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Abstract: This paper discusses the problem of stabilizing third order MIMO system whose constant high-frequency gain matrix contains elements of unknown signs. This scenario is commonly referred to as the control problem of systems with "uncertain control direction" and it has not been satisfactorily addressed yet in the MIMO setting. We develop a geometric framework to solve the underlying stabilization problem by a discontinuous feedback control inspired to the so-called "unit vector" method.

Adaptive stabilization of 3rd-order MIMO systems with uncertain control direction

G. BARTOLINI, A PILLONI, A. PISANO, E. USAI

Abstract— **This paper discusses the problem of stabilizing third order MIMO system whose constant high-frequency gain matrix contains elements of unknown signs. This scenario is commonly referred to as the control problem of systems with "uncertain control direction" and it has not been satisfactorily addressed yet in the MIMO setting. We develop a geometric framework to solve the underlying stabilization problem by a discontinuous feedback control inspired to the so-called "unit vector" method.**

I. Introduction

The control of single-input single-output (SISO) systems with high frequency gain (HFG) of unknown sign has been the subject of many investigations [7], [1]. The transition from single to multi-input system is far from being straightforward. Existing solutions are based upon the satisfaction of certain strong structural assumptions regarding the HFG matrix G (such as, e.g., $G > 0$, $G + G^T > 0$, G diagonally dominant). Often it is tautologically assumed to know in advance a constant matrix K that transforms the HFG control matrix through right-multiplication, in a matrix falling into one of the above classes which shrinks the considered class of uncertain dynamics considerably.

A major progress in the area has been the introduction of techniques that only requires the knowledge of an "unmixing set" for $\mathcal{D}(G)$, where $\mathcal{D}(G)$ is the set to which the HFG matrix G is assumed to belong (e.g., the set $\mathcal{D}(G) = GL(m, \Re)$ of real-valued nonsingular square matrices of dimension m).

A matrix set U is unmixing $\mathcal{D}(G)$ if for any $M \in \mathcal{D}(G)$ there is at least one element $K \in$ U such that $\sigma(MK) \subset C^+$, where $\sigma(\cdot)$ denotes spectrum and C^+ is the open right half complex plane. With reference to $\mathcal{D}(G) = GL(m, \Re)$, the existence of unmixing sets with finite cardinality (called "finite spectrum unmixing sets", FSUSs for brevity) has been proven in [5] for any $m \in \mathbb{N}$. In [9] it is assumed that a FSUS is known a-priori to the designer. The control is expressed as an high-gain adaptive output-feedback where the gain-matrix belongs to the finite spectrum unmixing set and a suitable strategy for cycling through the elements of this set is provided. Inherent to this strategy is the definition of a nonnegative and non decreasing scalar adaptive gain $k(t)$. This gain premultiplies the manipulable control (which can be proportional to the output vector $\mathbf{y}$ or, in a different "robust" realization of the approach, to the corresponding unit vector $\mathbf{y}/\|\mathbf{y}\|$) and it also drives the mechanism for cycling through the elements of the unmixing set.

The paper [8] also assumes the prior availability of an unmixing set, and makes use of the unit vector discontinuous control approach. The cycling mechanism through the elements of the unmixing set proposed in [8] is however radically different from that of [9], being based on so-called "monitoring functions". The construction of the monitoring functions requires that a lower bound to the norm of G must be known in advance.

Some examples of how to build a finite spectrum unmixing set for $\mathcal{D}(G) = GL(m, \Re)$ can be found in [5], [12]. Unfortunately, the cardinality of the unmixing sets given by the general construction in [5] is far too large than would be reasonable for applications, and hardly anything is known on the minimum cardinality of the unmixing sets for arbitrary values of m. For the case $m = 2$ an unmixing set of cardinality 6 is given in [5]. Zhu [12] has constructed an unmixing set having cardinality 32 for the case $m = 3$.

The contribution of the present work is twofold. First, based on a novel geometric reasoning where the quaternion based representation of rotations plays an important role, we construct an unmixing set for $\mathcal{D}(G) = GL(3, \Re)$ having cardinality 25, i.e. reduced cardinality compared to that proposed in [12]. Then, we propose a modified version of the adaptive strategy for cycling through the elements of the set as compared to that proposed in [9]. More precisely, we devise a method capable of identifying redundant elements of the unmixing set, namely elements $K_i \in U$ such that condition $\sigma(GK) \subset C^+$ is not verified. These elements K_i are then removed from the unmixing set in runtime, yielding improved transient behaviour of the closed loop system. It is worth to point out that the proposed method for identifying, and removing from the unmixing set, the mentioned redundant matrices shares some similarity with the monitoring functions proposed in [8].

The paper is structured as follows. Section II presents the geometric-driven derivation of the unmixing set for $\mathcal{D}(G) = GL(3, \Re)$. Section III illustrates the discontinuous adaptive control feedback law employing the constructed unmixing set and its modified version where specific redundant elements of the unmixing set can possibly be removed on line. Section IV discusses some simulation results. Section V makes a critical discussion of the scope and future perspectives of the present line of investigation whereas the Appendix Section I reports the coordinates of some characteristic points of the regular dodecahedron which are employed in the FSUS construction procedure.

The authors are with the Department of Electrical and Electronic Engineering (DIEE), University of Cagliari, Italy. Corr. author: A. Pisano (apisano@unica.it)

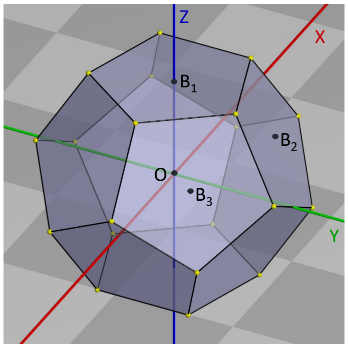

Fig. 1. Regular dodechaedron.

II. A FSUS FOR $GL(3,\Re)$

A. Preliminaries

Any matrix $G \in GL(3,\Re)$ can be factorized as follows (Polar Decomposition Theorem, [2])

$$G = P_G O_G, \tag{1}$$

with $P_G = P_G{}^T, \quad P_G > 0, \quad O_G \in O(3)$.

The symmetric positive definite (SPD) matrix P_G and the orthogonal matrix O_G take the explicit forms

$$P_G = [GG^T]^{1/2}, \qquad O_G = [GG^T]^{-1/2}G. \tag{2}$$

The so-called "axis-angle" representation of a special orthogonal matrix $R \in SO(3)$ (also referred to as the "Rodrigues formula") is reported as follows

$$R = R(\mathbf{r},\theta) = cos\,(\theta)\,I + (1 - cos\theta)\mathbf{r}\mathbf{r}^T + sin\theta S(\mathbf{r}), \tag{3}$$

where $\mathbf{r} = [r_x\ r_y\ r_z] \in R^3$, $\|\mathbf{r}\| = 1$, is the rotation axis, $\theta \in [-\pi,\ \pi]$ is the rotation angle, I is the identity matrix, and

$$S(\mathbf{r}) = \begin{bmatrix} 0 & -r_z & r_y \\ r_z & 0 & -r_x \\ -r_y & r_x & 0 \end{bmatrix}. \tag{4}$$

The relation

$$\frac{R(\mathbf{r},\theta) + R^T(\mathbf{r},\theta)}{2} = cos\,(\theta)\,I + (1 - cos\theta)\mathbf{r}\mathbf{r}^T \tag{5}$$

will also be used afterwards.

B. FSUS construction

The proposed construction method of the unmixing set for $GL(3,\Re)$ is strictly related to the structure and properties of a regular dodechaedron, that is depicted in the Figure 1. Let B_i $(i = 1,...,12)$ be the baricenters of the twelve pentagons which form the dodechaedron (the corresponding coordinates are given in the Appendix). Define

$$\mathbf{b}_i = \frac{(B_i - 0)}{\|B_i - 0\|}, \quad i = 1,...,12 \tag{6}$$

and

$$T = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}. \tag{7}$$

Lemma 1: A finite spectrum unmixing set for $GL(3,\Re)$ is the set of matrices $\{K_1, K_2, ..., K_{25}\}$ defined by

$$K_i = \begin{cases} R(\mathbf{b}_i,\beta^*) & i = 1,2,...,12 \\ TR(\mathbf{b}_{i-12},\beta^*) & i = 13,14,...,24 \end{cases} \tag{8}$$

where $\beta^* = -3\pi/4$ and $K_{25} = I$.

Proof of Lemma 1.

Exploiting well-known properties of matrix determinants one derives from the expression of O_G in (2) that $det(O_G) = sign(det(G))$. Then, if $det(G) > 0$ the polar decomposition (1) involves a special orthogonal matrix $O_G = R_G \in SO(3)$, i.e., a rotation matrix. Let us temporarily assume that $det(G) > 0$, and let $R_G = R(\mathbf{r}_G,\theta_G)$.

If $|\theta_G| < \pi/2$ then matrix $-G$ is Hurwitz. To prove this statement, consider the auxiliary dynamical system

$$\dot{\mathbf{y}} = -G\mathbf{y} = -P_G R_G(\mathbf{r}_G,\theta_G)\mathbf{y}. \tag{9}$$

Using the Lyapunov function $V = \frac{1}{2}\mathbf{y}^T P_G^{-1}\mathbf{y}$ one straightforwardly shows that its time derivative along the solutions of (9) is

$$\dot{V} = -\mathbf{y}^T R_G(\mathbf{r}_G,\theta_G)\mathbf{y} = -\mathbf{y}^T\frac{R_G + R_G^T}{2}\mathbf{y}. \tag{10}$$

By virtue of (5) one further manipulates (10) to get

$$\dot{V} = -cos\,(\theta_G)\,\|\mathbf{y}\|^2 - (1 - cos\,(\theta_G))\,(\mathbf{y}^T\mathbf{r}_G)^2. \tag{11}$$

It straightforwardly derives from (11) that if $|\theta_G| < \pi/2$ the linear system (9) is asymptotically stable, which implies that matrix $-G$ is Hurwitz. Hence, if $|\theta_G| < \pi/2$ the identity element $K_{25} = I$ meets the requirement of the FSUS since $\sigma\,(GK_{25}) \in C^+$, and from now on the case in which $\theta_G \in [-\pi,\pi/2)\bigcup(\pi/2,\pi]$ will be investigated only.

Now consider the matrices GK_i $(i = 1,2,...,12)$ under the temporary assumption that $det(G) > 0$ and by restricting the analysis to the case $\theta_G \in [-\pi,\pi/2)\bigcup(\pi/2,\pi]$. One has that

$$GK_i = P_G R_G(\mathbf{r}_G,\theta_G)R(\mathbf{b}_i,\beta^*), \quad i = 1,2,...,12. \tag{12}$$

By similar reasoning as that done before, we are going to demonstrate that a sufficient condition for the matrix $-GK_i$ to be Hurwitz is that the rotation angle θ_i^* of the matrix $R_G(\mathbf{r}_G,\theta_G)R(\mathbf{b}_i,\beta^*)$ is such that $|\theta_i^*| < \pi/2$. To prove this statement, let $R_G(\mathbf{r}_G,\theta_G)R(\mathbf{b}_i,\beta^*) = R_2(\mathbf{c}_i,\theta_i^*)$ and consider the dynamical system

$$\dot{\mathbf{y}} = -GK_i\mathbf{y} = -P_G R_2(\mathbf{c}_i,\theta_i^*)\mathbf{y}. \tag{13}$$

Using the same Lyapunov function $V = \frac{1}{2}\mathbf{y}^T P_G^{-1}\mathbf{y}$ it yields that the corresponding time derivative along the solutions of (13) is

$$\dot{V} = -\mathbf{y}^T R_2(\mathbf{c}_i,\theta_i^*)\mathbf{y} = -\mathbf{y}^T\frac{R_2 + R_2^T}{2}\mathbf{y}. \tag{14}$$

Again, by exploiting (5) one further manipulates (14) as

$$\dot{V} = -cos\,(\theta_i^*)\,\|\mathbf{y}\|^2 - (1 - cos\,(\theta_i^*))\,(\mathbf{y}^T\mathbf{c}_i)^2, \tag{15}$$

from which it is concluded that if $0 < cos(\theta_i^*) < 1$, i.e. if $|\theta_i^*| < \pi/2$, then the linear system (13) is asymptotically

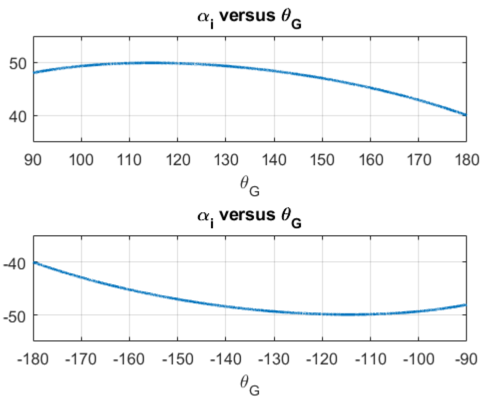

Fig. 2. Plot of α_i in (17)-(18) vs. θ_G

stable, which implies that matrix $-GK_i$ is Hurwitz and, therefore, that $\sigma\left(GK_i\right) \in C^+$.

To get an explicit representation of θ_i^*, which is the rotation angle of the composition between two rotation matrices, we take advantage of the properties of quaternions, particularly the fact that when quaternions are used to represent rotations, composition of rotations simply corresponds to multiplication of quaternions (see [11]). Lengthy but straightforward manipulations yield that

$$\begin{aligned} cos\left(\frac{\theta_i^*}{2}\right) &= \frac{1}{2}cos\left(\frac{\theta_G+\beta^*}{2}\right)(1+cos(\alpha_i)) \\ &+ \frac{1}{2}cos\left(\frac{\theta_G-\beta^*}{2}\right)(1-cos(\alpha_i)), \end{aligned} \tag{16}$$

where α_i is the angle between the rotation axes $\mathbf{r}_G$ and $\mathbf{b}_i$. Note that $\mathbf{r}_G$ and θ_G are uncertain, whereas $\mathbf{b}_i$ and β^* are design parameters. It must be shown that there exists at least a value of $i \in 1, 2, ..., 12$ for which $|\theta_i^*| < \pi/2$, i.e. for which the right hand side of (16) is larger then $\sqrt{2}/2$. By manipulating (16) one derives that

$$\alpha_i = arccos\left(\delta_i\right), \tag{17}$$

with

$$\delta_i = 2\frac{cos\left(\frac{\theta_i^*}{2}\right) - \frac{1}{2}cos\left(\frac{\theta_G+\beta^*}{2}\right) - \frac{1}{2}cos\left(\frac{\theta_G-\beta^*}{2}\right)}{cos\left(\frac{\theta_G+\beta^*}{2}\right) - cos\left(\frac{\theta_G-\beta^*}{2}\right)}. \tag{18}$$

Plugging the limit value $cos\left(\frac{\theta_i^*}{2}\right) = \sqrt{2}/2$ into the right hand side of (18) and analyzing the corresponding plot of α_i in the interval of interest $\theta_G \in [-\pi, \pi/2) \bigcup (\pi/2, \pi]$, one concludes that as long as the magnitude of angle α_i does not exceed the value of 40.06 the condition $cos\left(\frac{\theta_i^*}{2}\right) > \sqrt{2}/2$ is in force. Thus the problem boils down into that of finding a set of rotation axes $\mathbf{b}_i$ such that at least one of them forms an angle α_i with uncertain rotation axis $\mathbf{r}_G$ not exceeding (in magnitude) the value 40.06.

By exploiting geometrical properties of the regular dodechaedron [13] namely the fact that the angle between the position vector of the baricenter of a generic pentagon and the position vector of an arbitrary point of the same pentagon does not exceed the value of 37.37 degrees, it can be shown that for arbitrary $\mathbf{r}_G$ and $\theta_G \in [-\pi, \pi/2) \bigcup (\pi/2, \pi]$ there is at least one value of $i \in \{1, 2, ..., 12\}$ such that the right hand side of (16) is larger than $\sqrt{2}/2$, which implies that $|\theta_i^*| \le \pi/2$. It is thus demonstrated that the set $\{K_1, K_2, ..., K_{11}, K_{12}, K_{25}\}$ is unmixing the set of matrices belonging to $GL(3, \Re)$ and having positive determinant. To accomodate the case when matrix G has negative determinant, notice that the postmultiplication of any matrix by the strictly unimodular matrix T reverses the sign of its determinant. Hence, if matrix G has negative determinant then matrix GT has positive determinant. Analogous developments as those presented in the preceding part of the proof show that the set $\{K_{13}, K_{14}, ..., K_{23}, K_{24}, K_{25}\}$ is unmixing the set of matrices belonging to $GL(3, \Re)$ and having negative determinant. Lemma 1 is proved. □

III. Adaptive Stabilization

Consider a driftless 3rd-order square system

$$\dot{\mathbf{y}} = G\mathbf{u}, \quad \mathbf{y}, \mathbf{u} \in \Re^3, \quad G \in GL(3, \Re), \tag{19}$$

where $\mathbf{y} = [y_1, y_2, y_3]^T$ is the state vector, $\mathbf{u}$ is the manipulable input vector, and G is the uncertain nonsingular high-frequency gain (HFG) matrix. No additional information on G is assumed.

Inspired by [9], and exploiting the finite spectrum unmixing set presented in the Lemma 1, we consider the adaptive unit-vector control strategy

$$\mathbf{u} = -k(t)K_{\Sigma(k(t))}\frac{\mathbf{y}}{\|\mathbf{y}\|}, \tag{20}$$

$$\dot{k}(t) = \gamma\|\mathbf{y}\|^2, \qquad k(0) = 0, \qquad \gamma > 0, \tag{21}$$

where function $\Sigma : \Re \to \{1, 2, \ldots, N\}$, where $N = 25$ is the cardinality of the FSUS, is given by

$$\Sigma(\kappa) = i. \quad \kappa \in [\kappa_{(i-1)+jN}, \kappa_{i+jN}), \quad j = 0, 1, 2, ... \tag{22}$$

and the sequence $\{\kappa_h\}_{h=0,1,2,...}$ is defined as

$$\kappa_0 = 0, \;\; \kappa_1 > 1, \;\; \kappa_{h+1} = \kappa_h^{\rho}, \;\; \rho > 1, \;\; h = 1, 2, , ... \tag{23}$$

The control gain matrix $K_{\Sigma(k(t))}$ belongs to the unmixing set and a suitable strategy for cycling through the elements of this set is suggested.

The next Theorem is in force.

Theorem 1: Consider system (19) and the FSUS $\{K_1, K_2, ..., K_{25}\}$ described in Lemma 1. The application of the discontinuous feedback (20)-(23) guarantees that the closed-loop system approaches the origin asymptotically whereas $k(t)$ settles to a constant finite value k^*.

Proof of Theorem 1. The proof follows analogous arguments as those in the Proof of Theorem 1 in [9] □

The main drawback of the cycling mechanism in (20)-(23) between the matrices of the FSUS is that it can yield poor transient performance due to the large number of elements of the FSUS and to the progressively longer dwelling time at each element of the set. Thus, an improvement of the cycling mechanism is proposed where some element K_i can possibly be removed on line from the FSUS as long as a certain growth condition involving $\|\mathbf{y}\|$ is detected during operation.

The implementation of this procedure, however, requires the availability of some additional informations regarding the matrix P_G which constitutes the positive definite matrix entering the polar decomposition (2) of the HFG matrix G. This additional information requirement is specified by means of the following Assumption.

Assumption 1: Let a constant π be known in advance such that

$$\pi \geq \frac{\lambda_M(P_G)}{\lambda_m(P_G)} \tag{24}$$

where $\lambda_M(P_G)$ and $\lambda_m(P_G)$ denote respectively the largest and smallest eigenvalues of the symmetric positive-definite matrix P_G. □

The procedure for the possible on-line removal of selected elements of the FSUS is now outlined. Let $t_0 = \kappa_0 = 0$ and t_ℓ $(\ell = 1, 2, ...)$, be the sequence of time instants such that $k(t_\ell) = \kappa_\ell$. Since $k(t)$ is strictly increasing, it follows from (22) that

$$\Sigma(k(t)) = \Sigma(\kappa_\ell) \quad \forall t \in \mathcal{T}_\ell \equiv [t_\ell, t_{\ell+1}), \;\; \ell = 0, 1, 2, ... \tag{25}$$

The mentioned procedure involves the continuous monitoring of $\|\mathbf{y}(t)\|$ in all the time intervals $\mathcal{T}_\ell$ where the piecewise-constant function $\Sigma(k(t))$ remains constant, according to (25). If there exists a time instant $t_{s,\ell} \in \mathcal{T}_\ell$ such that

$$\|\mathbf{y}(t_{s,\ell})\|_2^2 > \pi \|\mathbf{y}(t_\ell)\|_2^2, \tag{26}$$

where π is the constant defined in the Assumption 1, then the matrix $-GK_{\Sigma(\kappa_\ell)}$ cannot be Hurwitz. This means that matrix $K_{\Sigma(\kappa_\ell)}$ can be removed from the unmixing set thus avoiding that the same "wrong" matrix is chosen at any future time instant. At the time instants $t_{s,\ell} \in \mathcal{T}_\ell$ (if any) where condition (26) is detected, the value of $k(t)$ is changed according to

$$k(t_{s,\ell}^+) = \Sigma(\kappa_{\ell+1}), \;\; \ell = 1, 2, ... \tag{27}$$

in such a way that the successive element of the FSUS is immediately selected thus speeding up the transient.

The cycling mechanism (22) has to be adjusted as follows to account for the variation of the cardinality of the FSUS

$$\Sigma(\kappa) = i, \quad \kappa \in [\kappa_{(i-1)+jN_\ell(t)}, \kappa_{i+jN_\ell(t)}), \;\; j = 0, 1, .. \tag{28}$$

where

$$N_\ell(t) = N - (\ell - 1), \; t_{s,\ell} \leq t < t_{s,\ell+1} \leq t, \; \ell = 0, 1, ... \tag{29}$$

and, by definition, $t_{s,0} = 0$.

The resulting closed-loop performance are analyzed in the next Theorem 2.

Theorem 2: Consider system (19) and the FSUS $\{K_1, K_2, ..., K_{25}\}$ described in Lemma 1. Let Assumption 1 holds, where P_G is the positive definite matrix entering the polar decomposition (2) of the HFG matrix G. Let the sequence $\{\kappa_h\}_{h=0,1,2,...}$ be defined as in (23). Apply the discontinuous unit-vector feedback (20), (28), (29) where $k(t)$ is governed by (21) and (27), where $t_0 = \kappa_0 = t_{s,0} = 0$, t_ℓ $(\ell = 1, 2, ...)$, is the sequence of time instants such that $k(t_\ell) = \kappa_\ell$ and $t_{s,\ell} \in \mathcal{T}_\ell \equiv [t_\ell, t_{\ell+1})$ is the sequence of time instants (if any) where condition (26) is detected. At any instants $t_{s,\ell}$ remove the matrix $K_{\Sigma(\kappa_\ell)}$ from the unmixing set. Then the closed-loop system approaches the origin asymptotically whereas $k(t)$ settles to a constant finite value k^*.

Proof of Theorem 2. If follows from (25) that when $t \in \mathcal{T}_\ell \equiv [t_\ell, t_{\ell+1})$ the active element of the unmixing set is that having index $\Sigma(\kappa_\ell)$. The closed loop dynamics in the time interval $t \in \mathcal{T}_\ell$ is therefore

$$\dot{\mathbf{y}} = -k(t)GK_\Sigma(\kappa_\ell)\frac{\mathbf{y}}{\|\mathbf{y}\|}. \tag{30}$$

If matrix $-GK_\Sigma(\kappa_\ell)$ is Hurwitz then the Lyapunov function $V = \frac{1}{2}\mathbf{y}^T P_G^{-1}\mathbf{y}$ has a negative definite time derivative in the time interval $t \in \mathcal{T}_\ell$, which means tat

$$V(t) \leq V(t_\ell) \;\; \forall t \in \mathcal{T}_\ell. \tag{31}$$

Clearly, the existence of a time instant $t_{s,\ell} \in \mathcal{T}_\ell$ such that relation (26) holds reveals that relation (31) is violated, thus implying that matrix $-GK_\Sigma(\kappa_\ell)$ is not Hurwitz. Thus, matrix $K_{\Sigma(\kappa_\ell)}$ can be removed from the unmixing set. The application of (27) guarantees that the successive element $K_\Sigma(\kappa_{\ell+1})$ of the unmixing set is activated from $t = t_{s,\ell}^+$ on. This removal of a "wrong" matrix from the unmixing set has no impact on the asymptotic convergence properties already demonstrated in Theorem 1. Theorem 2 is proved. □

IV. Simulation results

Consider system (19) with the HFG matrix

$$G = \begin{bmatrix} 1 & -2 & 0.3 \\ -1 & -1 & 0.5 \\ 0.5 & 0 & -2 \end{bmatrix}. \tag{32}$$

Matrix G is nonsingular ($det(G) = 5.65$) and the matrices P_G and $O_G = R_G$ of its spectral decomposition are

$$P_G = \begin{bmatrix} 2.233 & 0.317 & 0.009 \\ 0.317 & 1.398 & -0.44 \\ 0.009 & -0.44 & 2.013 \end{bmatrix}, \tag{33}$$

$$O_G = R_G = \begin{bmatrix} 2.233 & 0.317 & 0.009 \\ 0.317 & 1.398 & -0.44 \\ 0.009 & -0.44 & 2.013 \end{bmatrix}. \tag{34}$$

One has that

$$\sigma(G) = \{1.72, -1.42, -2.3\}, \tag{35}$$

$$\sigma(P_G) = \{2.41, 2.13, 1.09\}, \tag{36}$$

which yields that

$$\lambda_M(P_G)/\lambda_m(P_G) = 2.202. \tag{37}$$

The initial condition is $y(0) = [1, 2, 3]^T$. The FSUS $\{K_1, K_2, ..., K_{25}\}$ is constructed according to the procedure outlined in the Lemma 1. The coordinates of points B_i (i=1,2,...,12) are given in the Appendix Section with reference to the regular dodecahedron inscribed in the unitary sphere centered at the origin. The “good” matrices of the FSUS of Lemma 1, i.e. those matrices K_i such that $\sigma(GK_i) \in C^+$, correspond to the indexes $i \in \{6, 11, 12\}$.

In the first simulation (TEST 1) the control law discussed in Theorem 1 is implemented with parameters $\gamma = 0.0001$ and $\rho = 1.2$. Figure 3 show the time evolution of the vector $\mathbf{y}$ components, whereas Figures 4 and 5 depict, respectively, the cycling function $\Sigma(k(t))$ and the adaptive gain $k(t)$. It is seen that the state variables exhibit transient peaking. The cyclying function $\Sigma(k(t))$ settles to the value 12 which corresponds to one of the ”good” matrices of the unmixing set. The adaptive gain $k(t)$ grows monotonically until it settles to a constant value.

In the second simulation run (TEST 2) the control law discussed in Theorem 2 is implemented with the additional parameter $\pi = 5$, chosen according to (37). Condition (26) is detected at the time instant $t = t_{s,1} \approx 24.29$, and at that time instant the active control gain matrix K_1 is removed from the unmixing set. As a result, the state trajectory, shown in Figure 6 exhibit faster transient and reduced peaking as compared to TEST 1. At the same time, the steady state value of the adaptive gain, whose time evolution is shown in Figure 8, is also reduced, meaning that less control effort is employed. It should be noticed that the steady state value of the cycling function $\Sigma(k(t))$, shown in the Figure 7 and equal to 5, actually correspond to the “good” matrix K_6 of the original unmixing set, given that at $t = t_{s,1} \approx 24.29$ matrix K_1 is removed from the unmixing set.

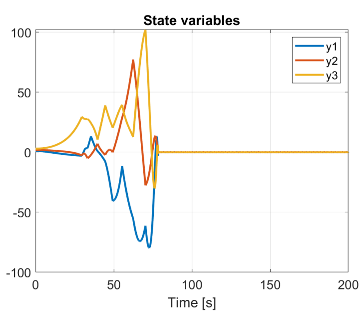

Fig. 3. State vector in Test 1.

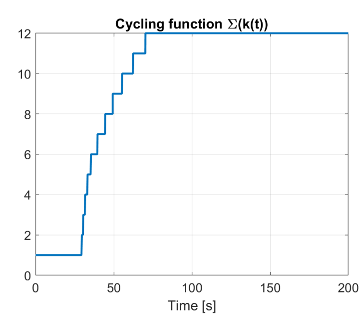

Fig. 4. Cycling function $\Sigma(k(t))$ in Test 1.

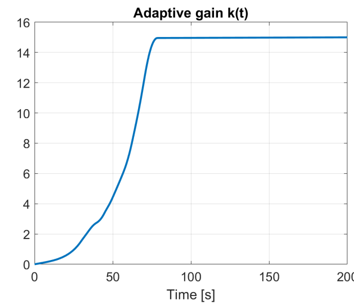

Fig. 5. Adaptive gain $k(t)$ in Test 1.

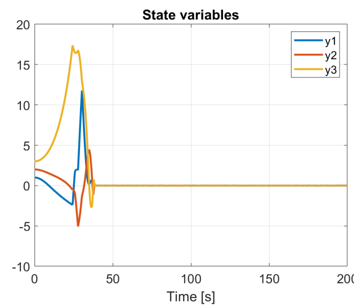

Fig. 6. State vector in Test 2.

V. CONCLUSIONS

The adaptive stabilization of a 3-dimensional MIMO integrator process modeled by (19) has been addressed under the unique restriction that the underlying uncertain HFG matrix G is nonsingular. We refer to the adaptive unit-vector control approach presented in [9] that only require the prior knowledge of an unmixing set of finite cardinality for the set $GL(3, \Re)$. The first novelty of the present work is the construction of an unmixing set for $\mathcal{D}(G) = GL(3, \Re)$ having cardinality 25, i.e. reduced cardinality compared to that proposed in [12]. The implementation of the adaptive control strategy proposed in [9] with an unmixing set of reduced cardinality yields improved transient performance. The second novel contribution of the present work is a on-line criterion for detecting, and removing from the unmixing set, redundant matrices of the set yielding unstable closed-loop behaviour, thus further improving the transient features of the closed loop system.

The class of controlled processes is intentionally kept very simple in the present work to keep the main focus of the treatment on the more significant novel contributions. The more general class of systems

$$\dot{\mathbf{y}} = \mathbf{\Phi}(\mathbf{y}, t) + G\mathbf{u}, \quad \mathbf{y}, \mathbf{u} \in \Re^3, \quad G \in GL(3, \Re), \qquad (38)$$

where $\|\mathbf{\Phi}(\mathbf{y}, t)\| \leq F(\mathbf{y}, t)$ for some known upperbounding function $F(\mathbf{y}, t)$ will be studied in next research works. Notice that the presence of the gain $k(t)$ as premultiplying factor in the control law (20) is redundant given that a control magnitude large enough to rejecting such an uncertain drift terms can be derived. However, the presence of the gain $k(t)$ as premultiplying factor is crucial in the convergence proof

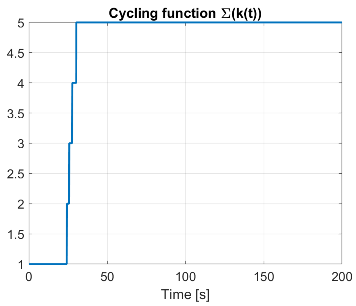

Fig. 7. Cycling function $\Sigma(k(t))$ in Test 2.

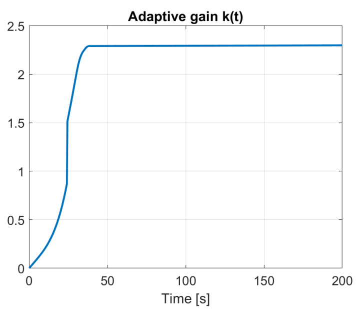

Fig. 8. Adaptive gain $k(t)$ in Test 2.

presented in [9], based on a certain contradiction argument. Therefore, in next research we will be investigating the properties of the closed-loop system with the modified control

$$\mathbf{u} \;=\; -\Gamma F(\mathbf{y},t) K_{\Sigma(k(t))} \frac{\mathbf{y}}{\|\mathbf{y}\|}, \tag{39}$$

with sufficiently large $\Gamma > 0$, aiming at counteracting the potentially large growth of the magnitude of the control vector $\mathbf{u}$. Besides this, it is also be of great interest to generalize to higher dimensions the proposed quaternion-based construction method of the FSUS for the set $GL(m, \Re)$ with $m > 3$ such that higher order MIMO processes could correspondingly be managed.

Appendix I
Coordinates of characteristic points of the regular dodecahedron

Define

$$A_1 = \sqrt{\frac{10-2\sqrt{5}}{15}}, \quad A_2 = \sqrt{\frac{5+2\sqrt{5}}{15}}, \tag{40}$$

$$A_3 = \frac{-1+\sqrt{5}}{4}, \quad A_4 = \sqrt{\frac{5+\sqrt{5}}{8}}, \tag{41}$$

$$A_5 = \frac{1+\sqrt{5}}{4}, \quad A_6 = \sqrt{\frac{5-\sqrt{5}}{8}}, \tag{42}$$

$$A_7 = \frac{10+2\sqrt{5}}{15}, \quad A_8 = \sqrt{\frac{5-2\sqrt{5}}{15}}, \tag{43}$$

The coordinates of points B_i (i=1,2,...,12) used in Lemma 1 for deriving the FSUS, namely the coordinates of the barycenters of the twelve pentagons that form the regular dodecahedron inscribed in the unit sphere centered at the origin, are given as follows (see [13]):

$$B_1 = \left[\frac{A_1(1+2A_3-2A5)}{5}, 0, A_2\right]^T \tag{44}$$

$$B_2 = \begin{bmatrix} \frac{A_1(1+A_3)+A_7(1+A_3+A_5)}{5} \\ \frac{A_4(A_1+A_7)+A_7A_6}{5} \\ \frac{2A_2+A_8}{5} \end{bmatrix} \tag{45}$$

$$B_3 = \begin{bmatrix} \frac{A_1(A_3-A_5)-A_5A_7}{5} \\ \frac{A_1(A_4+A_6)+A_7(2A_4+A_6)}{5} \\ \frac{2A_2+A_8}{5} \end{bmatrix} \tag{46}$$

$$B_4 = \begin{bmatrix} \frac{A_5(A_1+A_7)-A_1A_3}{5} \\ \frac{A_7(A_6+2A_4)+A_1(A_4+A_6)}{5} \\ -\frac{2A_2+A_8}{5} \end{bmatrix} \tag{47}$$

$$B_5 = \begin{bmatrix} \frac{A_7+2A_5(A_1+A_7)}{5} \\ 0 \\ -\frac{2A_2+A_8}{5} \end{bmatrix} \tag{48}$$

$$B_6 = \begin{bmatrix} \frac{(A_1+A_7)(1+A_3)+A_5A_7}{5} \\ \frac{-A_4(A_1+A_7)-A_6A_7}{5} \\ \frac{2A_2+A_8}{5} \end{bmatrix} \tag{49}$$

$$B_j = -B_{j-6}, \quad j = 7, 8., ...12 \tag{50}$$

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