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Foliated Schwarz symmetry of solutions to a cooperative system of equations involving nonlocal operators

Antonio Greco[1] ⬤ · Sven Jarohs[2]

Abstract

In this paper, we prove foliated Schwarz symmetry of solutions to a cooperatively coupled system of equations involving nonlocal operators. Here, the class of nonlocal operators covers in particular the case of the fractional Laplacian. Moreover, we give an explicit example of a nonlocal nonlinear system, in which our result can be applied.

Keywords Nonlocal operator · Axial symmetry · Maximum principle for systems

1 Introduction

In the following, we investigate symmetry properties of solutions of a system of equations in bounded radial domains. More precisely, for $m \in \mathbb{N}$ we investigate bounded continuous solutions of

$$\begin{cases} I_i u_i = f_i(|x|, u_1, \dots, u_m) & \text{in } \Omega, i = 1, \dots, m \\ u_1 = \cdots = u_m = 0 & \text{in } \mathbb{R}^N \backslash \Omega \end{cases} \tag{1}$$

where Ω is a bounded, radial, connected open subset of $\mathbb{R}^N$, $N \geq 2$. Moreover, $f_1, \dots, f_m \in C^1([0, \infty) \times \mathbb{R}^m)$ are nonlinearities to be specified later and I_i, $i = 1, \dots, m$ are nonlocal operators, which for $u \in C^2(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ are given by

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$$I_i u(x) := p.v. \int_{\mathbb{R}^N} (u(x) - u(y)) \, k_i(x - y) \, dy$$

$$:= \lim_{\epsilon \to 0^+} \int_{\mathbb{R}^N \setminus B_\epsilon(0)} (u(x) - u(y)) \, k_i(x - y) \, dy, \quad x \in \mathbb{R}^N. \tag{2}$$

Here, $k_i : \mathbb{R}^N \setminus \{0\} \to [0, \infty)$ for $i = 1, \dots, m$ are kernel functions, which are each given by $k_i(z) = k_{i,0}(|z|), z \in \mathbb{R}^N \setminus \{0\}$ for a monotone decreasing (in the broad sense) function $k_{i,0} : (0, \infty) \to [0, \infty)$ satisfying

$$\int_0^1 k_{i,0}(r) \, r^{N-1} \, dr = \infty, \quad \int_0^1 k_{i,0}(r) \, r^{N+1} \, dr < \infty, \quad \text{and} \quad \int_1^\infty k_{i,0}(r) \, r^{N-1} \, dr < \infty. \tag{3}$$

These conditions imply that the kernel functions k_i are not in $L^1(\mathbb{R}^N)$ and satisfy the usual Lévy condition $\int_{\mathbb{R}^N} \min\{1, |z|^2\} \, k_i(z) \, dz < \infty$. In particular, this implies the k_i satisfy a certain singular behavior for $z \to 0$. We emphasize moreover that the k_i can be different for each i. Hence, these assumptions on the k_i cover also the particular case, where $I_i = (-\Delta)^{s_i}$ with possibly different $s_i \in (0, 1)$ by setting $k_{i,0}(r) = c_{N,s_i} \, r^{-N-2s_i}$ for $i = 1, \dots, m$. Here $c_{N,s} > 0$ for $s \in (0, 1)$ is a normalization constant given by

$$c_{N,s} = \frac{2^{2s} s \, \Gamma(\frac{N}{2} + s)}{\pi^{\frac{N}{2}} \, \Gamma(1 - s)} = \left(\int_{\mathbb{R}^N} \frac{1 - \cos x_1}{|x|^{N+2s}} \, dx \right)^{-1}. \tag{4}$$

The value of $c_{N,s}$ is chosen to make the *fractional Laplacian* $(-\Delta)^s$ the pseudodifferential operator whose symbol is $|\xi|^{2s}$ (see e.g. [4, Section 3.1] or [8, Proposition 3.3] for details). The equality of the two values given in (4) is shown in [12, 16] (see also [14]). For further information on the operators I_i and the definition of *weak solution*, which we use in this paper, we refer to Sect. 2.1 below: see, in particular, (16) (see also [13, 19]).

Symmetry properties of solutions to nonlocal nonlinear problems have been studied for one or more equations in the case where I is the fractional Laplacian in [10, 11, 14, 18], while in [19] the question of symmetries to solutions was studied for a general class of nonlocal operators. However, if Ω is not a ball but rather an annulus or the solutions change sign, then in general it is no longer true that a solution of (1) must be radial even in the case where I is a local operator and $m = 1$. However, under some suitable assumptions on the equation or the system, some axial symmetry can still be achieved. In the case where $m = 1$ and I is a local operator this has been studied in [24, 25, 30], whereas symmetry for systems have been studied in [5, 6] (see also there references in there). For the nonlocal case, the axial symmetry of solutions has been studied in [17] for $m = 1$.

In the present paper we face the difficulties that arise when two or more equations interact with each other, thus dropping the restriction to the case of one scalar equation, which was in effect in [17]. Furthermore, unlike [10, 11, 14, 18, 19], we also take annular domains into consideration, and we allow the solutions to change sign. To be more specific, we consider a particular kind of axial symmetry

called foliated Schwarz symmetry, which was defined in [29, Definition 2.4], based on an idea in [26]. We refer to the general survey [33], in particular Section 2.3. Note that the literature on the subject is large and still growing: thus, our reference list is far from being complete.

Let $\Omega \subset \mathbb{R}^N$, $N \geq 2$, be a radial domain, and take a unit vector $p \in S^{N-1} := \{x \in \mathbb{R}^N : |x| = 1\}$. A function $u : \Omega \to \mathbb{R}$ is called *foliated Schwarz symmetric with respect to p in Ω*, if for every $r > 0$ with $rp \in \Omega$ and for every $c \in \mathbb{R}$, the restricted superlevel set $\{x \in r S^{N-1} : u(x) \geq c\}$ is equal to $r S^{N-1}$ or a geodesic ball in the sphere $r S^{N-1}$ centered at rp.

We simply call *u foliated Schwarz symmetric*, if u has this property for some $p \in S^{N-1}$.

We give an equivalent definition in Sect. 2.2.1 below (see also [30, Proposition 3.3]), which we use in our proof. Note that if $u : \mathbb{R}^N \to \mathbb{R}$ is such that $u|_\Omega$ is foliated Schwarz symmetric with respect to some p for some radial set $\Omega \subset \mathbb{R}^N$, then $u \chi_\Omega$ is axially symmetric with respect to the axis $\mathbb{R} \cdot p$ and nonincreasing in the polar angle $\theta = \arccos(\frac{x}{|x|} \cdot p)$.

Our main result on the symmetry properties of solutions of (1) is the following.

Theorem 1.1 *Let $\Omega \subset \mathbb{R}^N$ be a bounded radial domain and $m \in \mathbb{N}$. For $i = 1, \ldots, m$ assume that $f_i \in C^1([0, \infty) \times \mathbb{R}^m)$ and $f_i(r, u_1, \ldots, u_m)$ satisfies*

$$\frac{\partial}{\partial u_j} f_i > 0 \quad \text{on } (0, \infty) \times \mathbb{R}^m, \quad i, j \in \{1, \ldots, m\}, i \neq j. \tag{5}$$

Let $u_1, \ldots, u_m : \mathbb{R}^N \to \mathbb{R}$ be continuous bounded functions satisfying (1) in weak sense. If

$$u_j(x_1, x') \geq u_j(-x_1, x') \quad \text{for all } x_1 > 0, \ x' \in \mathbb{R}^{N-1}, \ j = 1, \ldots, m, \text{ and}$$
$$\text{there is } x_1 > 0, \ x' \in \mathbb{R}^{N-1} \text{ with } \quad u_1(x_1, x') > u_1(-x_1, x'), \tag{6}$$

then there is $p \in S^{N-1}$ such that $u_1, \ldots, u_m$ are foliated Schwarz symmetric in Ω with respect to p and strictly decreasing in the polar angle.

Clearly, assumption (5) is restricting, which is due to the fact that we did not assume any further connection between the f_i and u_j. In the following variant of Theorem 1.1 we weaken the assumption on f_i but assume positivity of the u_j.

Theorem 1.2 *Let $\Omega \subset \mathbb{R}^N$ be a bounded radial domain and $m \in \mathbb{N}$. For $i = 1, \ldots, m$ assume that $f_i \in C^1([0, \infty) \times \mathbb{R}^m)$ and $f_i(r, u_1, \ldots, u_m)$ satisfies*

$$\frac{\partial}{\partial u_j} f_i > 0 \quad \text{on } (0, \infty)^{m+1}, \quad i, j \in \{1, \ldots, m\}, i \neq j \tag{7}$$

Let $u_1, \ldots, u_m : \mathbb{R}^N \to \mathbb{R}$ be continuous bounded functions satisfying (1) in weak sense. If $u_j > 0$ in Ω for $j = 1, \ldots, m$ and (6) holds, then there is $p \in S^{N-1}$ such that

$u_1, \ldots, u_m$ *are foliated Schwarz symmetric in* Ω *with respect to p and strictly decreasing in the polar angle.*

The proof of Theorems 1.1 and 1.2 follows from the rotating plane method for nonlocal operators developed in [17], which we adjust here to the case of systems. Theorems 1.1 and 1.2 then follow from Theorem 4.1 below.

Remark 1.3 Some remarks are necessary on the assumptions in Theorems 1.1 and 1.2.

1. Assumption (5), or assumption (7) with the positivity of the u_j, can be weakened further. Indeed, we only need a kind of *strong coupling* condition on a linearized system connected with (1) (see Sect. 3 below).
2. Condition (6) is necessary in order that $u_1, \ldots, u_m$ are foliated Schwarz symmetric with respect to any $p = (p_1, \ldots, p_N)$ with $p_1 > 0$, and strictly decreasing in the polar angle. Of course, the second requirement in (6) is not satisfied if u_1 is foliated Schwarz symmetric with respect to some p with $p_1 = 0$.
3. Assumption (6) clearly can be rotated to an arbitrary hyperplane with respect to which the first inequality holds. Moreover, the function u_1 in the second assumption of (6) can be replaced by any u_j. For the general formulation, see Theorem 4.1 below.
4. We note that the connectedness of Ω is not necessary if the $k_{i,0}$ are strictly decreasing. Indeed, this assumption follows from the kind of strong maximum principle used in our proof (see also Remark 3.3).

It is not obvious under which circumstances a solution $u_1, \ldots, u_m$ can be found such that (6) is satisfied. In the following, we give an explicit example, covering the case of the fractional Laplacian, where Theorem 1.2 can be applied. For this, we need the first eigenvalue of the operator I_{k_i} in an open subset Ω of $\mathbb{R}^N$, given by

$$\lambda_{1,k_i}(\Omega) := \inf_{u \in \mathscr{D}_{k_i}(\Omega) \setminus \{0\}} \frac{\mathscr{E}_{k_i}(u, u)}{\|u\|^2_{L^2(\Omega)}}, \tag{8}$$

where $\mathscr{E}_{k_i}$ denotes the bilinear form associated to the kernel k_i and $\mathscr{D}_{k_i}(\Omega)$ is the associated *nonlocal Dirichlet domain*, see (11) and (12) below. Recall from [21] that $\lambda_{1,k_i}(\Omega) > 0$ for $i = 1, \ldots, m$ if Ω is bounded in one direction. The following existence statement is related to the Brézis–Nirenberg problem, which for systems with the fractional Laplacian has been studied in [9]. In the following statement, we consider a more general class of nonlocal operators, which includes the fractional Laplacian and deals with the geometry of the pair of solutions to the system.

Theorem 1.4 *For* $i = 1, 2$ *let the function* $k_{i,0}$ *in* (3) *be strictly decreasing and, for some* $0 < s \leq \sigma < 1, \gamma \in (0, 1),$ *and* $c > 0,$ *satisfy*

$$\frac{1}{c}\, r^{-1-2s} \le k_{i,0}(r) \le cr^{-1-2\sigma} \quad \text{for } r \in (0,1) \quad \text{and} \quad k_{i,0}(r) \le cr^{-1-2\gamma} \quad \text{for } r \ge 1. \tag{9}$$

Furthermore, let $\Omega \subset \mathbb{R}^N$ be a bounded radial domain, $a_1, a_2 \in L^\infty(\mathbb{R})$ with $\|a_i^+\|_{L^\infty(\mathbb{R})} < \lambda_{1,k_i}(\Omega)$ for $i = 1, 2$, and let $1 < q < \frac{N}{N-2s}$. If $a_1 \ne a_2$, then there are two continuous, bounded functions $u_1, u_2 : \mathbb{R}^N \to \mathbb{R}$, $u_1 \ne u_2$, which are positive in Ω and satisfy in weak sense

$$\begin{cases} I_1 u_1 = a_1(|x|)\, u_1 + |u_2|^q\, |u_1|^{q-2}\, u_1 & \text{in } \Omega \\ I_2 u_2 = a_2(|x|)\, u_2 + |u_1|^q\, |u_2|^{q-2}\, u_2 & \text{in } \Omega \\ u_1 = u_2 = 0 & \text{in } \mathbb{R}^N \backslash \Omega. \end{cases} \tag{10}$$

Moreover, u_1 and u_2 are foliated Schwarz symmetric with respect to some $p \in S^{N-1}$ and, if u_1 and u_2 are not radial, then they are strictly decreasing in the polar angle.

The paper is organized as follows. In Sect. 2, we present our notation and recall known statements on the nonlocal operators we use. Moreover, we introduce the notation for systems and recall the properties and definitions of foliated Schwarz symmetry. In Sect. 3, we state and prove variants of maximum principles, which we use in Sect. 4 to derive Theorems 1.1 and 1.2. The proof of Theorem 1.4 can be found in Sect. 5.

2 Notation and preliminaries

In the following we use $N \in \mathbb{N}$ to denote the dimension. For $A, B \subset \mathbb{R}^N$ non-empty measurable sets we denote by $\chi_A : \mathbb{R}^N \to \mathbb{R}$ the characteristic function and $|A|$ the Lebesgue measure. The notation $B \subset\subset A$ means that $\overline{B}$ is compact and contained in the interior of A. We denote $\operatorname{dist}(A, B) := \inf_{a \in A,\, b \in B} |a - b|$ and as usual $\operatorname{dist}(x, A) := \operatorname{dist}(\{x\}, A)$ for $x \in \mathbb{R}^N$. For $r > 0$ we denote $B_r(A) := \{x \in \mathbb{R}^N \ : \ \operatorname{dist}(x, A) < r\}$ and then $B_r(x)$ denotes the ball of radius r for $x \in \mathbb{R}^N$. Moreover, we fix $S^{N-1} := \partial B_1(0) = \{x \in \mathbb{R}^N \ : \ |x| = 1\}$ to denote the N-dimensional sphere.

As usual, for A open, $C^m(A)$ (resp. $C^m(\overline{A})$) denotes the space of m-times continuously differentiable functions in A (resp. $\overline{A}$) and $C^{0,1}(A)$ denotes the space of Lipschitz functions. $C_c^m(A)$ and $C_c^{0,1}(A)$ denotes respectively those functions in $C^m(A)$ or $C^{0,1}(A)$, which have compact support in A. In the following, if $X(A)$ is some function space and $u \in X(A)$ is a function, we always mean that $u : \mathbb{R}^N \to \mathbb{R}$ is such that $\chi_A u \in X(A)$ and $\chi_{\mathbb{R}^N \backslash A}\, u \equiv 0$. For instance, if $u \in L^2(A)$, then $u \in L^2(\mathbb{R}^N)$ and $u = 0$ on $\mathbb{R}^N \backslash A$.

Finally, for a function $u : A \to \mathbb{R}$ we use $u^+ := \max\{u, 0\}$ and $u^- := -\min\{u, 0\}$ to denote the positive and negative part of u respectively, so that $u = u^+ - u^-$.

2.1 On the operator and associated spaces

Let $k : \mathbb{R}^N \backslash \{0\} \to [0, \infty)$ be a radial and radially decreasing function. That is $k(z) = k_0(|z|)$, $z \in \mathbb{R}^N \backslash \{0\}$ for a monotone decreasing function $k_0 : (0, \infty) \to [0, \infty)$ satisfying (3). We denote formally the bilinear form associated to k by

$$\mathcal{E}_k(u, v) := \frac{1}{2} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} (u(x) - u(y)) (v(x) - v(y)) k(x - y) \, dy. \tag{11}$$

For $\Omega \subset \mathbb{R}^N$ open, this bilinear form is well-defined on

$$\mathcal{D}_k(\Omega) := \{u \in L^2(\Omega) \; : \; \mathcal{E}_k(u, u) < \infty\}. \tag{12}$$

It follows that $\mathcal{D}_k(\Omega)$ is a Hilbert space with scalar product

$$\langle u, v \rangle_k := \langle u, v \rangle_2 + \mathcal{E}_k(u, v), \quad u, v \in \mathcal{D}_k$$

where $\langle \cdot, \cdot \rangle_2$ denotes the usual L^2 scalar product. By standard methods (see e.g. [19, 20]) it follows that $\mathcal{E}_k$ is associated to a (nonlocal) operator I, which on $C^2(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ is represented by (2) and it holds

$$\langle Iu, v \rangle_2 = \int_{\mathbb{R}^N} Iu(x) \, v(x) \, dx = \mathcal{E}_k(u, v) \quad \text{for all} u \in C^2(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N), v \in \mathcal{D}_k(\mathbb{R}^N).$$

We note that the embedding $\mathcal{D}_k(\Omega) \to L^2(\Omega)$ is locally compact in the sense that $\mathcal{D}_k(\mathbb{R}^N) \ni u \mapsto \chi_B u \in L^2(\mathbb{R}^N)$ is compact for any bounded open set $K \subset \mathbb{R}^N$ (see [21, Theorem 1.1]). In the particular case, where Ω is bounded in one direction, say $\Omega \subset (-a, a) \times \Omega$ for some $a > 0$, we have (see [21, Proposition 1.7], [13, Lemma 2.7])

$$\lambda_{1,k}(\Omega) := \inf_{u \in \mathcal{D}_k(\Omega) \backslash \{0\}} \frac{\mathcal{E}_k(u, u)}{\|u\|_{L^2(\Omega)}^2} > 0 \tag{13}$$

and moreover (see [21, Proposition 1.7], [19, Lemma 2.1])

$$\lambda_{1,k}(\Omega) \to \infty \quad \text{for either} \, a \to 0 \text{ or } |\Omega| \to 0. \tag{14}$$

It hence follows that in this case $\mathcal{E}_k$ is a scalar product and the induced norm is equivalent to $\langle \cdot, \cdot \rangle_2$. In particular, if Ω is bounded, then $\mathcal{D}_k(\Omega) \to L^2(\Omega)$ is compact and $\lambda_1(\Omega)$ corresponds to the first eigenvalue of I.

In the following, we understand solutions in the weak sense, that is, given $f \in L^2(\mathbb{R}^N)$, we say that $u \in \mathcal{D}_k(\Omega)$ is a solution of

$$Iu = f \quad \text{in } \Omega \quad \text{and} \quad u = 0 \quad \text{in } \mathbb{R}^N \backslash \Omega \tag{15}$$

if for all $\phi \in \mathcal{D}_k(\Omega)$ we have

$$\mathcal{E}_k(u, \phi) = \int_\Omega f(x) \, \phi(x) \, dx.$$

Accordingly, the m-tuple $(u_1, \dots, u_m)$ is called *weak solution* of (1), if for $i = 1, \dots, m$ we have $u_i \in \mathscr{D}_{k_i}(\Omega)$ and

$$\mathscr{E}_{k_i}(u_i, \phi_i) = \int_\Omega f_i(|x|, u_1, u_2)\, \phi_i(x)\, dx \tag{16}$$

for all $\phi_i \in \mathscr{D}_{k_i}(\Omega)$, whenever the right-hand side is well defined.

Finally, in our analysis, we use the *rotating plane method* and linearize the system of equations. Our symmetry results then follow from an application of different maximum principles for supersolutions. For this, we introduce a function space extending $\mathscr{D}_k(\Omega)$. Let $\Omega \subset \mathbb{R}^N$ open and denote

$$\mathscr{V}_k(\Omega) := \left\{ u : \mathbb{R}^N \to \mathbb{R} \text{ measurable } : \right.$$

$$\left. \rho_k(u, \Omega) := \int_\Omega \int_{\mathbb{R}^N} (u(x) - u(y))^2\, k(x - y)\, dx\, dy < \infty \right\}.$$

Clearly by definition we have for $A \subset B \subset \mathbb{R}^N$ open

$$\mathscr{D}_k(A) \subset \mathscr{D}_k(B) \subset \mathscr{D}_k(\mathbb{R}^N) \subset \mathscr{V}_k(\mathbb{R}^N) \subset \mathscr{V}_k(B) \subset \mathscr{V}_k(A).$$

The following Lemma collects all information on $\mathscr{V}_k(\Omega)$ needed in this paper.

Lemma 2.1 (See [19] and Lemma 3.1, Lemma 3.2 in [20]) *Let $\Omega \subset \mathbb{R}^N$ open.*

1. *$\mathscr{E}_k$ is well-defined on $\mathscr{V}_k(\Omega) \times \mathscr{D}_k(\Omega)$ and*

   $$\mathscr{E}_k(u, v) \le (2 + \sqrt{2})\, \rho_k(u, \Omega)^{\frac{1}{2}}\, \mathscr{E}_k(v, v)^{\frac{1}{2}} \quad \text{for } u \in \mathscr{V}_k(\Omega), v \in \mathscr{D}_k(\Omega).$$

2. *$u \in \mathscr{V}_k(\Omega)$ implies $u^\pm, |u| \in \mathscr{V}_k(\Omega)$.*
3. *If $u \in \mathscr{D}_k(\Omega)$, then $\mathscr{E}_k(u^+, u^-)$ is well defined and*

   $$\mathscr{E}_k(u^+, u^-) \le 0 \quad \text{and also} \quad \mathscr{E}_k(|u|, |u|) \le \mathscr{E}_k(u, u).$$

   Moreover, if k_0 is strictly decreasing, then equality holds in these inequalities if and only if $u = u^+$ or $u = u^-$ a. e. in $\mathbb{R}^N$
   Furthermore, if Ω is in addition bounded and $u \in \mathscr{V}_k(\Omega)$, then

4. *$u\, \chi_{\mathbb{R}^N \setminus \Omega} \equiv 0$, then $u \in \mathscr{D}_k(\Omega)$.*
5. *$u \ge 0$ on $\mathbb{R}^N \setminus \Omega$, then $u^- \in \mathscr{D}_k(\Omega)$.*

The additional assertion in Lemma 2.1.3 follows immediately from the proof in [19]. Based on Lemma 2.1 we say $u \in \mathscr{V}_k(\Omega)$ satisfies for some $f \in L^2(\Omega)$ in weak sense

$$Iu \ge f \quad \text{in } \Omega \quad \text{and} \quad u \ge 0 \quad \text{in } \mathbb{R}^N \setminus \Omega \tag{17}$$

if $u \ge 0$ on $\mathbb{R}^N \setminus \Omega$ and for all $v \in \mathscr{D}_k(\Omega)$, $v \ge 0$ we have

$$\mathscr{E}_k(u, v) \geq \int_\Omega f(x)\, v(x)\, dx.$$

We also call u in this case a *supersolution* of (15). Similarly, we call u a *subsolution* of (15) if $-u$ satisfies in weak sense (17).

2.1.1 On the notation for systems

In the following, let M be any set, $m \in \mathbb{N}$, and $\Psi : M \to \mathbb{R}^m$, where we denote the coordinates of Ψ with $\psi_1, \ldots, \psi_m : M \to \mathbb{R}$. We say $\Psi \geq 0$ (or > 0), if $\psi_i \geq 0$ (or > 0) for $i = 1, \ldots, m$ and we say $\Psi \gneq 0$, if $\Psi \geq 0$ and there is $x \in M$ and $i \in \{1, \ldots, m\}$ such that $\psi_i(x) > 0$. Furthermore, we denote $\Psi^\pm := (\psi_1^\pm, \ldots, \psi_m^\pm)$.

We fix $k := (k_1, \ldots, k_m)$, where $k_i : \mathbb{R}^N \backslash \{0\} \to [0, \infty)$, $i = 1, \ldots, m$ are functions such that $k_i(z) = k_{i,0}(|z|)$ for a monotone decreasing function $k_{i,0} : (0, \infty) \to [0, \infty)$ satisfying (3). Denote for $\Omega \subset \mathbb{R}^N$ open

$$\boldsymbol{D}_k(\Omega) := \mathscr{D}_{k_1}(\Omega) \times \cdots \times \mathscr{D}_{k_m}(\Omega) \quad \text{and} \quad \boldsymbol{V}_k(\Omega) := \mathscr{V}_{k_1}(\Omega) \times \cdots \times \mathscr{V}_{k_m}(\Omega).$$

For $U = (u_1, \ldots, u_m) \in \boldsymbol{V}_k(\Omega)$, $V = (v_1, \ldots, v_m) \in \boldsymbol{D}_k(\Omega)$ we write

$$\mathscr{E}_k(U, V) := \sum_{i=1}^m \mathscr{E}_{k_i}(u_i, v_i).$$

and similarly, for $U \in (L^2(\Omega))^m$, $\|U\|_{L^2(\Omega)}^2 = \sum_{i=1}^m \int_\Omega (u_i)^2\, dx$.

Hence, a solution $(u_1, \ldots, u_m) \in \boldsymbol{D}_k(\Omega)$ of

$$\begin{cases} I_i u_i = f_i(|x|, u_1, \ldots, u_m) & \text{in } \Omega \\ u_i = 0 & \text{in } \mathbb{R}^N \backslash \Omega \end{cases} \tag{18}$$

for $i = 1, \ldots, m$, where $f_1, \ldots, f_m \in C^1([0, \infty) \times \mathbb{R}^m)$ can be rewritten in one equation by setting $U = (u_1, \ldots, u_m) \in \boldsymbol{D}_k(\Omega)$, $F(r, U) = \big(f_i(r, u_1, \ldots, u_m)\big)_{1 \leq i \leq m}$ and $IU = (I_1 u_1, \ldots, I_m u_m)$. The system (18) then reads

$$\begin{cases} IU = F(|x|, U) & \text{in } \Omega \\ U = 0 & \text{in } \mathbb{R}^N \backslash \Omega. \end{cases} \tag{19}$$

and U solves (19) in the weak sense if for all $V \in \boldsymbol{D}_k(\Omega)$ we have

$$\mathscr{E}_k(U, V) = \int_\Omega F(|x|, U(x)) \cdot V(x)\, dx,$$

whenever the right-hand side exists.

2.2 Notation for the reflection of a hyperplane

In the following let $\Omega \subset \mathbb{R}^N$ be an open radial set. For $e \in S^{N-1}$ we set $H_e := \{x \in \mathbb{R}^N : x \cdot e > 0\}$ and $\Omega_e := \Omega \cap H_e$. Moreover, we let $\sigma_e : \mathbb{R}^N \to \mathbb{R}^N$, $\sigma_e(x) := x_e := x - 2(x \cdot e)e$ be the reflection at $T_e := \partial H_e$; for a function $u : \mathbb{R}^N \to \mathbb{R}^m$, $m \in \mathbb{N}$ we let $u_e := u \circ \sigma_e$ be the *reflected function at T_e*.

For $U \in V_k(\Omega)$ we say that H_e is *dominant*, if $U \geq U_e$ in H_e and we say H_e is *strictly dominant*, if $U \gneq U_e$ in H_e. Moreover, we note that

Lemma 2.2 *Let $\Omega \subset \mathbb{R}^N$ be an open radial set, $e \in S^{N-1}$, and $U \in V_k(\Omega)$. Then*

1. $U_e \in V_k(\Omega)$.
2. *If $U \in D_k(\Omega)$ satisfies $U_e = -U$, then $\chi_{H_e} U \in D_k(\Omega_e)$.*
3. *Let $U \in V_k(\Omega)$ such that $U_e = -U$ and $U \geq 0$ on $H_e \backslash \Omega$. If Ω is bounded, then $\chi_{H_e} U^- \in D_k(\Omega_e)$ and*

$$\mathscr{E}_k(U, \chi_{H_e} U^-) \leq -\mathscr{E}_k(\chi_{H_e} U^-, \chi_{H_e} U^-) \leq - \min_{i \in \{1,\ldots,m\}} \lambda_{1,k_i}(\Omega_e) \|U^-\|^2_{L^2(\Omega_e)}.$$

Proof 1. follows immediately from the definition of the function space since $k(z_e) = k(z)$ for all $z \in \mathbb{R}^N$, $e \in S^{N-1}$. 2. and 3. follow from [17, Lemma 3.2] with (13) noting that we have

$$\mathscr{E}_k(U, \chi_{H_e} U^-) = \sum_{i=1}^m \mathscr{E}_{k_i}(u_i, \chi_{H_e} u_i^-) \leq - \sum_{i=1}^m \mathscr{E}_{k_i}(\chi_{H_e} u_i^-, \chi_{H_e} u_i^-)$$

$$\leq - \min_{i \in \{1,\ldots,m\}} \lambda_{1,k_i}(\Omega_e) \int_\Omega \sum_{i=1}^m (u_i^-)^2 \, dx = - \min_{i \in \{1,\ldots,m\}} \lambda_{1,k_i}(\Omega_e) \|U^-\|^2_{L^2(\Omega_e)}.$$

$\square$

Lemma 2.3 *Let $m \in \mathbb{N}$, $\Omega \subset \mathbb{R}^N$ be an open bounded radial set, $F \in C^1([0, \infty) \times \mathbb{R}^m)$, and let $U \in D_k(\Omega)$ with $u_1, \ldots, u_m \in L^\infty(\Omega)$ be a solution of*

$$IU = F(|x|, U) \quad \text{in } \Omega; \quad U = 0 \quad \text{in } \mathbb{R}^N \backslash \Omega.$$

Let $e \in S^{N-1}$ and $W := W_e := U - U_e$. Then $W_e \in D_k(\Omega)$ is a solution of the linear problem

$$\begin{cases} IW = C(x) \, W & \text{in } \Omega; \\ W = 0 & \text{in } \mathbb{R}^N \backslash \Omega, \end{cases} \tag{20}$$

which satisfies in addition $W = -W_e$. Here, $C(x) = (c_{ij}(x))_{1 \leq i,j \leq m}$ where $c_{ij} \in L^\infty(\Omega)$, $i, j = 1, \ldots, m$ is given by

$$c_{ij}(x) = \int_0^1 \frac{\partial}{\partial u_j} f_i(|x|, U_e + t(U - U_e)) \, dt, \quad x \in \Omega. \tag{21}$$

Proof Let $e \in S^{N-1}$, $W = (w_1, \ldots, w_m)$ as in the statement and fix $i \in \{1, \ldots, m\}$. Then clearly $W = -W_e$ and by Lemma 2.2 we have $W \in D_k(\Omega)$ and we have in weak sense in Ω

$$I_i \, w_i = f_i(|x|, U) - f_i(|x|, U_e) = \int_0^1 \frac{\partial}{\partial U} f_i(|x|, U_e + t(U(x) - U_e(x))) \, dt \cdot (U(x) - U_e(x))$$
$$= c_{i1}(x) \, w_1 + \cdots + c_{im}(x) \, w_m,$$

where we have used the mean value theorem. $\qquad\square$

2.2.1 Foliated Schwarz symmetry

Denote by $\mathscr{H}$ the set of open half spaces in $\mathbb{R}^N$. Give $u : \mathbb{R}^N \to \mathbb{R}$, the *polarization* $u_H : \mathbb{R}^N \to \mathbb{R}$ *of u* with respect to $H \in \mathscr{H}$ is given by

$$u_H(x) = \begin{cases} \max\{\, u(x), \, u(\sigma_H(x)) \,\} & x \in H; \\ \min\{\, u(x), \, u(\sigma_H(x)) \,\} & x \in \mathbb{R}^N \backslash H, \end{cases} \tag{22}$$

where $\sigma_H(x)$ denotes the reflection of x at ∂H. For $U = (u_1, \ldots, u_m) : \mathbb{R}^N \to \mathbb{R}^m$ and $H \in \mathscr{H}$ we denote similarly $U_H : \mathbb{R}^N \to \mathbb{R}^m$ by $U_H := ((u_1)_H, \ldots, (u_m)_H)$. Clearly, $U \geq U_e$ if and only if $U = U_{H_e}$. The following Proposition relates the polarization of a function with the property that this function is foliated Schwarz symmetric.

Proposition 2.4 (Proposition 3.3, [30]) *Let $\Omega \subset \mathbb{R}^N$ be an open radial set and let P be a set of functions $u : \mathbb{R}^N \to \mathbb{R}$, which are continuous. Moreover, let*

$$M := \{e \in S^{N-1} \; : \; u = u_{H_e} \quad \text{on } \Omega_e \text{ for all } U \in P\}.$$

Assume that there is $e_0 \in M$ such that the following is true:

For all two dimensional subspaces $V \subset \mathbb{R}^N$ with $e_0 \in V$ there are $e_+, e_- \in M \cap V$, $e_+ \neq e_-$, which are in the same connected component of $M \cap V$ and satisfy $u = u_{e_+}$ and $u = u_{e_-}$ for every $u \in P$.

Then there is $p \in S^{N-1}$ such that for every connected component D on Ω the functions $u \chi_D$ for $u \in P$ are foliated Schwarz symmetric with respect to p.

Proposition 2.4 is essential in our proofs and we apply it to the family $P = \{u_1, \ldots, u_m\}$, where $(u_1, \ldots, u_m) \in D_k(\Omega) \cap (C^0(\mathbb{R}^N))^m$ solves (1). The assumption of the proposition is verified with the rotating plane method based on the notation

of Sect. 2.2. We note that the polarization of a function in $\mathscr{D}_k(\Omega)$ remains in $\mathscr{D}_k(\Omega)$ —we include a statement of this fact for the reader's convenience in Lemma 5.7 below.

3 Linear systems and the maximum principle

In the following we collect maximum principles needed for our proofs for linear systems of equations. Here, the problems are stated in a half space and the definition of supersolution is adjusted to the oddness of the solution with respect to a hyperplane as presented in Lemma 20. In the following, let as above $\mathscr{H}$ be the set of half spaces in $\mathbb{R}^N$ and fix $H \in \mathscr{H}$ and $D \subset H$, an open bounded set. We denote the reflection at ∂H by σ_H. Moreover, let $c_{ij} \in L^\infty(D)$, $i,j = 1, \ldots, m$ be given and denote $C(x) := (c_{ij}(x))_{1 \le i,j \le m}$. The following maximum principles are for functions $U \in V_k(D)$ such that $U = -U \circ \sigma_H$ and

$$\mathscr{E}_k(U, V) \ge \int_\Omega C(x)\, U(x) \cdot V(x)\, dx.$$

We also say, that U satisfies in weak sense

$$\begin{cases} IU \ge C(x)\, U(x) & \text{in } D \\ U \ge 0 & \text{in } H \backslash D \\ U = U \circ \sigma_H & \text{in } \mathbb{R}^N \end{cases} \tag{23}$$

We call the linear system (23) *weakly coupled (in D)*, if

$$c_{ij} \ge 0 \quad \text{for all } i, j \text{ such that } i \ne j.$$

Moreover, we call the linear system (23) *strongly coupled*, if it is weakly coupled and

$$\text{for all } i,j \text{ there is a compact set } K \subset D \text{ with } |K| > 0 \text{ and } \quad \operatorname{essinf}_K c_{ij} > 0.$$

Proposition 3.1 (Small volume maximum principle for systems) *Let $c_\infty > 0$ and $H \in \mathscr{H}$. Then there is $\delta > 0$ such that for any $D \subset H$ open bounded with $|D| < \delta$ the following holds. If $c_{ij} \in L^\infty(D)$, $i,j = 1, \ldots, m$ are weakly coupled and with $c_{ij} \le c_\infty$ for $i,j = 1, \ldots, m$, then any function $U \in V_k(D)$ satisfying (23) satisfies $U \ge 0$ in D.*

Proof For $m = 1$ see [19, Proposition 3.5]. The general case follows similarly. Indeed, let $c_\infty > 0$ be given and by (14) we may fix $\delta > 0$ such that $\Lambda_1(D) := \min_{i \in \{1, \ldots, m\}} \lambda_{1,k_i}(D) > 2^{m-1} c_\infty$ for all $D \subset H$ with $|D| < \delta$. Moreover, by Lemma 2.2 we may choose $W = \chi_H U^- \in D_k(D)$ as a suitable test function and we have with Lemma 2.2 and the weak coupling assumption

$$-\Lambda_1(D)\,\|U^-\|^2_{L^2(D)} \ge -\mathscr{E}_k(\chi_H\,U^-,\,\chi_H\,U^-) \ge \mathscr{E}_k(U,\,\chi_H\,U^-) \ge \int_D C(x)\,U\cdot U^-\,dx$$

$$=\sum_{i,j=1}^m \int_D c_{ij}(x)\,u_i\,u_j^-\,dx$$

$$=-\sum_{i=1}^m \int_D c_{ii}(x)\,(u_i^-)^2\,dx + \sum_{\substack{i,j=1\\i\ne j}}^m \int_D c_{ij}(x)\,(u_i^+ - u_i^-)\,u_j^-\,dx$$

$$\ge -c_\infty \sum_{i=1}^m \int_D (u_i^-)^2\,dx - c_\infty \sum_{\substack{i,j=1\\i\ne j}}^m \int_D u_i^-\,u_j^-\,dx$$

$$=-c_\infty \int_D \Big(\sum_{i=1}^m u_i^-\Big)^2\,dx \ge -2^{m-1}\,c_\infty\,\|U^-\|^2_{L^2(D)}.$$

Hence

$$(2^{m-1}\,c_\infty - \Lambda_1(D))\,\|U^-\|^2_{L^2(D)} \ge 0,$$

which is only possible if $U^- = 0$ a.e. on D. $\qquad\square$

Proposition 3.2 (*Strong maximum principle for systems*) *Let $H \in \mathscr{H}$, $D \subset H$ be a domain, and let $c_{ij} \in L^\infty(D)$, $i,j = 1,\dots,m$ be strongly coupled. Then for any function $U \in V_k(D)$ satisfying in weak sense* (23) *with $U \ge 0$ in H we have either $U \equiv 0$ in D or $U > 0$ in D in the sense that*

$$\operatorname{essinf}_K u_i > 0 \quad \text{for } i = 1,\dots,m \text{ and all compact } K \subset D.$$

Proof For $m = 1$ see [19, Proposition 3.6]. For $m \in \mathbb{N}$ arbitrary, we first note that for any $i = 1,\dots,m$ we have in weak sense

$$I_i\,u_i \ge \sum_{j=1}^m c_{ij}(x)\,u_j \ge c_{ii}(x)\,u_i \quad \text{in} D,$$

so that $u_i \equiv 0$ in D or $u_i > 0$ in D by [19, Proposition 3.6]. If $U \not\equiv 0$ in D, then there is at least one $i \in \{1,\dots,m\}$ such that $u_i \not\equiv 0$ in D. But then $u_i > 0$ in D (in the essential sense). Next, let $j \in \{1,\dots,m\}$, $j \ne i$ and assume by contradiction that $u_j \equiv 0$ in D. Then for $v \in \mathscr{D}_{k_j}(D)$, $v \ge 0$ we have

$$\mathscr{E}_{k_j}(u_j,v) \ge \sum_{k=1}^m \int_D c_{kj}(x)\,u_k\,v\,dx \ge \int_D c_{jj}(x)\,u_j\,v + c_{ij}(x)\,u_i\,v\,dx = \int_D c_{ij}(x)\,u_i\,v\,dx.$$

$$(24)$$

Since there is a compact set $K \subset D$ with $|K| > 0$, $\operatorname{essinf}_K c_{ij} > 0$, and also $\operatorname{essinf}_K u_i > 0$, and moreover, there is $v \in \mathscr{D}_{k_j}(D) \cap C_c^2(D)$, $v \geq 0$ with $v \equiv 1$ on K, it follows that by (24) we have

$$
\begin{aligned}
0 < \mathscr{E}_{k_j}(u_j, v) &= - \int_D \int_{\mathbb{R}^N \setminus D} u_j(y) \, v(x) \, k(x - y) \, dy \, dx \\
&= - \int_D \int_{H \setminus D} u_j(y) \, v(x) \, k(x - y) \, dy \, dx - \int_D \int_H u_j(\sigma_H(y)) \, v(x) \, k(x - \sigma_H(y)) \, dy \, dx \\
&= - \int_D \int_{H \setminus D} v(x) \, u_j(y) \, [k(x - y) - k(x - \sigma_H(y))] \, dy \, dx \leq 0,
\end{aligned}
$$

where we have used that $u_j(x) = u_j(\sigma_H(x))$ for all $x \in H$ since U solves (23). Clearly, this however is a contradiction and hence $u_j \equiv 0$ in D is impossible. Thus $u_j > 0$ in D by [19, Proposition 3.6] and since j was arbitrary the statement of the Proposition follows. $\qquad\square$

Remark 3.3 We note that the connectedness of D in Proposition 3.2 is not needed, if for $i = 1, \ldots, m$ the functions $k_{i,0}$ are strictly decreasing and hence $k_i > 0$ in $\mathbb{R}^N \setminus \{0\}$.

Remark 3.4 We emphasize that the conclusions of Propositions 3.1 and 3.2 also follow if $U \in V_k(D)$ satisfies $IU \geq C(x)U$ in D and $U \geq 0$ on $\mathbb{R}^N \setminus D$. The proof in this case is similar, but simpler.

4 Proof of the symmetry result

Using the notation of the previous sections, Theorems 1.1 and 1.2 follow from

Theorem 4.1 *Let $m \in \mathbb{N}$, $\Omega \subset \mathbb{R}^N$ be a bounded radial domain, and assume that $F \in C^1([0, \infty) \times \mathbb{R}^m, \mathbb{R}^m)$. Let $U \in D_k(\Omega)$ be a bounded continuous solution of (19), and assume that there is $e_0 \in S^{N-1}$ such that $U \geq U_{e_0} = U \circ \sigma_{e_0}$ in Ω_{e_0} in the sense that $U(x) \geq U_{e_0}(x)$ for all $x \in \Omega_{e_0}$. Suppose, further, that there exist $i \in \{1, \ldots, m\}$ and $x \in \Omega_{e_0}$ such that $u_i(x) > u_i(\sigma_{e_0}(x))$, hence $U \gneq U_{e_0}$. If either (5) holds, or $U > 0$ in Ω and (7) is satisfied, then there is $p \in S^{N-1}$ such that U is foliated Schwarz symmetric with respect to p and strictly decreasing in the polar angle.*

Proof Denote $W_e := U - U_e$ for $e \in S^{N-1}$ and note that W satisfies in weak sense

$$
\begin{cases}
IW_e = C(x) \, W_e & \text{in } \Omega \\
W_e = 0 & \text{in } \mathbb{R}^N \setminus \Omega \\
W_e = -W_e \circ \sigma_e & \text{in } \mathbb{R}^N,
\end{cases}
\tag{25}
$$

where $C(x) = (c_{ij}(x))_{1 \leq i,j \leq m}$ with entries $c_{ij}(x)$ as in (21). Note that by our assumptions there is $c_\infty > 0$ such that

$$\max_{i,j} \sup_{\substack{x \in \Omega \\ e \in S^{N-1}}} |c_{ij}(x)| \leq c_\infty.$$

If (5) holds, or if $U > 0$ in Ω and (7) is satisfied, then the system (25) is strongly coupled.

Step 1: We claim that

$$W_{e_0} > 0 \quad \text{in } \Omega_{e_0}. \tag{26}$$

Note that $W_{e_0} \gneqq 0$ in Ω_{e_0} and for $W_{e_0} = (w_1, \ldots, w_m)$ we have for $i = 1, \ldots, m$

$$\begin{cases} I_i w_i = c_{ii} w_i + \displaystyle\sum_{\substack{j=1 \\ j \neq i}}^m c_{ij} w_j & \text{in } \Omega_{e_0} \\ w_i = 0 & \text{in } H_{e_0} \backslash \Omega_{e_0} \\ w_i = -w_i \circ \sigma_e & \text{in } \mathbb{R}^N, \end{cases}$$

Hence (26) follows by Proposition 3.2 (since $W_{e_0} \equiv 0$ is impossible by assumption).

Step 2: Next, by continuity of U and $e \mapsto \sigma_e$, there is for any $\delta > 0$ an $\epsilon > 0$ such that for any $e \in S^{N-1}$ with $|e - e_0| < \epsilon$ there is $K \subset \Omega_e \cap \Omega_{e_0}$ with

$$W_e \geq 0 \quad \text{in } K \quad \text{and} \quad |\Omega_e \backslash K| \leq \delta.$$

We claim that there is $\epsilon > 0$ such that

$$W_e \geq 0 \text{ in } \Omega_e \text{ for } e \in S^{N-1} \text{ with } |e - e_0| < \epsilon. \tag{27}$$

To see (27), we use Proposition 3.1. Fix $\delta > 0$ such that $\lambda_1(M) > c_\infty$ for any $A \subset \mathbb{R}^N$ with $|A| < \delta$. Let $\epsilon > 0$ be given by the above remark and fix $e \in S^{N-1}$ with $|e - e_0| < \epsilon$ and $K \subset \Omega_e \cap \Omega_{e_0}$ with $W_e \gneqq 0$ in K. Finally, let $A := \Omega_e \backslash K$. As before, let $W_e = (w_1, \ldots, w_m)$ and note that now for $i = 1, \ldots, m$

$$\begin{cases} I_i w_i = c_{ii} w_i + \displaystyle\sum_{\substack{j=1 \\ j \neq i}}^m c_{ij} w_j & \text{in } A \\ w_i = 0 & \text{in } H_{e_0} \backslash A \\ w_i = -w_i \circ \sigma_e & \text{in } \mathbb{R}^N, \end{cases}$$

Since by the assumptions on F (and U) this system is weakly coupled, Proposition 3.1 implies $W_e \geq 0$ in H_e. Whence (27) holds.

Step 3: Next, we fix $(-\pi, \pi) \ni \phi \to R(\phi) \in O(N)$, such that $R(\phi)$ is a rotation of angle ϕ in a fixed direction and put $e^\phi := R(\phi) e_0 \in S^{N-1}$. Denote

$$M := \{\phi \in (-\pi, \pi) \ : \ W_{e^\phi} \geq 0 \text{ in } \Omega_{e^\phi}\}.$$

and let

$$\phi_+ := \sup M \quad \text{and} \quad \phi_- := \inf M.$$

Clearly, $\phi_+ \in (\epsilon, \pi - \epsilon)$ for some $\epsilon > 0$ by Step 1 and similarly $\phi_- \in (-\pi + \epsilon, -\epsilon)$ for some $\epsilon > 0$. Let $e_+ := e^{\phi_+}$ and $e_- := e^{\phi_-}$. The proof is finished once we have shown (see Proposition 2.4)

$$W_{e_+} \equiv 0 \quad \text{in } \Omega_{e_+} \quad \text{and} \quad W_{e_-} \equiv 0 \quad \text{in } \Omega_{e_-}. \tag{28}$$

Note that then there must be $i \in \{1, \dots, m\}$ such that $w_{e_+,i} \equiv 0$ in Ω_{e_+}, because otherwise a similar argumentation as in Step 1 and Step 2 allows to continue rotating the hyperplanes which is a contradiction to the definition of e_+. Similarly, there must be $i \in \{1, \dots, m\}$ such that $w_{e_-,i} \equiv 0$ in Ω_{e_-}. Let $i \in \{1, \dots, m\}$ such that $w_{e_+,i} \equiv 0$ in Ω_{e_+}, and assume there is $j \in \{1, \dots, m\}$ such that $w_{e_+,j} \not\equiv 0$ in Ω_{e_+}. Then as in Step 1 it follows that this is impossible. Thus $W_{e_+} \equiv 0$ and similarly, also $W_{e_-} \equiv 0$. Hence (28) holds.

By Proposition 2.4 and equalities (28) it follows that there is $p \in S^{N-1}$ such that U is foliated Schwarz symmetric, since e_+ and e_- are clearly in the same two dimensional component of e's in which $U \geq U \circ \sigma_e$ and $e_+ \neq e_-$. The fact that U is strictly decreasing in the polar angle now follows because from Step 2 and with Proposition 3.2 we actually have that $W_{e^\phi} > 0$ in Ω_{e^ϕ} for $\phi \in (\phi_-, \phi_+)$. This finishes the proof. $\qquad \square$

5 An application

In the following, we consider the case $m = 2$. The system (18) is called *of gradient type* if there exists a scalar function $g(|x|, u_1, u_2)$ such that $f_j(|x|, u_1, u_2) = \frac{\partial g}{\partial u_j}(|x|, u_1, u_2)$ for $j = 1, 2$ (see [7, p. 3]). Let us consider the following system:

$$\begin{cases} I_1 u_1 = a_1(x) u_1 + |u_2|^q |u_1|^{q-2} u_1 & \text{in } \Omega \\ I_2 u_2 = a_2(x) u_2 + |u_1|^q |u_2|^{q-2} u_2 & \text{in } \Omega \\ u_1 = u_2 = 0 & \text{in } \mathbb{R}^N \backslash \Omega \end{cases} \tag{29}$$

where Ω is an open bounded set in $\mathbb{R}^N$, $N \geq 2$ with Lipschitz boundary, and we assume:

$$\begin{aligned} &\text{there is } c > 0 \text{ and } s \in (0, 1) \text{ such that} \\ &k_{i,0}(r) \geq cr^{-1-2s} \text{ for } r \in (0, 1) \text{ and } i = 1, 2, \end{aligned} \tag{30}$$

$$a_1, a_2 \in L^\infty(\mathbb{R}) \text{ with } a_1 \neq a_2 \text{ and } \|a_i^+\|_{L^\infty(\mathbb{R})} < \lambda_{1,k_i}(\Omega) \text{ for } i = 1, 2,$$

where $\lambda_{1,k_i}(\Omega) > 0$ is the first eigenvalue of I_i (see (8), (13)) for $i = 1, 2$. Moreover, we let $1 < q < \frac{N}{N-2s}$ (cf. [23], (2.2)]). Note that we clearly have $N > 2s$ since $s \in (0, 1)$ and that the kernel of the fractional Laplacian $(-\Delta)^s$ satisfies (30). Moreover, the system (29) is of gradient type, with g given by $g(u_1, u_2) = \frac{1}{q} |u_1 u_2|^q$. In the following let $f_i(u_1, u_2) = |u_{3-i}|^q |u_i|^{q-2} u_i$ for $i = 1, 2$, then we see immediately that

$$\frac{\partial}{\partial u_j} f_i = q |u_i u_j|^{q-2} u_i u_j \quad \text{for } i = 1, 2 \text{ and } j = 3 - i.$$

Hence, the system (29) is *weakly coupled* as long as the product $u_1 u_2$ is non-negative in Ω. A similar system is considered in [23] (see also [5, (4.1)]) with a local operator in place of I, and with the bounded set Ω replaced by the whole space $\mathbb{R}^N$. An existence proof of a pair of non-negative, radially symmetric solutions $u_1, u_2 \geq 0$ satisfying $u_1 + u_2 \not\equiv 0$ in $\mathbb{R}^N$ is given there. In the present paper, to keep the argument as transparent as possible, the nonlinearities in (29) are simpler than those in [23]. However, the parameter ω occurring there is replaced by the function $a_2(x)$. In order to prove Theorem 1.4, we begin with an existence statement.

Theorem 5.1 (Existence of non-trivial solutions) *Let $\Omega \subset \mathbb{R}^N$, $N \geq 2$ be a bounded open set with Lipschitz boundary, assume $k_{1,0}, k_{2,0}$ satisfy (30) for some $s \in (0, 1)$, and $1 < q < \frac{N}{N-2s}$. Then system (29) has a weak solution (u, v) satisfying $u, v \not\equiv 0$ in Ω and $u \not\equiv v$.*

The existence proof is based on the mountain-pass theorem (see, for instance, [1, Theorem 8.2] or [31, Chapter III, Theorem 6.1, p. 109]). More precisely, we consider the functional

$$J(u, v) = \frac{1}{2} \|(u, v)\|^2 - \frac{1}{q} \|uv\|_{L^q(\Omega)}^q, \tag{31}$$

where we have used the notation $\|(u, v)\|^2 = \mathscr{E}_{k_1}(u, u) - \int_\Omega (a_1 u^2 + a_2 v^2) \, dx + \mathscr{E}_{k_2}(v, v)$, for shortness. Note that it is easy to see that $\mathscr{D}_{k_i}(\Omega)$, $i = 1, 2$ are continuously embedded into $\mathscr{H}_0^s(\Omega) = \{u \in H^s(\mathbb{R}^N) : u = 0 \text{ on } \mathbb{R}^N \backslash \Omega\}$ and hence by the assumption $q < \frac{N}{N-2s}$, it follows by the Sobolev embedding that $\mathscr{D}_{k_i}(\Omega)$, $i = 1, 2$ are compactly embedded into $L^{2q}(\Omega)$ (see [8, Theorem 6.7]). Hence the product uv belongs to $L^q(\Omega)$, and the functional $J(u, v)$ is well defined on $\boldsymbol{D}_k(\Omega)$. The differential J' at (u, v) is the linear operator L given by

$$L(\varphi, \psi) = \mathscr{E}_k(u, \varphi) - \int_\Omega \left(a_1 u \varphi + a_2 v \psi \right) dx + \mathscr{E}_k(v, \psi)$$
$$- \int_\Omega |v|^q |u|^{q-2} u \varphi \, dx - \int_\Omega |u|^q |v|^{q-2} v \psi \, dx, \tag{32}$$

where (φ, ψ) ranges in $\boldsymbol{D}_k(\Omega)$. Hence the critical points of J are the weak solutions of (29). To apply the mountain pass theorem, we collect in the next section several

properties of the Nehari Manifold $\mathcal{N}$. For general applications of the mountain pass theorem to nonlocal operators, see also [27, 28].

5.1 The Nehari manifold

In the sequel we refer to the Nehari manifold $\mathcal{N}$ associated to the functional J. First define the functional

$$G(u, v) = \tfrac{1}{2} \, \|(u, v)\|^2 - \|uv\|_{L^q(\Omega)}^q,$$

and then let

$$\mathcal{N} = \{ \, (u, v) \in D_k(\Omega) \backslash (0, 0) \mid \text{the differential } J'(u, v) \text{ vanishes in the direction of } (u, v) \, \}$$
$$= \{ \, (u, v) \in D_k(\Omega) \backslash (0, 0) \mid G(u, v) = 0 \, \}. \tag{33}$$

The last equality is readily obtained by letting $(\varphi, \psi) = (u, v)$ in (32): this gives $L(u, v) = \|(u, v)\|^2 - 2 \, \|uv\|_{L^q(\Omega)}^q$, and (33) follows. In order to prove the existence of polarized solutions of system (29), we need

Lemma 5.2

1. *The Nehari manifold $\mathcal{N}$ is a C^1-manifold of codimension one in $D_k(\Omega)$.*
2. *If (u, v) belongs to $\mathcal{N}$, then the direction of (u, v) is non-tangential to $\mathcal{N}$.*
3. *The manifold $\mathcal{N}$ keeps far from the origin in the sense that there exists $r_0 > 0$ such that if $\|(u, v)\| < r_0$ then $(u, v) \notin \mathcal{N}$.*

Proof Choose a point $(u_0, v_0) \in \mathcal{N}_+$, and observe that the product $u_0 v_0$ cannot vanish identically (that would be in contrast with (33)). In a neighborhood of (u_0, v_0), the Nehari manifold is the set of zeros of the functional $G(u, v)$, whose differential is the linear functional $G'(u, v)$ given by

$$G'(u, v) : (\varphi, \psi) \mapsto \mathscr{E}_{k_1}(u, \varphi) - \int_\Omega \left(a_1 \, u \, \varphi + a_2 \, v \, \psi \right) dx + \mathscr{E}_{k_2}(v, \psi)$$
$$- q \int_\Omega |v|^q \, |u|^{q-2} \, u \, \varphi \, dx - q \int_\Omega |u|^q \, |v|^{q-2} \, v \, \psi \, dx.$$

To prove Claim 1 we show that the image of (φ, ψ) through $G'(u, v)$ does not vanish for every (φ, ψ). This is achieved by letting $(\varphi, \psi) = (u, v)$ and taking into account that $G(u, v) = 0$, i.e., $\|(u, v)\|^2 = 2 \, \|uv\|_{L^q(\Omega)}^q$. We obtain

$$G'(u, v) : (u, v) \mapsto \|(u, v)\|^2 - 2q \, \|uv\|_{L^q(\Omega)}^q = (1 - q) \, \|(u, v)\|^2 < 0,$$

which implies that $\mathcal{N}$ is a C^1-manifold of codimension 1, and the direction of (u, v) is non-tangential, thus proving Claims 1 and 2 at once. To prove the last claim, observe that by the Poincaré inequality and the Sobolev embedding we have

$$
\begin{aligned}
\|(u, v)\|^2 &\geq \mathcal{E}_k(u, u) - \|a_1^+\|_{L^\infty(\Omega)} \|u\|_{L^2(\Omega)}^2 + \mathcal{E}_k(v, v) - \|a_2^+\|_{L^\infty(\Omega)} \|v\|_{L^2(\mathbb{R})}^2 \\
&\geq C_0 \left(\mathcal{E}_{k_1}(u, u) + \mathcal{E}_{k_2}(v, v) \right) \geq C \left(\|u\|_{L^q(\Omega)}^2 + \|v\|_{L^q(\Omega)}^2 \right) \\
&\geq C_1 \|uv\|_{L^q(\Omega)},
\end{aligned}
$$

where $C_0, C, C_1 > 0$ are constants. Hence, we may write $\|uv\|_{L^q(\Omega)}^q \leq C_2 \|(u, v)\|^{2q}$, and therefore the inequality:

$$
G(u, v) \geq \|(u, v)\|^2 \left(\tfrac{1}{2} - C_2 \|(u, v)\|^{2(q-1)} \right) \geq \tfrac{1}{3} \|(u, v)\|^2 \tag{34}
$$

holds provided that $\|(u, v)\| < r_0$ with a conveniently small $r_0 > 0$. The last claim follows, and the proof is complete. $\qquad\square$

Proof of Theorem 5.1 Let us check that the functional (31) satisfies the assumptions of the mountain-pass theorem.

Step 1: The equality $J(0, 0) = 0$ holds, and there exists $r_0 > 0$ such that $J(u, v) > 0$ for all $u, v \in \mathcal{D}_k(\Omega)$ satisfying $0 < \|(u, v)\| < r_0$. Indeed, arguing as in the proof of the last claim of Lemma 5.2, and writing J in place of G, we arrive at $J(u, v) \geq \tfrac{1}{3} \|(u, v)\|^2$ for $\|(u, v)\| < r_0$ (cf. (34)). The same inequality also shows that $J(u, v) \geq r_0^2/3$ whenever $\|(u, v)\| = r_0$.

Step 2: The functional J is unbounded from below. To see this, fix a pair $(u, v) \in D_k(\Omega)$ satisfying $\|uv\|_{L^q(\Omega)}^q > 0$ in Ω. Since for every $t \geq 0$ we have

$$
J(tu, tv) = t^2 \left(\tfrac{1}{2} \|(u, v)\|^2 - \frac{t^{2(q-1)}}{q} \|uv\|_{L^q(\Omega)}^q \right), \tag{35}
$$

we see that $J(tu, tv) \to -\infty$ as $t \to \infty$, hence J is unbounded from below, as claimed.

Step 3: The last condition needed to apply the mountain-pass theorem is the Palais-Smale compactness condition. More precisely, assume that a sequence of pairs $(u_i, v_i) \in D_k(\Omega)$ satisfies $J(u_i, v_i) \to c \in (0, \infty)$ as $i \to \infty$ in the Euclidean topology of the real line, as well as $J'(u_i, v_i) \to 0$ as $i \to \infty$ in the strong topology of the dual space $(D_k(\Omega))'$. Then we have to prove the existence of a strongly convergent subsequence in $D_k(\Omega)$. To this purpose, observe that the differential J' at the point (u_i, v_i) is the linear functional $L_i(\varphi, \psi)$ given by

$$
L_i(\varphi, \psi) = \mathcal{E}_{k_1}(u_i, \varphi) - \int_\Omega \left(a_1 u_i \varphi + a_2 v_i \psi \right) dx + \mathcal{E}_{k_2}(v_i, \psi) \tag{36}
$$

$$
- \int_\Omega |v_i|^q |u_i|^{q-2} u_i \varphi \, dx - \int_\Omega |u_i|^q |v_i|^{q-2} v_i \psi \, dx, \tag{37}
$$

where (φ, ψ) ranges in $D_k(\Omega) \subset (L^{2q}(\Omega))^2$. In the special case when $(\varphi, \psi) = (u_i, v_i)$ we find $L_i(\phi, \psi) = \|(u_i, v_i)\|^2 - 2 \|uv\|_{L^q(\Omega)}^q$, and hence

$$2 \|uv\|_{L^q(\Omega)}^q = \|(u_i, v_i)\|^2 - L_i(u_i, v_i). \tag{38}$$

Let us combine the equality above with the assumption that $J'(u_i, v_i) \to 0$ as $i \to \infty$ strongly. Such an assumption implies $L_i(u_i, v_i) = o(1) \|(u_i, v_i)\|$ as $i \to \infty$: by plugging this into (38) we obtain

$$\|u_i v_i\|_{L^q(\Omega)}^q = \tfrac{1}{2} \|(u_i, v_i)\|^2 + o(1) \|(u_i, v_i)\| \quad \text{as } i \to \infty. \tag{39}$$

Now we are ready to prove the existence of a strongly convergent subsequence. As usual, the proof is divided into two parts.

Part i: The sequence (u_i, v_i) is bounded. Indeed, if we assume $\|(u_i, v_i)\| \to \infty$ for $i \to \infty$, then we reach a contradiction by the following argument. Taking (39) into account, we have

$$J(u_i, v_i) = \tfrac{1}{2}(1 - \tfrac{1}{q}) \|(u_i, v_i)\|^2 + o(1) \|(u_i, v_i)\| \to \infty,$$

which contradicts the assumption $J(u_i, v_i) \to c < \infty$ for $i \to \infty$. Hence the sequence (u_i, v_i) must be bounded, as claimed.

Part ii: Once we know that the sequence (u_i, v_i) is bounded in $\boldsymbol{D}_k(\Omega)$, the proof of the existence of a strongly converging subsequence is standard: see [1, p. 125] and [31, Proposition 2.2]. To be more precise, by the weak compactness theorem in Hilbert spaces there exists a subsequence, still denoted by (u_i, v_i), weakly convergent to some $(u, v) \in \boldsymbol{D}_k(\Omega)$. Furthermore, since $q < \frac{N}{N-2s}$, the set $\mathscr{D}_k(\Omega) \subset \mathscr{H}_0^s(\Omega)$ is compactly embedded in the Lebesgue space $L^{2q}(\Omega)$, hence we may assume that when $i \to \infty$ the sequences $(u_i), (v_i)$ converge to u, v, respectively, strongly in $L^{2q}(\Omega)$, and therefore $\|u_i v_i\|_{L^q(\Omega)}^q \to \|uv\|_{L^q(\Omega)}^q$. This and (39), taking the boundedness of the sequence (u_i, v_i) into account, imply

$$\lim_{i \to \infty} \|(u_i, v_i)\|^2 = 2 \|uv\|_{L^q(\Omega)}^q. \tag{40}$$

Consider the functional $L(\varphi, \psi)$ in (32). Taking (36) into account, and since $(u_i, v_i) \rightharpoonup (u, v)$ weakly in $\boldsymbol{D}_k(\Omega)$, and $(u_i, v_i) \to (u, v)$ strongly in $(L^{2q}(\Omega))^2$, we deduce

$$\lim_{i \to \infty} (L_i - L)(\varphi, \psi) = 0$$

for every $(\varphi, \psi) \in \boldsymbol{D}_k(\Omega)$: thus, we have proved the weak-$*$ convergence $L_i \overset{*}{\rightharpoonup} L$. But since $L_i \to 0$ strongly by assumption, we must have $L = 0$. In particular, $(u, v) \in \mathcal{N}$. By comparing (33) with (40) we deduce

$$\lim_{i \to \infty} \|(u_i, v_i)\| = \|(u, v)\|.$$

Finally, by recalling that the weak convergence in a Hilbert space together with the convergence of the norms to the norm of the limiting function implies the strong convergence, we conclude that $(u_i, v_i) \to (u, v)$ strongly in $\boldsymbol{D}_k(\Omega)$, which completes the proof of the Palais–Smale compactness condition.

At this point the mountain-pass theorem implies the existence of a critical point $(u, v) \neq (0, 0)$ of the functional J, which is therefore a weak solution $(u_1, u_2) = (u, v)$ of the system (29). By the mountain-pass theorem we also know that the two identities $u_1 \equiv 0$ and $u_2 \equiv 0$ cannot hold at once, but we may, in principle, have $u_2 \equiv 0$. However, if u_2 vanishes identically, then system (29) implies $Iu_1 = a_1 u_1$ in Ω, $u_1 = 0$ in $\mathbb{R}^N \backslash \Omega$, hence u_1 should also vanish identically by unique solvability and the maximum principle, a contradiction. A similar argument shows that $u_1 \not\equiv 0$, hence $u_1, u_2 \not\equiv 0$ in Ω. Finally, if $u_1 \equiv u_2$ in (29), then by comparing the two equations— recall $a_1 \neq a_2$—we obtain $u_1 \equiv 0$, which has been just excluded. Hence u_1, u_2 are distinct functions, and the proof is complete. □

5.2 Positivity

Let us now turn to show that the solutions u, v obtained so far do not change sign. To this aim we need to define the set of paths $\Gamma = \{ \gamma \in C^0([0, 1], D_k(\Omega)) \mid \gamma(0) = 0, \ J(\gamma(1)) < 0 \}$ and the two infima

$$c = \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} J(\gamma(t)), \qquad c_{\mathcal{N}} = \inf_{(u,v) \in \mathcal{N}} J(u, v).$$

Lemma 5.3 *The two values $c, c_{\mathcal{N}}$ defined above are positive and coincide.*

Proof The argument is similar to [34, Theorem 4.2] (for a scalar equation) and [23, Lemma 3.2] for a system of local equations. Let us verify that $c \leq c_{\mathcal{N}}$. Take $(u, v) \in \mathcal{N}$ and observe that $\|uv\|_{L^q(\Omega)}^q > 0$, otherwise we would reach a contradiction with (33). Then (35) applies, and the path $\gamma(t) = (tu, tv)$, $t \in [0, \infty)$, starts from the origin and satisfies $\lim_{t \to \infty} J(\gamma(t)) = -\infty$. Of course, we may find a reparametrization such that $J(\gamma(1)) < 0$, but we prefer to avoid unnecessary technicalities. Taking (33) into account, a straightforward computation shows that the real-valued function $f(t) = J(\gamma(t))$ of the real variable $t > 0$ (whose graph is outlined in Fig. 1) satisfies $f'(1) = 0$.

Fig. 1 The function $f(t)$
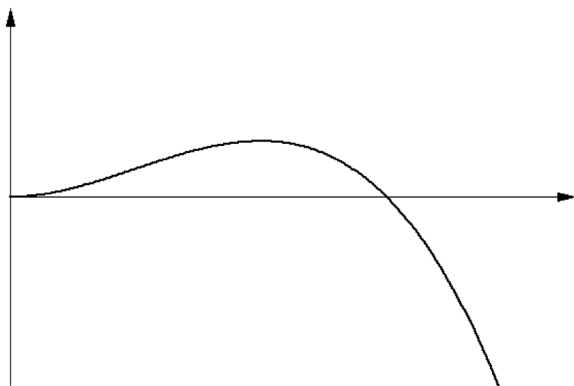
Furthermore, f attains its maximum (which is positive) at $t = 1$ and hence $c \leq \max_{t \geq 0} J(\gamma(t)) = J(u, v)$. Since $(u, v) \in \mathcal{N}$ is arbitrary, we may write $c \leq c_{\mathcal{N}}$. To prove the converse, recall that by Theorem 5.1 there exists $(u, v) \neq (0, 0)$ such that $J'(u, v) = 0$ and $J(u, v) = c > 0$ (this is a by-product of the mountain-pass theorem). But then $(u, v) \in \mathcal{N}$ and therefore $c_{\mathcal{N}} \leq J(u, v) = c$. The lemma follows. □

Proposition 5.4 *The two functions u, v obtained by Theorem 5.1 do not change sign.*

Proof The argument is based on the combination of three inequalities:

1. Since (u, v) is a critical point of the functional J, we have $G(u, v) = 0$ (see (33)), hence the function $f(t) = J(tu, tv) = \frac{t^2}{2} \|(u, v)\|^2 - \frac{t^{2q}}{q} \|uv\|_{L^q(\Omega)}^q$ satisfies $f'(1) = 0$. An elementary computation shows that

$$f(t) \leq f(1) = J(u, v)$$

   for all $t > 0$, with equality if and only if $t = 1$ (the graph of f is outlined in Fig. 1).

2. Since the graph of $g(t) = G(t|u|, t|v|) = \frac{t^2}{2} \|(|u|, |v|)\|^2 - t^{2q} \|uv\|_{L^q(\Omega)}^q$ has the same shape as the one of f, there exists $t_0 > 0$ such that $g(t_0) = 0$. Then, by (33) we have $(t_0|u|, t_0|v|) \in \mathcal{N}$, and by Lemma 5.3 we get

$$J(u, v) = c \leq J(t_0|u|, t_0|v|).$$

3. Using Lemma 2.1 we obtain $J(t_0|u|, t_0|v|) \leq J(t_0 u, t_0 v) = f(t_0)$.

In conclusion, we arrive at $J(t_0|u|, t_0|v|) = f(t_0) = J(u, v)$, whence we deduce that $t_0 = 1$ and $J(|u|, |v|) = J(u, v)$. This and Lemma 2.1 imply that either u^+ or u^- vanishes almost everywhere, and either v^+ or v^- vanishes almost everywhere. The claim follows. □

Corollary 5.5 (*Existence of positive solutions*) *Let $\Omega \subset \mathbb{R}^N$, $N \geq 2$ be a bounded open set with Lipschitz boundary, assume $k_{1,0}, k_{2,0}$ satisfy (30) for some $s \in (0, 1)$, and $1 < q < \frac{N}{N-2s}$. Then system (29) has a weak solution (u_1, u_2) satisfying $u_1, u_2 > 0$ in Ω and $u_1 \not\equiv u_2$.*

Proof Consider the non-negative functions $u_1 = |u|$ and $u_2 = |v|$, where (u, v) is the weak solution whose existence follows from Theorem 5.1. In view of Proposition 5.4, we must have either $u_1 = u$ or $u_1 = -u$, and either $u_2 = v$ or $u_2 = -v$. Therefore the pair (u_1, u_2) satisfies (29). But then (u_1, u_2) also satisfies the system of uncoupled inequalities

$$\begin{cases} I_1 u_1 \geq a_1(x) u_1 & \text{in } \Omega \\ I_2 u_2 \geq a_2(x) u_2 & \text{in } \Omega \\ u_1 = u_2 = 0 & \text{in } \mathbb{R}^N \backslash \Omega \end{cases}$$

By the strong maximum principle (see Proposition 3.2 and Remark 3.4) we have that for each $j = 1, 2$ either $u_j > 0$ in Ω or $u_j \equiv 0$ in $\mathbb{R}^N$, and the conclusion follows from Theorem 5.1. □

Remark 5.6

(i) Since the weak solution (u_1, u_2) whose existence is asserted by Corollary 5.5 minimizes the functional J over the Nehari manifold $\mathcal{N}$, we say that (u_1, u_2) is a *ground state*.

(ii) The pair $(-u_1, u_2)$ is also a weak solution, as well as $(u_1, -u_2)$ and $(-u_1, -u_2)$: the assertion follows by replacing u_j in (29) with $\pm u_j$, $j = 1, 2$.

5.3 Polarized solutions

The main result in this paragraph states that if Ω is symmetric, then system (29) admits a solution made up of two polarized functions. Before proceeding further, observe that in our notation we may write $u_{\sigma_H(H)}(x) = u_H(\sigma_H(x))$. Let us describe the effect of polarization on the functionals J and G:

Lemma 5.7 (Functionals reduced by polarization) *Let Ω be a bounded open set in $\mathbb{R}^N$, symmetric with respect to the hyperplane ∂H for some half-space H. Moreover, assume that a_i is symmetric with respect to ∂H for $i = 1, 2$. For every $(u, v) \in D_k(\Omega)$ satisfying $u, v \geq 0$ in $\mathbb{R}^N$ we have*

$$J(u_H, v_H) \leq J(u, v). \tag{41}$$

Furthermore, if $k_{1,0}, k_{2,0}$ are strictly decreasing and $J(u_H, v_H) = J(u, v)$ then:

1. *either $u = u_H$ or $u = u_{\sigma_H(H)}$;*
2. *either $v = v_H$ or $v = v_{\sigma_H(H)}$;*
3. *the following inequality is satisfied:*

$$\big(u(x) - u(\sigma_H(x))\big)\big(v(x) - v(\sigma_H(x))\big) \geq 0 \quad \text{a.e. in } \Omega. \tag{42}$$

The lemma also holds with G in place of J.

Proof By Proposition A.2 we have $\mathcal{E}_k(u_H, u_H) \leq \mathcal{E}_k(u, u)$ and $\mathcal{E}_k(v_H, v_H) \leq \mathcal{E}_k(v, v)$. Furthermore, by Cavalieri principle and since due to the symmetry of a_1, a_2 we also have $\int_\Omega a_1 u^2 \, dx = \int_\Omega (a_1)_H (u_H)^2 \, dx$ and $\int_\Omega a_2 v^2 \, dx = \int_\Omega (a_2)_H (v_H)^2 \, dx$. This and Proposition A.3 prove (41). Now suppose that (41) holds with equality. We may write

$$0 = J(u, v) - J(u_H, v_H) = \big(\mathcal{E}_k(u, u) - \mathcal{E}_k(u_H, u_H)\big) + \big(\mathcal{E}_k(v, v) - \mathcal{E}_k(v_H, v_H)\big)$$
$$- \frac{1}{q}\big(\|uv\|_{L^q(\Omega)}^q - \|u_H v_H\|_{L^q(\Omega)}^q\big)$$

hence

$$\left(\mathscr{E}_k(u,u) - \mathscr{E}_k(u_H, u_H)\right) + \left(\mathscr{E}_k(v,v) - \mathscr{E}_k(v_H, v_H)\right) = \frac{1}{q}\left(\|uv\|^q_{L^q(\Omega)} - \|u_H v_H\|^q_{L^q(\Omega)}\right) \le 0,$$

where the last inequality follows from Proposition A.3. Since the right-hand side cannot be negative by Proposition A.2, it must vanish. But then Proposition A.2 implies that either $u = u_H$ or $u = u_{\sigma_H(H)}$, and either $v = v_H$ or $v = v_{\sigma_H(H)}$, as claimed, and Proposition A.3 implies that (42) holds. The argument obviously applies to the functional G as well. □

Remark 5.8 When $u_H = u \ne u_{\sigma_H(H)}$ and $v_H \ne v = v_{\sigma_H(H)}$, and the product in (42) vanishes almost everywhere in Ω, the equality holds in (41) although u and v have opposite polarizations.

Theorem 5.9 (*Solutions are polarized*) *Let $\Omega \subset \mathbb{R}^N$, $N \ge 2$ be a bounded open set with Lipschitz boundary and assume Ω is symmetric with respect to some hyperplane ∂H of a half-space H. Assume further that $k_{1,0}, k_{2,0}$ satisfy (30) for some $s \in (0,1)$, and $1 < q < \frac{N}{N-2s}$. Then system (29) has a weak solution (u_1, u_2) satisfying $u_1, u_2 > 0$ in Ω, which satisfy either $u_j = (u_j)_H$ for both $j = 1,2$, or $u_j = (u_j)_{\sigma_H(H)}$ for both $j = 1,2$. Furthermore, if u_j is symmetric with respect to ∂H for some $j \in \{1,2\}$, then u_{3-j} is also symmetric.*

Proof *Step 1:* Construction of a polarized solution. Denote by $v_j = (u_j)_H$ the polarization of u_j for $j = 1,2$. By Lemma 5.7, we find $G(v_1, v_2) \le G(u_1, u_2) = 0$, hence the real-valued function $g(t) = \frac{t^2}{2}\|(v_1, v_2)\|^2 - t^{2q}\|v_1 v_2\|^q_{L^q(\Omega)}$, whose graph has the shape depicted in Fig. 1, satisfies $g(1) = G(v_1, v_2) \le 0$. Consequently, there exists $t_0 \in (0,1]$ such that $(t_0 v_1, t_0 v_2) \in \mathscr{N}$. We may write $J(t_0 v_1, t_0 v_2) \le J(t_0 u_1, t_0 u_2)$ by Lemma 5.7, and $J(t_0 u_1, t_0 u_2) \le J(u_1, u_2)$ because the function $f(t) = J(tu_1, tu_2)$ attains its maximum at $t = 1$, hence

$$J(t_0 v_1, t_0 v_2) \le J(t_0 u_1, t_0 u_2) \le J(u_1, u_2). \tag{43}$$

Furthermore, recall that the value $c = J(u_1, u_2)$ is the minimum of J constrained to $\mathscr{N}$ by Lemma 5.3: this and (43) imply $t_0 = 1$ and $J(u_1, u_2) = J(v_1, v_2)$. Hence the pair (v_1, v_2), which is made up of polarized functions, positive in Ω, is also a minimizer of the functional J constrained to $\mathscr{N}$, and therefore the intrinsic gradient, also called the tangential gradient, of the functional J on the manifold $\mathscr{N}$ vanishes there.

Let us prove that the normal component of the gradient vanishes as well. By Lemma 5.2 we know that the direction of (u, v) is non-tangential to $\mathscr{N}$. Furthermore, by (33), the differential J' at any $(u, v) \in \mathscr{N}$ vanishes in the direction of (u, v), hence the normal component of the gradient also vanishes, as claimed. But then $J'(v_1, v_2) = 0$, and therefore the pair (v_1, v_2) is a weak solution of system (29).

Step 2: Comparison between solutions. Since $J(u_1, u_2) = J(v_1, v_2)$, by Lemma 5.7 we have that either $u_1 = v_1$ or $u_1 = (u_1)_{\sigma_H(H)}$, and either $u_2 = v_2$ or $u_2 = (u_2)_{\sigma_H(H)}$. To prove the theorem we have to exclude two cases: the case when $u_1 \ne v_1$ and $u_2 = v_2$, and the case when $u_1 = v_1$ and $u_2 \ne v_2$. We examine the first case in detail, the second one being analogous. Suppose, by contradiction, that $u_1 \ne v_1$ and $u_2 = v_2$. Then $u_1 = (u_1)_{\sigma_H(H)}$ and there exists a set $X_1 \subset \Omega \cap H$ having positive

measure and such that $u_1(x) < u_1(\sigma_H(x)) = v_1(x)$ for every $x \in X_1$. We may assume that $u_1(x) = u_1(\sigma_H(x)) = v_1(x)$ in $(\Omega \cap H) \backslash X_1$. Recall that the pairs (u_1, u_2) and $(v_1, v_2) = (v_1, u_2)$ are both critical points of the functional J. Now the condition $J'(u_1, u_2) = J'(v_1, u_2) = 0$ comes into play: we have

$$\mathscr{E}_{k_1}(u_1, \varphi) - \int_\Omega \left(a_1 \, u_1 \, \varphi + a_2 \, u_2 \, \psi \right) dx + \mathscr{E}_k(u_2, \psi) = \int_\Omega u_2^q \, u_1^{q-1} \, \varphi \, dx + \int_\Omega u_1^q \, u_2^{q-1} \, \psi \, dx$$

for every $(\varphi, \psi) \in D_k(\Omega)$ (cf. (36)), and similarly

$$\mathscr{E}_{k_2}(v_1, \varphi) - \int_\Omega \left(a_1 \, v_1 \, \varphi + a_2 \, u_2 \, \psi \right) dx + \mathscr{E}_k(u_2, \psi) = \int_\Omega u_2^q \, v_1^{q-1} \, \varphi \, dx + \int_\Omega v_1^q \, u_2^{q-1} \, \psi \, dx.$$

Letting $\varphi = 0$ and $\psi = (u_1 - v_1)^-$, and subtracting the second equality from the first one, we obtain

$$0 = \int_\Omega (u_1^q - v_1^q) \, u_2^{q-1} \, \psi \, dx = - \int_{X_1} (u_1^q - v_1^q) \, u_2^{q-1} \, (u_1 - v_1) \, dx < 0.$$

This contradiction shows that it is impossible to have $u_1 \neq v_1$ and $u_2 = v_2$. The case when $u_1 = v_1$ and $u_2 \neq v_2$ is excluded similarly. Hence we must have either $(u_1, u_2) = (v_1, v_2)$ or $(u_1, u_2) = ((u_1)_{\sigma_H(H)}, (u_2)_{\sigma_H(H)})$, as claimed. To complete the proof, suppose that u_j is symmetric with respect to ∂H for some $j \in \{1, 2\}$. For instance, suppose that u_2 is symmetric, the other case being analogous. Then $u_2 = v_2$, and the preceding argument shows that $u_1 = v_1$. Now we replace the half-space H with $\sigma_H(H)$, and we apply the same reasoning again, thus proving that $u_1 = (u_1)_{\sigma_H(H)}$, hence u_1 is symmetric. The proof is complete. □

Remark 5.10 We note that if $u = u_H$, then either u is symmetric with respect to the reflection at ∂H or there is $x \in H$ such that $u(x) > u(\sigma_H(x))$. Moreover, there exist non radial functions $u : \Omega \to \mathbb{R}$, defined in a radial set Ω, polarized with respect to every half-space H. A two-dimensional example is given by $\Omega = B_1(0) \subset \mathbb{R}^2$ and $u(x_1, x_2) = x_1 \, (1 - |x|^2)$.

5.4 Proof of the existence of solutions with axial symmetry

In the following, we finish the proof of Theorem 1.4. For this we assume $k_{1,0}, k_{2,0}$ satisfy (9) with $c > 0$ and $0 < s \leq \sigma < 1$. Let Ω be an open, bounded, radial domain in $\mathbb{R}^N$, $N \geq 2$, and let $1 < q < \frac{N}{N-2s}$. Moreover, we let $a_1, a_2 \in L^\infty(\mathbb{R})$ with $a_1 \neq a_2$ and $\|a_i^+\|_{L^\infty(\mathbb{R})} < \lambda_{1,k_i}(\Omega)$ for $i = 1, 2$.

Proof of Theorem 1.4 completed Note that by Theorem 5.1 and Corollary 5.5 it follows that there are $u_1, u_2 \in \mathscr{D}_k(\Omega)$, $u_1, u_2 > 0$ satisfying (10) with $u_1 \neq u_2$. Moreover, by Theorem 5.9 and the radiality of Ω and a_1, a_2, it follows that for every half-space H with $0 \in \partial H$ we have either

- $u_1 = (u_1)_H$ and $u_2 = (u_2)_H$, or
- $u_1 = (u_1)_{\sigma_H(H)}$ and $u_2 = (u_2)_{\sigma_H(H)}$.

Hence, if either u_1 or u_2 is not radial, it follows that after a rotation —and a renumbering if necessary— the assumption (6) is satisfied. Since clearly the right-hand sides of (10) satisfy (7) the statement of Theorem 1.4 follows from Theorem 1.2 once we have shown that

$$u_1 \text{ and } u_2 \text{ are bounded and continuous in } \Omega. \tag{44}$$

The boundedness of the solution pair follows indeed by a standard iteration argument using the Sobolev embedding theorem. We give the details of this argument in the appendix (see Lemma B.1 and Corollary B.3). Having the boundedness of u_1 and u_2, the continuity of u_1 and u_2 in Ω follow e.g. from [22]. Thus (44) holds and the statement of Theorem 1.4 follows from Theorem 1.2 as mentioned before. □

Appendix A: On the polarization of a function in the nonlocal setting

Recall the polarization of a function u with respect to an open half space defined in (22). Moreover, we use the notation of Section 5.

In the next proposition we show that polarization reduces the energy, with special care to the equality case (see also [2, Theorem 2] and [32, Proposition 8]). In the proof we will need the following (somehow surprising) identity:

Lemma A.1 (Functional identity and inequality) *Let H be a half-space in $\mathbb{R}^N$, and let $u : \mathbb{R}^N \to \mathbb{R}$ be any real-valued function. Define*

$$f(x_1, x_2) = u_H(x_1) u_H(x_2) + u_H(\sigma_H(x_1)) u_H(\sigma_H(x_2)) - u(x_1) u(x_2) - u(\sigma_H(x_1)) u(\sigma_H(x_2)),$$
$$g(x_1, x_2) = u_H(x_1) u_H(\sigma_H(x_2)) + u_H(\sigma_H(x_1)) u_H(x_2) - u(x_1) u(\sigma_H(x_2)) - u(\sigma_H(x_1)) u(x_2).$$

For every $x_1, x_2 \in H$ we have $f(x_1, x_2) = -g(x_1, x_2) \geq 0$. Furthermore, $f(x_1, x_2) = 0$ if and only if

$$\left(u(x_1) - u(\sigma_H(x_1)) \right) \left(u(x_2) - u(\sigma_H(x_2)) \right) \geq 0. \tag{45}$$

Proof Define $\xi_j = \frac{1}{2} \left(u(x_j) + u(\sigma_H(x_j)) \right)$ and $\eta_j = \frac{1}{2} \left(u(x_j) - u(\sigma_H(x_j)) \right)$, $j = 1, 2$, so that

$$\begin{cases} u(x_j) = \xi_j + \eta_j \\ u(\sigma_H(x_j)) = \xi_j - \eta_j \end{cases} \quad \begin{cases} u_H(x_j) = \xi_j + |\eta_j| \\ u_H(\sigma_H(x_j)) = \xi_j - |\eta_j| \end{cases}$$

where we have used the assumption that $x_1, x_2 \in H$. With this notation, we may write $f(x_1, x_2) = 2 |\eta_1 \eta_2| - 2 \eta_1 \eta_2$ and $g(x_1, x_2) = 2 \eta_1 \eta_2 - 2 |\eta_1 \eta_2|$, while inequality (45) reduces to $\eta_1 \eta_2 \geq 0$. The lemma follows. □

Proposition A.2 (*Polarization reduces the energy*) *Let Ω be an open set in $\mathbb{R}^N$, $k : \mathbb{R}^N \setminus \{0\} \to [0, \infty)$ be given by $k(z) = k_0(|z|)$ for a monotone decreasing function $k_0 : (0, \infty) \to [0, \infty)$ satisfying (3). Let $u \in \mathscr{D}_k(\Omega)$. For every half-space $H \subset \mathbb{R}^N$ we have $\mathscr{E}_k(u_H, u_H) \le \mathscr{E}_k(u, u)$. Furthermore, if k_0 strictly decreasing, then the equality $\mathscr{E}_k(u_H, u_H) = \mathscr{E}_k(u, u)$ holds if and only if either $u = u_H$, or $u = u_{\sigma_H(H)}$.*

Proof We start by giving a convenient expression of $\mathscr{E}_k(u, u)$. Since the integral is additive with respect to the domain of integration, we can split

$$\mathscr{E}_k(u, u) = \int_{H \times \mathbb{R}^N} (u(x_1) - u(z))^2 \, k(x_1 - z) \, dx_1 \, dz$$
$$+ \int_{\sigma_H(H) \times \mathbb{R}^N} (u(x) - u(y))^2 \, k(x - y) \, dx \, dy.$$

The last integral, by the change of variables $x = \sigma_H(x_1)$ and $y = \sigma_H(z)$, satisfies

$$\int_{\sigma_H(H) \times \mathbb{R}^N} (u(x) - u(y))^2 \, k(x - y) \, dx \, dy$$
$$= \int_{H \times \mathbb{R}^N} (u(\sigma_H(x_1)) - u(\sigma_H(z)))^2 \, k(x_1 - z) \, dx_1 \, dz$$

and therefore we may write

$$\mathscr{E}_k(u, u) = \int_{H \times \mathbb{R}^N} \left((u(x_1) - u(z))^2 + (u(\sigma_H(x_1)) - u(\sigma_H(z))^2 \right) k(x_1 - z) \, dx_1 \, dz.$$

Let us repeat the argument once more: we split

$$\mathscr{E}_k(u, u) = \int_{H \times H} \left((u(x_1) - u(z))^2 + (u(\sigma_H(x_1)) - u(\sigma_H(z))^2 \right) k(x_1 - z) \, dx_1 \, dz$$
$$+ \int_{H \times \sigma_H(H)} \left((u(x_1) - u(z))^2 + (u(\sigma_H(x_1)) - u(\sigma_H(z))^2 \right) k(x_1 - z) \, dx_1 \, dz.$$

Now in the first integral we write x_2 in place of z, and in the last integral we let $x_2 = \sigma_H(z)$, thus obtaining

$$\mathscr{E}_k(u, u) = \int_{H \times H} \left((u(x_1) - u(x_2))^2 + (u(\sigma_H(x_1)) - u(\sigma_H(x_2))^2 \right) k(x_1 - x_2) \, dx_1 \, dx_2$$
$$+ \int_{H \times H} \left((u(x_1) - u(\sigma_H(x_2)))^2 + (u(\sigma_H(x_1)) - u(x_2)^2 \right) k(x_1 - \sigma_H(x_2)) \, dx_1 \, dx_2.$$

By a similar procedure we also obtain

$$\mathscr{E}_k(u_H, u_H) = \int_{H \times H} \left((u_H(x_1) - u_H(x_2))^2 + (u_H(\sigma_H(x_1)) - u_H(\sigma_H(x_2))^2 \right)$$
$$k(x_1 - x_2) \, dx_1 \, dx_2$$
$$+ \int_{H \times H} \left((u_H(x_1) - u_H(\sigma_H(x_2)))^2 + (u_H(\sigma_H(x_1)) - u_H(x_2)^2 \right)$$
$$k(x_1 - \sigma_H(x_2)) \, dx_1 \, dx_2.$$

To go further, observe that $u^2(x_j) + u^2(\sigma_H(x_j)) = u_H^2(x_j) + u_H^2(\sigma_H(x_j))$ for $j = 1, 2$. Hence

$$\frac{1}{2} \left(\mathscr{E}_k(u, u) - \mathscr{E}_k(u_H, u_H) \right) = \int_{H \times H} f(x_1, x_2) \, k(x_1 - x_2) \, dx_1 \, dx_2$$
$$+ \int_{H \times H} g(x_1, x_2) \, k(x_1 - \sigma_H(x_2)) \, dx_1 \, dx_2$$

where f and g are as in Lemma A.1. Since $f = -g$, we may write

$$\frac{1}{2} \left(\mathscr{E}_k(u, u) - \mathscr{E}_k(u_H, u_H) \right) = \int_{H \times H} f(x_1, x_2) \left(k(x_1 - x_2) - k(x_1 - \sigma_H(x_2)) \right) dx_1 \, dx_2.$$

When the pair (x_1, x_2) ranges in the domain of integration $H \times H$, the distance from x_1 to x_2 cannot be larger than the distance from x_1 to $\sigma_H(x_2)$ (see (46)). Since k_0 is monotone decreasing by assumption, it follows that $k(x_1 - x_2) - k(x_1 - \sigma_H(x_2)) \geq 0$, which implies $\mathscr{E}_k(u_H, u_H) \leq \mathscr{E}_k(u, u)$ because f is non-negative. To manage the special case when k_0 is strictly decreasing, we need the equality

$$|x_1 - \sigma_H(x_2)|^2 - |x_1 - x_2|^2 = 4 \, d_1 \, d_2, \tag{46}$$

where $d_j \geq 0$ denotes the distance from x_j to ∂H, $j = 1, 2$. Equality (46) is established as follows. Let $\pi_j \in \partial H$ be the projection of x_j onto ∂H, $j = 1, 2$. Then by the Pythagorean theorem (see Fig. 2) we have $|x_1 - \sigma_H(x_2)|^2 = (d_1 + d_2)^2 + |\pi_1 - \pi_2|^2$ as well as $|x_1 - x_2|^2 = (d_1 - d_2)^2 + |\pi_1 - \pi_2|^2$, and (46) follows.

Equality (46) shows that $k(x_1 - x_2) - k(x_1 - \sigma_H(x_2)) > 0$ for all x_1, x_2 in the (open) half-space H. But then the equality $\mathscr{E}_k(u_H, u_H) = \mathscr{E}_k(u, u)$ holds if and only if $f(x_1, x_2) = 0$ a.e. in $H \times H$. By Lemma A.1, this occurs if and only if (45) holds a.e. in $H \times H$. Clearly, if $u = u_H$ or $u(x) = u_H(\sigma_H(x))$ almost everywhere in $\mathbb{R}^N$, then both

Fig. 2 Finding $|x_1 - x_2|$ and $|x_1 - \sigma_H(x_2)|$

factors in (45) have the same sign and therefore the inequality holds. Conversely, assume that (45) holds true. Then u may be symmetric with respect to ∂H. Otherwise there exists a non-negligible set $X \subset H$ such that either $u(x_1) - u(\sigma_H(x_1)) > 0$ in X, or $u(x_1) - u(\sigma_H(x_1)) < 0$ in X. In the first case, (45) implies that $u(x_2) - u(\sigma_H(x_2)) \geq 0$ a.e. in H, hence $u = u_H$. In the second case, (45) implies that $u = u_{\sigma_H(H)}$. The proof is complete.   □

The proposition above, which deals with the energy functional, is used in combination with the following, which deals with the L^q-norm of the product of two given functions. Contrary to what one may expect, it turns out that polarization increases the norm:

Proposition A.3 (On the L^q-norm of a product) *Let $\Omega \subset \mathbb{R}^N$ be an open, nonempty set, symmetric with respect to the boundary ∂H of some half-space H. Take two nonnegative functions $u, v \in L^q(\Omega)$ for some $q \in [1, \infty)$. Then*

$$\|u_H \, v_H\|_{L^q(\Omega)} \geq \|u \, v\|_{L^q(\Omega)}. \tag{47}$$

Furthermore, equality holds in (47) if and only if (42) is satisfied.

Proof In order to prove (47), we split

$$\int_\Omega u^q(x) \, v^q(x) \, dx = \int_{\Omega \cap H} u^q(x) \, v^q(x) \, dx + \int_{\Omega \cap \sigma_H(H)} u^q(y) \, v^q(y) \, dy$$

and perform the change of variable $y = \sigma_H(x)$ in the last integral. Since $\Omega = \sigma_H(\Omega)$, we obtain

$$\int_\Omega u^q(x) \, v^q(x) \, dx = \int_{\Omega \cap H} \left(u^q(x) \, v^q(x) + u^q(\sigma_H(x)) \, v^q(\sigma_H(x)) \right) dx.$$

By a similar procedure we also obtain

$$\int_\Omega u_H^q(x) \, v_H^q(x) \, dx = \int_{\Omega \cap H} \left(u_H^q(x) \, v_H^q(x) + u_H^q(\sigma_H(x)) \, v_H^q(\sigma_H(x)) \right) dx.$$

Thus, it is enough to prove that for all $y \in \Omega \cap H$ we have

$$u_H^q(x) \, v_H^q(x) + u_H^q(\sigma_H(x)) \, v_H^q(\sigma_H(x)) \geq u^q(x) \, v^q(x) + u^q(\sigma_H(x)) \, v^q(\sigma_H(x)), \quad (48)$$

which is readily obtained from the rearrangement inequality [15, (10.2.1)]. To manage with the equality case, we prefer to let

$$\begin{cases} u^q(x) = \xi_u + \eta_u \\ u^q(\sigma_H(x)) = \xi_u - \eta_u \end{cases} \qquad \begin{cases} v^q(x) = \xi_v + \eta_v \\ v^q(\sigma_H(x)) = \xi_v - \eta_v \end{cases}$$

Thus, for $y \in \Omega \cap H$ we have

$$\begin{cases} u_H^q(x) = \xi_u + |\eta_u| \\ u_H^q(\sigma_H(x)) = \xi_u - |\eta_u| \end{cases} \qquad \begin{cases} v_H^q(x) = \xi_v + |\eta_v| \\ v_H^q(\sigma_H(x)) = \xi_v - |\eta_v| \end{cases}$$

and (48) reduces to $|\eta_u \eta_v| \geq \eta_u \eta_v$, which obviously holds. Equality is achieved in (47) if and only if $\eta_u \eta_v \geq 0$ a.e. in $\Omega \cap H$, which is equivalent to (42). $\qquad\square$

Remark A.4 If u is symmetric with respect to ∂H, for instance if u is constant, then (42) holds for every v.

Appendix B: On the boundedness of solutions

In the following, $N \geq 2$, and we assume that $k_{1,0}, k_{2,0}$ satisfy (9) for some $s, \gamma, \sigma \in (0, 1)$, and $c > 0$. Moreover, Ω is an open bounded set in $\mathbb{R}^N$ with Lipschitz boundary, $\mathscr{E}_{k_i}(u, v)$ and $\mathscr{D}_{k_i}(\Omega)$, $i = 1, 2$ are defined as in Sect. 2.1. For a related result with the fractional Laplacian, see also [9, Lemma 2.3].

Lemma B.1 *Let A be a non-negative constant, and $1 \leq q < \frac{2_s}{2}$ with $2_s := \frac{2N}{N-2s}$. If u_1, u_2 are two functions in $\mathscr{D}_k(\Omega)$ satisfying*

$$\left| \mathscr{E}_{k_i}(u_i, \varphi_i) \right| \leq \int_\Omega \left(A \, |u_i| + |u_{3-i}|^q \, |u_i|^{q-1} \right) \varphi_i \, dx \quad \text{for all } \varphi_i \in \mathscr{D}_{k_i}(\Omega), \varphi_i \geq 0, i = 1, 2, \tag{49}$$

then $u_1, u_2 \in L^\infty(\Omega)$.

Remark B.2 Since $u_1, \varphi_1 \in \mathscr{D}_{k_1}(\Omega)$ and $u_2, \varphi_2 \in \mathscr{D}_{k_2}(\Omega)$, it follows immediately by the Sobolev embedding theorem that $u_1, u_2, \varphi_1, \varphi_2 \in L^m(\Omega)$ for every $m \in [1, 2_s]$. Since $q \leq \frac{2_s}{2}$ the integral in (49) converges. The strict inequality $q < \frac{2_s}{2}$ is needed in the Moser iteration (see below).

Proof of Lemma B.1 We follow the idea of Moser's iteration presented in [3] to show the claim. Moreover, k represents either k_1 or k_2 in order to present general inequalities, which follow in a similar manner.

Step 1: Preliminaries. Let $g \in W^{1,1}_{\text{loc}}(\mathbb{R})$ be nondecreasing and define

$$G : \mathbb{R} \to \mathbb{R}, \quad G(t) = \int_0^t \sqrt{g'(\tau)} \, d\tau. \tag{50}$$

Then we have (see also [3, Lemma A.2]) for $a, b \in \mathbb{R}$ using Hölder's inequality

$$(G(a) - G(b))^2 = \left(\int_b^a G'(t) \, dt \right)^2 \leq |b - a| \int_{\min\{a,b\}}^{\max\{a,b\}} G'(t)^2 \, dt = |b - a| \int_{\min\{a,b\}}^{\max\{a,b\}} g'(t) \, dt$$
$$= (a - b)(g(a) - g(b)).$$

Hence, if $g : \mathbb{R} \to \mathbb{R}$ is a nondecreasing Lipschitz function, that is, we have for some $L_g > 0$, $|g(a) - g(b)| \le L_g |a - b|$ for all $a, b \in \mathbb{R}$, it follows that $\mathscr{E}_k(g(v), g(v)) \le L_g^2 \, \mathscr{E}_k(v, v)$ and

$$\mathscr{E}_k(G(v), G(v)) \le \mathscr{E}_k(v, g(v)) \le L_g \, \mathscr{E}_k(v, v) \quad \text{for all } v \in \mathscr{D}_k(\Omega). \tag{51}$$

In particular, we see that both $g(v)$ and $G(v)$ belong to $\mathscr{D}_k(\Omega)$.

Step 2: A convenient Lipschitz function. To apply Moser's iteration, define for $L > 0, r \ge 2$ the Lipschitz function

$$g : \mathbb{R} \to \mathbb{R}, \quad g(t) = \begin{cases} 0 & t \le 0, \\[2mm] \dfrac{t^{r-1}}{r-1} & 0 < t < L, \\[2mm] \dfrac{L^{r-1}}{r-1} & t \ge L. \end{cases}$$

Then $g'(t) = 0 = G(t)$ for $t < 0$, where G is defined as in (50), and for $t > 0$ we have

$$g'(t) = \begin{cases} t^{r-2} & 0 < t < L, \\ 0 & t > L, \end{cases} \quad \text{and hence} \quad G(t) = \begin{cases} \dfrac{2\, t^{\frac{r}{2}}}{r} & 0 < t < L, \\[2mm] \dfrac{2\, L^{\frac{r}{2}}}{r} & t \ge L. \end{cases}$$

By the definition of g, it follows that if $v \ge 0$ then $g(v) \ge 0$.

Step 3: Energy estimate from above. We perform a suitable truncation of the kernels k_i and the solutions u_i: our purpose is to get rid of the linear term $A\,|u_i|$ in (49), thus proving (52). Let

$$k_{\delta,i} := \chi_{B_\delta(0)} \, k_i \quad \text{and} \quad j_{\delta,i} := k_i - k_{\delta,i}.$$

Note that k_{δ_i} satisfies the same assumptions as k_i and, in particular, we have $\mathscr{D}_{k_{\delta,i}}(\Omega) = \mathscr{D}_k(\Omega) \subset L^{2_s}(\Omega)$ for all $\delta > 0$. Moreover, by our assumptions on $k_{i,0}$, $i = 1, 2$, we have $j_{\delta,i} \in L^1(\mathbb{R}^N) \cap L^2(\mathbb{R}^N)$ for all $\delta > 0$ and $J_\delta := \min_{i=1,2} \|j_{\delta,i}\|_{L^1(\mathbb{R}^N)} \to \infty$ for $\delta \to 0$, hence we can fix some $\delta > 0$ such that

$$J_\delta > A.$$

With the Cauchy–Schwarz inequality and $v_{in} = (u_i - n)^+$ for $n \in \mathbb{N}_0$ we have, since $k_{\delta,i} = k_i - j_{\delta,i}$ and taking (49) into account,

$$\mathscr{E}_{k_{\delta,i}}(u_i, g(v_{in})) \le \mathscr{E}_{k_i}(u_i, g(v_{in})) - J_\delta \int_\Omega u_i(x)\, g(v_{in}(x))\, dx + \int_\Omega g(v_{in}(x)) \int_{\mathbb{R}^N} u_i(y) j_\delta(x-y)\, dy\, dx$$

$$\le \int_\Omega \left\{ (A\,|u_i(x)| - J_\delta\, u_i(x)) + |u_{3-i}(x)|^q |u_i(x)|^{q-1} + \int_{\mathbb{R}^N} u_i(y) j_\delta(x-y)\, dy \right\} g(v_{in}(x))\, dx$$

$$= \int_\Omega \left\{ (A - J_\delta)\, u_i(x) + |u_{3-i}(x)|^q |u_i(x)|^{q-1} + \int_{\mathbb{R}^N} u_i(y) j_\delta(x-y)\, dy \right\} g(v_{in}(x))\, dx$$

$$\le \int_\Omega \left\{ (A - J_\delta)\, n + |u_{3-i}(x)|^q |u_i(x)|^{q-1} + \|u_i\|_{L^2(\mathbb{R}^N)} \|j_\delta\|_{L^2(\mathbb{R}^N)} \right\} g(v_{in}(x))\, dx,$$

where we have used that $u_i \geq n$ in the set $\{g(v_{in}) > 0\}$. Since $J_\delta > A$ and $u_i \in L^2(\Omega)$, we can fix from now on some $n \in \mathbb{N}$ large such that $(A - J_\delta) n + \|u_i\|_{L^2(\mathbb{R}^N)} \|j_{\delta,i}\|_{L^2(\mathbb{R}^N)} \leq 0$ for $i = 1, 2$, and therefore

$$\mathscr{E}_{k_{\delta,i}}(u_i, g(v_{in})) \leq \int_\Omega |u_{3-i}(x)|^q |u_i(x)|^{q-1} g(v_{in}(x)) \, dx, \quad i = 1, 2. \tag{52}$$

Let $p \in [2_s, \infty)$ be such that $u_1, u_2 \in L^p(\Omega)$, and observe that for $x \in \{g(v_{in}) > 0\}$ we have

$$|u_i(x)|^{q-1} = u_i(x)^{q-1} = (n + v_{in})^{q-1} \leq C_q (n^{q-1} + v_{in}^{q-1})$$

for a suitable constant C_q. Hence, from (52) we get

$$\mathscr{E}_{k_{\delta,i}}(u_i, g(v_{in})) \leq C_q \|u_{3-i}\|_p^q \left(n^{q-1} \|g(v_{in})\|_\kappa + \|v_{in}^{q-1} g(v_{in})\|_\kappa \right), \tag{53}$$

where $\kappa = \frac{p}{p-q} \in (1, 2)$ is the conjugate exponent to $\frac{p}{q}$. Here and in the following, for $\beta \geq 1$ we let $\| \cdot \|_\beta = \| \cdot \|_{L^\beta(\Omega)}$.

Step 4: Energy estimate from below. To estimate $\mathscr{E}_{k_\delta}(u_i, g(v_{in}))$ from below, note that we have with Lemma 2.1, (51), and with the Sobolev embedding $\mathscr{D}_{k_{\delta,i}}(\Omega) \hookrightarrow L^{2_s}(\Omega)$

$$\mathscr{E}_{k_{\delta,i}}(u_i, g(v_{in})) = \mathscr{E}_{k_{\delta,i}}(u_i - n, g(v_{in})) \geq \mathscr{E}_{k_{\delta,i}}(v_{in}, g(v_{in})) \geq \mathscr{E}_{k_{\delta,i}}(G(v_{in}), G(v_{in})) \geq \varepsilon_0 \|G(v_{in})\|_{2_s}^2,$$

where $\varepsilon_0 > 0$ is a suitable constant.

Step 5: We show that $u_1, u_2 \in L^p(\Omega)$ for all $p \in [1, \infty)$. Combining the above inequality with (53), we have

$$\|G(v_{in})\|_{2_s}^2 \leq C \|u_{3-i}\|_p^q \left(\|g(v_{in})\|_\kappa + \|v_{in}^{q-1} g(v_{in})\|_\kappa \right),$$

where C depends on n, Ω, k, δ, q. Hence, with the monotone convergence theorem, we have for $L \to \infty$ and for every $r \geq 2$

$$\frac{4}{r^2} \|v_{in}\|_{2_s \frac{r}{2}}^r \leq \frac{C}{r-1} \|u_{3-i}\|_p^q \left(\|v_{in}\|_{(r-1)\kappa}^{r-1} + \|v_{in}\|_{(r+q-2)\kappa}^{r+q-2} \right).$$

Here and in the sequel it is understood that the norms of v_{in} may attain the value ∞. However, if the right-hand side is finite, then the left-hand side is also finite, and the inequalities hold. Furthermore, by Hölder's inequality the L^β-norm in the bounded domain Ω dominates the norm in $L^\alpha(\Omega)$ for $\alpha \in [1, \beta]$ in the sense that

$$\|f\|_\alpha \leq |\Omega|^{\frac{\beta-\alpha}{\alpha\beta}} \|f\|_\beta. \tag{54}$$

Using (54) with $f = v_{in}$, $\alpha = (r-1)\kappa$ and $\beta = (r+q-2)\kappa$ we get

$$\|v_{in}\|_{(r-1)\kappa} \leq |\Omega|^{\frac{q-1}{(r-1)(r+q-2)\kappa}} \|v_{in}\|_{(r+q-2)\kappa} = |\Omega|^{\frac{(q-1)(p-q)}{(r-1)(r+q-2)p}} \|v_{in}\|_{(r+q-2)\kappa}.$$

In view of the subsequent application, it is relevant that the coefficient $|\Omega|^{\frac{(q-1)(p-q)}{(r-1)(r+q-2)p}}$ keeps bounded when $p, r \to \infty$. Thus, by suitably modifying the constant C introduced before we may write

$$\frac{4}{r^2} \|v_{in}\|_{2_s\frac{r}{2}}^r \leq \frac{C}{r-1} \|u_{3-i}\|_p^q \left(\|v_{in}\|_{(r+q-2)\kappa}^{r-1} + \|v_{in}\|_{(r+q-2)\kappa}^{r+q-2} \right)$$

$$\leq \frac{2C}{r-1} \|u_{3-i}\|_p^q \max\left\{ 1, \|v_{in}\|_{(r+q-2)\kappa}^{r+q-2} \right\}.$$

The last inequality follows from the fact that $r - 1 \leq r + q - 2$. Hence

$$\|v_{in}\|_{2_s\frac{r}{2}} \leq \left(\frac{C}{2} \frac{r^2}{r-1} \right)^{\frac{1}{r}} \|u_{3-i}\|_p^{\frac{q}{r}} \max\left\{ 1, \|v_{in}\|_{(r+q-2)\kappa}^{1+\frac{q-2}{r}} \right\}$$

$$\leq (Cr)^{\frac{1}{r}} \|u_{3-i}\|_p^{\frac{q}{r}} \max\left\{ 1, \|v_{in}\|_{(r+q-2)\kappa}^{1+\frac{q-2}{r}} \right\} \tag{55}$$

$$\leq C' \|u_{3-i}\|_p^{\frac{q}{r}} \max\left\{ 1, \|v_{in}\|_{(r+q-2)\kappa}^{1+\frac{q-2}{r}} \right\},$$

where C' has the same dependencies as C by using that $(Cr)^{\frac{1}{r}} \to 1$ for $r \to \infty$, so that we can bound this quantity independently of $r \in [2, \infty)$. Notice that $v_{in} \leq u_i^+ \leq n + v_{in}$ in Ω, hence u_i^+ belongs to some $L^p(\Omega)$ if and only if v_{in} does. To manage with u_1^-, u_2^-, note that assumption (49) continues to hold if we replace u_i with $-u_i$. Hence, following the Steps 3, 4 and the above argumentation with $w_{in} = (-u_i - n)^+$ in place of u_i for $i = 1, 2$ and n as above, we also find

$$\|w_{in}\|_{2_s\frac{r}{2}} \leq C' \|u_{3-i}\|_p^{\frac{q}{r}} \max\left\{ 1, \|w_{in}\|_{(r+q-2)\kappa}^{1+\frac{q-2}{r}} \right\}. \tag{56}$$

Clearly, for $p \in [1, \infty]$ we have $u_i \in L^p(\Omega)$ if and only if $w_{in}, v_{in} \in L^p(\Omega)$. In order to use (55) and (56) iteratively (for both $i = 1$ and $i = 2$), we start from $p_0 = 2_s$, $\kappa_0 = \frac{2_s}{2_s-q}$ and r_0 such that

$$(r_0 + q - 2)\kappa_0 = 2_s, \quad \text{i.e. } r_0 := 2_s - 2q + 2 > 2.$$

Moreover, for $m \in \mathbb{N}$ we define $p_m = 2_s \frac{r_{m-1}}{2}$, $\kappa_m = \frac{p_m}{p_m-q}$ and we let r_m be obtained from r_{m-1} through the equality $2_s \frac{r_{m-1}}{2} = (r_m + q - 2)\kappa_m$. In other terms, we define $v = \frac{2_s}{2}$ and

$$r_m := v\, r_{m-1} - 2(q-1) = 2\, v^{m+1} - 2(q-1) \sum_{k=0}^{m} v^k = 2\, v^{m+1} \frac{v-q}{v-1} + 2 \frac{q-1}{v-1}.$$

Since $1 \leq q < v$ by assumption, it follows that $r_m \nearrow \infty$ for $m \to \infty$ and therefore $u_1, u_2 \in L^p$ for every $p \in [1, \infty)$. Indeed, using the notation introduced above, and letting $r = r_m$ in (55) we obtain

$$\|v_{in}\|_{p_{m+1}} \le C' \|u_{3-i}\|_{p_m}^{\frac{q}{r_m}} \max\left\{1, \|v_{in}\|_{p_m}^{1+\frac{q-2}{r_m}}\right\} \quad \text{for } i = 1, 2, \ m \in \mathbb{N} \text{ and}$$

$$\|w_{in}\|_{p_{m+1}} \le C' \|u_{3-i}\|_{p_m}^{\frac{q}{r_m}} \max\left\{1, \|w_{in}\|_{p_m}^{1+\frac{q-2}{r_m}}\right\} \quad \text{for } i = 1, 2, \ m \in \mathbb{N}$$

and we may inductively apply the inequalities above together with (54) to prove $v_{in}, w_{in} \in L^p(\Omega)$ for all $p \in [1, \infty)$, but then $u_1, u_2 \in L^p(\Omega)$ for $p \in [1, \infty)$ as claimed.

Step 6: Conclusion. To show that indeed we have $u_1, u_2 \in L^\infty(\Omega)$, we fix $i \in \{1, 2\}$ and $p_0 > \frac{N}{2s}$. Since the product $f_i := |u_{3-i}|^q |u_i|^{q-1}$ belongs to $L^{p_0}(\Omega)$, from (52) we get in place of (53) the estimate

$$\mathscr{E}_{k_{\delta,i}}(u_i, g(v_{in})) \le \|f_i\|_{p_0} \|g(v_{in})\|_{p'_0},$$

where $p'_0 = \frac{p_0}{p_0-1}$ and $v_{in} = (u_i - n)^+$ as above. Using again Step 4 and the subsequent argument, we arrive at

$$\|v_{in}\|_{2_s \frac{r}{2}} \le (C \|f_i\|_{p_0} r)^{\frac{1}{r}} \|v_{in}\|_{(r-1)p'_0}^{1-\frac{1}{r}} \le (C \|f_i\|_{p_0} r)^{\frac{1}{r}} \max\{1, \|v_{in}\|_{(r-1)p'_0}\}, \quad (57)$$

where C depends on n, Ω, k, and δ. In order to use (57) iteratively, we define $r_0 = 2$, $v = \frac{2_s}{2p'_0} > 1$ and

$$r_m := v r_{m-1} + 1 = \frac{2v - 1}{v - 1} v^m - \frac{1}{v - 1} \quad \text{for } m \in \mathbb{N}. \quad (58)$$

Note that $\alpha_m := 2_s \frac{r_m}{2} = (r_{m+1} - 1)p'_0$ for $m \in \mathbb{N}_0$, and $r_m \nearrow \infty$ for $m \to \infty$. Letting $r = r_{m+1}$ and $M \ge C \|f_i\|_{p_0}$ in (57) we get $\|v_{in}\|_{r_{m+1}} \le (M r_{m+1})^{\frac{1}{r_{m+1}}} a_m$, where $a_m = \max\{1, \|v_{in}\|_{\alpha_m}\} \ge 1$. Without loss of generality we take $M \ge \frac{1}{2}$, so that $M r_{m+1} \ge M r_0 \ge 1$, and therefore we may write $a_{m+1} \le \max\left\{1, (M r_{m+1})^{\frac{1}{r_{m+1}}} a_m\right\} = (M r_{m+1})^{\frac{1}{r_{m+1}}} a_m$ for $m \in \mathbb{N}_0$. Thus, by induction we obtain

$$a_m \le a_0 \prod_{j=1}^m (M r_j)^{\frac{1}{r_j}} \le a_0 \prod_{j=1}^\infty (M r_j)^{\frac{1}{r_j}}.$$

Using (58), it is readily seen that the infinite product in the right-hand side converges to a (finite) limit, hence we have

$$\|v_{in}\|_\infty = \lim_{m \to \infty} \|v_{in}\|_{\alpha_m} < \infty.$$

Since the argument above also holds with $w_{in} = (-u_i - n)^+$ in place of v_{in} we conclude that $u_1, u_2 \in L^\infty(\Omega)$. $\qquad\square$

Corollary B.3 *Let* $u_1 \in \mathscr{D}_{k_1}(\Omega)$, $u_2 \in \mathscr{D}_{k_2}(\Omega)$ *satisfy the system* (29) *for some* $a_1, a_2 \in L^\infty(\Omega)$ *and* q *such that* $1 \le q < \frac{N}{N-2s}$. *Then* $u_1, u_2 \in L^\infty(\Omega)$.

Proof Let $A := \max\{\|a_i\|_{L^\infty(\Omega)} \ : \ i = 1, 2\}$, then for $i = 1, 2$ and any $\varphi_i \in \mathscr{D}_{k_i}(\Omega)$, $\varphi_i \geq 0$ we have

$$\left| \mathscr{E}_{k_i}(u_i, \varphi) \right| \leq \int_\Omega \left(A |u_i| + |u_{3-i}|^q |u_i|^{q-1} \right) \varphi_i \, dx,$$

so the statement follows from Lemma B.1. □

References

1. Ambrosetti, A., Malchiodi, A.: Nonlinear Analysis and Semilinear Elliptic Problems. Cambridge Studies in Advanced Mathematics, vol. 104. Cambridge University Press, Cambridge (2006)
2. Baernstein, A.: A Unified Approach to Symmetrization. Partial Differential Equations of Elliptic Type (Cortona, 1992). Sympos. Math. XXXV, pp. 47–91. Cambridge University Press, Cambridge (1994)
3. Brasco, L., Parini, E.: The second eigenvalue of the fractional p-Laplacian. Adv. Calc. Var. **9**(4), 323–355 (2016). https://doi.org/10.1515/acv-2015-0007
4. Cabré, X., Sire, Y.: Nonlinear equations for fractional Laplacians I: regularity, maximum principles, and Hamiltonian estimates. Ann. Inst. H. Poincaré Anal. Non Linéaire **367**, 911–941 (2014)
5. Damascelli, L., Pacella, F.: Symmetry results for cooperative elliptic systems via linearization. SIAM J. Math. Anal. **45**(3), 1003–1026 (2013)
6. Damascelli, L., Pacella, F.: Sectional symmetry of solutions of elliptic systems in cylindrical domains. Discret. Contin. Dyn. Syst. **40**(6), 3305–3325 (2020)
7. de Figueiredo, D.G.: Semilinear Elliptic Systems: Existence, Multiplicity, Symmetry of Solutions, Handbook of Differential Equations: Stationary Partial Differential Equations. Vol. V (2008), pp. 1–48
8. Di Nezza, E., Palatucci, G., Valdinoci, E.: Hitchhiker's guide to the fractional Sobolev spaces. Bull. Sci. Math. **136**, 521–573 (2012)
9. Faria, L.F.O., Miyagaki, O.H., Pereira, F., Squassina, M., Zhang, C.: The Brezis–Nirenberg problem for nonlocal systems. Adv. Nonlinear Anal. **5**(1), 85–103 (2016). https://doi.org/10.1515/anona-2015-0114

10. Felmer, P., Quaas, A., Tan, J.: Positive solutions of nonlinear Schrödinger equation with the fractional Laplacian. Proc. R. Soc. Edinburgh Sect. A **142**(2), 1237–1262 (2012)
11. Felmer, P., Wang, Y.: Radial symmetry of positive solutions involving the fractional Laplacian. Commun. Contemp. Math (2013). https://doi.org/10.1142/S0219199713500235
12. Felsinger, M.: Parabolic equations associated with symmetric nonlocal operators. Doctoral thesis (2013)
13. Felsinger, M., Kassmann, M., Voigt, P.: The Dirichlet problem for nonlocal operators. Math. Z. **279**, 779–809 (2015)
14. Greco, A., Mascia, V.: Non-local sublinear problems: existence, comparison, and radial symmetry. Discret. Contin. Dyn. Syst. **39**, 503–519 (2019). https://doi.org/10.3934/dcds.2019021
15. Hardy, G.H., Littlewood, J.E., Pólya, G.: Inequalities. Cambridge University Press, Cambridge (1934)
16. Jarohs, S.: Symmetry via maximum principles for nonlocal nonlinear boundary value problems. Doctoral thesis (2015)
17. Jarohs, S.: Symmetry of solutions to nonlocal nonlinear boundary value problems in radial sets. NoDEA Nonlinear Differ. Equ. Appl. **23**(3), 1–22 (2016)
18. Jarohs, S., Weth, T.: Asymptotic symmetry for a class of nonlinear fractional reaction–diffusion equations. Discret. Contin. Dyn. Syst. **34**(6), 2581–2615 (2014). https://doi.org/10.3934/dcds.2014.34.2581
19. Jarohs, S., Weth, T.: Symmetry via antisymmetric maximum principles in nonlocal problems of variable order. Ann. Mat. Pura Appl.(4) **195**(1), 273–291 (2016)
20. Jarohs, S., Weth, T.: On the strong maximum principle for nonlocal operators. Math. Z. **293**(1), 81–111 (2019). https://doi.org/10.1007/s00209-018-2193-z
21. Jarohs, S., Weth, T.: Local compactness and nonvanishing for weakly singular nonlocal quadratic forms. Nonlinear Anal. **193**, 111431 (2020). https://doi.org/10.1016/j.na.2019.01.021
22. Kassmann, M., Mimica, A.: Intrinsic scaling properties for nonlocal operators. J. Eur. Math. Soc. **19**(4), 983–1011 (2017)
23. Maia, L.A., Montefusco, E., Pellacci, B.: Positive solutions for a weakly coupled nonlinear Schrödinger system. J. Differ. Equ. **229**, 743–767 (2006)
24. Pacella, F.: Symmetry results for solutions of semilinear elliptic equations with convex nonlinearities. J. Funct. Anal. **192**, 271–282 (2002)
25. Pacella, F., Weth, T.: Symmetry of solutions to semilinear elliptic equations via Morse index. Proc. Am. Math. Soc. **135**, 1753–1762 (2007)
26. Pólya, G.: Sur la symétrisation circulaire. Comp. Rend. Hebdomadaires Séances l'Académie Sci. **230**, 25–27 (1950)
27. Servadei, R., Valdinoci, E.: Mountain Pass solutions for non-local elliptic operators. J. Math. Anal. Appl. **389**, 887–898 (2012)
28. Servadei, R., Valdinoci, E.: Variational methods for non-local operators of elliptic type. Discret. Contin. Dyn. Syst. **33**(5), 2105–2137 (2013)
29. Smets, D., Willem, M.: Partial symmetry and asymptotic behavior for some elliptic variational problems. Calc. Var. Partial Differ. Equ. **18**(1), 57–75 (2003)
30. Saldaña, A., Weth, T.: Asymptotic axial symmetry of solutions of parabolic equations in bounded radial domains. J. Evol. Equ. **12**(3), 697–712 (2012)
31. Struwe, M.: Variational Methods. Applications to Nonlinear Partial Differential Equations and Hamiltonian Systems, 4th edn. Springer, Berlin (2008)
32. Van Schaftingen, J., Willem, M.: Set Transformations, Symmetrizations and Isoperimetric Inequalities, Nonlinear Analysis and Applications to Physical Sciences, pp. 135–152. Springer Italia, Milan (2004)
33. Weth, T.: Symmetry of solutions to variational problems for nonlinear elliptic equations via reflection methods. Jahresber. Deutsch. Math.-Ver. **112**, 119–158 (2010)
34. Willem, M.: Minimax Theorems, Progress in Nonlinear Differential Equations and Their Applications, vol. 24. Birkhäuser, Boston (1996)