

**Commitment under uncertainty: Production-inventory policies
associated with environmental considerations**

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Abstract

We consider a manufacturer that produces in response to a stochastic demand and emits pollution during the production process. Industrial pollutants released into the air are characterized by spatial variability and special heterogeneity, and the precision of instruments for measuring stochastic pollution stocks varies widely across pollutants. Consequently, commitment (open-loop) strategies to control production and associated emissions are considered more practical than contingent (feedback) approaches when manufacturers are concerned about environmental consequences. In this paper, we derive an optimal control policy over a time horizon and show that such a policy induces asymptotic convergence of expected inventory and pollution stock trajectories to a unique steady state in terms of mean pollutant assimilation and assimilation volatility. Moreover, we find that the variance of the pollution stock also converges asymptotically to a unique steady state, which is critical for prevention of irreversible environment consequences. Notably, we show that the variance of the pollution stock increases with the volatility of pollutant assimilation while the level of the long-run expected pollution stock is not affected. On the other hand, environmental uncertainty does affect inventories, leading in the long run to lower expected steady-state inventory stocks.

1. Introduction

Environmental pollution worldwide is at an all-time high, and global leaders have convened in Glasgow and New York to discuss climate commitments and related actions. Many countries are establishing Greening Government Commitments that set out actions they plan to take to reduce their impacts on the environment (GOV.UK, 2022). At the corporate level, around 80% of chief executive officers believe in the importance of demonstrating a commitment to society through environmental sustainability (DHL, 2022).

Industrial processes account for almost 30% of greenhouse gas emissions worldwide while direct manufacturing and construction are responsible for about six billion tons of gas emissions annually (Ritchie and Roser, 2020). The problem of industrial emissions has generally been approached by governments in

two ways. One is market-based pollution permits, which allow manufacturers who pollute less to sell surplus permits to other producers. The other is taxation, which can be applied to emissions and to ambient environmental quality. Examples of such charges can be found in Title IV of the U.S. Clean Air Act Amendments and the National Ambient Air Quality Standards, which regulate sulfur dioxide (SO₂) emissions and ambient SO₂ concentrations, respectively. Industrial waste is also subject to numerous fees, including emission taxes, landfill taxes, threat to water quality and bay protection fees, and toxic cleanup fees.

In accord with these governmental approaches, the operations management literature generally has addressed production-inventories and associated pollution dynamics. Specifically, Wirl (1991) modifies the classical optimal-control model considered by Arrow and Karlin (1958) by introducing a linear charge for pollution per produced unit over a finite time horizon. He shows that, when abatement costs are low, it is likely that the pollution charge will smooth production, causing the firm to hold larger inventories. Dobos (2005) also follows the Arrow and Karlin (1958) framework to extend Wirl's (1991) formulation to incorporate tradable emission permits. In particular, Dobos (2005) assumes convex production costs and linear procurement emission costs and finds that the optimal production-inventory strategy under emission trading is smoother and that inventories are larger when tradable permits are applied. Li (2013) employs a similar approach and shows that inventories increase in size under emission permit banking.

The common feature of this research stream is employment of a deterministic model of production-inventory policies over limited time horizons. In real life, however, production and environmental dynamics are stochastic and associated emissions can have long-term consequences. To overcome these modeling drawbacks, Kogan *et al* (2017) introduce uncertainty regarding the production and environmental dynamics over an infinite time horizon. They determine the optimal feedback production policy and show that, unlike the short-term effect of production smoothing on inventories (described in deterministic studies), environmental uncertainty leads, in the long run, to both lower expected levels of steady-state pollution and smaller inventory stocks while not affecting the expected steady-state production rate. The high volatility of pollution and inventory stocks, however, makes implementation of production feedback policies extremely difficult. In particular, inventory inaccuracy is common. Indeed, the more transactions the firm has, the greater the probability of discrepancies in inventory data, including misplacement, shrinkage, damaged products, wrong scanning, and flaws in inventory control procedures. These events lead to differences between the inventory reflected in the information technology (IT) system records and inventory available for sale. Raman *et al.* (2001), for example, find from a case study that information in the inventory management system was inaccurate (the information system inventory and physical inventory were different) for more than 65% of the stock keeping units (SKUs).

More severe problems arise when measuring pollution levels. Industrial pollutant releases into the air include greenhouse gases such as carbon dioxide (CO₂) and acidifying pollutants such as sulfur oxides (SO_x) that affect the quality of ambient air. However, local air quality is also affected by how the pollutants disperse in the atmosphere. Consequently, variances in instrument precision and errors associated with spatial variability and special heterogeneity in air pollutant concentrations lead to variability in measures of ambient air pollution (Goldman et al, 2010; Butland et al, 2019). Thus, because errors in evaluating levels of inventory and pollution stocks at any point in time can be sizeable, the commitment open-loop strategy (rather than the contingent feedback policy) is more practical and is widely applied for industries.

We thus follow the stream of work that evaluates commitment strategies to derive optimal production control under uncertainty (e.g., Van der Ploeg and de Zeeuw, 1992; Mäler *et al*, 2003). Specifically, we adopt the stochastic model (Kogan *et al*, 2017), which is characterized by linear pollution-stock dynamics with Gaussian noise (e.g., Zemel, 2012; Zhang *et al*, 1997) and stochastic consumer demand (e.g., Kogan 2009, Haurie 1995, Ghosh *et al* 1993, and Tapiero 1988). The goal is to understand optimal commitment dynamics and how demand and pollution uncertainty affect the long-run inventory and pollution stocks.

2. Statement of the problem

Consider a manufacturer that produces a single product-type at a rate of $u(t) \geq 0$ and that its production process causes polluting emissions at a rate of $eu(t)$. Demand for the product at each time point t is affected by a random noise described by the standard Wiener process $W(t)$. The classical inventory stock $X(t)$ dynamics are then described by the stochastic differential equation presenting the difference between the production rate and the demand rate:

$$dX(t) = (u(t) - D)dt - \sigma dW(t) \quad (1)$$

where $X(0)$ is known and D and σ are the mean and volatility of demand, respectively.

We assume that the pollution stock $P(t)$ is proportional to the emission rate $eu(t)$ associated with production $u(t)$ and inversely proportional to the natural environmental capacity of pollutant assimilation, $P(t)(\gamma dt + \varepsilon dS(t))$, where $\gamma P(t)$ and $\varepsilon P(t)$ represent mean assimilation and volatility of pollutant assimilation, respectively, and $S(t)$ is a standard Wiener process not correlated with $W(t)$. Specifically, the pollution stock is described by the stochastic process (see, for example, Kogan *et al*, 2017 and Zemel, 2012):

$$dP(t) = (eu(t) - \gamma P(t))dt - \varepsilon P(t)dS(t), \quad (2)$$

where $P(0)$ is known. The manufacturer bears an emission/production-associated cost $C(u(t))$ and ambient pollution costs/taxes $A(P(t))$ for preventing immediate and cumulative damage to human health and/or the environment (Palmer and Nursey-Bray, 2007). In addition, an inventory cost $I(X(t))$ is incurred. It represents the inventory holding cost when $X(t) > 0$ and a backlog cost when there is a shortage. The objective is to minimize the total discounted expected cost:

$$\min_{u(\cdot) \geq 0} J = \min_{u(\cdot) \geq 0} E \left\{ \int_0^{\infty} [I(X(t)) + C(u(t)) + A(P(t))] e^{-\delta t} dt \right\}. \quad (3)$$

Following the convex cost of the Arrow-Karlin approach adopted in previous studies, we assume second-order polynomial costs, $I(X(t)) = hX(t)^2$, $C(u(t)) = c(u(t) - u_0)^2$, $A(P(t)) = aP(t)^2$, where u_0 is the nominal capacity of the production facility. For convenience, all notations employed in the formulation are collected in Table 1.

Table 1. Main Notations

<i>Parameters</i>	
D	mean demand rate
σ	demand volatility
e	emission per unit produced
γ	mean pollutant unit assimilation rate
ε	volatility of pollutant unit assimilation rate
h	inventory holding cost coefficient
u_0	nominal capacity of the production facility
a	pollution cost coefficient
δ	discount rate
$W(t), S(t)$	uncorrelated Wiener processes
<i>Independent variables</i>	
t	time
<i>Control variables</i>	
$u(t)$	production rate at time t
<i>State variables</i>	
$X(t)$	inventory stock level at time t
$P(t)$	pollution stock level at time t
$x(t)$	expected stock level at time t
$p(t)$	Expected pollution stock level at time t
$y(t)$	$y(t) = E[P^2(t)]$.

3. Equivalent formulation and steady-state conditions

We start off by introducing a new state variable, $x(t) = X^0 + \int_0^t (u(s) - D) ds$, i.e., $x(t) = E[X(t)]$, and thus, from equation (1), $X^2(t) = x^2(t) + 2x(t) \int_0^t \sigma dW + \left(\int_0^t \sigma dW \right)^2$ and $E[X^2(t)] = x(t)^2 + \sigma^2 t$.

Next, using the Ito formula with respect to equation (2), we obtain

$$dP^2(t) = 2P(t)eu(t)dt - 2\gamma P^2(t)dt - 2\varepsilon P^2(t)dS + \frac{2\varepsilon^2 P^2(t)}{2}dt. \quad (4)$$

Consequently,

$$dE[P^2(t)] = 2E[P(t)]eu(t)dt - (2\gamma - \varepsilon^2)E[P^2(t)]dt. \quad (5)$$

Accounting for the fact that $\int_0^\infty \sigma^2 t e^{-\delta t} dt$ is a constant that does not affect optimization and introducing two more variables, $y(t) = E[P^2(t)]$ and $p(t) = E[P(t)]$, we conclude:

Lemma 1. Given the production rate, $\{u(t), t \geq 0\}$ is an open-loop control. Then, the problem,

$$\min_{u(\cdot) \geq 0} J = \min_{u(\cdot) \geq 0} \left\{ \int_0^\infty [hx^2(t) + c(u(t) - u_0)^2 + ay]e^{-\delta t} dt \right\}. \quad (6)$$

s.t.

$$dy(t) = 2p(t)eu(t)dt - (2\gamma - \varepsilon^2)y(t)dt, y(0) = P^2(0); \quad (7)$$

$$dp(t) = (eu(t) - \gamma p(t))dt, p(0) = P(0); \quad (8)$$

$$dx(t) = (u(t) - D)dt, x(0) = X(0), \quad (9)$$

is equivalent to the problem in (1)-(3). ■

We henceforth omit the independent variable t whenever its presence is obvious. To study the problem in (6)-(9), we construct the Hamiltonian:

$$H = -hx^2 - c(u - u_0)^2 - ay + \psi_x(u - D) + \psi_y(2peu - (2\gamma - \varepsilon^2)y) + \psi_p(eu - \gamma p) \quad (10)$$

where the co-states ψ_x , ψ_y , and ψ_p satisfy the following equations:

$$\dot{\psi}_x = 2hx + \delta\psi_x, \lim_{t \rightarrow \infty} e^{-\delta t} \psi_x = 0; \quad (11)$$

$$\dot{\psi}_y = a + (2\gamma - 2\frac{\varepsilon^2}{2} + \delta)\psi_y, \lim_{t \rightarrow \infty} e^{-\delta t} \psi_y = 0; \quad (12)$$

$$\dot{\psi}_p = -2eu\psi_y + (\gamma + \delta)\psi_p, \lim_{t \rightarrow \infty} e^{-\delta t} \psi_p = 0. \quad (13)$$

The optimal production rate, u , arises from the first-order optimality condition applied to (10), i.e., if $2pe\psi_y + \psi_x + 2cu_0 + \psi_p e > 0$, then

$$2pe\psi_y + \psi_x - 2cu + 2cu_0 + \psi_p e = 0. \quad (14)$$

Otherwise, no production is optimal ($u = 0$). According to (14), the optimal production rate depends on the two co-states, ψ_x and ψ_p , which represent the marginal revenue/loss from increasing inventory and pollution stocks, respectively. From (12), it immediately follows that, for $2\gamma - \varepsilon^2 + \delta \neq 0$,

$$\psi_y = -\frac{a}{2\gamma - \varepsilon^2 + \delta} + C_3 e^{(2\gamma - \varepsilon^2 + \delta)t}. \quad (15)$$

Equation (15) explicitly determines when a steady co-state solution of (12) in terms of $\dot{\psi}_y = 0$ can be stable. Indeed, when $\delta \leq 2\gamma - \varepsilon^2 + \delta$, i.e., $2\gamma \geq \varepsilon^2$, $C_3 = 0$ must hold to meet the transversality condition from (12), which also ensures stability because the co-state becomes a constant. Otherwise, when $2\gamma < \varepsilon^2$, $C_3 \neq 0$, and the transversality condition $\lim_{t \rightarrow \infty} e^{-\delta t} \psi_y = 0$ implies that the steady state of the co-state differential (equation (12)) is unstable when $\delta > 2\gamma - \varepsilon^2 + \delta > 0$, as explicitly observed from (15) or, equivalently, from (12) with $\frac{\partial \psi_y}{\partial \psi_y} > 0$. Note that a steady-state solution to (12) does not exist when $2\gamma - \varepsilon^2 + \delta = 0$ and $\psi_y = at + C_3$, which meets the transversality condition. On the other hand, when $0 > 2\gamma - \varepsilon^2 + \delta$, C_3 again does not need to be zero to meet the transversality condition, and the steady state of (12) is stable. Summarizing our observations:

Lemma 2. The steady state of the co-state equation (12) can be stable only when either $2\gamma \geq \varepsilon^2$ or $2\gamma < \varepsilon^2 - \delta$. Otherwise, when $\varepsilon^2 > 2\gamma > \varepsilon^2 - \delta > 0$, it is unstable. ■

Inequality ($2\gamma < \varepsilon^2 - \delta$) implies extreme conditions associated with mean pollutant assimilation γ being so low that equation (2) no longer adequately represents the pollution dynamic, leading to $y^{ss} < 0$ as subsequently shown. Therefore, we next focus on the condition $2\gamma \geq \varepsilon^2$, which does not depend on the discounting rate δ . We show that a feasible steady state exists only under a stricter condition, $2\gamma > \varepsilon^2$.

Specifically, by setting $\dot{x} = 0$, $\dot{p} = 0$, $\dot{\psi}_y = 0$, $\dot{\psi}_x = 0$, and $\dot{\psi}_p = 0$, we find that $u = D$; $p = e \frac{D}{\gamma}$; $\psi_y = -\frac{a}{2\gamma - \varepsilon^2 + \delta}$; $\psi_x = -2 \frac{h}{\delta} x$; $\frac{2e \frac{D^2}{\gamma}}{2\gamma - \varepsilon^2} = y$; and $2eu\psi_y = (\gamma + \delta)\psi_p = 0$, respectively. Accordingly, $\psi_p = -\frac{2eD}{(\gamma + \delta)}$ and, from (14), we find the steady-state inventory level as hereafter summarized.

Lemma 3. Let $2\gamma > \varepsilon^2$. Then, the expected steady state of the three state variables (x , p , and y) is unique and is given by

$$p^{ss} = e \frac{D}{\gamma}, y^{ss} = \frac{2e^2 D^2}{2\gamma - \varepsilon^2} \text{ and } x^{ss} = \frac{\delta}{h} \left(cu_0 - e^2 \frac{D}{\gamma} \frac{a}{2\gamma - \varepsilon^2 + \delta} - cD - \frac{e^2 D}{(\gamma + \delta)2\gamma - \varepsilon^2 + \delta} \frac{a}{\gamma} \right). \quad \blacksquare \quad (16)$$

Several observations follow from Lemma 3. In particular, increased mean demand causes the expected long-run pollution level to increase and the expected long-run inventory stock to decrease. We also observe that taxation decreases the inventory. However, in the absence of an abatement pollution effort that would reduce the emission rate (e), the emission tax and ambient tax do not reduce the long-run pollution stock. Notably, the emission rate per product unit, which has a natural proportional influence on the expected long-run pollution level, has a disproportionately increasing effect on the pollution stock variance and a decreasing effect on the long-run inventory stock. Indeed, since $y = E[P^2]$, i.e., $y = Var[P] + E[P]^2 = Var[P] + p^2$, we observe that $Var[P]^{ss} = \frac{2e^2 D^2}{2\gamma - \varepsilon^2} - e^2 \frac{D^2}{\gamma^2} = \frac{D^2 e^2 \varepsilon^2}{(2\gamma - \varepsilon^2)\gamma^2}$. Accordingly, demand and pollution-related uncertainty (volatility of pollutant assimilation) increasingly elevate the variance in the pollution stock. The effect of mean pollutant assimilation, γ , is quite the opposite; it decreasingly reduces the long-run expected pollution stock p^{ss} and increasingly reduces its variance.

4. Optimal production rate

To study the properties of the steady states determined by Lemma 3, we next derive optimal inventory and pollution dynamics for a transient state. Specifically, by differentiating the optimality condition (14), $2\dot{p}e\psi_y + \dot{\psi}_x - 2c\dot{u} + \dot{\psi}_p e = 0$, which, when accounting for (8), (11), (13), and (15), results in

$$\gamma\psi_x - 2\gamma cu - 2c\dot{u} + e(2\gamma + \delta)\psi_p + 2hx + \delta\psi_x + 2\gamma cu_0 = 0. \quad (17)$$

Differentiating (17) again and substituting the respective state and co-state differential equations we have:

$$(\gamma + \delta)(2hx + \delta\psi_x) - 2\gamma c\dot{u} - 2c\ddot{u} + e(2\gamma + \delta)(-2eu\psi_y + (\gamma + \delta)\psi_p) + 2h(u - D) = 0. \quad (18)$$

and substituting in addition for ψ_x from (17) to (18),

$$2\gamma hx - 2\delta\gamma cu_0 + 2c(\delta - \gamma)\dot{u} - 2c\ddot{u} + 2[-e(2\gamma + \delta)e\psi_y + h + \delta\gamma c]u + e(2\gamma + \delta)\gamma\psi_p - 2hD = 0. \quad (19)$$

Similarly, differentiating (19), we obtain

$$2\gamma h(u - D) + 2c(\delta - \gamma)\ddot{u} - 2c\ddot{u} + 2[-e(2\gamma + \delta)e\psi_y + h + \delta\gamma c]\dot{u} - e(2\gamma + \delta)\gamma 2eu\psi_y + e\gamma(2\gamma + \delta)(\gamma + \delta)\psi_p = 0. \quad (20)$$

Substituting ψ_p from (19) into (20) results in

$$2\gamma h(u - D) + 2c(\delta - \gamma)\ddot{u} - 2c\ddot{u} + 2[-e(2\gamma + \delta)e\psi_y + h + \delta\gamma c]\dot{u} - e(2\gamma + \delta)\gamma 2eu\psi_y - (\gamma + \delta)(2\gamma hx - 2\delta\gamma cu_0 + 2c(\delta - \gamma)\dot{u} - 2c\ddot{u} + 2[-e(2\gamma + \delta)e\psi_y + h + \delta\gamma c]u - 2hD) = 0. \quad (21)$$

Finally, differentiating (21), we obtain the fourth-order linear differential equation with constant coefficients:

$$-2c\left(\frac{d^4u}{dt^4}\right) + 4c\delta\left(\frac{d^3u}{dt^3}\right) + (-2c\delta^2 + (-2e^2\psi_y + 2c\gamma)\delta - 4e^2\gamma\psi_y + 2c\gamma^2 + 2h)\left(\frac{d^2u}{dt^2}\right) - 2\delta\left((-e^2\psi_y + c\gamma)\delta - 2e^2\gamma\psi_y + c\gamma^2 + h\right)\left(\frac{du}{dt}\right) + 2h\gamma(\gamma + \delta)(D - u) = 0. \quad (22)$$

Lemma 4. Let $2\gamma > \varepsilon^2$. Then, the dynamics of the optimal production control is given by (22) where $\psi_y = -\frac{a}{2\gamma - \varepsilon^2 + \delta}$. ■

5. Explicit state and control trajectories

Equation (22) can be represented as a system of four linear differential equations. Let $m_1 = u$, $m_2 = \dot{m}_1 = \dot{u}$, $m_3 = \dot{m}_2 = \ddot{u}$, and $m_4 = \dot{m}_3 = \dddot{u}$. Then, (22) transforms into

$$\begin{cases} \dot{m}_1 = m_2 \\ \dot{m}_2 = m_3 \\ \dot{m}_3 = m_4 \\ \dot{m}_4 = -\frac{1}{2c}\left(-4c\delta m_4 - (-2c\delta^2 + (-2e^2\psi_y + 2c\gamma)\delta - 4e^2\gamma\psi_y + 2c\gamma^2 + 2h)m_3 + \right) \\ \quad + 2\delta((-e^2\psi_y + c\gamma)\delta - 2e^2\gamma\psi_y + c\gamma^2 + h)m_2 - 2h\gamma(\gamma + \delta)(D - m_1) \end{cases} \quad (23)$$

The Jacobian matrix for the system (23) is

$$J = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \zeta_1 & \zeta_2 & \zeta_3 & 2\delta \end{bmatrix}, \quad (24)$$

where $\zeta_1 = -\frac{\gamma h(\gamma + \delta)}{c}$, $\zeta_2 = -\frac{\delta((-e^2\psi_y + c\gamma)\delta - 2e^2\gamma\psi_y + c\gamma^2 + h)}{c}$ and $\zeta_3 = -\frac{2c\delta^2 - (-2e^2\psi_y + 2c\gamma)\delta + 4e^2\gamma\psi_y - 2c\gamma^2 - 2h}{2c}$.

The eigenvalues for J are:

$$\lambda_{1,2,3,4} = \frac{\delta}{2} \pm \sqrt{\left(\frac{\delta}{2}\right)^2 - \frac{\Psi}{2} \pm \frac{1}{2}\sqrt{\Psi^2 - 4|J|}} = \frac{\delta}{2} \pm \sqrt{\left(\frac{\delta}{2}\right)^2 + \frac{e^2a(2\gamma + \delta) + (2\gamma - \varepsilon^2 + \delta)(\gamma c(\gamma + \delta) + h)}{2c(2\gamma - \varepsilon^2 + \delta)} \pm \frac{1}{2}\sqrt{\Omega}}, \quad (25)$$

where

$$|J| = \frac{\gamma h(\gamma + \delta)}{c}, \quad (26)$$

$$\Omega = \frac{e^4 a^2}{(2\gamma - \varepsilon^2 + \delta)^2} (\delta + 2\gamma)^2 + \frac{2e^2 a}{2\gamma - \varepsilon^2 + \delta} (\delta + 2\gamma)(c\delta\gamma + c\gamma^2 + h) + (\gamma c(\gamma + \delta) - h)^2, \text{ and} \quad (27)$$

$$\Psi = -\frac{e^2 a(2\gamma + \delta) + (2\gamma - \varepsilon^2 + \delta)(\gamma c(\gamma + \delta) + h)}{c(2\gamma - \varepsilon^2 + \delta)}. \quad (28)$$

That is, the solution to (22) is given by

$$u(t) = D + A_1 e^{\lambda_1 t} + A_2 e^{\lambda_2 t} + A_3 e^{\lambda_3 t} + A_4 e^{\lambda_4 t}. \quad (29)$$

Note that, since $|J| > 0$, $\Psi < 0$, $\Psi^2 - 4|J| > 0$, and $0 < \Psi^2 - 4|J| < \Psi^2$, all eigenvalues are real numbers, implying smooth convergence to a steady state if one exists:

$$\left(\frac{\delta}{2}\right)^2 - \frac{\Psi}{2} \pm \frac{1}{2}\sqrt{\Psi^2 - 4|J|} > 0, \left(\frac{\delta}{2}\right)^2 < \left(\frac{\delta}{2}\right)^2 - \frac{\Psi}{2} \pm \frac{1}{2}\sqrt{\Psi^2 - 4|J|}$$

and the eigenvalues are given by

$$\lambda_{3,4} = \frac{\delta}{2} + \sqrt{\left(\frac{\delta}{2}\right)^2 - \frac{\Psi}{2} \pm \frac{1}{2}\sqrt{\Psi^2 - 4|J|}} > \delta \text{ and } \lambda_{1,2} = \frac{\delta}{2} - \sqrt{\left(\frac{\delta}{2}\right)^2 - \frac{\Psi}{2} \pm \frac{1}{2}\sqrt{\Psi^2 - 4|J|}} < 0. \quad (30)$$

From (30), we conclude that the Jacobian admits two negative eigenvalues and two positive eigenvalues. Consequently, the general solution of the system in (23) implies that the steady state has a saddle point property and that the path to the steady state is smooth (e.g., Dockner, 1985). However, not every path that satisfies (23) meets the transversality conditions. To verify this, we note that, when differentiating the transversality condition from (11), we must have $\lim_{t \rightarrow \infty} (-\delta e^{-\delta t} \psi_x + e^{-\delta t} \dot{\psi}_x) = 0$ and, since, $\lim_{t \rightarrow \infty} e^{-\delta t} \psi_x = 0$, we find that $\lim_{t \rightarrow \infty} e^{-\delta t} \dot{\psi}_x = 0$ must also hold. Similarly, we find from (12) and (13) that $\lim_{t \rightarrow \infty} e^{-\delta t} \dot{\psi}_y = 0$ and $\lim_{t \rightarrow \infty} e^{-\delta t} \dot{\psi}_p = 0$ must hold. Then, multiplying the optimality condition in (14) by $e^{-\delta t}$, we obtain

$$2pe \lim_{t \rightarrow \infty} e^{-\delta t} \dot{\psi}_y + \lim_{t \rightarrow \infty} e^{-\delta t} \dot{\psi}_x - 2c \lim_{t \rightarrow \infty} e^{-\delta t} u + 2cu_0 \lim_{t \rightarrow \infty} e^{-\delta t} + \lim_{t \rightarrow \infty} e^{-\delta t} \dot{\psi}_p e = 0,$$

which leads to $\lim_{t \rightarrow \infty} e^{-\delta t} u = 0$ when accounting for the transversality conditions. In addition, by differentiating the optimality condition in (14) and multiplying it by $e^{-\delta t}$, we find that $\lim_{t \rightarrow \infty} e^{-\delta t} \dot{u} = 0$; that is, from $\lambda_{3,4} > \delta$ we conclude that $A_3 = A_4 = 0$ must hold to meet the transversality conditions.

It is worth noting that, unlike under the $2\gamma > \varepsilon^2$ condition, the transversality conditions can admit non-stable solutions to our problem. For example, when $\varepsilon^2 - \delta < 2\gamma < \varepsilon^2$, the co-state ψ_y tends to infinity with t while the condition $\lim_{t \rightarrow \infty} e^{-\delta t} \psi_y = 0$ holds (see Lemma 2). We summarize our findings:

Lemma 5. Let $2\gamma > \varepsilon^2$. Then, the optimal production rate is given by

$$u(t) = D + A_1 e^{\lambda_1 t} + A_2 e^{\lambda_2 t} \quad (31)$$

where $\lambda_1, \lambda_2 < 0$ are determined by (30) and A_1, A_2 are the integration constants defined by the system of linear equations (A1.3) and (A1.4) in two unknowns as shown in Appendix 1. ■

We denote the solution to (A1.3) and (A1.4) in A_1 and A_2 by \tilde{A}_1 and \tilde{A}_2 , respectively. Next, we describe the dynamics of the state variables: $y(t)$, $p(t)$, and $x(t)$ (see (7)-(9), respectively). Substituting the obtained control, $u(t) = D + \tilde{A}_1 e^{\lambda_1 t} + \tilde{A}_2 e^{\lambda_2 t}$, into (8) and (9) and accounting for the known initial conditions of the state variables, $p(0)$ and $x(0)$, respectively, we find that

$$p(t) = e^{\left(\frac{\tilde{A}_1 e^{\lambda_1 t}}{\gamma + \lambda_1} + \frac{\tilde{A}_2 e^{\lambda_2 t}}{\gamma + \lambda_2} + \frac{D}{\gamma}\right)} + p(0)e^{-\gamma t} - e^{\left(\frac{\tilde{A}_1}{\gamma + \lambda_1} + \frac{\tilde{A}_2}{\gamma + \lambda_2} + \frac{D}{\gamma}\right)} e^{-\gamma t} \quad (32)$$

and

$$x(t) = \frac{\tilde{A}_1 e^{\lambda_1 t}}{\lambda_1} + \frac{\tilde{A}_2 e^{\lambda_2 t}}{\lambda_2} + x^{ss}. \quad (33)$$

Next, we once again substitute the optimal control with the solution (32) for the state variable $p(t)$ into (7). Then, accounting for $y(0) = p^2(0)$ from (7) yields

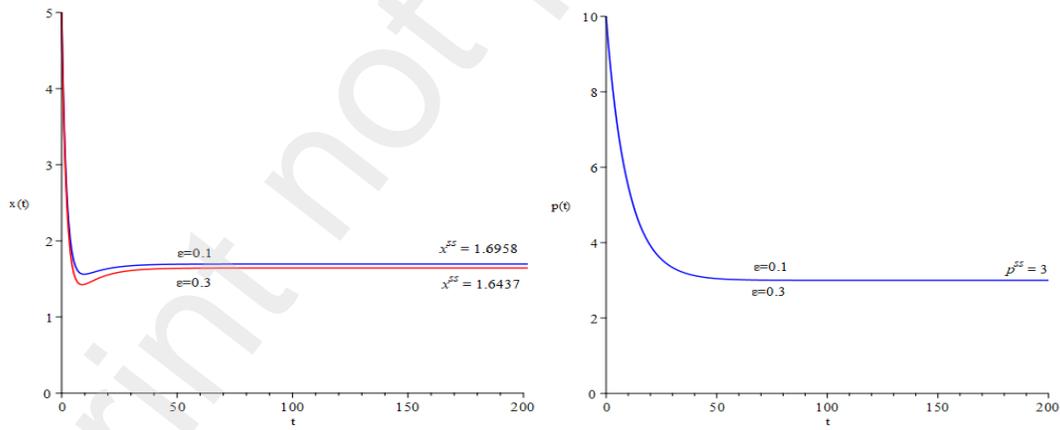
$$y(t) = B(t) + (y(0) - \check{B})e^{(\varepsilon^2 - 2\gamma)t} \quad (34)$$

where $B(t)$ and \check{B} are as shown in Appendix 2. By letting t tend to infinity, it is straightforward to verify that all state variables tend to the corresponding steady-state levels defined by Lemma 3. Considering that $y = Var[P] + p^2$, we summarize our main results as follows.

Theorem 1. If $2\gamma > \varepsilon^2$, the optimal trajectories of the expected inventory (x) and pollution (p) stocks are given by (32) and (33) while the trajectory of the variance of the pollution stock is given by $Var[P] = y - p^2$ where y is determined by (34). The three state variables (x , p , and y) and, therefore, $Var[P]$ also converge asymptotically to their expected steady-state values: x^{ss} , p^{ss} , and y^{ss} and $(Var[P]^{ss})$, respectively. This is achieved by the means of the optimal production control determined by Lemma 5, which also asymptotically converges to its steady state: $u^{ss} = D$. ■

Note, that asymptotic convergence of the pollution stock variance to a steady state is of special importance. If this does not happen thereby implying very high environmental absorption volatility, the resulting pollution path can have irreversible negative environmental consequences by going through the so-called *Soylent Green area* with negative environmental absorption efficiency (El Ouardighi and Benchekroun, 2015).

The results of Theorem 1 are illustrated in Figure 1 for $\gamma = 0.1$, $\varepsilon = 0.1$, and $\varepsilon = 0.3$ (thereby, $2\gamma > \varepsilon^2$), $e = 0.05$, $\delta = 0.05$, $h = 0.5$, $D = 6$, $a = 1$, $u_0 = 15$, $c = 2$, $X(0) = 5$, and $P(0) = 10$; therefore, $y(0) = 100$. Specifically, Figures 1a and 1b show dynamic convergence of expected inventory and pollution stocks to steady states for two levels of pollution-related uncertainty, and we observe that the volatility of pollutant assimilation affects only the inventory dynamic and the expected inventory steady state. In particular, the greater the pollution-related uncertainty, the smaller the inventory held in stock. Though pollution uncertainty does not affect the dynamics of the expected pollution stock (Figure 1b), it increases the variance of the pollution stock as shown in Figure 1c. Finally, Figure 1d simulates the stochastic evolution of the pollution stock repeatedly (fifteen samples) by generating Wiener processes $S(t)$ in equation (2) with the *ItoProcess* function of Maple 2022. From this figure, we again observe convergence of the expected pollution stock and its variance to their steady states. Overall, the initial pollution and inventory stocks in Figure 1 quickly reduce toward their expected steady states while causing an initial hike in the pollution stock variance because of the initially significant pollution level.



(a)

(b)

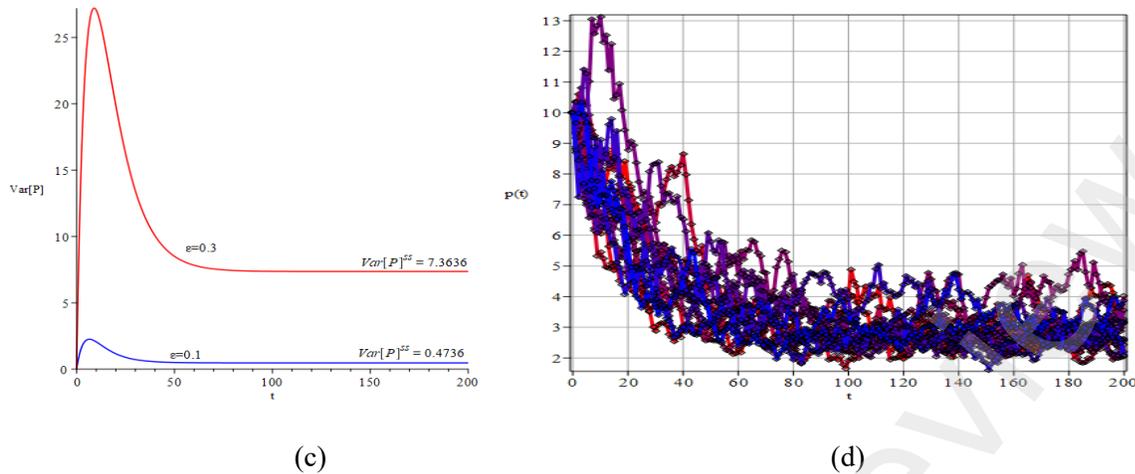


Figure 1. Optimal trajectories of (a) $x(t)$, (b) $p(t)$, (c) $Var[P]$, and (d) fifteen sample paths of $P(t)$

6. Conclusions

Difficulties in continuously evaluating inventory and pollution stocks make commitment open-loop strategies practical, and they are widely used in industry, especially when environmental concerns are a focus. We consider a production system characterized by a commitment strategy that accounts for associated emissions and thereby for environmental consequences. We derive an optimal production control policy over an entire time horizon and show that, in the long run in this production system, a unique expected steady state exists when mean pollutant assimilation is high relative to the volatility of pollutant assimilation. If that condition does not hold, implying very high environmental absorption volatility, the resulting pollution path can have irreversible negative environmental consequences by going through the *Soylent Green area* with negative environmental absorption efficiency (El Ouardighi and Benchekroun, 2015).

The common issue associated with commitment strategies is that they can expose unstable long-term outcomes and they are not necessarily affected by environmental taxes when no abatement policy is adopted by the polluting firm. Our results confirm the latter. Our results confirm the latter in terms of long-run environmental consequences but not in terms of transient dynamics. In terms of unstable long-term outcomes, we show that, when mean pollutant absorption is high enough, the production system has a unique steady state characterized not only by the expected inventory and pollution stocks but also by asymptotic convergence of the expected inventory and pollution stock trajectories to their respective steady states. Moreover, variance of the pollution stock also converges asymptotically to a unique steady state.

We find that growing mean demand increases expected long-term pollution but decreases expected long-term inventory stocks. We further find that demand volatility (demand-related uncertainty) only increases inventory variances and, therefore, associated inventory holding costs. Demand volatility does not affect expected stocks or expected pollution levels. The effect of mean pollutant absorption is quite the opposite, it reduces decreasingly the long-run pollution stock and increasingly reduces its variance. Importantly, the greater the volatility of pollutant assimilation, the greater the variance of the pollution stock while levels of the long-run expected pollution stocks are not affected (unlike with a contingent strategy (e.g., Kogan *et al*, 2017). However, as in the contingent approach (Kogan *et al*, 2017) and in contrast to finite-time-horizon deterministic approaches (Dobos 2005, Li 2013), ambient pollution taxes and environmental uncertainty affect inventories, leading in the long run to smaller expected steady-state inventory stocks. Notably, the emission rate per product unit, which has a natural proportional influence on expected long-run pollution levels, causes a disproportionately strong contraction of long-run expected inventory stocks.

Appendices

Appendix 1 – Determining the unknown constants of integration A_1 and A_2

To determine the two unknown constants of integration A_1 and A_2 , we first substitute optimal control (31) into (13) and then solve it for $\psi_p(t)$, which results in

$$\psi_p(t) = \left(\frac{2ea \left(\frac{A_1 e^{-t(\gamma + \delta - \lambda_1)}}{-\gamma - \delta + \lambda_1} + \frac{A_2 e^{-t(\gamma + \delta - \lambda_2)}}{-\gamma - \delta + \lambda_2} + \frac{De^{-t(\gamma + \delta)}}{-\gamma - \delta} \right)}{-\varepsilon^2 + \delta + 2\gamma} + C_{11} \right) e^{t(\gamma + \delta)},$$

wherein $C_{11} = 0$ is due to the transversality condition $\lim_{t \rightarrow \infty} e^{-\delta t} \psi_p = 0$, and, accordingly,

$$\psi_p(0) = \frac{2ea \left(\frac{A_1}{-\gamma - \delta + \lambda_1} + \frac{A_2}{-\gamma - \delta + \lambda_2} + \frac{D}{-\gamma - \delta} \right)}{-\varepsilon^2 + \delta + 2\gamma}. \quad (\text{A1.1})$$

Next, by differentiating (11), we have $\dot{\psi}_x = 2h(u - d) + \delta\psi_x$. Substituting (31) and solving this equation, we obtain

$$\psi_x = -\frac{1}{(\delta - \lambda_1)(\delta - \lambda_2)} \left(C_{12} e^{\delta t} \left(\lambda_1 + \lambda_2 - \delta - \frac{\lambda_1 \lambda_2}{\delta} \right) + \frac{2hA_1}{\lambda_1} e^{\lambda_1 t} (\delta - \lambda_2) + \frac{2hA_2}{\lambda_2} e^{\lambda_2 t} (\delta - \lambda_1) \right) + C_{13},$$

which, with respect to $\lim_{t \rightarrow \infty} e^{-\delta t} \psi_x = 0$, leads to $C_{12} = 0$ and $\dot{\psi}_x(0) = 2hx(0) + \delta\psi_x(0)$.

Accordingly, $\dot{\psi}_x = -\frac{1}{(\delta - \lambda_1)(\delta - \lambda_2)} \left(2hA_1 e^{\lambda_1 t} (\delta - \lambda_2) + 2hA_2 e^{\lambda_2 t} (\delta - \lambda_1) \right)$ and the constant of integration C_{13} is found from

$$\begin{aligned}
& -\frac{1}{(\delta - \lambda_1)(\delta - \lambda_2)}(2hA_1(\delta - \lambda_2) + 2hA_2(\delta - \lambda_1)) \\
& = 2hx(0) - \frac{\delta}{(\delta - \lambda_1)(\delta - \lambda_2)}\left(\frac{2hA_1}{\lambda_1}(\delta - \lambda_2) + \frac{2hA_2}{\lambda_2}(\delta - \lambda_1)\right) + \delta C_{13}
\end{aligned}$$

and

$$\psi_x(0) = -\frac{2hx(0)}{\delta} - \frac{1}{\delta(\delta - \lambda_1)(\delta - \lambda_2)}(2hA_1(\delta - \lambda_2) + 2hA_2(\delta - \lambda_1)). \quad (A1.2)$$

Then, based on the resulting solution in (A1.1)-(A1.2), the unknown constants of integration, A_1 and A_2 can be straightforwardly found from equations (14) and (17) in which $t = 0$. In particular, accounting for (15), (A1.1), (A1.2), and control (31), equation (14) at $t = 0$ transforms into

$$\begin{aligned}
& 2p(0)e\left(-\frac{a}{2\gamma - \varepsilon^2 + \delta}\right) + \left(-\frac{2hx(0)}{\delta} - \frac{1}{\delta(\delta - \lambda_1)(\delta - \lambda_2)}(2hA_1(\delta - \lambda_2) + 2hA_2(\delta - \lambda_1))\right) - 2c(D + A_1 + A_2) \\
& + 2cu_0 + e\left(\frac{2ea\left(\frac{A_1}{-\gamma - \delta + \lambda_1} + \frac{A_2}{-\gamma - \delta + \lambda_2} + \frac{D}{-\gamma - \delta}\right)}{-\varepsilon^2 + \delta + 2\gamma}\right) = 0 \quad (A1.3)
\end{aligned}$$

whereas equation (17) becomes

$$\begin{aligned}
& \gamma\left(-\frac{2hx(0)}{\delta} - \frac{1}{\delta(\delta - \lambda_1)(\delta - \lambda_2)}(2hA_1(\delta - \lambda_2) + 2hA_2(\delta - \lambda_1))\right) - 2\gamma c(D + A_1 + A_2) - 2c(A_1\lambda_1 + A_2\lambda_2) + e \\
& (2\gamma + \delta)\left(\frac{2ea\left(\frac{A_1}{-\gamma - \delta + \lambda_1} + \frac{A_2}{-\gamma - \delta + \lambda_2} + \frac{D}{-\gamma - \delta}\right)}{-\varepsilon^2 + \delta + 2\gamma}\right) + 2hx(0) + \delta \\
& \left(-\frac{2hx(0)}{\delta} - \frac{1}{\delta(\delta - \lambda_1)(\delta - \lambda_2)}(2hA_1(\delta - \lambda_2) + 2hA_2(\delta - \lambda_1))\right) + 2\gamma cu_0 = 0. \quad (A1.4)
\end{aligned}$$

Equations (A1.3) and (A1.4) constitute a linear system of two equations in two unknown constants, A_1 and A_2 .

Appendix 2 – $B(t)$ and \check{B} for equation (34)

$$\begin{aligned}
B(t) = & \frac{1}{(\gamma + \lambda_1)(\gamma + \lambda_2)\gamma} \left[\frac{2D^2 e^2 (\gamma + \lambda_1)(\gamma + \lambda_2)}{-\varepsilon^2 + 2\gamma} + \frac{2e^2 \gamma \check{A}_1^2 (\gamma + \lambda_2) e^{2\lambda_1 t}}{-\varepsilon^2 + 2\gamma + 2\lambda_1} + \frac{2e^2 \gamma \check{A}_2^2 (\gamma + \lambda_1) e^{2\lambda_2 t}}{-\varepsilon^2 + 2\gamma + 2\lambda_2} \right. \\
& + \frac{2e^2 D \check{A}_2 (\gamma + \lambda_1)(2\gamma + \lambda_2) e^{\lambda_2 t}}{-\varepsilon^2 + 2\gamma + \lambda_2} + \frac{2e^2 \gamma \check{A}_1 \check{A}_2 (2\gamma + \lambda_2 + \lambda_1) e^{(\lambda_2 + \lambda_1)t}}{-\varepsilon^2 + 2\gamma + \lambda_2 + \lambda_1} + G\left(\frac{De^{-\gamma t}}{-\varepsilon^2 + \gamma}\right) \\
& \left. + \frac{\check{A}_1 e^{(-\gamma + \lambda_1)t}}{-\varepsilon^2 + \gamma + \lambda_1} + \frac{\check{A}_2 e^{(-\gamma + \lambda_2)t}}{-\varepsilon^2 + \gamma + \lambda_2} + \frac{2e^2 D \check{A}_1 (\gamma + \lambda_2)(2\gamma + \lambda_1) e^{\lambda_1 t}}{-\varepsilon^2 + 2\gamma + \lambda_1} \right],
\end{aligned}$$

$$\check{B} = \frac{1}{(\gamma + \lambda_1)(\gamma + \lambda_2)\gamma} \left[\frac{2e^2 D^2 (\gamma + \lambda_1)(\gamma + \lambda_2)}{-\varepsilon^2 + 2\gamma} + \frac{2e^2 \gamma \tilde{A}_1^2 (\gamma + \lambda_2)}{-\varepsilon^2 + 2\gamma + 2\lambda_1} + \frac{2e^2 \gamma \tilde{A}_2^2 (\gamma + \lambda_1)}{-\varepsilon^2 + 2\gamma + 2\lambda_2} \right. \\ \left. + \frac{2e^2 D \tilde{A}_2 (\gamma + \lambda_1)(2\gamma + \lambda_2)}{-\varepsilon^2 + 2\gamma + \lambda_2} + \frac{2e^2 \gamma \tilde{A}_1 \tilde{A}_2 (2\gamma + \lambda_2 + \lambda_1)}{-\varepsilon^2 + 2\gamma + \lambda_2 + \lambda_1} + G \left(\frac{D}{-\varepsilon^2 + \gamma} + \frac{\tilde{A}_1}{-\varepsilon^2 + \gamma + \lambda_1} \right. \right. \\ \left. \left. + \frac{\tilde{A}_2}{-\varepsilon^2 + \gamma + \lambda_2} \right) + \frac{2e^2 D \tilde{A}_1 (\gamma + \lambda_2)(2\gamma + \lambda_1)}{-\varepsilon^2 + 2\gamma + \lambda_1} \right],$$

and the constant G in expressions for $B(t)$ and \check{B} is defined by

G

$$= 2e(\gamma^3 p(0) - ((D + \tilde{A}_1 + \tilde{A}_2)e - p(0)(\lambda_2 + \lambda_1))\gamma^2 - (((\lambda_2 + \lambda_1)D + \lambda_2 \tilde{A}_1 + \lambda_1 \tilde{A}_2)e - \lambda_1 \lambda_2 p(0))\gamma - De\lambda_1 \lambda_2).$$

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