ADDENDUM TO THE PAPER "REFINED CRITERIA TOWARD BOUNDEDNESS IN AN ATTRACTION-REPULSION CHEMOTAXIS SYSTEM WITH NONLINEAR PRODUCTIONS"

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Abstract. These notes aim to provide a deeper insight on the specifics of the paper "Refined criteria toward boundedness in an attraction-repulsion chemotaxis system with nonlinear productions" by A. Columbu, S. Frassu and G. Viglialoro [*Appl. Anal.* 2024, 103:2, 415–431].

1. Aim of the paper

In this report we focus on [1, Theorem 2.2] where an attraction-repulsion chemotaxis model is formulated as follow:

$$(1)\qquad \begin{cases} u_t = \Delta u - \chi \nabla \cdot (u \nabla v) + \xi \nabla \cdot (u \nabla w) & \text{in } \Omega \times (0, T_{max}),\\ v_t = \Delta v - \beta v + f(u) & \text{in } \Omega \times (0, T_{max}),\\ w_t = \Delta w - \delta w + g(u) & \text{in } \Omega \times (0, T_{max}),\\ u_\nu = v_\nu = w_\nu = 0 & \text{on } \partial\Omega \times (0, T_{max}),\\ u(x,0) = u_0(x),\ v(x,0) = v_0(x),\ w(x,0) = w_0(x) & x \in \bar{\Omega}. \end{cases}$$

Herein, Ω of $\mathbb{R}^n$, with $n \geq 2$, is a bounded and smooth domain, $\chi, \xi, \beta, \delta > 0$, and $f = f(s)$ and $g = g(s)$ sufficiently regular functions in their argument $s \geq 0$, essentially behaving as s^k and s^l for some $k, l > 0$. Moreover, further regular initial data $u_0(x), v_0(x), w_0(x) \geq 0$ are fixed, u_ν (and similarly v_ν and w_ν) indicates the outward normal derivative of u on $\partial\Omega$, whereas T_{max} identifies the maximum time up to which solutions to the system can be extended.

Once these hypotheses are fixed

$$(2)\qquad \begin{cases} f, g \in C^1([0,\infty)) \quad \text{with} \quad 0 \leq f(s) \leq \alpha s^k \text{ and } \gamma_0(1+s)^l \leq g(s) \leq \gamma_1(1+s)^l, \quad \text{for some } \alpha, k, l > 0, \gamma_1 \geq \gamma_0 > 0,\\ (u_0, v_0, w_0) \in (W^{1,\infty}(\Omega))^3, \text{ with } u_0, v_0, w_0 \geq 0 \text{ on } \bar{\Omega}, \end{cases}$$

[1, Theorem 2.2] establishes that problem (1) admits a unique global and uniformly bounded classical solution (i.e., $T_{max} = \infty$ and there exists $C > 0$ such that $\|u(\cdot, t)\|_{L^\infty(\Omega)} \leq C$ for all $t \in (0, \infty)$) whenever

(i) $l, k \in \left(0, \frac{1}{n}\right]$;
(ii) $l \in \left(\frac{1}{n}, \frac{1}{n} + \frac{2}{n^2+4}\right)$ and $k \in \left(0, \frac{1}{n}\right]$, or $k \in \left(\frac{1}{n}, \frac{1}{n} + \frac{2}{n^2+4}\right)$ and $l \in \left(0, \frac{1}{n}\right]$;
(iii) $l, k \in \left(\frac{1}{n}, \frac{1}{n} + \frac{2}{n^2+4}\right)$.

From the one hand, we mention that the above conditions have been improved in the recent paper [2]; in the specific, [2, Theorem 2.2] ensures boundedness under the more relaxed assumption $k, l \in \left(0, \frac{2}{n}\right)$.

As to our contribution, we aim at providing a further scenario toward boundedness involving also coefficients connected to g in (2); essentially we will show that

solutions to model (1) are uniformly bounded in time whenever $k < l$ and under a largeness assumption on γ_0.

Remark 1 (On the origins and the meaning of model (1)). *The interested reader can find motivations connected to biological phenomena described by system (1) exactly in [1], and references therein mentioned. Also known results in close contexts are collected.*

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2. Presentation of the main theorem

As an essential tool in order to mathematically formulate our main theorem, we have first to recall the following consequence of Maximal Sobolev regularity results ([4] or [3, Theorem 2.3]):

Proposition 1. *For $n \in \mathbb{N}$, let $\Omega \subset \mathbb{R}^n$ be a bounded domain with smooth boundary, $\rho > 0$ and $q > \max\{1, \frac{1}{\rho}\}$. Then there is $C_\rho = C_\rho(\Omega, n, q) > 0$ such that the following holds: Whenever $T \in (0, \infty]$, $I = [0, T)$, $h \in L^q(I; L^q(\Omega))$ and $\psi_0 \in W^{2,q}_{\frac{\partial}{\partial\nu}}(\Omega) = \{\psi_0 \in W^{2,q}(\Omega) : \partial_\nu \psi_0 = 0 \text{ on } \partial\Omega\}$, every solution $\psi \in W^{1,q}_{loc}(I; L^q(\Omega)) \cap L^q_{loc}(I; W^{2,q}(\Omega))$ of*

$$\psi_t = \Delta\psi - \rho\psi + h \quad \text{in} \quad \Omega \times (0, T); \quad \partial_\nu \psi = 0 \quad \text{on} \quad \partial\Omega \times (0, T); \quad \psi(\cdot, 0) = \psi_0 \quad \text{on} \quad \Omega$$

satisfies

$$\int_0^t e^s \int_\Omega \left(|\psi(\cdot, s)|^q + |\psi_t(\cdot, s) + \frac{\psi(\cdot, s)}{q}|^q + |\Delta\psi(\cdot, s)|^q \right) ds \leq 2^{q-1} C_\rho^q \left[\|\psi_0\|^q_{q, 1-\frac{1}{q}} + \int_0^t e^s \int_\Omega |h(\cdot, s)|^q ds \right] \quad \text{for all } t \in (0, T).$$

Proof. The proof is based on the classical result in [8]; for an appropriate adaptation to our case see details, for instance, in [5]. □

Remark 2 (On the constant C_ρ and the norm $\|\psi_0\|_{q,1-\frac{1}{q}}$ in Proposition 1)**.** *The key role of Proposition 1 is the existence of the constant C_ρ, which remains defined once n, Ω and q are set. In particular (see [8, Theorem 2.5]), C_ρ does not depend on the initial configuration ψ_0 and the source h.*

As to $\|\psi_0\|_{q,1-\frac{1}{q}}$, it represents the norm of ψ_0 in the interpolation space $(L^q(\Omega), W^{2,q}_{\frac{\partial}{\partial\nu}}(\Omega))_{1-\frac{1}{q},q}$. (See, for instance, [7, §1].)

Exactly in view of what said, we can now give the claim of our

Theorem 2.1. *For $n \in \mathbb{N}$, let Ω be a bounded domain of $\mathbb{R}^n$ with smooth boundary, $0 < k < l$, $\delta, \alpha, \beta > 0$ and*

$$\bar{p} = \max\left\{ \frac{n}{2}, k\left(\frac{1}{\beta} - 1\right), l\left(\frac{1}{\delta} - 1\right) \right\} + 1. \tag{3}$$

Additionally, let us set

$$\mathcal{A} = 2^{-\frac{l(\bar{p}+l-1)+\bar{p}}{\bar{p}+l}} \left(\frac{\bar{p} + l}{\bar{p} + 2l + \delta(\bar{p} + l)} \right). \tag{4}$$

Then there exists $\mathcal{C} = \mathcal{C}(n, \Omega, l, k, \delta, \beta) > 0$ such that if $\mathcal{C} < \mathcal{A}$, it is possible to find $\gamma_1, \gamma_0 > 0$ fulfilling

$$\gamma_1 \geq \gamma_0 > \mathcal{A}^{-1}\mathcal{C}\gamma_1 \tag{5}$$

and with this property: Whenever f, g, u_0, v_0, w_0 are taken as in (2), problem (1) admits a global and uniformly bounded solution $(u, v, w) \in (C^0(\bar{\Omega} \times [0, \infty)) \cap C^{2,1}(\bar{\Omega} \times (0, \infty)))^3$.

3. Existence of local-in-time solutions and a pricniple for boundedness

The arguments concerning the forthcoming local existence issue and the boundedness criterion are standard; details are achievable in [9] and [10, Appendix A.].

3.1. Local existence statement.

Once $\chi, \xi, \beta, \delta > 0$ and f, g, u_0, v_0 are fixed as in (2), from here henceforth, with (u, v, w) we will refer to the classical and nonnegative solution to problem (1); u, v, w are defined for all $(x, t) \in \bar{\Omega} \times [0, T_{max})$, for some finite T_{max}.

3.2. Boundedness criterion.

As explained in the next lines, if we establish that $u \in L^\infty((0, T_{max}); L^p(\Omega))$, for some $p > \frac{n}{2}$, we can exploit the boundedness criterion below and directly obtain that, indeed, $u \in L^\infty((0, \infty); L^\infty(\Omega))$; as an immediate consequence of that, well-known parabolic regularity results applied to the equations of v and w entail that also v, w belong to $L^\infty((0, \infty); L^\infty(\Omega))$.

Definitely, globality and boundedness of (u, v, w), in the sense that

$$u, v, w \in C^0(\bar{\Omega} \times [0, \infty)) \cap C^{2,1}(\bar{\Omega} \times (0, \infty)) \cap L^\infty((0, \infty); L^\infty(\Omega))$$

are achieved whenever this *boundedness criterion* applies:

$$\text{If } \exists\, L > 0, p > \tfrac{n}{2} \;\Big|\; \int_\Omega u^p \leq L \text{ on } (0, T_{max}) \Rightarrow (u, v, w) \in (L^\infty((0, \infty); L^\infty(\Omega)))^3. \tag{6}$$

Subsequently, Theorem 2.1 is established once (6) is derived.

4. A priori bounds; proof of the main result

From now on we will tacitly assume that all the appearing constants below c_i, $i = 1, 2, \ldots,$ are positive.

4.1. **Some preparatory tools.** Let us start with this necessary result:

Lemma 4.1. *Let $A, B \geq 0$ and $p \geq 1$. Then we have*

$$(A+B)^p \leq 2^{p-1}(A^p + B^p). \tag{7}$$

Proof. The proof is available [6, Theorem 1]. □

4.2. **Achieving the boundedness criterion.** We have this sequence of results, valid for any constant $\Xi > 0$, which will be properly chosen later on, in the proof of our theorem.

The following lemma is valid for a general class of proper functions. Despite that, we contextualize it to the local solution (u, v, w) to problem (1).

Lemma 4.2. *For any $p > 1$ and all $t \in (0, T_{max})$, we have*

$$(p-1)\xi \int_\Omega u^p |w_t| \leq (p-1)\xi\Xi \int_\Omega |w_t + \frac{l}{p+l} w|^{\frac{p+l}{l}} + \left[\xi p \frac{p-1}{p+l}\left(\Xi \frac{p+l}{l}\right)^{-\frac{l}{p}}\left(1 + \frac{l}{p+l}\right)\right] \int_\Omega u^{p+l} + l\xi\Xi \frac{p-1}{p+l}\int_\Omega w^{\frac{p+l}{l}},$$

and

$$(p-1)\xi\delta \int_\Omega u^p w \leq (p-1)\xi\delta\Xi \int_\Omega w^{\frac{p+l}{l}} + \xi p \delta \frac{p-1}{p+l}\left(\Xi \frac{p+l}{l}\right)^{-\frac{l}{p}} \int_\Omega u^{p+l}.$$

Proof. Let, for commodity but also for reasons which will be clearer later, $q = \frac{p+l}{l}$. From the evident relation $|w_t| \leq |w_t + \frac{1}{q}w| + |\frac{1}{q}w|$, we obtain

$$(p-1)\xi \int_\Omega u^p |w_t| \leq (p-1)\xi \int_\Omega u^p |w_t + \frac{w}{q}| + \xi \frac{p-1}{q}\int_\Omega u^p w \quad \text{on} \quad (0, T_{max}),$$

so that thanks to the Young inequality for all $t \in (0, T_{max})$ it is seen

$$(p-1)\xi \int_\Omega u^p |w_t| \leq (p-1)\xi\Xi \int_\Omega |w_t + \frac{w}{q}|^q + \frac{(p-1)p\xi}{p+l}(\Xi q)^{-\frac{l}{p}} \int_\Omega u^{p+l} + \Xi \frac{p-1}{q}\xi \int_\Omega w^q + \xi p \frac{p-1}{q(p+l)}(\Xi q)^{-\frac{l}{p}} \int_\Omega u^{p+l},$$

and the first claim is established.

As to the other relation, it can be derived in the same flavor. □

Lemma 4.3. *For any $p > \max\left\{1, l\left(\frac{1}{\delta} - 1\right)\right\}$ and $t \in (0, T_{max})$ it holds that*

$$(p-1)\xi\Xi \int_0^t e^s \left(\int_\Omega |w_t + \frac{l}{p+l}w|^{\frac{p+l}{l}}\right) ds \leq (p-1)\xi\Xi 2^{\frac{p}{l}} \mathcal{C}_\delta^{\frac{p+l}{l}} \times \left[\|w_0\|^{\frac{p+l}{l}}_{\frac{p+l}{l}, \frac{p}{p+l}} + \gamma_1^{\frac{p+l}{l}} 2^{p+l-1} \int_0^t e^s \left(\int_\Omega u^{p+l} ds\right) + \gamma_1^{\frac{p+l}{l}} 2^{p+l-1} |\Omega| \int_0^t e^s ds\right], \tag{8}$$

and

$$(p-1)\xi\Xi \left(\frac{l}{p+l} + \delta\right) \int_0^t e^s \left(\int_\Omega w^{\frac{p+l}{l}}\right) \leq (p-1)\xi\Xi 2^{\frac{p}{l}} \left(\frac{l}{p+l} + \delta\right) \mathcal{C}_\delta^{\frac{p+l}{l}} \times \left[\|w_0\|^{\frac{p+l}{l}}_{\frac{p+l}{l}, \frac{p}{p+l}} + \gamma_1^{\frac{p+l}{l}} 2^{p+l-1} \int_0^t e^s \left(\int_\Omega u^{p+l} ds\right) + \gamma_1^{\frac{p+l}{l}} 2^{p+l-1} |\Omega| \int_0^t e^s ds\right]. \tag{9}$$

Proof. We can derive (8) (and similarly (9)) by invoking Proposition 1 with $\psi = w$, $h = g$ and $\rho = \delta$; indeed, for $q = \frac{p+l}{l}$ as before, it is $q > \max\{1, \frac{1}{\delta}\}$ so that

$$(p-1)\xi\Xi \int_0^t e^s \left(\int_\Omega |w_t + \frac{w}{q}|^q\right) ds \leq (p-1)\xi\Xi \mathcal{C}_\delta^q 2^{q-1} \left[\|w_0\|^q_{q, 1-\frac{1}{q}} + \int_0^t e^s \left(\int_\Omega g(u)^q\right) ds\right],$$

and the conclusion is attained by virtue of the upper bound (2) for g and (7), in the form $(u+1)^{p+l} \leq 2^{p+l-1}(u^{p+l} + 1)$ (naturally $p + l > 1$). □

The next two results, indeed, provide properties of local solutions (u, v, w) to model (1) and are based on applications of Proposition 1.

Lemma 4.4. *For any $p > \max\left\{1, k\left(\frac{1}{\beta} - 1\right)\right\}$ and $t \in (0, T_{max})$ there is c_1 such that*

$$c_1 \int_0^t e^s \left(\int_\Omega |\Delta v|^{\frac{p+k}{k}}\right) ds \leq c_1 \mathcal{C}_\beta^{\frac{p+k}{k}} \left[\|v_0\|^{\frac{p+k}{k}}_{\frac{p+k}{k}, \frac{p}{p+k}} + \alpha^{\frac{p+k}{k}} \int_0^t e^s \left(\int_\Omega u^{p+k}\right) ds\right].$$

Proof. The proof follows from analogous arguments used in Lemma 4.3; in this case, in particular, Proposition 1 is exploited with $q=\frac{p+k}{k}>\max\{1,\frac{1}{\beta}\}$, $\psi=v, h=f$ and $\rho=\beta$. $\square$

With the aim of ensuring $u\in L^\infty((0,T_{max});L^p(\Omega))$ for some $p>\frac{n}{2}$, let us study the evolution in time of $t\mapsto\int_\Omega u^p$; this will be done by means of testing procedures.

Lemma 4.5. *For any $p>1$ and all $t\in(0,T_{max})$ the following relation is satisfied:*

$$
\begin{split}
\frac{d}{dt}\int_\Omega u^p \leq & \int_\Omega u^{p+k}+c_1\int_\Omega|\Delta v|^{\frac{p+k}{k}}+(p-1)\xi\Xi\int_\Omega|w_t+\frac{l}{p+l}w|^{\frac{p+l}{l}}\\
&+(p-1)\xi\Xi\left(\delta+\frac{l}{p+l}\right)\int_\Omega w^{\frac{p+l}{l}}+(p-1)\xi\left[\frac{p}{p+l}\left(\frac{p+l}{l}\right)^{-\frac{l}{p}}\Xi^{-\frac{l}{p}}\left(1+\delta+\frac{l}{p+l}\right)-\gamma_0\right]\int_\Omega u^{p+l}.
\end{split}
\tag{10}
$$

Proof. By testing the first equation of problem (1) with pu^{p-1}, using its boundary conditions and taking into account the second and the third equation, we have thanks to the Young inequality

$$
\begin{split}
\frac{d}{dt}\int_\Omega u^p&=p\int_\Omega u^{p-1}u_t=-p(p-1)\int_\Omega u^{p-2}|\nabla u|^2-(p-1)\chi\int_\Omega u^p\Delta v+(p-1)\xi\int_\Omega u^p\Delta w\\
&\leq\int_\Omega u^{p+k}+c_1\int_\Omega|\Delta v|^{\frac{p+k}{k}}+(p-1)\xi\int_\Omega u^p(w_t+\delta w-g(u))\quad\text{on }(0,T_{max}).
\end{split}
$$

Now, we recall the properties of g given in (2) so to deduce, by using Young's inequality, again the relation $|w_t|\leq|w_t+\frac{l}{p+l}w|+|\frac{l}{p+l}w|$, and Lemma 4.2

$$
\begin{split}
\frac{d}{dt}\int_\Omega u^p&\leq\int_\Omega u^{p+k}+c_1\int_\Omega|\Delta v|^{\frac{p+k}{k}}+(p-1)\xi\int_\Omega u^p|w_t|+\delta(p-1)\xi\int_\Omega u^p w-\xi\gamma_0(p-1)\int_\Omega u^{p+l}\\
&\leq\int_\Omega u^{p+k}+c_1\int_\Omega|\Delta v|^{\frac{p+k}{k}}+(p-1)\xi\Xi\int_\Omega|w_t+\frac{l}{p+l}w|^{\frac{p+l}{l}}\\
&\quad+\frac{l(p-1)\xi\Xi}{p+l}\int_\Omega w^{\frac{p+l}{l}}+\frac{p(p-1)\xi}{p+l}\left(\Xi\frac{p+l}{l}\right)^{-\frac{l}{p}}\left(1+\frac{l}{p+l}\right)\int_\Omega u^{p+l}\\
&\quad+(p-1)\xi\delta\Xi\int_\Omega w^{\frac{p+l}{l}}+\frac{p(p-1)\xi\delta}{p+l}\left(\Xi\frac{p+l}{l}\right)^{-\frac{l}{p}}\int_\Omega u^{p+l}-\xi(p-1)\gamma_0\int_\Omega u^{p+l}\quad\text{on }(0,T_{max}).
\end{split}
$$

The claim is achieved by collecting terms. $\square$

Lemma 4.6. *Let $k<l$. Then for every $p>\max\left\{\frac{n}{2},k\left(\frac{1}{\beta}-1\right),l\left(\frac{1}{\delta}-1\right)\right\}$ we have that for all $t<T_{max}$*

$$
\begin{split}
e^t\int_\Omega u^p\leq& c_2+c_3\int_0^t e^s ds+\\
&(p-1)\xi\left[\left(1+\delta+\frac{l}{p+l}\right)\left(\Xi C_\delta^{\frac{p+l}{l}}\gamma_1^{\frac{p+l}{l}}2^{\frac{p}{l}+p+l-1}+\frac{p}{p+l}\left(\frac{p+l}{l}\right)^{-\frac{l}{p}}\Xi^{-\frac{l}{p}}\right)+\varepsilon-\gamma_0\right]\int_0^t e^s\left(\int_\Omega u^{p+l}\right)ds.
\end{split}
$$

Proof. Let us start with these estimates, fruit of the application of Young's inequality: for all $\varepsilon>0$, $\hat{c}>0$, $p>1$ and $0<k<l$ it holds that

$$
\hat{c}\int_\Omega u^p\leq\frac{\varepsilon}{2}\int_\Omega u^{p+l}+c_4\quad\text{for all }t\in(0,T_{max}),
\tag{11}
$$

and

$$
\hat{c}\int_\Omega u^{p+k}\leq\frac{\varepsilon}{2}\int_\Omega u^{p+l}+c_5\quad\text{for all }t\in(0,T_{max}).
\tag{12}
$$

By adding to both sides of relation (10) the term $\int_\Omega u^p$, estimate (11) leads to this inequality, valid on $(0,T_{max})$.

$$
\begin{split}
\frac{d}{dt}\int_\Omega u^p+\int_\Omega u^p\leq&\int_\Omega u^{p+k}+c_1\int_\Omega|\Delta v|^{\frac{p+k}{k}}+(p-1)\xi\Xi\int_\Omega|w_t+\frac{l}{p+l}w|^{\frac{p+l}{l}}\\
&+(p-1)\xi\Xi\left(\delta+\frac{l}{p+l}\right)\int_\Omega w^{\frac{p+l}{l}}+(p-1)\xi\left[\frac{p}{p+l}\left(\frac{p+l}{l}\right)^{-\frac{l}{p}}\Xi^{-\frac{l}{p}}\left(1+\delta+\frac{l}{p+l}\right)+\frac{\varepsilon}{2}-\gamma_0\right]\int_\Omega u^{p+l}+c_6.
\end{split}
\tag{13}
$$

Successively, we multiply (13) by e^t and integrate on $(0,t)$. From the identity $\frac{d}{dt}(e^t\int_\Omega u^p)=e^t\frac{d}{dt}\int_\Omega u^p+e^t\int_\Omega u^p$, we get

$$
\begin{split}
e^t\int_\Omega u^p\leq&\int_\Omega u_0^p+\int_0^t e^s\left\{\int_\Omega u^{p+k}+c_1\int_\Omega|\Delta v|^{\frac{p+k}{k}}+(p-1)\xi\Xi\int_\Omega|w_t+\frac{l}{p+l}w|^{\frac{p+l}{l}}\right.\\
&\left.+(p-1)\xi\Xi\left(\delta+\frac{l}{p+l}\right)\int_\Omega w^{\frac{p+l}{l}}+(p-1)\xi\left[\frac{p}{p+l}\left(\frac{p+l}{l}\right)^{-\frac{l}{p}}\Xi^{-\frac{l}{p}}\left(1+\delta+\frac{l}{p+l}\right)+\frac{\varepsilon}{2}-\gamma_0\right]\int_\Omega u^{p+l}+c_6\right\}ds.
\end{split}
$$

The term involving $\int_0^t e^s(\int_\Omega |\Delta v|^{\frac{p+k}{k}})ds$ can be essentially controlled by $\int_0^t e^s(\int_\Omega u^{p+k})ds$, thanks to Lemma 4.4; additionally, $\int_\Omega u^{p+k}$ is treated through (12). These two operations provide

$$e^t\int_\Omega u^p \leq \int_\Omega u_0^p + \int_0^t e^s\left\{(p-1)\xi\Xi\int_\Omega |w_t + \frac{l}{p+l}w|^{\frac{p+l}{l}} + (p-1)\xi\Xi\left(\delta+\frac{l}{p+l}\right)\int_\Omega w^{\frac{p+l}{l}}\right.$$
$$\left. +(p-1)\xi\left[\frac{p}{p+l}\left(\frac{p+l}{l}\right)^{-\frac{l}{p}}\Xi^{-\frac{l}{p}}\left(1+\delta+\frac{l}{p+l}\right)+\varepsilon-\gamma_0\right]\int_\Omega u^{p+l} + c_7\right\}ds.$$

By invoking (8) and (9) we conclude by virtue of a reorganization of the involved terms. □

4.3. **Proof of Theorem 2.1.**

With the above preparations we are now in a position to establish what anticipated.

Proof. For $0 < k < l$, $\delta, \alpha, \beta > 0$, let $\bar{p}$ be as in (3) and, additionally, for $C_\delta\left(\Omega, n, \frac{\bar{p}+l}{l}\right)$ being the constant provided by Proposition 1, when it is applied to the equation for w in model (1), let also set

$$\mathcal{C} = C_\delta\left(\Omega, n, \frac{\bar{p}+l}{l}\right).$$

Since by assumptions $\mathcal{C} < \mathcal{A}$, where $\mathcal{A}$ is defined in (4), we can find $\gamma_1 \geq \gamma_0$ complying with (5); in these positions, let

$$\Xi = \frac{l}{\bar{p}+l}\mathcal{C}_\delta^{-\frac{\bar{p}}{l}}\gamma_1^{-\frac{\bar{p}}{l}}2^{-\frac{\bar{p}(\bar{p}+(\bar{p}+l-1)l)}{l(\bar{p}+l)}},$$

and let f, g, u_0, v_0 and w_0 obey (2). Some computations show that for proper small $\varepsilon > 0$

$$\gamma_0 > \mathcal{A}^{-1}\mathcal{C}\gamma_1 \Rightarrow \left(1+\delta+\frac{l}{\bar{p}+l}\right)\left(\Xi C_\delta^{\frac{\bar{p}+l}{l}}\gamma_1^{\frac{\bar{p}+l}{l}}2^{\frac{\bar{p}}{l}+\bar{p}+l-1} + \frac{\bar{p}}{\bar{p}+l}\left(\frac{\bar{p}+l}{l}\right)^{-\frac{l}{\bar{p}}}\Xi^{-\frac{l}{\bar{p}}}\right)+\varepsilon-\gamma_0 \leq 0,$$

and henceforth hypothesis (5) allows to exploit Lemma 4.6 and obtain

$$e^t\int_\Omega u^{\bar{p}} \leq c_2 + c_3\int_0^t e^s ds \quad \text{for all } t \in (0, T_{max}),$$

or also $\int_\Omega u^{\bar{p}} \leq L$ on $(0, T_{max})$. The claim follows from the extensibility criterion (6). □

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