\right] = -\frac{1}{\tau}\,C_5\,t_{\langle\delta\beta\rangle_3}\,,\\
&h_{\beta\delta}\,\dot U^\beta\Big(\rho\theta_{0,2}c^2 + \frac{2}{3}\rho c^2\theta_{1,2} + \frac{1}{4\,c^2}\,\Delta - \frac{1}{2\,c^2}\,\frac{N^\Delta}{D_4}\,\Delta - 2\,\frac{N^\Pi}{D_4}\,\Pi\Big) + h_{\beta\delta}\,\frac{N_3}{D_3}\,\dot q^\beta - q_\delta\left(\frac{N_3}{D_3}\right)^{\bullet} +\\
&\qquad + 2\,C_5\,t_{\langle\delta\gamma\rangle_3}\,\dot U^\gamma - h^\mu_\delta\,\partial_\mu\left(\frac{1}{3}\rho c^4\theta_{1,2} - \frac{1}{4}\,\frac{N^\Delta}{D_4}\,\Delta - \frac{N^\Pi}{D_4}\,c^2\,\Pi\right)\\
&\qquad - \left(\frac{N_3}{D_3} + \frac{1}{5}\,\frac{N_{31}}{D_3}\right)\left(q^\mu\,\partial_\mu\,U_\delta + q_\delta\,\partial_\alpha\,U^\alpha\right)\\
&\qquad + \frac{1}{5}\,\frac{N_{31}}{D_3}\,h^\mu_\delta\,q^\gamma\,\partial_\mu\,U_\gamma + h^\mu_\alpha\,h_{\delta\beta}\,\partial_\mu\left(C_5\,c^2\,t^{\langle\alpha\beta\rangle_3}\right) = \frac{1}{\tau}\left(\frac{N_3}{D_3} - \frac{\theta_{1,3}}{2\,\theta_{1,2}}\right) q_\delta\,,\\
&\left(\rho\theta_{0,2}c^4 + \frac{1}{4}\,\Delta\right)^{\bullet} + \partial_\alpha\,U^\alpha\cdot\left(\rho\theta_{0,2}c^4 + \frac{2}{3}\rho c^4\theta_{1,2} + \frac{1}{4}\,\Delta - \frac{1}{2}\,\frac{N^\Delta}{D_4}\,\Delta - 2\,\frac{N^\Pi}{D_4}\,\Pi\,c^2\right)\\
&\qquad - 3\,\frac{N_3}{D_3}\,q^\alpha\,\dot U_\alpha - h^\mu_\alpha\,\partial_\mu\left(\frac{N_3}{D_3}\,c^2\,q^\alpha\right) - 2\,C_5c^2\,t^{\langle\mu\gamma\rangle_3}\,\partial_\mu U_\gamma = -\frac{1}{4\,\tau}\,\Delta\,.
\end{aligned}
\tag{19}
$$

In [30], it has been also shown that, as the result of the first iteration of the Maxwellian iteration procedure [33], the closed set of the 15 field equations include, as its parabolic limit, the Eckart theory for 5 fields in which the closed field equations are given by the conservation laws $(19)_{1,2,3}$ and the remaining equations reduce to

$$
\begin{aligned}
&q_\beta = -\chi\,h^\alpha_\beta\left[\partial_\alpha T - \frac{T}{c^2}U^\mu\partial_\mu U_\alpha\right],\\
&\Pi = -\nu\,\partial_\alpha U^\alpha,\\
&t_{\langle\beta\delta\rangle_3} = 2\mu\,h^\alpha_\beta\,h^\mu_\delta\partial_{\langle\alpha}U_{\mu\rangle}.
\end{aligned}
\tag{20}
$$

As the result of this iterative procedure, it is possible to find the relationship between the phenomenological coefficients, that are, the heat conductivity χ, the shear viscosity μ and the bulk viscosity ν, and the relaxation time τ as follows:

$$
\begin{aligned}
&\chi = -\frac{2\rho c^2}{3B^qT}\left[3\theta_{0,2} + \theta_{1,2}(1-\omega\,\gamma)\right],\\
&\nu = -\frac{\rho c^2}{3B^\Pi_2}\left\{\frac{2}{3}\theta_{1,2} - \frac{\theta'_{1,2}}{\gamma\omega'} + 3\frac{N^\Delta}{D_4}\left(\frac{2}{3}\theta_{1,2} - \frac{\theta'_{0,2}}{\gamma\omega'}\right)\right\},\\
&\mu = -\frac{\rho c^2}{3B^t}\theta_{1,2}\,,
\end{aligned}
\tag{21}
$$

where $\theta'_{k,j} = \frac{d\theta_{k,j}}{d\gamma}$ and, as it has been proved in [32], we have

$$
\theta'_{k,j} = \omega\theta_{k,j} - \frac{j+2-2k}{j+2}\theta_{k,j+1}. \tag{22}
$$

[1] Some typos in the expression of the closed field equations are corrected comparing to the ones shown in [30]

Furthermore, although Δ does not appear in the closed field equations of the Eckart theory, we obtain its expression in the parabolic limit as follows:

$$\Delta = \sigma\, \partial_\alpha U^\alpha \qquad \text{with} \qquad \sigma = \frac{\rho}{B_1^\Delta}\left(\frac{2}{3}\theta_{1,2} - \frac{\theta'_{0,2}}{\gamma\omega'}\right). \tag{23}$$

4. Singular limit to monatomic gas

The monatomic limit of the relativistic model for polyatomic gases presented in the previous section is delicate. This is because the relativistic monatomic gases can be modeled by using 14 independent variables, as described by the RET$_{14}^{R,M}$ model [3] while the corresponding model for polyatomic gases, RET$_{15}^{R}$, instead, has 15 independent variables. Consequently, at the monatomic limit under an appropriate initial condition compatible with the monatomic gases, one of the independent variables is no longer independent from the other fields.

To study the limit to monatomic gases, we introduce a particular expression of the function $\phi(\mathcal{I})$ which appears in the definition of the moments (7). In paper [25], the following expression has been found:

$$\phi(\mathcal{I}) = \mathcal{I}^a, \tag{24}$$

where $a = \frac{D-5}{2}$ is constant and $D = 3 + f_i$ is related to the degrees of freedom of the molecule, given by the sum of the space dimensions 3 for the translational motion and the contribution of the internal degrees of freedom $f_i \geq 0$ related to molecular rotation and vibration. This expression is introduced in order to recover the polytropic caloric equation of state in the classical limit $\gamma \to \infty$. In fact, by taking into account $e = \rho c^2 + \rho\varepsilon$, where ε is the internal energy, the expression of the measure (24) ensures the following caloric equation of state for polytropic gases holds

$$\lim_{\gamma\to\infty} \varepsilon = \frac{D}{2}\frac{k_B}{m}T. \tag{25}$$

Monatomic gases, having only translational modes, are characterized by the number of degrees of freedom $D = 3$, which means $a = -1$. For the proof, the limit is considered by taking that a is continuous and reduces to -1 as in the case of classical gases [14,39].

For this value of a for monatomic gases, the integral of moments is not convergent, so we have to calculate the limit and, thanks to the above mentioned expression of $\phi(\mathcal{I})$, it is possible to use some of the Lemmas proved in paper [38], where the monatomic limit of the theory with previous moments (6) has been studied. However, compared to [38], due to the exponent in the moments (7), we now need to take into account the non-linear contributions of $\mathcal{I}$.

For completeness, we report here the properties demonstrated in [38] we use in the sequel:

Lemma 1. *We have:*

$$\int_0^{+\infty} J_{m,n}^*\, \mathcal{I}^a\, d\,\mathcal{I} = \frac{1}{a+1}\frac{\gamma}{mc^2}\int_0^{+\infty} J_{m,n+1}^*\, \mathcal{I}^{a+1}\, d\,\mathcal{I} \tag{26}$$

and

$$\begin{aligned}
&\lim_{a\to-1} \frac{\int_0^{+\infty} J_{m,n}^*\, \mathcal{I}^a\, d\,\mathcal{I}}{\int_0^{+\infty} J_{2,1}^*\, \mathcal{I}^a\, d\,\mathcal{I}} = \frac{J_{m,n}(\gamma)}{J_{2,1}(\gamma)},\\
&\lim_{a\to-1} \frac{\int_0^{+\infty} J_{m,n}^*\, \mathcal{I}^{a+1}\, d\,\mathcal{I}}{\int_0^{+\infty} J_{2,1}^*\, \mathcal{I}^a\, d\,\mathcal{I}} = 0,\\
&\lim_{a\to-1} \frac{\int_0^{+\infty} J_{m,n}^*\, \mathcal{I}^{a+n}\, d\,\mathcal{I}}{\int_0^{+\infty} J_{2,1}^*\, \mathcal{I}^a\, d\,\mathcal{I}} = 0.
\end{aligned} \tag{27}$$

The convergence of the energy and pressure is the same of [38]. For completeness, we summarize these results. The energy of polyatomic gas $(9)_2$ converges, in the limit $a \to -1$, to the corresponding monatomic expression, i.e., $e = \frac{1}{3}\rho c^2 \frac{J_{2,2}}{J_{2,1}}$. We may recognize it as the Synge energy

$$e = \rho c^2 \omega^{Mono}(\gamma) \quad \text{with} \quad \omega^{Mono}(\gamma) = G - \frac{1}{\gamma},$$

where $G = K_3(\gamma)/K_2(\gamma)$ being K_n the modified Bessel function.

Similarly, the pressure of polyatomic gas $(9)_1$ converges, in the limit $a \to -1$, to the corresponding monatomic expression, i.e., $p = \frac{1}{3}\rho c^2 \frac{J_{4,0}}{J_{2,1}}$. As the proof was not explicitly shown in the previous study [38], we report it here for completeness. By using Eq. (11), we have that $p = \rho c^2 \frac{1}{\gamma}$. Moreover, thanks to eq. $(7.4)_6$ of paper [30], we obtain

$$p = \rho c^2 \theta_{1,1} = \rho c^2 \frac{1}{3} \frac{\int_0^{+\infty} J_{4,0}^* \left(1 + \frac{\mathcal{I}}{mc^2}\right) \mathcal{I}^\alpha\, d\,\mathcal{I}}{\int_0^{+\infty} J_{2,1}^*\, \mathcal{I}^\alpha\, d\,\mathcal{I}}.$$

By using $(27)_{1,2}$, we have that the above expression converts into the corresponding one for monatomic gases.

4.1. Monatomic limit of equilibrium variables

Due to the existence of the 15th variable Δ and of the non-linear contribution of $\mathcal{I}$ in the definition of the moments, the coefficients of the closed system are different from the ones analyzed in [38]. Therefore, we study the monatomic limit of the coefficients present in the closure of system (1) with (7), given by Eqs. (8), (9) and (15).

We have the following theorem:

Theorem 1. *The following limits hold:*

$$\begin{aligned}
&\lim_{a\to -1} \rho\,\theta_{0,2} = \frac{1}{2}\left(C_1^0 + nm^2\right),\\
&\lim_{a\to -1} \rho\,c^2\,\theta_{1,2} = \frac{c^2}{2}\left(C_1^0 - nm^2\right),\\
&\lim_{a\to -1} 3C_5 = 3C_5^{Mono},\\
&\lim_{a\to -1} \frac{N_3}{D_3} = -5C_3,\\
&\lim_{a\to -1} \frac{N_{31}}{D_3} = -5C_3,\\
&\lim_{a\to -1} \left(-\frac{6}{c^2}\frac{N^{\Pi}}{D_4+3N^{\Delta}}\right) = C_1^{\Pi}.
\end{aligned} \tag{28}$$

In the right-hand sides of the above equalities we use the notation of $RET_{14}^{R,M}$ *[3], in particular, we refer to eqs. (7.4) and (7.7).*[2]

The proof of the theorem is shown in Appendix A.

Corollary 1. *As a consequence of the above Theorem we have:*

$$\lim_{a\to -1} \frac{N_3}{D_3} = \lim_{a\to -1} \frac{N_{31}}{D_3} \qquad \lim_{a\to -1} \left(\theta_{0,2}-\theta_{1,2}\right) = 1, \quad \lim_{a\to -1} \left(\theta'_{0,2}-\theta'_{1,2}\right) = 0. \tag{29}$$

Proof. The first two relations derives immediately from the results of Theorem 1, while the third one can be proved with the following procedure. From (22), we obtain

$$\theta'_{0,2} - \theta'_{1,2} = \omega\left(\theta_{0,2}-\theta_{1,2}\right) - \left(\theta_{0,3} - \frac{1}{2}\theta_{1,3}\right).$$

With $(27)_{2,3}$ and (26), by taking into account (12), we can prove that

$$\lim_{a\to -1} \omega(\theta_{0,2}-\theta_{1,2}) = \omega^{Mono}, \quad \lim_{a\to -1}\left(\theta_{0,3} - \frac{1}{2}\theta_{1,3}\right) = \omega^{Mono}.$$

Then, the relation $(29)_3$ is satisfied.

4.2. Monatomic limit of RET_{15}^{R}

Let us now investigate the monatomic limit ($a \to -1$) of the system (19). By using the results of Theorem 1, we find a system of 15 equations of monatomic gas. However, we need to recall that monatomic gases must satisfy the trace conditions (4) according to the kinetic theory. To prove that the trace conditions hold in the monatomic limit, we introduce the following balance equation

$$\partial_\alpha\left(A^{\alpha\beta\gamma} g_{\beta\gamma} - c^2 V^\alpha\right) = I^{\beta\gamma} g_{\beta\gamma}, \tag{30}$$

that is an identity if and only if the trace conditions (4) are satisfied.

Recalling (18), the above Eq. (30) can be rewritten with the material derivative as

$$\begin{aligned}
I^{\beta\gamma} g_{\beta\gamma} &= g_{\beta\gamma}\partial_\alpha A^{\alpha\beta\gamma} - c^2\partial_\alpha V^\alpha = g_{\beta\gamma}\left(\frac{U_\alpha}{c^2}\,\dot{A}^{\alpha\beta\gamma} - h^\mu_\alpha\,\partial_\mu\,A^{\alpha\beta\gamma}\right) - c^2\partial_\alpha V^\alpha\\
&= \left(-h_{\beta\gamma} + \frac{U_\beta U_\gamma}{c^2}\right)\left(\frac{U_\alpha}{c^2}\,\dot{A}^{\alpha\beta\gamma} - h^\mu_\alpha\,\partial_\mu\,A^{\alpha\beta\gamma}\right) - c^2\partial_\alpha V^\alpha\\
&= h_{\beta\theta}h_{\gamma\delta}g^{\theta\delta}\left(\frac{U_\alpha}{c^2}\,\dot{A}^{\alpha\beta\gamma} - h^\mu_\alpha\,\partial_\mu\,A^{\alpha\beta\gamma}\right) + \frac{U_\beta U_\gamma}{c^2}\left(\frac{U_\alpha}{c^2}\,\dot{A}^{\alpha\beta\gamma} - h^\mu_\alpha\,\partial_\mu\,A^{\alpha\beta\gamma}\right) - c^2\partial_\alpha V^\alpha.
\end{aligned} \tag{31}$$

We note that the above equation is the sum of $(19)_4$, $(19)_7$ divided by c^2 and $(19)_1$ multiplied by $-c^2$. By introducing the variable

$$Y(\mathbf{x},t) = \left(A^{\alpha\beta}{}_{\beta} - A^{\alpha\beta}{}_{\beta}|_E\right)\frac{U_\alpha}{c^2} = \frac{1}{4c^2}\frac{D_4+3N^{\Delta}}{D_4}\Delta + 3\frac{N^{\Pi}}{D_4}\Pi, \tag{32}$$

[2] The dimension of $A^{\alpha\beta\gamma}$ defined in [3] is same with the one defined in (7) divided by m. For this reason, here and hereafter, we refer to the coefficients of monatomic case in [3] divided by m.

we obtain that, in the limit of monatomic gas, i.e., $a \to -1$, Eq. (31) becomes

$$\dot{Y}^{Mono} + Y^{Mono}\,\partial_\alpha U^\alpha = -\frac{1}{\tau} Y^{Mono}, \tag{33}$$

where $Y^{Mono} = \lim_{a\to -1} Y$. Recalling $(29)_2$, we have that $\lim_{a\to-1} A^{\alpha\beta}{}_{\beta}|_E = c^2 V^\alpha$ and therefore

$$Y^{Mono} = \lim_{a\to -1} \left(A^{\alpha\beta}{}_{\beta} - c^2 V^\alpha \right) \frac{U_\alpha}{c^2}. \tag{34}$$

In the limit from polyatomic gases to monatomic gases, it is natural to impose an initial condition compatible with the monatomic gases in which the trace conditions are satisfied. Therefore, in this limit, we assume that $A^{\alpha\beta}{}_{\beta} = c^2 V^\alpha$, which is equivalent to $I^{\beta}{}_{\beta} = 0$, is satisfied at the initial state, i.e.,

$$Y^{Mono}(\mathbf{x}, 0) = 0. \tag{35}$$

With this initial condition and assuming the uniqueness of the solutions, we have that the only solution of (33) is

$$Y^{Mono}(\mathbf{x}, t) = 0 \qquad \text{for any } t. \tag{36}$$

Therefore, under the initial condition (35), Y^{Mono} vanishes and moreover Eq. (30) is identically zero, then the trace conditions are satisfied for any time.

We emphasize that we can interpret that Y is the part of Δ that characterizes polyatomic gases. In fact, from (32), we have

$$\Delta = 4c^2 \frac{D_4}{D_4 + 3N^\Delta} \left(Y - 3\frac{N^\Pi}{D_4} \Pi \right), \tag{37}$$

and, the remaining part of Δ, represented by Π, persists in the monatomic limit. This indicates that the initial condition (35) implies setting the part of the polyatomic gas effect in Δ to zero at the initial state. In the monatomic limit, since Y^{Mono} is identically zero, Δ is expressed as

$$\Delta = \lim_{a\to -1} \left(-12c^2 \frac{N^\Pi}{D_4 + 3N^\Delta} \Pi \right) = 2\,c^4 C_1^\Pi\, \Pi\,, \tag{38}$$

where we used Eq. $(28)_6$. In this way, in the monatomic limit, Δ is expressed by Π and the number of independent fields is now 14.

With this singular limit, we can prove the convergence of the closure for $A^{\alpha\beta\gamma}$.

Theorem 2. *The triple tensor $A^{\alpha\beta\gamma}$ converges to the corresponding tensor of $RET_{14}^{R,M}$ when $a \to -1$.*

Proof. It is sufficient to use the results of Theorem 1 and Eq. (38) on Eq. (8) to prove that the triple tensor converges to that present in eq. $(4.3)_2$ of paper [3].

We show now that the 15 equations of RET_{15}^R coincide with the 14 equations for $\{\rho, U^\alpha, e, \Pi, t^{\langle\alpha\beta\rangle_3}, q^\alpha\}$ of $\mathrm{RET}_{14}^{R,M}$. By inserting (36), or (38), into system (19) it converts into

$$\begin{aligned}
&\dot\rho + \rho\,\partial_\alpha U^\alpha = 0\,,\\
&-\frac{e+p+\Pi}{c^2}\dot U^\delta + \frac{1}{c^2} h^\delta_\beta \dot q^\beta + \frac{1}{c^2} t^{\langle\alpha\delta\rangle_3}\dot U_\alpha - h^{\delta\mu}\partial_\mu(p+\Pi) - \frac{1}{c^2} q^\mu\partial_\mu U^\delta - \frac{1}{c^2} q^\delta \partial_\alpha U^\alpha - h^\delta_\beta h^\mu_\alpha \partial_\mu t^{\langle\alpha\beta\rangle_3} = 0\,,\\
&\dot e + 2\frac{U_\alpha}{c^2}\dot q^\alpha + (e+p+\Pi)\partial_\alpha U^\alpha - h^\mu_\alpha \partial_\mu q^\alpha - t^{\langle\alpha\beta\rangle_3}\partial_\alpha U_\beta = 0\,,\\
&\frac{c^2}{2}\left(C_1^0 - nm^2 + C_1^\Pi\,\Pi\right)^\bullet + \frac{15\,C_3}{c^2} q_\gamma \dot U^\gamma + 5\frac{c^2}{6}\left(C_1^0 - nm^2 + C_1^\Pi\,\Pi\right)\partial_\alpha U^\alpha\\
&\qquad - 5\,q^\mu\partial_\mu C_3 + 5\,C_3 h^\mu_\alpha \partial_\mu q^\alpha - 2C_5^{Mono} t^{\langle\mu}{}_{\gamma\rangle_3}\partial_\mu U^\gamma = -\frac{c^2}{2\,\tau} C_1^\Pi\,\Pi\,,\\
&C_5^{Mono} h_{\gamma\langle\delta} h_{\beta\rangle_3\theta} \dot t^{\langle\gamma\theta\rangle_3} + t_{\langle\delta\beta\rangle_3}\dot C_5^{Mono} - \frac{12\,C_3}{c^2} q_{\langle\delta}\dot U_{\beta\rangle_3} - \frac{c^2}{3}\left(C_1^0 - nm^2 + C_1^\Pi\,\Pi\right) h_{\gamma\langle\delta} h^\mu{}_{\beta\rangle_3}\partial_\mu U^\gamma\\
&\qquad - 2\left(q_{\langle\delta} h^\mu{}_{\beta\rangle_3}\right)\partial_\mu C_3 + 2C_3\left(h_{\gamma\langle\delta} h^\mu{}_{\beta\rangle_3}\partial_\mu q^\gamma\right)\\
&\qquad + C_5^{Mono}\left[t_{\langle\delta\beta\rangle_3}\partial_\alpha U^\alpha + 2\,t^{\langle\mu\gamma\rangle_3} h_{\gamma\langle\beta} h_{\delta\rangle_3\nu}\partial_\mu U^\nu\right] = -\frac{1}{\tau} C_5^{Mono} t_{\langle\delta\beta\rangle_3}\,,\\
&h_{\beta\delta}\dot U^\beta\left(\frac{c^2}{6}\left(5\,C_1^0 + nm^2\right) + \frac{5}{6}c^2 C_1^\Pi\,\Pi\right) - h_{\beta\delta}\,5\,C_3\,\dot q^\beta + q_\delta\left(5\,C_3\right)^\bullet + 2\,C_5^{Mono} t_{\langle\delta\gamma\rangle_3}\dot U^\gamma\\
&\qquad - h^\mu_\delta\partial_\mu\left(\frac{c^4}{6}\left(C_1^0 - nm^2\right) + \frac{c^4}{6}C_1^\Pi\,\Pi\right) + 6\,C_3\left(q^\mu\partial_\mu U_\delta + q_\delta\partial_\alpha U^\alpha\right)\\
&\qquad - C_3 h^\mu_\delta q^\gamma\partial_\mu U_\gamma + h^\mu_\alpha\partial_\mu\left(C_5^{Mono} c^2 t^{\langle\alpha}{}_{\delta\rangle_3}\right) = \frac{1}{\tau}\left(-5\,C_3 - \frac{\theta_{1,3}}{2\,\theta_{1,2}}\right) q_\delta\,,
\end{aligned} \tag{39}$$

that is exactly the system of $\mathrm{RET}_{14}^{R,M}$, i.e. eqs. (7.16) of [3] (where we used the values found in paper [38] for the coefficients in the right-hands sides because they were not explicitly present in [3]).

4.3. The phenomenological coefficients

Concerning the limit of the production term $I^{\beta\mu}$, the values of the coefficients appearing in (15) are strictly related to the heat conductivity and viscosities via Maxwellian iteration, in particular through Eqs. (21). The explicit expression of such coefficients in monatomic gases appeared for the first time in paper [27] and we refer to those results to make a comparison. Then, we have the following theorem.

Theorem 3. *The phenomenological coefficients, presented in Eqs. (21), converge to the corresponding one of the monatomic theory when* $a \to -1$.

The proof of the theorem is shown in Appendix B.

As we have seen in the previous section, the variable Y is the part of Δ characterizing polyatomic gases. In this sense, we may adopt Y as an independent field instead of Δ. Then, in the parabolic limit by conducting the Maxwellian iteration, we can obtain the expression of Y by inserting $(20)_2$, $(21)_2$ and (23) into (32) as follows:

$$Y = \sigma^Y \partial_\alpha U^\alpha,$$

where

$$\sigma^Y = \tau \frac{c^2 \rho}{\gamma \omega'} \left(\theta'_{0,2} - \theta'_{1,2} \right). \tag{40}$$

In the monatomic limit, recalling $(29)_3$, we obtain

$$\sigma^Y = 0. \tag{41}$$

5. Monatomic limit of the principal subsystems

In RET the principal subsystems are models with fewer moments than the original system but that retain the properties of convexity of the entropy and positivity of the entropy production. These subsystems can be obtained by eliminating some of the balance equations and this procedure corresponds to considering some components of the main field as constant (see [23] for details).

In this section, we consider the monatomic limit of RET_{14}^R and RET_6^R which are obtained as principle subsystems RET_{15}^R.

5.1. Monatomic limit of RET_{14}^R

As proved in paper [30], the principal subsystem with 14 moments is characterized by the following condition:

$$\Delta^{(14)} = 4 \frac{N_a}{D_a} c^2 \Pi \qquad \text{with} \qquad \frac{N_a}{D_a} = \frac{D_4^{44} + D_4^{43}}{D_4^{34} + D_4^{33}}$$

that reduces the number of independent variables to 14. The symbol D_4^{ij} represents the determinant of the matrix obtained by eliminating the ith row and the jth column of D_4 defined in Eq. $(13)_1$.

The closure of this principal subsystem is given by

$$\begin{aligned} A^{\alpha\beta\gamma} = & \left(\rho\, \theta_{0,2} - \frac{3}{c^2} \frac{N_1^\pi}{D_1^\pi} \Pi \right) U^\alpha U^\beta U^\gamma + \left(\rho\, c^2 \theta_{1,2} - 3 \frac{N_{11}^\pi}{D_1^\pi} \Pi \right) U^{(\alpha} h^{\beta\gamma)} + \\ & + \frac{3}{c^2} \frac{N_3}{D_3} q^{(\alpha} U^\beta U^{\gamma)} + \frac{3}{5} \frac{N_{31}}{D_3} q^{(\alpha} h^{\beta\gamma)} + 3\, C_5\, t^{(\langle\alpha\beta\rangle_3} U^{\gamma)}, \end{aligned} \tag{42}$$

and by the production term

$$I^{\langle\beta\gamma\rangle} = - \frac{1}{c^2 \tau} \frac{3N_1^\pi + N_{11}^\pi}{D_1^\pi} \Pi\, U^{\langle\beta} U^{\gamma\rangle} + \frac{1}{c^2 \tau} \left(\frac{\theta_{1,3}}{\theta_{1,2}} - 2 \frac{N_3}{D_3} \right) q^{(\beta} U^{\gamma)} - \frac{1}{\tau} C_5\, t^{\langle\beta\gamma\rangle_3}, \tag{43}$$

with

$$\frac{N_1^\pi}{D_1^\pi} = -\frac{1}{3} \frac{N_a}{D_a}, \qquad \frac{N_{11}^\pi}{D_1^\pi} = \frac{1}{D_4} \left(\frac{N_a}{D_a} N^\Delta + N^\Pi \right) = -\frac{N_b}{D_a}, \qquad N_b = N^{\Delta 34} + N^{\Delta 33},$$

and the other symbols defined in Eqs. (13) and (14). The notation $N^{\Delta ij}$ represents the determinant of the matrix obtained by eliminating the ith row and the jth column of N^Δ. By using the same technique adopted in Theorem 1, it is possible to prove that

$$\lim_{a \to -1} \frac{N_a}{D_a} = \frac{c^2}{2} C_1^\Pi, \qquad \lim_{a \to -1} \frac{N_b}{D_a} = \frac{c^2}{6} C_1^\Pi,$$

and, as a consequence, the monatomic limit of Eq. (42) corresponds to eq. $(4.3)_2$ of paper [3] and the monatomic limit of Eq. (43) gives the expression of the production term for monatomic gases obtained in paper [26].

The above results show that the present subsystem RET_{14}^R has the same monatomic limit of the system of RET_{15}^R, that is, $\mathrm{RET}_{14}^{R,M}$.

5.2. Monatomic limit of RET_6^R

The principal subsystem RET_6^R is characterized by the following conditions [30]

$$q_\mu = 0, \qquad t_{\langle\mu\nu\rangle_3} = 0, \qquad \Delta^{(6)} = 4c^2 \frac{D_4^{44} - 3D_4^{43}}{D_4^{34} - 3D_4^{33}} \Pi .$$

The relation between Δ and Π is different from the subsystem with 14 moments.

The closure of the principal subsystem with 6 moments is given by the following expressions of the tensors $T^{\alpha\beta}$, $A^{\alpha\beta}{}_{\beta}$ and I^{β}_{β}:

$$T^{\alpha\beta} = \frac{e}{c^2} U^\alpha U^\beta + \left(p + \Pi\right) h^{\alpha\beta}, \qquad A^{\alpha\beta}{}_{\beta} = \left\{\rho c^2 (\theta_{0,2} - \theta_{1,2}) + A_1 \Pi\right\} U^\alpha, \qquad I^{\beta}_{\beta} = -\frac{A_1}{\tau}\Pi, \tag{44}$$

with

$$A_1 = \frac{D_4^{44} - 3D_4^{43} - 3N^{\Delta 34} + 9N^{\Delta 33}}{D_4^{34} - 3D_4^{33}}, \tag{45}$$

where all the elements of the matrices involved in the definition of A_1 are functions of $\omega(\gamma)$ (see [30] for details). We see that the closure is determined in terms of $\theta_{0,2}$, $\theta_{1,2}$ and A_1. The field equation of Π is given by

$$\left(A_1 \Pi\right)^\bullet + \rho c^2 \left(\theta'_{0,2} - \theta'_{1,2}\right) \dot{\gamma} + A_1\, \Pi\, \partial_\alpha U^\alpha = -\frac{1}{\tau} A_1\, \Pi . \tag{46}$$

It is important to note that the bulk viscosity, derived through Maxwellian iteration, is obtained as shown in [30], and, by employing σ^Y, as defined in (40), the bulk viscosity can be represented in a more compact form:

$$\nu_6 = -\frac{\sigma^Y}{A_1}. \tag{47}$$

We remark that the expression of the bulk viscosity is different from the ones of RET_{15}^R, $(21)_2$, and of RET_{14}^R [30].

In the monatomic limit, recalling $(29)_2$, the field equation of Π (46) is expressed as follows

$$\left(A_1^{Mono} \Pi\right)^\bullet + A_1^{Mono}\, \Pi\, \partial_\alpha U^\alpha = -\frac{1}{\tau} A_1^{Mono}\, \Pi, \tag{48}$$

where $A_1^{Mono} = \lim_{a\to -1} A_1$. Utilizing $(27)_{1,2}$, it is determined that A_1^{Mono} results in the indeterminate form $\left[\frac{0}{0}\right]$. This form cannot be resolved using the method employed in proving the last point of Theorem 1, as it leads to the same indeterminate form. Additionally, substituting ω^{Mono} into Eq. (45) does not offer a solution. It is observed that the ω value for polyatomic gases differs from that of monatomic gases, represented as $\omega(\gamma) = \omega^{Mono}(\gamma) + \delta$. To derive the monatomic value of ω, we calculate the limit as δ approaches 0. Through this limiting process and cumbersome calculations, it is shown that both the numerator and denominator of A_1 vary linearly with δ, resulting in

$$A_1^{Mono} = \lim_{\delta\to 0} A_1 = 3\left\{-\frac{1}{\gamma} + \frac{3[\gamma + G(5 - G\gamma)]}{5 + 2\gamma[-5G + \gamma(-1 + G^2)]}\right\} \neq 0. \tag{49}$$

Similar to (33), (48) is a first-order partial differential equation in the variable $A_1^{Mono}\Pi$. As an initial condition compatible with monatomic gases, we assume that the trace conditions (4), $A^{\alpha\beta}{}_{\beta} = c^2 V^\alpha$, is satisfied at the initial time. In other words, since we have, from $(44)_2$,

$$\lim_{a\to -1} A^{\alpha\beta}{}_{\beta} = \left(\rho c^2 + A_1^{Mono}\, \Pi\right) U^\alpha = c^2 V^\alpha + A_1^{Mono}\, \Pi\, U^\alpha, \tag{50}$$

and therefore

$$A_1^{Mono}\, \Pi = \frac{U_\alpha}{c^2} \lim_{a\to -1} \left(A^{\alpha\beta}{}_{\beta} - c^2 V^\alpha\right), \tag{51}$$

we consider as an initial data that

$$(A_1^{Mono}\, \Pi)(\mathbf{x}, 0) = 0. \tag{52}$$

This condition also indicates that $I^{\beta}_{\beta} = -\frac{1}{\tau} A_1^{Mono}\, \Pi = 0$ is satisfied at the initial state. Then, the unique solution of (48) is

$$(A_1^{Mono}\, \Pi)(\mathbf{x}, t) = 0, \tag{53}$$

and the trace conditions are satisfied for any time in the monatomic limit. Since $A_1^{Mono} \neq 0$ (see Eq. (49)), the dynamic pressure vanishes.

We note that, in the present subsystem, different from the case of RET_{15}^R, the characteristic features of polyatomic gases are captured by Π, which tends to zero in the monatomic limit. Therefore, the initial condition (52) indicates that the gases initially demonstrate monatomic gas behavior.

The system now comprises five independent fields, $\{\rho, U^\alpha, \gamma\}$, with their closed field equations converging to those of the Euler system for a non-dissipative fluid. As indicated by (41) and (47), we observe that $\nu_6 = 0$ in this limit.

6. Summary and concluding remarks

The present study demonstrated that the system of RET_{15}^{R} converges to the system of $\mathrm{RET}_{14}^{R,M}$ under the singular limit of D approaching 3. Furthermore, it was shown that RET_{14}^{R}, which is a principal subsystem of RET_{15}^{R}, also aligns with the same monatomic theory, i.e., $\mathrm{RET}_{14}^{R,M}$. On the other hand, RET_{6}^{R} converges to the relativistic Euler theory for monatomic gases. As already shown in Fig. 1, these findings have clarified the relationships between various RET theories for relativistic polyatomic and monatomic gases that were previously unclear.

Finally, this paper has dealt with polytropic gases; however, the monatomic limit in non-polytropic gases remains an issue for future work. Additionally, while beyond the scope of this paper, the following topics remain as open questions: (i) As seen in Fig. 1, in the 14-variable model, the results differ depending on whether the monatomic limit or the classical limit is taken first. However, the results obtained are tied to the principal subsystem. The difference is due to the dynamic pressure existing even in relativistic monatomic gases, but a more detailed study is necessary. (ii) The phenomenological coefficients differ in RET_{15}^{R}, RET_{14}^{R} and RET_{6}^{R} (see [30] for details), while in the classical case, such differences do not appear. It is important to note that the form of the phenomenological coefficients' expression strongly depends on the collisional term. Further study is needed to understand these differences, considering the general study of Maxwellian iteration [40]. Furthermore, although the dependence of these coefficients on γ is discussed in [30], to compare these coefficients with experimental data, it is necessary to evaluate the undetermined relaxation time using different microscopic or mesoscopic theories. (iii) As argued in [41], including higher-order moments allows for the construction of more detailed models. Although the classical case is discussed in [42], the model for relativistic gases is not revealed.

CRediT authorship contribution statement

Takashi Arima: Conceptualization, Formal analysis, Investigation, Methodology, Supervision, Validation, Writing – original draft, Writing – review & editing. **Maria Cristina Carrisi:** Conceptualization, Formal analysis, Investigation, Methodology, Supervision, Validation, Writing – original draft, Writing – review & editing.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

No data was used for the research described in the article.

Acknowledgments

The authors wish to acknowledge Profs. Tommaso Ruggeri and Sebastiano Pennisi, whose insights and expertise greatly enhanced this research.

The work has been partially supported by JSPS KAKENHI Grant Numbers JP18K13471, JP22K03912 (TA), GNFM/INdAM (MCC) and National Recovery and Resilience Plan (NRRP) – Project Title eINS Ecosystem of Innovation for Next Generation Sardinia – CUP F53C22000430001 funded by the European Union – NextGenerationEU, Project Code ECS0000038 (MCC).

This research was conducted during the sabbatical leave of author T. Arima at the University of Bologna.

Appendix A. Proof of Theorem 1

The convergence of the coefficients in Theorem 1 is proved as follows.

- From Eq. (10) we have that

$$\theta_{0,2} = \frac{\int_0^{+\infty} J_{2,3}^* \, \mathcal{I}^a \, d\,\mathcal{I} + \frac{2}{mc^2} \int_0^{+\infty} J_{2,3}^* \, \mathcal{I}^{a+1} \, d\,\mathcal{I} + \frac{1}{(mc^2)^2} \int_0^{+\infty} J_{2,3}^* \, \mathcal{I}^{a+2} \, d\,\mathcal{I}}{\int_0^{+\infty} J_{2,1}^* \, \mathcal{I}^a \, d\,\mathcal{I}} \,.$$

  By using (27), we deduce that

$$\lim_{a \to -1} \theta_{0,2} = \frac{J_{2,3}(\gamma)}{J_{2,1}(\gamma)} \,.$$

  By using the recurrence relation (12) and eqs. $(7.4)_{1,4}$ of [3], we prove the first point of the Theorem.

- From Eqs. (10) we have that

$$\theta_{1,2} = \frac{\int_0^{+\infty} J^*_{4,1}\, \mathcal{I}^a\, d\,\mathcal{I} + \frac{2}{mc^2}\int_0^{+\infty} J^*_{4,1}\, \mathcal{I}^{a+1}\, d\,\mathcal{I} + \frac{1}{(mc^2)^2}\int_0^{+\infty} J^*_{4,1}\, \mathcal{I}^{a+2}\, d\,\mathcal{I}}{\int_0^{+\infty} J^*_{2,1}\, \mathcal{I}^a\, d\,\mathcal{I}}\,.$$

By using $(27)_{1,2}$, we deduce that

$$\lim_{a\to -1} \theta_{1,2} = \frac{J_{4,1}(\gamma)}{J_{2,1}(\gamma)}\,.$$

By using eqs. $(7.4)_{1,4}$ of [3] we prove the second point of the Theorem.

- From Eq. $(14)_4$ and (10), we have

$$\lim_{a\to -1} C_5 = \lim_{a\to -1} \frac{\theta_{2,4}}{5\,\theta_{2,3}} = \lim_{a\to -1} \frac{\dfrac{\int_0^{+\infty} J^*_{6,1}\left(1+\frac{\mathcal{I}}{mc^2}\right)^4 \mathcal{I}^a\, d\,\mathcal{I}}{\int_0^{+\infty} J^*_{2,1}\, \mathcal{I}^a\, d\,\mathcal{I}}}{\dfrac{\int_0^{+\infty} J^*_{6,0}\left(1+\frac{\mathcal{I}}{mc^2}\right)^3 \mathcal{I}^a\, d\,\mathcal{I}}{\int_0^{+\infty} J^*_{2,1}\, \mathcal{I}^a\, d\,\mathcal{I}}} = \frac{J_{6,1}(\gamma)}{J_{6,0}(\gamma)}\,,$$

where $(27)_{1,2}$ have been used in the last passage.

- From Eq. $(14)_{1,2}$, (10) and by using $(27)_{1,2}$ we have

$$\lim_{a\to -1} D_3 = \lim_{a\to -1} \left(\theta_{1,1}\cdot \frac{3}{2}\theta_{1,3} - \theta_{1,2}\cdot\theta_{1,2}\right) = \frac{1}{\left(J_{2,1}(\gamma)\right)^2} \begin{vmatrix} J_{4,0}(\gamma) & J_{4,1}(\gamma) \\ J_{4,1}(\gamma) & J_{4,2}(\gamma) \end{vmatrix}\,,$$

$$\lim_{a\to -1} N_3 = \lim_{a\to -1} \frac{1}{2}\left[\theta_{1,1}\cdot \frac{9}{5}\theta_{1,4} - \theta_{1,2}\cdot\theta_{1,3}\right] = \frac{1}{\left(J_{2,1}(\gamma)\right)^2} \begin{vmatrix} J_{4,0}(\gamma) & J_{4,1}(\gamma) \\ J_{4,2}(\gamma) & J_{4,3}(\gamma) \end{vmatrix}\,.$$

Hence $$\lim_{a\to -1} \frac{N_3}{D_3} = \frac{\begin{vmatrix} J_{4,0}(\gamma) & J_{4,1}(\gamma) \\ J_{4,2}(\gamma) & J_{4,3}(\gamma) \end{vmatrix}}{\begin{vmatrix} J_{4,0}(\gamma) & J_{4,1}(\gamma) \\ J_{4,1}(\gamma) & J_{4,2}(\gamma) \end{vmatrix}}$$ as in [3].

- From Eq. $(14)_3$, (10), by using $(27)_{1,2}$ and the recurrence relation (12), we have

$$\lim_{a\to -1} N_{31} = \lim_{a\to -1} \left(\theta_{1,1}\cdot 3\theta_{2,4} - \theta_{1,2}\cdot 5\theta_{2,3}\right) = \frac{1}{\left(J_{2,1}(\gamma)\right)^2} \begin{vmatrix} J_{4,0}(\gamma) & J_{4,1}(\gamma) \\ J_{4,2}(\gamma) & J_{4,3}(\gamma) \end{vmatrix}\,.$$

Hence $$\lim_{a\to -1} \frac{N_{31}}{D_3} = \frac{\begin{vmatrix} J_{4,0}(\gamma) & J_{4,1}(\gamma) \\ J_{4,2}(\gamma) & J_{4,3}(\gamma) \end{vmatrix}}{\begin{vmatrix} J_{4,0}(\gamma) & J_{4,1}(\gamma) \\ J_{4,1}(\gamma) & J_{4,2}(\gamma) \end{vmatrix}}$$ as in [3].

- Let us recall (13). By using the definition (10), we see that the determinants D_4, N^Δ and N^Π are made of terms on which it is possible to apply directly $(27)_{1,2}$, and calculate the limit of the individual elements. As a result, we obtain matrices in which the fourth column is a combination of the third and of the first ones by means of the recurrence relation (12) and so they have 0 determinant, giving

$$\lim_{a\to -1} -\frac{6}{c^2}\frac{N^\Pi}{D_4 + 3N^\Delta} = \left[\frac{0}{0}\right].$$

In order to solve such an indeterminate form, we proceed by observing that with the previous approach (the use of $(27)_{1,2}$) we involve only the terms of order a with respect to $\mathcal{I}$, so we use now (26), in order to have higher order terms. By applying (26) recursively, and with the use of (11), we obtain

$$\int_0^{+\infty} J^*_{m,n+k}\, \mathcal{I}^{a+k}\, d\,\mathcal{I} = (a+1)(a+2)\cdots(a+k)(k_BT)^k \int_0^{+\infty} J^*_{m,n}\, \mathcal{I}^a\, d\,\mathcal{I}. \tag{A.1}$$

Let us transform the determinants D^4, N^Δ and N^Π in the limit that $a\to -1$. We replace the fourth column of the determinant with the result of subtracting the third column from the fourth column and then adding the first column. With the use of (12), (A.1) and finally $(27)_1$, we obtain

$$\lim_{a\to -1}\frac{D_4}{a+1}=\frac{1}{9\left(J_{2,1}(\gamma)\right)^4}\begin{vmatrix} J_{2,1}(\gamma) & J_{2,2}(\gamma) & J_{2,3}(\gamma) & -\frac{2}{\gamma}J_{2,0}-\frac{1}{\gamma^2}J_{2,-1} \\ J_{2,2}(\gamma) & J_{2,3}(\gamma) & J_{2,4}(\gamma) & -\frac{2}{\gamma}J_{2,1}-\frac{3}{\gamma^2}J_{2,0}-\frac{2}{\gamma^3}J_{2,-1} \\ J_{2,3}(\gamma) & J_{2,4}(\gamma) & J_{2,5}(\gamma) & -\frac{2}{\gamma}J_{2,2}-\frac{5}{\gamma^2}J_{2,1}-\frac{8}{\gamma^3}J_{2,0}-\frac{6}{\gamma^4}J_{2,-1} \\ J_{4,0}(\gamma) & J_{4,1}(\gamma) & J_{4,2}(\gamma) & -\frac{2}{\gamma}J_{4,-1}-\frac{3}{\gamma^2}J_{4,-2}-\frac{2}{\gamma^3}J_{4,-3} \end{vmatrix}, \tag{A.2}$$

$$\lim_{a\to -1}\frac{N^{\Delta}}{a+1}=-\frac{1}{27\left(J_{2,1}(\gamma)\right)^4}\begin{vmatrix} J_{2,1}(\gamma) & J_{2,2}(\gamma) & J_{2,3}(\gamma) & -\frac{2}{\gamma}J_{2,0}-\frac{1}{\gamma^2}J_{2,-1} \\ J_{2,2}(\gamma) & J_{2,3}(\gamma) & J_{2,4}(\gamma) & -\frac{2}{\gamma}J_{2,1}-\frac{3}{\gamma^2}J_{2,0}-\frac{2}{\gamma^3}J_{2,-1} \\ J_{4,0}(\gamma) & J_{4,1}(\gamma) & J_{4,2}(\gamma) & -\frac{2}{\gamma}J_{4,-1}-\frac{3}{\gamma^2}J_{4,-2}-\frac{2}{\gamma^3}J_{4,-3} \\ J_{4,1}(\gamma) & J_{4,2}(\gamma) & J_{4,3}(\gamma) & -\frac{2}{\gamma}J_{4,0}-\frac{5}{\gamma^2}J_{4,-1}-\frac{8}{\gamma^3}J_{4,-2}-\frac{6}{\gamma^4}J_{4,-3} \end{vmatrix},$$

$$\lim_{a\to -1}\frac{N^{\Pi}}{a+1}=-\frac{1}{9\left(J_{2,1}(\gamma)\right)^4}\begin{vmatrix} J_{2,1}(\gamma) & J_{2,2}(\gamma) & J_{2,3}(\gamma) & -\frac{2}{\gamma}J_{2,0}-\frac{1}{\gamma^2}J_{2,-1} \\ J_{2,2}(\gamma) & J_{2,3}(\gamma) & J_{2,4}(\gamma) & -\frac{2}{\gamma}J_{2,1}-\frac{3}{\gamma^2}J_{2,0}-\frac{2}{\gamma^3}J_{2,-1} \\ J_{2,3}(\gamma) & J_{2,4}(\gamma) & J_{2,5}(\gamma) & -\frac{2}{\gamma}J_{2,2}-\frac{5}{\gamma^2}J_{2,1}-\frac{8}{\gamma^3}J_{2,0}-\frac{6}{\gamma^4}J_{2,-1} \\ J_{4,1}(\gamma) & J_{4,2}(\gamma) & J_{4,3}(\gamma) & -\frac{2}{\gamma}J_{4,0}-\frac{5}{\gamma^2}J_{4,-1}-\frac{8}{\gamma^3}J_{4,-2}-\frac{6}{\gamma^4}J_{4,-3} \end{vmatrix}.$$

For the last equation, by replacing the fourth row of the determinant with the result of subtracting the third row from the fourth column, and then adding the first row, and by using again the recurrence relation (12), we obtain

$$\lim_{a\to -1}\frac{N^{\Pi}}{a+1}=-\frac{1}{9\left(J_{2,1}(\gamma)\right)^4}\begin{vmatrix} J_{2,1}(\gamma) & J_{2,2}(\gamma) & J_{2,3}(\gamma) & -\frac{2}{\gamma}J_{2,0}-\frac{1}{\gamma^2}J_{2,-1} \\ J_{2,2}(\gamma) & J_{2,3}(\gamma) & J_{2,4}(\gamma) & -\frac{2}{\gamma}J_{2,1}-\frac{3}{\gamma^2}J_{2,0}-\frac{2}{\gamma^3}J_{2,-1} \\ J_{2,3}(\gamma) & J_{2,4}(\gamma) & J_{2,5}(\gamma) & -\frac{2}{\gamma}J_{2,2}-\frac{5}{\gamma^2}J_{2,1}-\frac{8}{\gamma^3}J_{2,0}-\frac{6}{\gamma^4}J_{2,-1} \\ 0 & 0 & 0 & -\frac{4}{\gamma^2}J_{2,-1}-\frac{8}{\gamma^3}J_{2,-2}-\frac{6}{\gamma^4}J_{2,-3} \end{vmatrix}. \tag{A.3}$$

In order to calculate

$$\lim_{a\to -1}\frac{D_4+3N^{\Delta}}{a+1},$$

we observe that the matrices of D_4 and N^{Δ} have the first two rows in common and the third row of D_4 is the fourth row of N^{Δ}. We can change the sign of the third row of D_4 and then exchange its last two rows obtaining the same determinant

$$\lim_{a\to -1}\frac{D_4+3N^{\Delta}}{a+1}=\frac{1}{9\left(J_{2,1}(\gamma)\right)^4}\cdot\begin{vmatrix} J_{2,1}(\gamma) & J_{2,2}(\gamma) & J_{2,3}(\gamma) & -\frac{2}{\gamma}J_{2,0}-\frac{1}{\gamma^2}J_{2,-1} \\ J_{2,2}(\gamma) & J_{2,3}(\gamma) & J_{2,4}(\gamma) & -\frac{2}{\gamma}J_{2,1}-\frac{3}{\gamma^2}J_{2,0}-\frac{2}{\gamma^3}J_{2,-1} \\ J_{4,0}(\gamma) & J_{4,1}(\gamma) & J_{4,2}(\gamma) & -\frac{2}{\gamma}J_{4,-1}-\frac{3}{\gamma^2}J_{4,-2}-\frac{2}{\gamma^3}J_{4,-3} \\ -J_{2,3}(\gamma)+J_{4,1}(\gamma) & -J_{2,4}(\gamma)+J_{4,2}(\gamma) & -J_{2,5}(\gamma)+J_{4,3}(\gamma) & \begin{array}{c}\frac{2}{\gamma}J_{2,2}+\frac{5}{\gamma^2}J_{2,1}+\frac{8}{\gamma^3}J_{2,0}\\ +\frac{6}{\gamma^4}J_{2,-1}-\frac{2}{\gamma}J_{4,0}-\frac{5}{\gamma^2}J_{4,-1}\\ -\frac{8}{\gamma^3}J_{4,-2}-\frac{6}{\gamma^4}J_{4,-3}\end{array} \end{vmatrix}.$$

By using the same procedure already used for N^{Π}, we obtain

$$\lim_{a\to -1}\frac{D_4+3N^{\Delta}}{a+1}=\frac{1}{9\left(J_{2,1}(\gamma)\right)^4}\begin{vmatrix} J_{2,1}(\gamma) & J_{2,2}(\gamma) & J_{2,3}(\gamma) & -\frac{2}{\gamma}J_{2,0}-\frac{1}{\gamma^2}J_{2,-1} \\ J_{2,2}(\gamma) & J_{2,3}(\gamma) & J_{2,4}(\gamma) & -\frac{2}{\gamma}J_{2,1}-\frac{3}{\gamma^2}J_{2,0}-\frac{2}{\gamma^3}J_{2,-1} \\ J_{4,0}(\gamma) & J_{4,1}(\gamma) & J_{4,2}(\gamma) & -\frac{2}{\gamma}J_{4,-1}-\frac{3}{\gamma^2}J_{4,-2}-\frac{2}{\gamma^3}J_{4,-3} \\ 0 & 0 & 0 & \frac{4}{\gamma^2}J_{2,-1}+\frac{8}{\gamma^3}J_{2,-2}+\frac{6}{\gamma^4}J_{2,-3} \end{vmatrix}. \tag{A.4}$$

In such way, we conclude that

$$\lim_{a\to -1}-\frac{6}{c^2}\frac{N^{\Pi}}{D_4+3N^{\Delta}}=\lim_{a\to -1}-\frac{6}{c^2}\frac{N^{\Pi}}{a+1}\frac{a+1}{D_4+3N^{\Delta}}=-\frac{6}{c^2}\frac{\begin{vmatrix} J_{2,1}(\gamma) & J_{2,2}(\gamma) & J_{2,3}(\gamma) \\ J_{2,2}(\gamma) & J_{2,3}(\gamma) & J_{2,4}(\gamma) \\ J_{2,3}(\gamma) & J_{2,4}(\gamma) & J_{2,5}(\gamma) \end{vmatrix}}{\begin{vmatrix} J_{2,1}(\gamma) & J_{2,2}(\gamma) & J_{2,3}(\gamma) \\ J_{2,2}(\gamma) & J_{2,3}(\gamma) & J_{2,4}(\gamma) \\ J_{4,0}(\gamma) & J_{4,1}(\gamma) & J_{4,2}(\gamma) \end{vmatrix}}.$$

Using the recurrence relation on the last row of the denominator, changing the sign to the last row, and shifting the rows down we obtain

$$\lim_{a\to-1} -\frac{6}{c^2}\frac{N^\Pi}{D_4+3N^\Delta} = -\frac{6}{c^2}\frac{\begin{vmatrix} J_{2,1}(\gamma) & J_{2,2}(\gamma) & J_{2,3}(\gamma)\\ J_{2,2}(\gamma) & J_{2,3}(\gamma) & J_{2,4}(\gamma)\\ J_{2,3}(\gamma) & J_{2,4}(\gamma) & J_{2,5}(\gamma)\end{vmatrix}}{\begin{vmatrix} J_{2,0}(\gamma) & J_{2,1}(\gamma) & J_{2,2}(\gamma)\\ J_{2,1}(\gamma) & J_{2,2}(\gamma) & J_{2,3}(\gamma)\\ J_{2,2}(\gamma) & J_{2,3}(\gamma) & J_{2,4}(\gamma)\end{vmatrix}},$$

and the Theorem is proved.

Appendix B. Proof of Theorem 3

The convergence of the phenomenological coefficients in Theorem 3 is proved as follows.

- By using Eq. (21), (16) and Theorem 1, we prove that

$$\lim_{a\to-1}\mu = \lim_{a\to-1} -\frac{nmc^2\theta_{1,2}}{3B^t} = \frac{\tau}{3C_5^{Mono}}nmc^2\frac{J_{4,1}}{J_{2,1}},$$

as in paper [27].
- By using Eq. (21), (9), (10), (16) and Theorem 1 we prove that

$$\lim_{a\to-1}\chi = \lim_{a\to-1} -\frac{2}{3B^q}\frac{nmc^2}{T}\left[3\theta_{0,2}+\theta_{1,2}\frac{p-e}{p}\right] = -\frac{nmc^4\,\tau}{T\left(\frac{J_{4,2}}{J_{4,1}} - \frac{\begin{vmatrix} J_{4,0}(\gamma) & J_{4,1}(\gamma)\\ J_{4,2}(\gamma) & J_{4,3}(\gamma)\end{vmatrix}}{\begin{vmatrix} J_{4,0}(\gamma) & J_{4,1}(\gamma)\\ J_{4,1}(\gamma) & J_{4,2}(\gamma)\end{vmatrix}}\right)}\left(\frac{J_{2,3}}{J_{2,1}}+\frac{p-e}{p}\frac{J_{4,1}}{3\,J_{2,1}}\right),$$

as in paper [27].
- By using Eq. (21), (9), (10), (16), (A.2), (A.3), (A.4), $(29)_3$ and Theorem 1, we prove that

$$\lim_{a\to-1}\nu = \lim_{a\to-1} -\frac{nmc^2}{3B_2^\Pi}\left\{\frac{2}{3}\theta_{1,2}-\frac{\theta'_{1,2}}{\gamma\omega'}+3\frac{N^\Delta}{D_4}\left(\frac{2}{3}\theta_{1,2}-\frac{\theta'_{0,2}}{\gamma\omega'}\right)\right\}$$
$$= -\tau\frac{nmc^2}{3}\left(\lim_{a\to-1}\frac{D_4}{N_\Pi}\right)\left(\lim_{a\to-1}\frac{D_4+3N^\Delta}{D_4}\right)\lim_{a\to-1}\left(\frac{2}{3}\theta_{1,2}-\frac{\theta'_{1,2}}{\gamma\omega'}\right)$$
$$= -\tau\rho c^2\left(\lim_{a\to-1}\frac{\frac{D_4}{a+1}}{\frac{N_\Pi}{a+1}}\right)\left(\lim_{a\to-1}\frac{\frac{D_4+3N^\Delta}{a+1}}{\frac{D_4}{a+1}}\right)\left[\frac{2}{9}\frac{J_{4,1}}{J_{2,1}}-\frac{1}{3}\frac{p}{\partial_\gamma e}\partial_\gamma\left(\frac{J_{4,1}}{J_{2,1}}\right)\right]$$
$$= -\tau\, nmc^2\frac{\begin{vmatrix} J_{2,0}(\gamma) & J_{2,1}(\gamma) & J_{2,2}(\gamma)\\ J_{2,1}(\gamma) & J_{2,2}(\gamma) & J_{2,3}(\gamma)\\ J_{2,2}(\gamma) & J_{2,3}(\gamma) & J_{2,4}(\gamma)\end{vmatrix}}{\begin{vmatrix} J_{2,1}(\gamma) & J_{2,2}(\gamma) & J_{2,3}(\gamma)\\ J_{2,2}(\gamma) & J_{2,3}(\gamma) & J_{2,4}(\gamma)\\ J_{2,3}(\gamma) & J_{2,4}(\gamma) & J_{2,5}(\gamma)\end{vmatrix}}\left[-\frac{1}{4}\frac{p}{\partial_T e}\partial_T\left(\frac{J_{2,3}}{J_{2,1}}+\frac{J_{4,1}}{3\,J_{2,1}}\right)+\frac{2\,J_{4,1}}{9\,J_{2,1}}\right]$$

as in paper [27].
In the second step, the equality $\gamma\,\omega' = \frac{\partial_\gamma e}{p}$ is used, derived from Eqs. (9) and (11).

References

[1] I. Müller, T. Ruggeri, Rational Extended Thermodynamics, second ed., Springer, New York, NY, USA, 1998.
[2] T. Ruggeri, M. Sugiyama, Classical and Relativistic Rational Extended Thermodynamics of Gases, Springer Heidelberg New York Dordrecht London, ISBN: 978-3-030-59143-4, 2021.
[3] I.-S. Liu, I. Müller, T. Ruggeri, Ann. Physics 169 (1986) 191–219.
[4] N.A. Chernikov, Acta Phys. Polonica 27 (1964) 465–489.
[5] Synge J.L. and, The Relativistic Gas, Amsterdam North Holland, 1957.
[6] C. Cercignani, G.M. Kremer, The Relativistic Boltzmann Equation: Theory and Applications, Birkhäuser Verlag, Basel-Boston, 2002.
[7] W. Dreyer, in: I. Müller, T. Ruggeri (Eds.), Maximisation of Entropy for Relativistic and Degenerate Gases in Non-Equilibrium, in: Kinetic Theory and Extended Thermodynamics (Bologna 1987), Pitagora, 1987, pp. 107–123.
[8] M.N. Kogan, Rarefied Gas Dynamics, Vol. I, Academic Press, New York, 1967, pp. 359–368.

[9] W. Dreyer, J. Phys. A Math. Gen. 20 (1987) 6505–6517.
[10] I. Müller, T. Ruggeri, Extended Thermodynamic, first ed., in: Springer Tracts in Natural Philosophy, vol. 37, Springer-Verlag, New York, 1993.
[11] W. Dreyer, W. Weiss, Ann. Inst. Henri Poincaré 45 (1986) 401–418.
[12] G.M. Kremer, Ann. Inst. Henri. Poincaré 45 (1986) 419–440.
[13] T. Arima, S. Taniguchi, M. Sugiyama, T. Ruggeri, Contin. Mech. Thermodyn. 24 (2012) 271–292.
[14] T. Arima, S. Taniguchi, T. Ruggeri, M. Sugiyama, Phys. Lett. A 377 (2013) 2136–2140.
[15] H. Grad, Comm. Pure Appl. Math. 2 (1949) 331–407.
[16] T. Arima, S. Taniguchi, T. Ruggeri, M. Sugiyama, Contin. Mech. Thermodyn. 25 (2013) 727–737.
[17] S. Taniguchi, T. Arima, T. Ruggeri, M. Sugiyama, Phys. Rev. E 89 (2014) 013025.
[18] Arima T., Barbera E., Brini B., M. Sugiyama, Phys. Lett. A, 378 (2014) 2695–2700.
[19] Arima T., M. Sugiyama, Phys. Lett. A 476 (2023) 5.
[20] F. Brini, L. Seccia, Proc. R. Soc. A 478 (2267) (2022) 20220246.
[21] E. Barbera, A. Pollino, Mathematics 10 (16) (2022) 14.
[22] T. Arima, S. Taniguchi, T. Ruggeri, M. Sugiyama, Phys. Lett. A 376 (2012) 2799–2803.
[23] G. Boillat, T. Ruggeri, Arch. Ration. Mech. Anal. 137 (1997) 305–320.
[24] S. Taniguchi, T. Arima, T. Ruggeri, M. Sugiyama, Phys. Fluids 26 (2014) 016103.
[25] S. Pennisi, T. Ruggeri, Ann. Physics 377 (2017) 415–445.
[26] M.C. Carrisi, S. Pennisi, T. Ruggeri, Ann. Physics 405 (2019) 298–307.
[27] M.C. Carrisi, S. Pennisi, T. Ruggeri, Ann. Phys. 428 (2021) 168447.
[28] S. Weinberg, Agron. J. 168 (1971) 175–194.
[29] L. Rezzolla, O. Zanotti, Relativistic Hydrodynamics, Oxford University Press, USA, 2013.
[30] T. Arima, M.C. Carrisi, S. Pennisi, T. Ruggeri, Entropy 24 (1) (2022) 30.
[31] S. Pennisi, T. Ruggeri, J. Phys. Conf. Ser. 1035 (2018) 012005.
[32] S. Pennisi, J. Nat. Sci. Tech. 2 (2021) 12–21.
[33] E. Ikenberry, C. Truesdell, J. Ration. Mech. Anal. 5 (1956) 1–54.
[34] T. Arima, M.C. Carrisi, S. Pennisi, T. Ruggeri, Int. J. Non-Linear Mech 137 (2021) 11.
[35] S. Pennisi, T. Ruggeri, J. Math. Phys. 59 (2018) 043102.
[36] C. Borgnakke, P.S. Larsen, J. Comput. Phys. 18 (1975) 405–420.
[37] J.-F. Bourgat, L. Desvillettes, P. Le Tallec, B. Perthame, Eur. J. Mech. B Fluids 13 (1994) 237–254.
[38] M.C. Carrisi, S. Pennisi, T. Ruggeri, Contin. Mech. Thermodyn. 31 (2019) 401–412.
[39] T. Arima, T. Ruggeri, M. Sugiyam, S. Taniguchi, Ann. Physics 372 (2016) 83–109.
[40] T. Ruggeri, Phil. Trans. Roy. Soc. A 381 (2263) (2023) 20220371.
[41] G. Boillat, T. Ruggeri, Contin. Mech. Thermodyn. 9 (1997) 205–212.
[42] T. Arima, A. Mentrelli, T. Ruggeri, Ann. Physics 345 (2014) 111–140.