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Relativistic Extended Thermodynamics of Polyatomic gases in the Landau and Lifshitz description

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Abstract: In this article the 15 moments model is found for polyatomic gases, expressed in terms of the physical variables previousy used in the Landau-Lifshitz description for monoatomic gases. It is also proved that the expression of the collision term Q recently proposed by Pennisi and Ruggeri as a variant of the Anderson and Witting model is equal, up to first order with respect to equilibrium, to the original one proposed by Anderson and Witting but which uses the four-veocity U_L^α in the Landau-Lifshitz description. The same thing can be said for all the balance equations, so that the two approaches are equivalent but only up to first order with respect to equilibrium.

Keywords: Rational Extended Thermodynamics; Rarefied Polyatomic Gases; Relativistic Fluids

1 Introduction

The idea behind this article comes from a comparison of two different models for monoatomic gases; one was presented in [1] and the other in [2]. The first one has been implemented in [3] for polyatomic gases and the same thing is here obtained by implementing, in the next section, the model of [2].
The principal difference between [1] and [2] is that they use different physical variables. In fact, both of them use the field equations

$$\partial_\alpha V^\alpha = 0 \, , \, \partial_\alpha T^{\alpha\beta} = 0 \, , \, \partial_\alpha A^{\alpha\beta\gamma} = I^{\beta\gamma} \, . \tag{1}$$

The first one of these is the conservation law of mass, while the second one is the conservation law of momentum-energy. In both models the independent variables are V^α and $T^{\alpha\beta}$, but they are decomposed in a different way. In fact, [1] uses the decomposition

$$V^\alpha = m\, n\, U^\alpha \, , \, T^{\alpha\beta} = \frac{e}{c^2}\, U^\alpha U^\beta \, + (p+\pi)\, h^{\alpha\beta} \, + \, \frac{2}{c^2}\, U^{(\alpha} q^{\beta)} \, + \, t^{<\alpha\beta>_3} \, , \tag{2}$$

where m is the rest mass of the particle, n is the particle number density, U^α is the four-velocity, e is the energy, c the light speed, p the pressure, π the dynamical pressure, $h^{\alpha\beta} = -\, g^{\alpha\beta} + \frac{1}{c^2}\, U^\alpha U^\beta$ is the projector into the subspace orthogonal to U^α, $g^{\alpha\beta} = \mathrm{diag}(1\,,\, -1\,,\, -1\,,\, -1)$ the metric tensor, q^α the heat flux, $t^{<\alpha\beta>_3}$ is the viscous deviatoric stress and they are constraned by

$$U^\alpha\, U_\alpha = c^2\,,\; q^\alpha\, U_\alpha = 0\,,\; t^{<\alpha\beta>_3}\, U_\alpha = 0\,,\; t^{<\alpha\beta>_3}\, g_{\alpha\beta} \,= 0\,. \tag{3}$$

On the other hand, [2] uses the decomposition

$$V^\alpha = m\, n^{CK}\, U_L^\alpha - \frac{m\, n^{CK}}{e^{CK} + p^{CK}}\, q^{CK\,\alpha}\,,\; T^{\alpha\beta} = \frac{e^{CK}}{c^2}\, U_L^\alpha U_L^\beta \,-\, (p^{CK} + \pi^{CK})\, \Delta^{\alpha\beta} \,+\, P^{<\alpha\beta>_3}\,, \tag{4}$$

(See eqs. (3)-(5) of cite2)where we have substituted en with e^{CK} and $\overline{\omega}$ with π^{CK} to compact the notations of the two articles), where the suffix CK has been introduced to distinguish the variables in [1] from those in [2]. Moreover, $\Delta^{\alpha\beta} = g^{\alpha\beta} - \frac{1}{c^2}\, U_L^\alpha U_L^\beta$ is the projector into the subspace orthogonal to U_L^α and the variables are constrained by

$$U_L^\alpha\, U_{L\alpha} = c^2\,,\; q^{CK\alpha}\, U_{L\alpha} = 0\,,\; P^{<\alpha\beta>_3}\, U_{\alpha L} = 0\,,\; P^{<\alpha\beta>_3}\, g_{\alpha\beta} \,= 0\,, \tag{5}$$

The transformation law between the two sets of variables isn't linear, as we will see in section 3, while in the next section we will find the model in the Landau-Lifshitz description extending the results of [2] to the case of polyatomic gases.
The physical meaning of the decomposition (2), (3) is self-evident, while that of (4), (5) can be found in the Landau-Lifshitz book [4]. It is important because it implies automatically zero production terms of mass and energy-momentum if the following model for the Boltzmann equation is adopted

$$p^\alpha\, \partial_\alpha\, f = Q\,, \quad \text{with} \quad Q = -\, \frac{U^{L\alpha}\, p_\alpha}{c^2\, \tau}\, (f \,-\, f_E)\,. \tag{6}$$

This model was proposed by Anderson and Witting [5], after having realized that the expression $Q = -\, \frac{m}{\tau}\, (f \,-\, f_E)$, previously proposed by Marle [6], implies a relaxation time τ tending to infinity in the limiting case of particles with zero rest-mass.
Both the expressions of Q in [5] and [6] in the non relativistic limit lead to the so-called BGK model which was formulated in [7] and [8] to simplify the structure of the collision term which otherwise depends on the product of distribution functions.
On the other hand, the decomposition (2), (3) jointly with $Q = -\, \frac{U^\alpha\, p_\alpha}{c^2\, \tau}\, (f \,-\, f_E)$ doesn't imply automatically zero production terms of mass and energy-momentum; this isn't due to the presence of $U_{L\alpha}$ in (6) because $f - f_E$ is of first order so that, to have a first order expression for Q, we must calculate $U_{L\alpha}$ at equilibrium where it is equal to U_α. So the problem is due to the different definitions of the first order deviations from equilibrium. This problem has been overcome in [9] by adding an extra term of Q in the framework of the Anderson-Witting model, both in the monoatomic and in the polyatomic case. It was useful in [10] to find the production term $I^{\beta\gamma}$ of (1).

In [11] it has been shown how a similar result is obtained in the framework of the Marle model. In sct. 4 we will prove that the expression (6) of Q in the Landau-Lifshitz description, if linearized with respect to equilibrium defined as in [3], gives the same expression found in [9]. The same thing can be said for all eqs. (1) and this will be proved in sect. 5.

2 The extension of the model in the Landau-Lifshitz description to the case of polyatomic gases.

Before going into this discussion, we want to take a look at the background in which it fits. For many years the works in Extended Thermodynamics were referred to monatomic gases and, for the case with an arbitrary but fixed number of moments in the classical context, the equations were used

$$\partial_t F^{i_1\cdots i_r} + \partial_k F^{k i_1\cdots i_r} = P^{i_1\cdots i_r}\,, \tag{7}$$

where the moments $F^{i_1\cdots i_r}$ and their fluxes are expressed in terms of the distribution function f by

$$F^{i_1\cdots i_r} = m\int_{\Re^3} f\,\xi^{i_1}\cdots\xi^{i_r}\,d\vec{\xi}. \tag{8}$$

Equation (7) with $r = 0, 1$ gives the conservation laws of mass and momentum, while that of energy is the trace of (7) with $r = 2$-
Countless articles have been written in this context, as can be seen from the book [12] and its bibliography.
An important step for the extension of the model to polyatomic gases was taken with the article [13] where the authors assumed that the distribution function depends on the time t, the space variables x^i , the microscopic velocity ξ^i and on a variable $\mathcal{I}$ representing the energy of the internal modes. This model led to the formulation of a generalized Boltzmann equation in [14]. Between these two works and inspired by the first of them, the article [15] was published; the idea developed there is that of two blocks of equations: In addition to the aforementioned eq. (7) (which represents the block of mass), there is another another one which represents the block of energy

$$\partial_t G^{i_1\cdots i_s} + \partial_k G^{k i_1\cdots i_s} = Q^{i_1\cdots i_s}\,. \tag{9}$$

Moreover, the moments and their fluxes are expressed in terms of the distribution function f by

$$\begin{aligned} F^{i_1\cdots i_r} &= m\int_{\Re^3}\int_0^{+\infty} f\,\xi^{i_1}\cdots\xi^{i_r}\phi(\mathcal{I})\,d\mathcal{I}\,d\vec{\xi}, \\ G^{i_1\cdots i_s} &= m\int_{\Re^3}\int_0^{+\infty} f\,\xi^{i_1}\cdots\xi^{i_s}\,\frac{2\mathcal{I}}{m}\,\phi(\mathcal{I})\,d\mathcal{I}\,d\vec{\xi}. \end{aligned} \tag{10}$$

In this way the conservation law of energy is no more the trace of (7) with $r = 2$, but eq. (9) with $s = 0$.

These works gave a breakthrough to Extended Thermodynamics and gave way to many other works some of which are summarized or cited in the book [16]. (See also [17] which describes the results obtained in this context, also in the relativistic framework as in [3], and which contains up to 539 references).
Working in this context and in particular in the relativistic context that has begun with [3], we will prove now that the closure for the field equations (1) in the Landau-Lifshitz description (which is different from that in [3]) is

$$A^{\alpha\beta\gamma} = A_1^0\, U_L^\alpha U_L^\beta U_L^\gamma - 3\, A_{11}^0\, \Delta^{(\alpha\beta} U_L^{\gamma)} \,+\, \frac{\Delta}{c^2}\, U_L^\alpha U_L^\beta U_L^\gamma - 3\, \Delta^{(\alpha\beta} U_L^{\gamma)} \left(\frac{N^\Delta}{D}\, \Delta \,+\, \frac{N^\pi}{D}\, \pi \right) - \frac{3}{c^4}\, \frac{m\, n}{e+p}\, \frac{N_1}{D_3}\, q^{(\alpha} U_L^\beta U_L^{\gamma)} + \frac{1}{5}\, \frac{m\, n}{e+p}\, \frac{N_2}{D_3}\, \Delta^{(\alpha\beta} q^{\gamma)} \,+ 3\, C_5\, P^{(<\alpha\beta>_3} U_L^{\gamma)}\,, \tag{11}$$

where A_1^0, A_{11}^0, D_3 and C_5 are the same of [3] and expressed in terms $\gamma = \frac{mc^2}{k_B T}$ (with T the absolute temperature), $\gamma^* = \gamma \left(1 + \frac{\mathcal{I}}{mc^2}\right)$ and of the functions

$$J_{m,n}(\gamma) = \int_0^{+\infty} \sinh^m s \cosh^n s \quad e^{-\gamma \cosh s}\, d\, s\,,$$

After that, the above expressions are

$$p = \frac{nmc^2}{\gamma}\,, \quad e = nmc^2\, \frac{\int_0^{+\infty} J_{2,2} \left(\gamma + \frac{\gamma \mathcal{I}}{mc^2}\right) \left(1 + \frac{\mathcal{I}}{mc^2}\right)\, \phi(\mathcal{I})\, d\,\mathcal{I}}{\int_0^{+\infty} J_{2,1}^*\, \phi(\mathcal{I})\, d\,\mathcal{I}}\,,$$

$$A_1^0 = n\, \frac{\int_0^{+\infty} J_{2,3}^* \left(1 + \frac{2\mathcal{I}}{mc^2}\right)\, \phi(\mathcal{I})\, d\,\mathcal{I}}{\int_0^{+\infty} J_{2,1}^*\, \phi(\mathcal{I})\, d\,\mathcal{I}}\,,\; A_{11}^0 = \frac{1}{3}\, n\, c^2\, \frac{\int_0^{+\infty} J_{4,1}^* \left(1 + \frac{2\mathcal{I}}{mc^2}\right)\, \phi(\mathcal{I})\, d\,\mathcal{I}}{\int_0^{+\infty} J_{2,1}^*\, \phi(\mathcal{I})\, d\,\mathcal{I}}\,, \tag{12}$$

$$D_3 = \begin{vmatrix} \frac{p}{m} & 2\, \frac{A_{11}^0}{m} \\ \frac{1}{3} B_4\, c^2 & \frac{2}{3} B_2\, c^2 \end{vmatrix} \quad , \quad C_5 = \frac{B_6}{B_1}\,.$$

The new coefficents appearing in (11) are

$$D = \begin{vmatrix} n\, c^2 & \frac{e}{m} & \frac{A_1^0}{m}\, c^2 & \frac{A_{11}^0}{m}\, c^2 \\ \frac{e}{m}\, c^2 & c^4\, B_5 & c^4\, B_3 & \frac{1}{3}\, c^4\, B_2 \\ \frac{p}{m} & \frac{1}{3}\, B_4 & \frac{1}{3}\, B_2 & \frac{1}{9}\, B_1 \\ \frac{A_1^0}{m}\, c^4 & c^4\, B_3 & c^4\, B_8 & \frac{1}{3}\, c^4\, B_7 \end{vmatrix}\,,$$

$$N^{\Delta} = \begin{vmatrix} n\,c^2 & \frac{e}{m} & \frac{A_1^0}{m}\,c^2 & \frac{A_{11}^0}{m}\,c^2 \\ \frac{e}{m}\,c^2 & c^4\,B_5 & c^4\,B_3 & \frac{1}{3}\,c^4\,B_2 \\ \frac{p}{m} & \frac{1}{3}\,B_4 & \frac{1}{3}\,B_2 & \frac{1}{9}\,B_1 \\ \frac{A_{11}^0}{m} & \frac{1}{3}\,B_2 & \frac{1}{3}\,B_7 & \frac{1}{9}\,B_6 \end{vmatrix} ,$$

$$N^{\pi} = - \begin{vmatrix} n\,c^2 & \frac{e}{m} & \frac{A_1^0}{m}\,c^2 & \frac{A_{11}^0}{m}\,c^2 \\ \frac{e}{m}\,c^2 & c^4\,B_5 & c^4\,B_3 & \frac{1}{3}\,c^4\,B_2 \\ \frac{A_{11}^0}{m} & \frac{1}{3}\,B_2 & \frac{1}{3}\,B_7 & \frac{1}{9}\,B_6 \\ \frac{A_1^0}{m}\,c^4 & c^4\,B_3 & c^4\,B_8 & \frac{1}{3}\,c^4\,B_7 \end{vmatrix} ,$$

$$N_1 = \begin{vmatrix} \frac{1}{3}\,B_2\,c^4 & \frac{2}{3}\,B_7 c^4 \\ \frac{1}{3}\,c^2\,B_4 & \frac{2}{3}\,c^2\,B_2 \end{vmatrix} ,$$

$$N_2 = \begin{vmatrix} B_1 & 2\,B_6 \\ \frac{1}{3}\,c^2\,B_4 & \frac{2}{3}\,c^2\,B_2 \end{vmatrix} .$$

The scalars $B_{\dots}$ appearing in the above expressions are those of eqs. (A.6)-(A.9) of [3], i.e.,

$$B_1 = n\,c^4\,\frac{\overline{J^*_{6,0}\left(1+\frac{\mathcal{I}}{mc^2}\right)\left(1+\frac{2\mathcal{I}}{mc^2}\right)}}{\overline{J^*_{2,1}}} ,\ B_2 = n\,c^2\,\frac{\overline{J^*_{4,2}\left(1+\frac{\mathcal{I}}{mc^2}\right)\left(1+\frac{2\mathcal{I}}{mc^2}\right)}}{\overline{J^*_{2,1}}} ,$$

$$B_3 = n\,\frac{\overline{J^*_{2,4}\left(1+\frac{\mathcal{I}}{mc^2}\right)\left(1+\frac{2\mathcal{I}}{mc^2}\right)}}{\overline{J^*_{2,1}}} ,\ B_4 = n\,c^2\,\frac{\overline{J^*_{4,1}\left(1+\frac{\mathcal{I}}{mc^2}\right)^2}}{\overline{J^*_{2,1}}} ,$$

$$B_5 = n\,\frac{\overline{J^*_{2,3}\left(1+\frac{\mathcal{I}}{mc^2}\right)^2}}{\overline{J^*_{2,1}}} ,\ B_6 = n\,c^4\,\frac{\overline{J^*_{6,1}\left(1+\frac{2\mathcal{I}}{mc^2}\right)^2}}{\overline{J^*_{2,1}}} ,$$

$$B_7 = n\,c^2\,\frac{\overline{J^*_{4,3}\left(1+\frac{2\mathcal{I}}{mc^2}\right)^2}}{\overline{J^*_{2,1}}} ,\ B_8 = n\,\frac{\overline{J^*_{2,5}\left(1+\frac{2\mathcal{I}}{mc^2}\right)^2}}{\overline{J^*_{2,1}}} ,$$

where overlined terms denote thath they are multiplied by $\phi(\mathcal{I})$ and, after that, integrated in $d\mathcal{I}\, d\vec{P}$.
To prove this we note that some passages followed in [3] still hold also in the present case. For example, the expression of the distribution function f in terms of the Lagrange multipliers λ, λ_β, $\lambda_{\beta\gamma}$ is

$$f = \exp\left(-1 - \frac{\chi}{k_B}\right) \tag{13}$$

with

$$\chi = m\lambda + \left(1 + \frac{\mathcal{I}}{mc^2}\right)\lambda_\beta p^\beta + \frac{1}{m}\left(1 + \frac{2\mathcal{I}}{mc^2}\right)\lambda_{\beta\gamma} p^\beta p^\gamma \,.$$

However, we want to consider here the case of the 15 moments model because as outlined in [18] this case is physically more significant. In any case, for monoatomic gases the trace of eq. $(1)_3$ is $(1)_1$ multiplied times a constant non zero coefficient; so the 15 moments model reduces in the monoatomic limit to the models in [1] and [2]. More than that, if we assume that the Lagrange multiplier $\lambda_{\beta\gamma}$ is traceless and take away the trace of $(1)_3$, we obtain also for polyatomic gases the model in [3]. Now eq. (13) **at equilibrium** reduces to

$$f = e^{-1-\frac{1}{k_B}\left[m\lambda + \left(1+\frac{\mathcal{I}}{mc^2}\right)\lambda_\beta p^\beta\right]} \,.$$

By substituting this distribution function in the definitions

$$\begin{aligned}
V^\alpha &= mc \int_{\Re^3}\int_0^{+\infty} f p^\alpha \phi(\mathcal{I})\, d\vec{P}\, d\mathcal{I}\,,\\
T^{\alpha\beta} &= c \int_{\Re^3}\int_0^{+\infty} f \left(1 + \frac{\mathcal{I}}{mc^2}\right) p^\alpha p^\beta\, \phi(\mathcal{I})\, d\vec{P}\, d\mathcal{I}\,,\\
A^{\alpha\beta\gamma} &= \frac{c}{m} \int_{\Re^3}\int_0^{+\infty} f \left(1 + \frac{2\,\mathcal{I}}{mc^2}\right) p^\alpha p^\beta p^\gamma \phi(\mathcal{I})\, d\vec{P}\, d\mathcal{I}\,,
\end{aligned} \tag{14}$$

we obtain

$$\begin{aligned}
&n_E = 4\pi m^3 c^3 e^{-1-\frac{m}{k}\lambda} \int_0^{+\infty} J^*_{2,1}\, \phi(\mathcal{I})\, d\mathcal{I}\,,\ V^\alpha_E = m\, n\, U^\alpha_L,\\
&T^{\alpha\beta}_E = -\,p\Delta^{\alpha\beta} + \frac{e}{c^2}\, U^\alpha_L U^\beta_L\,,\ A^{\alpha\beta\gamma}_E = A^0_1\, U^\alpha_L U^\beta_L U^\gamma_L - 3A^0_{11}\, \Delta^{(\alpha\beta} U^{\gamma)}_L
\end{aligned} \tag{15}$$

We note that these expressions are the same of (26), (41), (42), (48)-(50), jointly with $(36)_1$ of [3], excpt that now we have U^α_L instead of U^α and $-\,\Delta^{\alpha\beta}$ instead of $h^{\alpha\beta}$. They are also the extension of [2], at equiibrium, to the polyatomic case with 15 moments.
In order to obtain the **first order deviation from equilibrium** we have to consider the following system

$$\begin{aligned}
&V^\alpha_E(\lambda - \lambda_E) + T^{\alpha\mu}_E\left(\lambda_\mu - \frac{U_{L\mu}}{T}\right) + A^{\alpha\mu\nu}_E \lambda_{\mu\nu} = \frac{k_B\, n}{e+p}\, q^\alpha\,,\\
&T^{\alpha\beta}_E(\lambda - \lambda_E) + m\, A^{\alpha\beta\mu}_{11}\left(\lambda_\mu - \frac{U_{L\mu}}{T}\right) + m\, A^{\alpha\beta\mu\nu}_{12}\lambda_{\mu\nu} = -\frac{k_B}{m}\left(-\,\pi\,\Delta^{\alpha\beta} \,+\, P^{<\alpha\beta>_3}\right)\,,\\
&A^{\alpha\beta\gamma}_E(\lambda - \lambda_E) + m\, A^{\alpha\beta\gamma\mu}_{12}\left(\lambda_\mu - \frac{U_{L\mu}}{T}\right) + m\, A^{\alpha\beta\gamma\mu\nu}_{22}\lambda_{\mu\nu} = -\frac{k_B}{m}\left(A^{\alpha\beta\gamma} - A^{\alpha\beta\gamma}_E\right)\,,
\end{aligned} \tag{16}$$

where the new tensors appear

$$
\begin{aligned}
A_{11}^{\alpha\beta\mu} &= \frac{c}{m}\int_{\Re^3}\int_0^{+\infty} f_E p^\alpha p^\beta p^\mu \left(1+\frac{\mathcal{I}}{mc^2}\right)^2 \phi(\mathcal{I})\, d\vec{P}\, d\mathcal{I}\,,\\
A_{12}^{\alpha\beta\mu\nu} &= \frac{c}{m^2}\int_{\Re^3}\int_0^{+\infty} f_E p^\alpha p^\beta p^\mu p^\nu \left(1+\frac{\mathcal{I}}{mc^2}\right)\left(1+\frac{2\,\mathcal{I}}{mc^2}\right) \phi(\mathcal{I})\, d\vec{P}\, d\mathcal{I}\,,\\
A_{22}^{\alpha\beta\gamma\mu\nu} &= \frac{1}{m^5c^3}\int_{\Re^3}\int_0^{+\infty} f_E p^\alpha p^\beta p^\gamma p^\mu p^\nu \left(1+\frac{2\,\mathcal{I}}{mc^2}\right) \phi(\mathcal{I})\, d\vec{P}\, d\mathcal{I}\,.
\end{aligned}
\tag{17}
$$

Their expressions is reported in (A.6)-(A.8) of [3] and is

$$
\begin{aligned}
A_{11}^{\alpha\beta\mu} &= B_4\, h^{(\alpha\beta}U^{\gamma)} \,+\, B_5\, U^\alpha\, U^\beta\, U^\gamma\,,\\
A_{12}^{\alpha\beta\mu\nu} &= \frac{1}{5}\, B_1\, h^{(\alpha\beta}h^{\mu\nu)} \,+\, 2\, B_2\, h^{(\alpha\beta}U^\mu U^{\nu)} \,+\, B_3\, U^\alpha\, U^\beta\, U^\mu\, U^\nu\,,\\
A_{22}^{\alpha\beta\gamma\mu\nu} &= B_6\, h^{(\alpha\beta}h^{\gamma\mu}\, U^{\nu)} \,+\, \frac{10}{3}\, B_7\, h^{(\alpha\beta}U^\gamma\, U^\mu U^{\nu)} \,+\, B_8\, U^\alpha\, U^\beta\, U^\gamma U^\mu\, U^\nu\,.
\end{aligned}
\tag{18}
$$

By comparing the system (16) with (54) of [3], we see that only the right hand sides are different and this is due to the different definition of the non-equilibrium variables. This system is useful to give the first order deviation from equilibrium of the Lagrange multipliers in terms of physical variables. Since we are considering a 15 moments model, we have to take another independent variable besides n. U_L^α, γ, π, q^α, $P^{<\alpha\beta>_3}$ and we assume that it is

$$
\Delta = \frac{1}{c^4}\, U_{L\alpha}U_{L\beta}U_{L\gamma}\left(A^{\alpha\beta\gamma} - A_E^{\alpha\beta\gamma}\right)\,. \tag{19}
$$

If we contract $(16)_1$ with $U_{L\alpha}$, $(16)_2$ with $U_{L\alpha}U_{L\beta}$ and $\Delta_{\alpha\beta}$, $(16)_3$ with $U_{L\alpha}U_{L\beta}U_{L\gamma}$, we obtain

$$
\begin{aligned}
&nc^2(\lambda-\lambda_E) + \frac{e}{m}\, U^{L\mu}\left(\lambda_\mu - \frac{U_{L\mu}}{T}\right) + \frac{A_1^0}{m}\, c^2\, U^{L\mu}U^{L\nu}\lambda_{\mu\nu} \,-\, \frac{A_{11}^0}{m}\, c^2\, \Delta^{\mu\nu}\lambda_{\mu\nu} = 0\,,\\
&\frac{e}{m}\, c^2(\lambda-\lambda_E) + c^4 B_5\, U^{L\mu}\left(\lambda_\mu - \frac{U_{L\mu}}{T}\right) +\, B_3 c^4\, U^{L\mu}U^{L\nu}\lambda_{\mu\nu} \,-\, \frac{1}{3}\, B_2\, c^4\, \Delta^{\mu\nu}\lambda_{\mu\nu} = 0\,,\\
&\frac{p}{m}\,(\lambda-\lambda_E) + \frac{1}{3}B_4 U^{L\mu}\left(\lambda_\mu - \frac{U_{L\mu}}{T}\right) +\, \frac{1}{3}B_2\, U^{L\mu}U^{L\nu}\lambda_{\mu\nu} -\, \frac{1}{9}\, B_1\, \Delta^{\mu\nu}\lambda_{\mu\nu} = -\,\frac{k_B}{m^2}\,\pi\,,\\
&\frac{A_1^0}{m}\, c^4\,(\lambda-\lambda_E) +\, B_3\, c^4 U^{L\mu}\left(\lambda_\mu - \frac{U_{L\mu}}{T}\right) +\, B_8\, c^4\, U^{L\mu}U^{L\nu}\lambda_{\mu\nu} \,-\, \frac{1}{3}\, B_7\, c^4\, \Delta^{\mu\nu}\lambda_{\mu\nu} = -\,\frac{k_B}{m^2}\,\Delta\,,
\end{aligned}
\tag{20}
$$

where the coefficients of the unknowns $(\lambda-\lambda_E)$, $U^{L\mu}\left(\lambda_\mu - \frac{U_\mu}{T}\right)$, $U^{L\mu}U^{L\nu}\lambda_{\mu\nu}$, $-\,\Delta^{\mu\nu}\lambda_{\mu\nu}$ are the same reported in [3]. We see that this system is the same as what would have been obtained with the same calculations but using decomposition (2), (3) so also the value of the above unknowns is the same and reads

$$
\begin{aligned}
&\lambda-\lambda_E = -\,\frac{k_B}{m^2}\left(\frac{N_{31}}{D}\,\pi \,+\, \frac{N_{41}}{D}\,\Delta\right)\,, \quad U^{L\mu}\left(\lambda_\mu - \frac{U_\mu}{T}\right) = -\,\frac{k_B}{m^2}\left(\frac{N_{32}}{D}\,\pi \,+\, \frac{N_{42}}{D}\,\Delta\right)\,,\\
&U^{L\mu}U^{L\nu}\lambda_{\mu\nu} = -\,\frac{k_B}{m^2}\left(\frac{N_{33}}{D}\,\pi \,+\, \frac{N_{43}}{D}\,\Delta\right)\,, \quad \Delta^{\mu\nu}\lambda_{\mu\nu} = \frac{k_B}{m^2}\left(\frac{N_{34}}{D}\,\pi \,+\, \frac{N_{44}}{D}\,\Delta\right)\,,
\end{aligned}
\tag{21}
$$

where D is the determinant $(12)_5$ and N_{ij} is the algebraic complement in D of its element in the line i and coulumn j.

We contract now $(16)_1$ with Δ^δ_α, and $(16)_2$ with $\Delta^\delta_\alpha U_{L\beta}$; so we obtain

$$\frac{p}{m}\,\Delta^{\delta\mu}\left(\lambda_\mu - \frac{U_{L\mu}}{T}\right) + 2\frac{A^0_{11}}{m}\,\Delta^{\delta\mu}U^\nu_L\,\lambda_{\mu\nu} = -\,\frac{k_B\,n}{m\,(e+p)}\,q^\delta\,,$$
$$\frac{1}{3}B_4 c^2\,\Delta^{\delta\mu}\left(\lambda_\mu - \frac{U_{L\mu}}{T}\right) + \frac{2}{3}B_2\,c^2\,\Delta^{\delta\mu}U^\nu_L\,\lambda_{\mu\nu} = 0\,.$$

We see that this system isn't the same as what would have been obtained with the same calculations but using decomposition (2), (3) (the difference is present in the right hand sides); in any case it gives the above unknowns and they are

$$\begin{aligned}\Delta^{\delta\mu}\left(\lambda_\mu - \frac{U_{L\mu}}{T}\right) &= -\,\frac{2}{3}\,\frac{k_B\,n}{m\,(e+p)}\,B_2\,c^2\,\frac{q^\delta}{D_3}\,,\\ \Delta^{\delta\mu}U^\nu_L\,\lambda_{\mu\nu} &= \frac{1}{3}\,\frac{k_B\,n}{m\,(e+p)}\,B_4\,c^2\,\frac{q^\delta}{D_3}\,,\end{aligned} \tag{22}$$

where D_3 is the above reported determinant $(12)_5$.
We contract now $(16)_2$ with $\Delta^{<\delta}_\alpha\Delta^{\phi>_3}_\beta$; so we obtain

$$\frac{2}{15}\,B_1\,\Delta^{\mu<\delta}\Delta^{\phi>_3\nu}\,\lambda_{\mu\nu} = -\,\frac{k_B}{m^2}\,P^{<\delta\phi>_3} \;\rightarrow\quad \Delta^{\mu<\delta}\Delta^{\phi>_3\nu}\,\lambda_{\mu\nu} = -\,\frac{15}{2\,B_1}\,\frac{k_B}{m^2}\,P^{<\delta\phi>_3}\,. \tag{23}$$

We substitute now the Lagrange multipliers given by (21), (22), (23) in $(16)_3$; so we obtain the closure (11).

• **The production terms.**
Eq. $(6)_1$ multiplied by $mc\,\phi(\mathcal{I})$, $c\,\left(1+\frac{\mathcal{I}}{mc^2}\right)p^\beta\,\phi(\mathcal{I})$, $\frac{c}{m}\,\left(1+\frac{2\mathcal{I}}{mc^2}\right)p^\beta\,p^\gamma\,\phi(\mathcal{I})$ and integrated in $d\mathcal{I}\,d\vec{P}$ gives respectively

$$\begin{aligned}
I &= -\,\frac{U_{L\alpha}}{c^2\tau}\,(V^\alpha - V^\alpha_E) = \frac{U_{L\alpha}}{c^2\tau}\,\frac{m\,n\,q^\alpha}{e+p} = 0 \quad \text{(Mass production)},\\
I^\beta &= -\,\frac{U_{L\alpha}}{c^2\tau}\,\left(T^{\alpha\beta} - T^{\alpha\beta}_E\right) = -\,\frac{U_{L\alpha}}{c^2\tau}\,\left[\pi\,\Delta^{\alpha\beta} + P^{<\alpha\beta>_3}\right] = 0\,\text{(Momentum-energy production)},\\
I^{\beta\gamma} &= -\,\frac{U_{L\alpha}}{c^2\tau}\,\left(A^{\alpha\beta\gamma} - A^{\alpha\beta\gamma}_E\right) = -\,\frac{1}{c^2\tau}\,\left[\Delta\,U^\beta_L U^\gamma_L - \,\Delta^{\beta\gamma}\,c^2\,\left(\frac{N^\Delta}{D}\,\Delta\,+\,\frac{N^\pi}{D}\,\pi\right) - \right.\\
&\qquad \left.\frac{2}{c^2}\,\frac{m\,n}{e+p}\,\frac{N_1}{D_3}\,U^{L(\beta}\,q^{\gamma)}_L\,+C_5\,c^2\,P^{(<\beta\gamma>_3}\right]\,.
\end{aligned} \tag{24}$$

So the closure of (1) is completed.

3 The transformation law between the sets of variables in the two descriptions,

Let us firstly see how to obtain the variables in [1] in terms those in the Landau-Lifshitz description.
From $(2)_1$ and $(4)_1$ we have $m\,n\,U^\alpha = m\,n^{CK}\,U^\alpha_L - \frac{m\,n^{CK}}{e^{CK}+p^{CK}}\,q^{CK\,\alpha}$; this equation, contracted with itself gives

$$n = n^{CK}\,\left(1\,+\,\frac{q^{CK\,\alpha}q_{CK\,\alpha}}{c^2\,(e^{CK}+p^{CK})^2}\right)^{\frac{1}{2}}\,. \tag{25}$$

After that, the equation under consideration gives

$$\begin{aligned} U^\alpha &= \left(1\,+\,\frac{q^{CK\,\alpha}q_{CK\,\alpha}}{c^2\,(e^{CK}+p^{CK})^2}\right)^{-\frac{1}{2}}\,\left(U^\alpha_L\,-\,\frac{q^{CK\,\alpha}}{e^{CK}+p^{CK}}\right)\,, \\ p &= n^{CK}\,k_B\,T\,\left(1\,+\,\frac{q^{CK\,\alpha}q_{CK\,\alpha}}{c^2\,(e^{CK}+p^{CK})^2}\right)^{\frac{1}{2}}\,, \end{aligned} \tag{26}$$

where the last equation is a consequence of $p = n\,k_B\,T$. From (25) and (26) we see that the transformation law between the sets of variables in the two descriptions is not linear. Moreover, from (26) we can desume the expression of $h^{\alpha\beta}$; from $(2)_2$ and $(4)_2$ we have

$$\frac{e}{c^2}\,U^\alpha U^\beta\,+(p+\pi)\,h^{\alpha\beta}\,+\,\frac{2}{c^2}\,U^{(\alpha}q^{\beta)}\,+\,t^{<\alpha\beta>_3} = \frac{e^{CK}}{c^2}\,U^\alpha_L U^\beta_L\,-\,(p^{CK}+\pi^{CK})\,\Delta^{\alpha\beta}\,+\,P^{<\alpha\beta>_3}\,, \tag{27}$$

which, contracted with $\frac{1}{c^2}\,U_\alpha U_\beta,\ h_{\alpha\beta},\ U_\alpha h^\delta_\beta,\ h^\gamma_\alpha h^\delta_\beta\,-\,\frac{1}{3}\,h_{\alpha\beta}h^{\gamma\delta}$ gives $e,\ p+\pi,\ q^\alpha,\ t^{<\alpha\beta>_3}$ respectively, in tems of the other variables. These expressions are not linear (for example, q^α is a linear combination of U^α_L, $q^{CK\,\alpha}$, $P^{<\alpha\beta>_3}q_{CK\,\beta}$ through non linear scalar coefficients). For the sake of simplicity we will limit ourselves to write here only the linear parts with respect to the equilibrium state defined in [2] and which we will denote with the suffix E to distinguish it from that defined in [1] which we will denote with the suffix eq.
Well, eqs. (25) and (26) give

$$n = n^{CK}\,,\ p = p^{CK}\,,\ U^\alpha = U^\alpha_L\,-\,\frac{q^{CK\,\alpha}}{e^{CK}+p^{CK}}\,,\ \text{up to first order with respect to equilibrium}\,. \tag{28}$$

After that, (27) contracted with $\frac{U_\alpha U_\beta}{c^2}$ and $\frac{h_{\alpha\beta}}{3}$ gives

$$e = e^{CK}\,,\ \pi = \pi^{CK}\,,\ \text{up to first order with respect to equilibrium}\,. \tag{29}$$

Finally, (27) contracted with $U_\alpha h^\gamma_\beta$ and $h^{<\gamma}_\alpha\,h^{\delta>_3}_\beta$ gives

$$q^\gamma = \frac{p^{CK}}{e^{CK}+p^{CK}}\,q^{CK\,\gamma}\,,\ t^{<\gamma\delta>_3} = P^{<\gamma\delta>_3}\,,\ \text{up to first order with respect to equilibrium}\,. \tag{30}$$

• Let us consider now the transformation in the inverse sense to find the variables of the Landau-Lifshiz description in terms of the commonly used variables. To this end we note that $(4)_2$ contracted with $U_{L\beta}$ gives

$$\left(T^{\alpha\beta} \,-\, e^{CK} g^{\alpha\beta}\right) U_{L\beta} = 0\,, \tag{31}$$

i.e., e^{CK} is an eigenvalue of $T^{\alpha\beta}$ and $U_{L\beta}$ is a corresponding eigenvector. Since eigenvectors are defined except for a coefficient of proportionality, this coefficient must be chosen in order to satisfy the condition $(5)_1$.
Once $U^{L\alpha}$ is known, eq. $(4)_1$ contacted with $U^{L\alpha}$ gives

$$\begin{aligned} n^{CK} &= \frac{n}{c^2}\, U^{L\alpha}\, U_\alpha\,, \quad \text{and its remaining part gives} \quad \frac{q^{CK\,\alpha}}{e^{CK}+p^{CK}} = U_L^\alpha \,-\, \frac{n}{n^{CK}}\, U^\alpha\,, \\ p^{CK} &= \frac{p}{c^2}\, U^{L\alpha}\, U_\alpha\,, \end{aligned} \tag{32}$$

where the third equation comes from $p^{CK} = n^{CK}\, k_B\, T$. (Note that $q^{CK\,\alpha}$ plays a role in the above equations only through $\frac{q^{CK\,\alpha}}{e^{CK}+p^{CK}}$).
After this, eq. (27) contracted with $g_{\alpha\beta}$ and $\Delta_\alpha^{<\gamma}\,\Delta_\beta^{\delta>_3}$ gives respectively,

$$\begin{aligned} \pi^{CK} &= p\left(1 \,-\, \frac{U^{L\alpha}\, U_\alpha}{c^2}\right) \,+\, \pi \,+\, \frac{1}{3}\,\left(e^{CK} - e\right)\,, \\ P^{<\gamma\delta>_3} &= \Delta_\alpha^{<\gamma}\,\Delta_\beta^{\delta>_3}\, t^{<\alpha\beta>_3} \,+\, \frac{e \,+\, p \,+\, \pi}{c^2}\,\Delta_\alpha^{<\gamma}\,\Delta_\beta^{\delta>_3}\, U^\alpha U^\beta\,. \end{aligned} \tag{33}$$

- For example, **at equilibrium** defined as in [1], eq. (31) becomes

$$\left[\left(e_{eq}^{CK} + p\right) h^{\alpha\beta} \,+\, \frac{e \,-\, e_{eq}^{CK}}{c^2}\, U^\alpha U^\beta\right] U_{L\beta} = 0\,.$$

  This has the eigenvalue $e_{eq}^{CK} = e$ and the eigenvector $U_{L\beta} = U_\beta$ which satisfies also the condition $(5)_1$. Eq. (32) gives

$$n_{eq}^{CK} = n\,, \quad p_{eq}^{CK} = p\,, \quad q_{eq}^{CK\,\alpha} = 0\,.$$

  Finally, eq. (33) gives

$$\pi_{eq}^{CK} = 0\,, \quad P_{eq}^{<\gamma\delta>_3} = 0\,.$$

- Let us consider now **the first order homogeneous part of our equations with respect to equilibrium**. Eq. (31) becomes

$$\begin{aligned} &\left(T_{eq}^{\alpha\beta} \,-\, e\, g^{\alpha\beta}\right) U_{L\beta}^{(1)} + \left(T^{\alpha\beta} \,-\, T_{eq}^{\alpha\beta} \,-\, e^{CK(1)}\, g^{\alpha\beta}\right) U_\beta = 0 \,\rightarrow \\ &\qquad \rightarrow\, (e+p)\, h^{\alpha\beta}\, U_{L\beta(1)} \,+\, q^\alpha \,-\, e^{CK(1)}\, U^\alpha = 0\,. \end{aligned}$$

This equation contracted with U_α gives

$$e^{CK(1)} = 0\,, \quad \text{and there remains} \quad h^{\alpha\beta}\, U_{L\beta(1)} = -\,\frac{q^\alpha}{e+p}\,.$$

There is still to impose the condition $(5)_1$ which, at first order becomes $U^\beta\, U_{L\beta(1)} = 0$. Jointly with the previous result, this equation gives

$$U^{L\alpha}_{(1)} = \frac{q^\alpha}{e+p}\,.$$

The first order part of eq. (32) gives

$$n^{CK(1)} = 0\,,\, p^{CK(1)} = 0\,,\, \frac{q^{CK(1)\,\alpha}}{e+p} = U^{L(1)\alpha} = \frac{q^\alpha}{e+p} \;\rightarrow\; q^{CK(1)\,\alpha} = q^\alpha\,.$$

The first order part of (33) is

$$\pi^{CK(1)} = \pi\,,\, P^{<\delta\gamma>_3(1)} = t^{<\delta\gamma>_3}\,.$$

So we have obtained that, up to first order

$$\begin{aligned} & n^{CK} = n\,,\, p^{CK} = p\,,\, \pi^{CK} = \pi\,,\, U^{L\alpha} = U^\alpha + \frac{q^\alpha}{e+p}\,,\, e^{CK} = e\,,\, q^{CK\,\alpha} = q^\alpha\,, \\ & P^{<\delta\gamma>_3(1)} = t^{<\delta\gamma>_3}\,. \end{aligned} \tag{34}$$

We note that these eqs. are the inverse of (28)-(30) except for $(30)_1$ and $(34)_6$.

4 The linearization of Q defined by (6) in the variables (2), (3).

We firstly see that $U^{L\alpha}$ in (6) is multplied by a first order term $f - f_E$ with respect to equilibrium. So in a linerized theory it can be replaced by U^α. After that, we see that in the expression of Q proposed in [9], there is too a factor $f - f_{eq}$; so we evaluate now their difference

$$(f - f_E) \,-\, (f - f_{eq}) = f_{eq}\left(1 \,-\, \frac{f_E}{f_{eq}}\right) = f_{eq}\left(1 \,-\, e^{-\frac{1}{k_B}\left[m(\lambda_E - \lambda_{eq}) + \left(1 + \frac{\mathcal{I}}{mc^2}\right)\left(\lambda^E_\beta - \lambda^{eq}_\beta\right) p^\beta\right]}\right)\,. \tag{35}$$

But λ_E and λ_{eq} are defined by

$$e^{-1 - \frac{1}{k_B}\lambda_E} = \frac{1}{4\,\pi\, m^3 c^3}\,\frac{n^{ck}}{\int_0^{+\infty} J^*_{2,1}\,\phi(\mathcal{I})\,d\mathcal{I}}\,, \quad e^{-1 - \frac{1}{k_B}\lambda_{eq}} = \frac{1}{4\,\pi\, m^3 c^3}\,\frac{n}{\int_0^{+\infty} J^*_{2,1}\,\phi(\mathcal{I})\,d\mathcal{I}}\,.$$

Since n^{ck} and n are equal, up to first order with respect to equilibrium, we have also that $\lambda_E = \lambda_{eq}$ up to the same order. So (35) becomes

$$(f - f_E) \, - \, (f - f_{eq}) = f_{eq} \left(1 \, - \, e^{-\frac{1}{k_B}\left(1+\frac{\mathcal{I}}{mc^2}\right)\left(\lambda^E_\beta - \lambda^{eq}_\beta\right) p^\beta} \right) . \tag{36}$$

Moreover, we have that

$$\lambda^E_\beta = \frac{U_{L\beta}}{T} \, , \quad \lambda^{eq}_\beta = \frac{U_\beta}{T} \; \rightarrow \quad \lambda^E_\beta \, - \, \lambda^{eq}_\beta = \frac{U_{L\beta} \, - \, U_\beta}{T} = \frac{q_\beta}{(e+p)T} \, ,$$

where in the last passage we have used $(34)_4$. So eq. (36) becomes

$$(f - f_E) \, - \, (f - f_{eq}) = f_{eq} \left(1 \, - \, e^{-\frac{1}{k_B}\left(1+\frac{\mathcal{I}}{mc^2}\right)\frac{q_\beta \, p^\beta}{(e+p)T}} \right) \approx f_{eq} \, \frac{1}{k_B} \left(1 + \frac{\mathcal{I}}{mc^2} \right) \frac{q_\beta \, p^\beta}{(e+p)T} \, ,$$

where the last passage holds up to first order with respect to quilibrium. So we have obtained that $(6)_2$, up to first order with respect to quilibrium becomes

$$Q = - \frac{U^\alpha \, p_\alpha}{c^2 \, \tau} \left[f \, - \, f_{eq} \, + \, f_{eq} \, \frac{1}{k_B} \left(1 + \frac{\mathcal{I}}{mc^2} \right) \frac{q_\beta \, p^\beta}{(e+p)T} \right] , \tag{37}$$

while in [9] it was proposed

$$Q = - \frac{U^\alpha \, p_\alpha}{c^2 \, \tau} \left[f \, - \, f_{eq} \, + 3 \, f_{eq} \left(1 + \frac{\mathcal{I}}{mc^2} \right) \frac{q_\beta \, p^\beta}{B_4 m^2 c^2} \right] . \tag{38}$$

To prove that (37) and (38) are equal, it suffices to prove that

$$m \, B_4 \, \gamma = 3(e+p) \, , \quad \text{i.e., that} \tag{39}$$

$$m \, n \, c^2 \, \frac{\gamma \int_0^{+\infty} J^*_{4,1} \, \left(1 + \frac{\mathcal{I}}{mc^2}\right)^2 \, \phi(\mathcal{I}) \, d\mathcal{I}}{\int_0^{+\infty} J^*_{2,1} \, \phi(\mathcal{I}) \, d\mathcal{I}} = m \, n \, c^2 \, \frac{\int_0^{+\infty} \, \left(3 \, J^*_{2,2} \, + \, J^*_{4,0}\right) \, \left(1 + \frac{\mathcal{I}}{mc^2}\right) \, \phi(\mathcal{I}) \, d\mathcal{I}}{\int_0^{+\infty} J^*_{2,1} \, \phi(\mathcal{I}) \, d\mathcal{I}} \, ,$$

and this relation is identically satisfied because from $(7.6)_2$ of [1] we have

$$\gamma \, J_{4,1}(\gamma) = - \, J_{2,0}(\gamma) + 4 J_{2,2}(\gamma) = J_{4,0}(\gamma) + 3 J_{2,2}(\gamma) \, ,$$

where in the last passage we have used $(7.6)_2$ of [1]. From this result it follows

$$\gamma \left(1 + \frac{\mathcal{I}}{mc^2} \right) J^*_{4,1} = J^*_{4,0} + 3 J^*_{2,2} \, .$$

After this, it is easy to see that the above identity holds.

5 Comparison of the closures coming from the two approaches.

We have seen that the production terms coming from the two approaches are equal up to the first order with respect to equilibrium. We prove now that also the left hand sides of (1) are equal up to the first order with respect to equilibrium. To this end we note that, by using $U^{L\alpha} = U^\alpha + \frac{q^\alpha}{e+p}$, we have

$$\Delta^{\alpha\beta} = g^{\alpha\beta} - \frac{1}{c^2}\left(U^\alpha + \frac{q^\alpha}{e+p}\right)\left(U^\beta + \frac{q^\beta}{e+p}\right) \approx -\,h^{\alpha\beta} - \frac{2}{(e+p)\,c^2}\,U^{(\alpha}\,q^{\beta)}\,,$$

where the second order term has been omitted. We have also

$$\begin{aligned} A_E^{\alpha\beta\gamma} &= A_1^0\,U_L^\alpha U_L^\beta U_L^\gamma - 3\,A_{11}^0\,\Delta^{(\alpha\beta}U_L^{\gamma)} \approx A_1^0\left(U^\alpha U^\beta U^\gamma + \frac{3}{e+p}\,U^{(\alpha}U^\beta q^{\gamma)}\right) + \\ &+ \frac{3A_{11}^0}{e+p}\,h^{(\alpha\beta}q^{\gamma)} + 3A_{11}^0\,h^{(\alpha\beta}U^{\gamma)} + \frac{6A_{11}^0}{(e+p)c^2}\,U^{(\alpha}U^\beta q^{\gamma)} = \\ &= A_1^0\,U^\alpha U^\beta U^\gamma + 3A_{11}^0\,h^{(\alpha\beta}U^{\gamma)} + \frac{3}{(e+p)}\left(A_1^0 + \frac{2}{c^2}\,A_{11}^0\right)U^{(\alpha}U^\beta q^{\gamma)} + \frac{3A_{11}^0}{e+p}\,h^{(\alpha\beta}q^{\gamma)}\,. \end{aligned}$$

(Here too, higher order terms have been omitted). This results shows that also the equilibrium term of $A^{\alpha\beta\gamma}$ in the Landau-Lifshitz description produces a first order term when it is converted in the usual variables. By substituting this expression in (11), we see that there are now two terms in $U^{(\alpha}U^\beta q^{\gamma)}$ and two terms in $h^{(\alpha\beta}q^{\gamma)}$. Consequently, we see that the two linearized closures are equal if and only if the following two relations hold

$$\begin{aligned} &-\,\frac{m\,n}{c^2}\,N_1 + \left(A_1^0\,c^2 + 2\,A_{11}^0\right)\,D_3 = N_3\,(e+p)\,, \\ &-\,m\,n\,N_2 + 15\,A_{11}^0\,D_3 = 3\,N_{31}\,(e+p)\,. \end{aligned} \tag{40}$$

The first one of these can be written as

$$\begin{vmatrix} \frac{m\,n}{c^2} & \frac{p}{m} & \frac{2\,A_{11}^0}{m} \\ A_1^0\,c^2 + 2\,A_{11}^0 & \frac{1}{3}\,B_2\,c^4 & \frac{2}{3}\,B_7\,c^4 \\ \frac{e+p}{c^2} & \frac{1}{3}\,B_4\,c^2 & \frac{2}{3}\,B_2\,c^2 \end{vmatrix} = 0\,,$$

and this is an identity because the second coulumn is equal to the first one multiplied by $\frac{k_B\,T\,c^2}{m^2}$. This is easy to verify for the first element and for the third one thanks to (39). To prove that it holds also for the second element, let us consider the expression of B_2 reported in [3] and have that

$$\gamma\,B_2 = n\,\frac{\gamma\,\int_0^{+\infty} J_{4,2}^*\,\left(1+\frac{\mathcal{I}}{mc^2}\right)\,\left(1+\frac{2\mathcal{I}}{mc^2}\right)\,\phi(\mathcal{I})\,d\,\mathcal{I}}{\int_0^{+\infty} J_{2,1}^*\,\phi(\mathcal{I})\,d\,\mathcal{I}}\,.$$

But, from $(7.6)_2$ of [1] we have

$$\gamma\, J_{4,2}(\gamma) = -2\, J_{2,1}(\gamma) + 5 J_{2,3}(\gamma) = 2\, J_{4,1}(\gamma) + 3 J_{2,3}(\gamma)\,,$$

where in the last passage we have used $(7.6)_2$ of [1]. From this result it follows

$$\gamma \left(1 + \frac{\mathcal{I}}{mc^2}\right) J^*_{4,2} = 2\, J^*_{4,1} + 3 J^*_{2,3}\,.$$

By using this, the above expression of $\gamma\, B_2$ becomes

$$\gamma\, B_2 = n\, \frac{\int_0^{+\infty} 2\, \left(J^*_{4,1} + 3 J^*_{2,3}\right)\, \left(1 + \frac{2\mathcal{I}}{mc^2}\right)\, \phi(\mathcal{I})\, d\mathcal{I}}{\int_0^{+\infty} J^*_{2,1}\, \phi(\mathcal{I})\, d\mathcal{I}} = \frac{3}{m}\, \left(A^0_1\, c^2 +\, 2\, A^0_{11}\right)\,.$$

Thank to this result, also the second element in the first coulumn of the above matrix, multiplied by $\frac{k_B\, T\, c^2}{m^2}$ becomes equal to the corresponding one in the second coulumn.
The second equation of (40) can be written as

$$\begin{vmatrix} m\, n & \frac{p}{m} & \frac{2\, A^0_{11}}{m} \\ 15\, A^0_{11} & B_1 & 2\, B_6 \\ e + p & \frac{1}{3}\, B_4\, c^2 & \frac{2}{3}\, B_2\, c^2 \end{vmatrix} = 0\,,$$

and this is an identity because the second coulumn is equal to the first one multiplied by $\frac{k_B\, T}{m^2}$. This is easy to verify for the first element and for the third one thanks to (39). To prove that it holds also for the second element, let us consider the expression of B_1 reported in [3] and have that

$$\gamma\, B_1 = n\, c^4\, \frac{\gamma\, \int_0^{+\infty} J^*_{6,0}\, \left(1 + \frac{\mathcal{I}}{mc^2}\right)\, \left(1 + \frac{2\mathcal{I}}{mc^2}\right)\, \phi(\mathcal{I})\, d\mathcal{I}}{\int_0^{+\infty} J^*_{2,1}\, \phi(\mathcal{I})\, d\mathcal{I}}\,.$$

But, from $(7.6)_2$ of [1] we have

$$\gamma\, J_{6,0}(\gamma) = 5\, J_{4,1}(\gamma) \quad \rightarrow \quad \gamma \left(1 + \frac{\mathcal{I}}{mc^2}\right) J^*_{6,0} = 5\, J^*_{4,1}\,.$$

By using this, the above expression of $\gamma\, B_1$ becomes

$$\gamma\, B_1 = 5\, n\, c^4\, \frac{\int_0^{+\infty} J^*_{4,1}\, \left(1 + \frac{2\mathcal{I}}{mc^2}\right)\, \phi(\mathcal{I})\, d\mathcal{I}}{\int_0^{+\infty} J^*_{2,1}\, \phi(\mathcal{I})\, d\mathcal{I}} = \frac{15\, c^2}{m}\, A^0_{11}\,.$$

Thank to this result, also the second element in the first coulumn of the above matrix, multiplied by $\frac{k_B\, T}{m^2}$ becomes equal to the corresponding one in the second coulumn.

Obviously, also the closure of the left hand sides of $(1)_{1,2}$ is the same up to first order terms. In fact, we have

$$V_E^\alpha = m\,n\,U_L^\alpha = m\,n\left(U^\alpha + \frac{q^\alpha}{e+p}\right) \quad \rightarrow$$
$$V^\alpha = V_E^\alpha + (V^\alpha - V_E^\alpha) = m\,n\left(U^\alpha + \frac{q^\alpha}{e+p}\right) - \frac{m\,n}{e+p}\,q^\alpha = m\,n\,U^\alpha\,.$$

and

$$T_E^{\alpha\beta} = -\,p\,\Delta^{\alpha\beta} + \frac{e}{c^2}\,U_L^\alpha U_L^\beta = \frac{e+p}{c^2}\left(U^\alpha + \frac{q^\alpha}{e+p}\right)\left(U^\beta + \frac{q^\beta}{e+p}\right) - p\,g^{\alpha\beta} \approx$$
$$\approx \frac{e+p}{c^2}\left(U^\alpha\,U^\beta + \frac{2}{e+p}\,U^{(\alpha}\,q^{\beta)}\right) - p\,g^{\alpha\beta} = \frac{e}{c^2}\,U^\alpha\,U^\beta + p\,h^{\alpha\beta} + \frac{2}{c^2}\,U^{(\alpha}\,q^{\beta)}\,.$$

From this expression it follows

$$T^{\alpha\beta} = T_E^{\alpha\beta} - \pi\,\Delta^{\alpha\beta} + P^{<\alpha\beta>_3} \approx \frac{e}{c^2}\,U^\alpha\,U^\beta + (p+\pi)\,h^{\alpha\beta} + \frac{2}{c^2}\,U^{(\alpha}\,q^{\beta)} + t^{<\alpha\beta>_3}\,.$$

We concude with the closure of the right hand sides of (1).

• **The production terms**.

We conclude our proof by analizing the production term $(24)_3$ and comparing it with its expression which comes out with the usual variables. In the article for the 15 moments obtained in the usual variables there is the variable Δ which we call now Δ_{15} and see that $\Delta_{15} = c\,\Delta$, where Δ is the 15th variable in the present article. Moreover, in the calculations with the usual variables there is a determinant D which we call now D_{15} and see that the third line of D_{15} is equal to the fourth line of the present D divided by $\frac{1}{c}$, while the fourth line of D_{15} is equal to the third line of the present D. So we have $D_{15} = -\,\frac{1}{c}\,D$. By taking into account these particulars, we see that the two expressions of $I^{\beta\gamma}$ are equal if and only if

$$\frac{B_2}{B_4} - \frac{N_3}{D_3} = \frac{m\,n}{(e+p)\,c^2}\,\frac{N_1}{D_3}\,.$$

By substituting here $e+p$ from eq. (39), this condition becomes

$$\begin{vmatrix} \frac{3\,n}{c^2\,\gamma} & \frac{p}{m} & \frac{2\,A_{11}^0}{m} \\ B_2 & \frac{1}{3}\,B_2\,c^4 & \frac{2}{3}\,B_7\,c^4 \\ \frac{1}{c^2}\,B_4 & \frac{1}{3}\,B_4\,c^2 & \frac{2}{3}\,B_2\,c^2 \end{vmatrix} = 0\,,$$

which surely holds because the second coulumn is equal to the first one multiplied by $\frac{1}{3}\,c^4$. So our proof is complete.

6 Conclusions

In this paper we have generalized the relativistic 14 moments model for polyatomic gases by Pennisi and Ruggeri to a 15 moments model. This was necessary because the same authors showed in a subsequent article that the 15 moments model is more suitable for describing such gases. However, we have limited ourselves here to obtaining this result using the variables of the Landau-Lifshitz description (used, for example, by Cercignani and Kremer) and which are less intuitive than the usual one; the corresponding model that makes use of the usual variables is left for further research and study which are now in progress jointly with prof. Ruggeri and other possible cooperators. The transformation law between the two sets of variables was also found here; this made it possible to compare the models that come out with the two sets of variables and to deduce that they are practically the same as long as we limit ourselves to the first order with respect to equilibrium; for this purpose it was necessary to clarify what this "first order with respect to equilibrium" means, since it is different in the two descriptions. As a bonus we discovered that the expression proposed by Pennisi and Ruggeri for the collisional term modifying a little that of Anderson-Witting, in reality doesnt differ from the original unmodified one, but with the presence of the Landau-Lifshitz four velocity U_L^α (Obviously, also this is valid within the approximation we talked about earlier).

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