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LOCC convertibility of entangled states in infinite-dimensional systems

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Abstract

We advance on the conversion of bipartite quantum states via local operations and classical communication (LOCC) for infinite-dimensional systems. We introduce δ-LOCC convertibility based on the observation that any pure state can be approximated by a state with finite-support Schmidt coefficients. We show that δ-LOCC convertibility of bipartite states is fully characterized by a majorization relation between the sequences of squared Schmidt coefficients, providing a novel extension of Nielsen's theorem for infinite-dimensional systems. Hence, our definition is equivalent to the one of ϵ-LOCC convertibility (Owari *et al* 2008 *Quantum Inf. Comput.* **8** 0030), but deals with states having finitely supported sequences of Schmidt coefficients. Additionally, we discuss the notions of optimal common resource and optimal common product in this scenario. The optimal common product always exists, whereas the optimal common resource depends on the existence of a common resource. This highlights a distinction between the resource-theoretic aspects of finite versus infinite-dimensional systems. Our results rely on the order-theoretic properties of majorization for infinite sequences, applicable beyond the LOCC convertibility problem.

1. Introduction

The purpose of this article is to explore the complexities that may arise for the infinite-dimensional quantum systems when dealing with the convertibility of entangled states by local operations and classical communication (LOCC) [1]. For example, it may be the case that a state cannot be converted by LOCC to a target state but can be converted to another state arbitrarily close to the former. To avoid such discontinuity, the notion of ϵ-convertibility under LOCC (ϵ-LOCC) was introduced [2]. Roughly speaking, $|\psi\rangle$ is ϵ-LOCC convertible to $|\phi\rangle$ if, for any neighborhood of $|\phi\rangle$, there exists a LOCC operation that takes $|\psi\rangle$ to a state in that neighborhood of $|\phi\rangle$. Furthermore, ϵ-LOCC convertibility is completely characterized in terms of a majorization relation between the sequences formed by the squared Schmidt coefficients [2, 3], which can be viewed as an extension of Nielsen's theorem [4] to the infinite-dimensional case. Additionally, a generalization of this result applies to quantum systems represented by commuting semi-finite von Neumann algebras [5].

The study of infinite-dimensional scenarios is essential both from a purely theoretical perspective and from practical applications to real systems. The qubits currently being used in various quantum computing platforms are ultimately embedded in infinite-dimensional systems, whether trapped ions or superconducting qubits. Also, many other applications of interest involve continuous-variable systems, meaning they are inherently of infinite dimension. A more comprehensive discussion on this point can be found, for example, in a recent work where the authors extend a known result on entanglement cost to the infinite-dimensional case [6].

Studying LOCC convertibility offers the advantage of operationally comparing entangled resources without initially specifying an entanglement measure. Specifically, if one state can be converted to another under any of the discussed LOCC transformations, the former state contains at least as much entanglement as the latter. While this method has some limitations, as state convertibility does not typically establish a total order, it still provides valuable insights.

Our contribution involves the introduction and discussion of a new definition of approximate LOCC convertibility for infinite-dimensional systems, which we refer to as δ-LOCC *convertibility*. This concept relies on the observation that, for any bipartite pure state, there exists a state that is arbitrarily close to it (in terms of the trace distance) and whose Schmidt coefficients have finite support. We will demonstrate that this approach turns out to be equivalent to ϵ-LOCC convertibility, while offering the added advantage of dealing with states whose sequences of Schmidt coefficients have finite support.

Additionally, we consider the following problem: suppose that two separated parties have to perform a series of quantum information tasks that require different entangled states. Rather than sharing multiple states, they aim to use a single entangled state, manipulating it to suit each task. Thus, the question arises: for any given set of target states, is there a minimal entangled state that can be locally transformed into any other target state using LOCC? This state, if exists, is known as an *optimal common resource* of the set [7]. Similarly, we also explore the existence of a maximal entangled state that can be obtained from any state of the original set by LOCC. This state, if exists, is referred to as an *optimal common product* of the set [8]. Understanding these problems is crucial for quantum resource theories and entanglement [9]. Exploring these issues in an infinite-dimensional setting provides insights into the fundamental properties of entanglement and its role as a resource in quantum information [10].

We recall that, in the case of pure bipartite finite-dimensional systems, the existence of an optimal common resource and an optimal common product has been established using the link between LOCC convertibility and majorization, as shown by Nielsen's theorem [4], and the fact that majorization forms a complete lattice [11, 12].

Here, we exploit the characterization of δ-LOCC (or, equivalently ϵ-LOCC) in terms of majorization in order to describe the optimal common resource and optimal common product for infinite-dimensional systems. Unlike the finite-dimensional case, we obtain that the existence of the optimal common resource is conditioned to the existence of a common resource of the set state under consideration, which does not always exist. In particular, we provide two families of states, created by applying a two-mode squeezer to the product of a Fock state and the vacuum [13], for which the optimal common resource does not exist. This poses a novel distinction in the entanglement resource theories of finite versus infinite-dimensional quantum systems. On the other hand, we show that the optimal common product always exists. These results stem from our characterization of the majorization lattice fan or infinite-dimensional setting, which is a result of mathematical interest in itself and can be applied beyond the scope of the LOCC convertibility problem addressed here.

The rest of the paper is organized as follows. In section 2, we present some important definition and results on majorization theory for infinite sequences. Also, we introduce the notion of majorization for infinite sequences based on finitely supported approximations. In section 3, we recall the definition of ϵ-convertibility and introduce the concept of δ-convertibility. Also, we state an extension of Nielsen's theorem based on our definition, and prove the equivalence between the two notions. In section 4, we show some applications of these ideas. Finally, in section 4, we provide some concluding remarks.

2. Majorization for infinite sequences

In this section, we present two results regarding the concept of majorization for infinite sequences, which will be useful to discuss the notion of LOCC convertibility. At the same time, they hold mathematical interest in their own right. For references regarding the finite-dimensional case, we recommend consulting the following sources [11, 12, 14].

To ensure clarity in our discussion, we introduce some notations. We consider the space $\ell^1([0,1]) \equiv \ell^1$ of sequences whose series is absolutely convergent, $\ell^1([0,1]) = \left\{(x_n)_{n\in\mathbb{N}} \in [0,1]^{\mathbb{N}} : \sum_{n\in\mathbb{N}} x_n < \infty\right\}$. Additionally, we define the space ℓ_1 (ℓ with one as sub-index) as the set of sequences $(x_n)_{n\in\mathbb{N}} \in \ell^1([0,1])$ that can be rearranged into non-increasing sequences. Accordingly, we define $x^{\downarrow}$ as a sequence whose components are rearranged in non-increasing order, i.e. $x_n \geqslant x_{n+1}$ for all $n \in \mathbb{N}$, and $\ell_1^{\downarrow}$ as the set of correspondingly rearranged sequences.

We also introduce the space Δ_∞ as the set of sequences on ℓ_1 that satisfy the normalization condition $\sum_{n=1}^{\infty} x_n = 1$. This is nothing else that the set of denumerable probability vectors. We use $\Delta_\infty^{\downarrow}$ to denote the set denumerable probability vectors whose components are sorted in non-increasing order. In addition, we consider the subset of denumerable probability vectors with finite support, denoted as Δ'_∞.

We recall the notion of weak submajorization, which is defined as follows [15].

Definition 1. Let $x, y \in \ell_1$. Then, x is said to be weakly submajorized by y, denoted as $x \preceq_w y$, if

$$\sum_{n=1}^{k} x_n^\downarrow \leqslant \sum_{n=1}^{k} y_n^\downarrow, \ \forall k \in \mathbb{N}. \tag{1}$$

In addition to weak submajorization, we are interested in the notion of majorization in infinite dimensions [16]. More precisely, if x and y are sequences on Δ_∞ such that $x \preceq_w y$ then x is said to be *majorized* by y, and denoted as $x \preceq y$.

2.1. Majorization lattice for infinite sequences

We now present our first result.

Proposition 2. *The poset $\langle \ell_1^\downarrow, \preceq_w, \mathbf{0}\rangle$ is a lattice with bottom element $\mathbf{0} = (0, 0, \ldots)$. Moreover, it is $\bigwedge$-complete and conditionally $\bigvee$-complete.*

Proof. First, notice that the binary operation $\preceq_w$ gives to $\ell_1^\downarrow$ a structure of a partially ordered set (poset). Indeed, $\preceq_w$ is reflexive and transitive and by an inductive argument, it follows that $\preceq_w$ is anti-symmetric.

Now, let us prove the lattice structure of $\ell_1^\downarrow$. Let S be a non-empty subset and let M be a fixed positive integer. Consider the infimum of the M-partial sums of the elements in S,

$$s_M = \inf\{s_M(x) \,:\, x \in S\}.$$

The sequence of partial sums $\{s_M\}_{M\in\mathbb{N}}$ is increasing and, given that we are dealing with non-increasing ordered sequences, it satisfies [11]

$$2s_M \geqslant s_{M-1} + s_{M+1}.$$

Also for any $x \in S$, $\lim_{M\to\infty} s_M \leqslant \lim_{M\to\infty} s_M(x) < \infty$. Hence, the sequence $\{m_M\}_{M\in\mathbb{N}}$, where $m_M = s_M - s_{M-1}$, is in $\ell_1^\downarrow$ and clearly is the infimum of S, that is, $\bigwedge S = \{m_M\}_{M\in\mathbb{N}}$. In particular, $\bigwedge \ell_1^\downarrow = \mathbf{0} \in \ell_1^\downarrow$.

Let $S' \subseteq \ell_1^\downarrow$ be another non-empty, upper-bounded subset, and consider the set $Up(S')$ of all upper bounds of S'. The result follows by recalling that the supremum of S' equals the infimum of $Up(S')$, that is, $\bigvee S' = \bigwedge Up(S')$. Thus, the lattice is conditionally $\bigvee$-complete.

□

We present the following observation of the order-structure of the set $\Delta_\infty^\downarrow$, which arise as a peculiarity in the infinite-dimensional context.

Observation 3. The set $\Delta_\infty^\downarrow$ is not bounded from below.

In other words, there is no analog to the uniform probability vector for infinite-dimensional systems. An instance of this situation is presented in the example 7. On the other side, it can be proved that any finite subset of Δ_∞ is bounded from below.

Lemma 4. *Let us consider the poset $\langle \Delta_\infty^\downarrow, \preceq, \mathbf{1}\rangle$, where $\mathbf{1} = (1, 0, 0, \ldots)$. Then, for each non-empty finite subset $\mathcal{S}$ of $\Delta_\infty^\downarrow$, $\mathcal{S}$ admits a lower bound, that is, there exists $z \in \Delta_\infty$ such that $z \preceq x$ for all $x \in \mathcal{S}$.*

Proof. Without loss of generality, we can assume $\mathcal{S} = \{x, y\}$. Let $s_M = \min\{s_M(x), s_M(y)\}$ be the minimum of the M-partial sums. The sequence $\{s_M\}_{M\in\mathbb{N}}$ is increasing and satisfies

$$2s_M \geqslant s_{M-1} + s_{M+1}.$$

It remains to check that $\lim_{M\to\infty} s_M = 1$, but this is direct since $\lim_{M\to\infty} s_M(x) = 1$ and $\lim_{M\to\infty} s_M(y) = 1$. □

Lemma 5. *Let $\mathcal{S}$ be a non-empty subset of $\Delta_\infty^\downarrow$ and assume that $z \in \Delta_\infty^\downarrow$ is a lower bound. Then, there exists the infimum of $\mathcal{S}$.*

Proof. Let $s \in \ell_1^\downarrow$ be the infimum of $\mathcal{S}$. Let us check that s is in fact in $\Delta_\infty^\downarrow$. Given that $z \preceq s$ we have for all $k \in \mathbb{N}$,

$$\sum_{n=1}^{k} z_n \leqslant \sum_{n=1}^{k} s_n \leqslant 1.$$

Taking $k \to \infty$, we get $s \in \Delta^{\downarrow}_{\infty}$. □

We are now able to state the main result of this section, in which we demonstrate the lattice structure of the poset $\langle \Delta^{\downarrow}_{\infty}, \preceq, 1 \rangle$, and its completeness properties.

Proposition 6. *The poset $\langle \Delta^{\downarrow}_{\infty}, \preceq, 1 \rangle$ is a lattice with top element $1 = (1, 0, 0, \ldots)$. Moreover, it is $\bigvee$-complete and conditionally $\bigwedge$-complete.*

Proof. It is straightforward to observe that lemma 4 guarantees that $\Delta^{\downarrow}_{\infty}$ is a lattice. Moreover, by lemma 5 the lattice is conditionally $\bigwedge$-complete. Let us prove now that $\Delta^{\downarrow}_{\infty}$ is indeed $\bigvee$-complete. Let $\mathcal{S}'$ be another non-empty subset and let $Up(S')$ be the non-empty set of upper bounds. Notice that any element of $\mathcal{S}'$ is a lower bound of $Up(S')$. Hence, from the previous lemma, $Up(S')$ has an infimum in $\Delta^{\downarrow}_{\infty}$. In other words, the supremum of S' is in $\Delta^{\downarrow}_{\infty}$. □

Let us explore two illustrative examples that shed light on these results (later on, we will discuss the physical relevance of these examples). In the first case, we present two different families of sequences which infima do not exist, while in the second example, the infimum is clearly defined.

Example 7. Consider the families of sequences $\{x^{(k)}(\lambda)\}_{k\in\mathbb{N}_0}$ and $\{x^{(k)}(\lambda)\}_{\lambda\in(0,1)}$, where

$$x^{(k)}_n(\lambda) = \binom{n+k}{k} \left(1-\lambda^2\right)^{k+1} \lambda^{2n}, \; n = 0, 1, \ldots. \tag{2}$$

Let us show that the infima $\bigwedge\{x^{(k)}(\lambda)\}_{k\in\mathbb{N}_0}$ and $\bigwedge\{x^{(k)}(\lambda)\}_{\lambda\in(0,1)}$ do not exist, whereas the suprema are given by $\bigvee\{x^{(k)}(\lambda)\}_{k\in\mathbb{N}_0} = x^{(0)}(\lambda)$ and $\bigvee\{x^{(k)}(\lambda)\}_{\lambda\in(0,1)} = (1, 0, \ldots, 0)$. First, we prove that each component of $x^{(k)}(\lambda)$ tends to zero by proving that some of its factors tends to zero and the others remain bounded. Let $r > n$ be such that $\delta := (1-\lambda^2)(n/r+1) < 1$ and let $k \to \infty$, then

$$\begin{aligned}
\left(1-\lambda^2\right)^k \binom{n+k}{k} &= \left(1-\lambda^2\right)^k \frac{n+k}{k} \frac{n+k-1}{k-1} \cdots \frac{n+1}{1} \\
&= \left(1-\lambda^2\right)^k \left(\frac{n}{k}+1\right)\left(\frac{n}{k-1}+1\right)\cdots\left(\frac{n}{1}+1\right) \\
&\leqslant \left(1-\lambda^2\right)^k \left(\frac{n}{r}+1\right)^{k-r}\left(\frac{n}{1}+1\right)^{r} \\
&= \left(\left(1-\lambda^2\right)\left(\frac{n}{r}+1\right)\right)^{k-r}\left(\left(1-\lambda^2\right)\left(\frac{n}{1}+1\right)\right)^{r} \\
&< \delta^{k-r}(n+1)^r \to 0.
\end{aligned}$$

Then $x^{(k)}_n(\lambda) \to 0$ when $k \to \infty$. It is easy to check that $x^{(k)}_n(\lambda) \to 0$ when $\lambda \to 1$. Then, it follows the non-existence of the infima for both sets.

The form of the suprema follows from the fact that $x^{(k+1)}(\lambda) \preceq x^{(k)}(\lambda)$ and $x^{(k)}(\lambda) \preceq x^{(k)}(\lambda')$ with $\lambda' \leqslant \lambda$, see [13].

The following example is a family of incomparable sequences that admits an infimum.

Example 8. Consider the family of sequences $\{x^{(k)}\}_{k\in\mathbb{N}_{\geqslant 3}}$ defined as

$$x^{(k)} = \left(1 - \frac{1}{\log k}, \overbrace{\frac{1}{k\log k}, \ldots, \frac{1}{k\log k}}^{k}, 0, \ldots, 0\right). \tag{3}$$

First, we can prove that the infimum $\bigwedge\{x^{(k)}\}_{k\in\mathbb{N}_{\geqslant 3}}$ exists. The M-partial sum of the sequence $x^{(k)}$ with $k \geqslant 3$ is given by

$$s_M\left(x^{(k)}\right) = \begin{cases} 1 - \frac{1}{\log k} + (M-1)\frac{1}{k\log k} & \text{if } 1 \leqslant M \leqslant k+1, \\ 1 & \text{if } M \geqslant k+1. \end{cases}$$

In order to compute $\inf_{k\geqslant 3} s_M(x^{(k)})$ we are going to use some techniques from calculus. For a fixed M, consider the function $s(\omega)$,

$$s(\omega) = 1 - \frac{1}{\log\omega} + (M-1)\frac{1}{\omega\log\omega}, \quad \omega \geqslant 3.$$

For $M = 1,2$, we have $s'(\omega) > 0$, so the minimum is attained for $\omega = 3$. For $M \geqslant 3$, taking derivatives and equating to 0, it follows that $s(\omega)$ has only one minimum at ω_0, where ω_0 satisfies

$$\frac{\omega_0}{1+\log\omega_0} = M-1.$$

Given that $s(\omega)$ has only one critical point, the number k such that $s_M(x^{(k)})$ is minimum happens at $k = \lfloor\omega_0\rfloor$ or at $k = \lceil\omega_0\rceil$. In other words, given $M \geqslant 3$, there exists k_0 such that

$$\inf_{k\geqslant 3} s_M\left(x^{(k)}\right) = s_M\left(x^{(k_0)}\right).$$

It can be shown directly for $M = 1,2,3$ that

$$\inf_{k\geqslant 3} s_1\left(x^{(k)}\right) = 1-\frac{1}{\log 3},\quad \inf_{k\geqslant 3} s_2\left(x^{(k)}\right) = 1-\frac{2}{3\log 3},\quad \inf_{k\geqslant 3} s_3\left(x^{(k)}\right) = 1-\frac{3}{5\log 5}.$$

The value ω_0 can be computed (if necessary) with a fixed-point iteration,

$$r_0 = 1,\quad r_{i+1} = (M-1)(1+\log r_i),\quad i\geqslant 1.$$

Notice that $r_1 = M-1$ and if $r_i \geqslant 1$, the value of r_{i+1} is always greater than $M-1$. This implies that the function $(M-1)(1+\log x)$ is a contraction implying the convergence of the method to ω_0.

Finally, it is easy to observe that the supremum of this family is $\bigvee\{x^{(k)}\}_{k\in\mathbb{N}_{\geqslant 3}} = (1,0,\ldots)$.

2.2. Approximate majorization in terms of finite support probability vectors

We will now proceed to define a notion of majorization in the infinite-dimensional case, based on approximations of the original sequences by sequences with finite support. With that purpose in mind, we first prove lemma 9 that provides an upper bound for the trace distance between two sequences in $\Delta_\infty^\downarrow$ coinciding in the first N components.

Lemma 9. *Let $x, x' \in \Delta_\infty^\downarrow$ such that $x_n = x'_n$ for all $n \leqslant N$ and $\sum_{n=1}^N x_n = s_N$. Then,*

$$d_{\mathrm{tr}}(x,x') \leqslant \sqrt{2(1-s_N)}, \tag{4}$$

where $d_{\mathrm{tr}}(x,y) = \sqrt{1-\left(\sqrt{x}\cdot\sqrt{y}\right)^2}$ with $\sqrt{x} = (\sqrt{x_n})_{n\in\mathbb{N}}$ and $\sqrt{y} = (\sqrt{y_n})_{n\in\mathbb{N}}$.

Proof. By direct calculation of the trace distance between x and its finite support counterpart, x', we have

$$\begin{aligned}
d_{\mathrm{tr}}(x,x')^2 &= 1-\left(\sqrt{x}\cdot\sqrt{x'}\right)^2\\
&= \left(1+\sqrt{x}\cdot\sqrt{x'}\right)\left(1-\sqrt{x}\cdot\sqrt{x'}\right)\\
&\leqslant 2\left(1-\sqrt{x}\cdot\sqrt{x'}\right)\\
&= 2\left(1-\sum_{n=1}^{N} x_n - \sum_{n=N+1}^{\infty}\sqrt{x_n}\sqrt{x'_n}\right)\\
&\leqslant 2\left(1-\sum_{n=1}^{N} x_n\right).
\end{aligned}$$

□

Building on the previous lemma, we can now demonstrate that any sequence $x \in \Delta_\infty^\downarrow$ can be approximated by another finite-support sequence $x' \in \Delta_\infty'^\downarrow$, which is arbitrarily close to x and majorizes the latter.

Proposition 10. *Let $x \in \Delta_\infty^\downarrow$. For any $\delta \in (0,1)$ and any $K \in \mathbb{N}$, there exists $x' \in \Delta_\infty'^\downarrow$ such that*

$$x'_n = x_n \text{ for } 1 \leqslant n \leqslant K,\ x \preceq x' \text{ and } d_{\mathrm{tr}}(x,x') \leqslant \delta. \tag{5}$$

Proof. In order to prove this result, we are going to construct one such x' that fulfills the requirements. Given $x \in \Delta_\infty^\downarrow$, $\delta \in (0,1)$ and $K \in \mathbb{N}$, there exist $N, M \geqslant K$ and $x' = (x'_n)_{n\in\mathbb{N}} \in \Delta_\infty'^\downarrow$ where

(i) N is such that $s_N \geqslant 1 - \frac{\delta^2}{2}$ with $s_N = \sum_{n=1}^{N} x_n$,
(ii) M is such that $M = \left\lfloor \frac{1-s_N}{x_N} \right\rfloor + N$,
(iii) $x'_n = x_n$ for $1 \leqslant n \leqslant N$,
(iv) $x'_n = x_N$ for $N+1 \leqslant n \leqslant M$,
(v) $x'_n = 1 - s_N - (M-N)x_N$ for $n = M+1$
(vi) $x'_n = 0$ for $n > M+1$,

First, let us observe that $x' \in \Delta'^{\downarrow}_{\infty}$. By construction, $\sum_{n=1}^{\infty} x'_n = 1$. Thus, all that remains is to demonstrate that $x_N \geqslant x'_{M+1}$, which is also directly satisfied by construction, since $M+1 \geqslant \frac{1-s_N}{x_N} + N$. Then, it follows that $s_k(x) = s_k(x')$ for $1 \leqslant k \leqslant N$, $s_k(x) \leqslant s_k(x')$ for $k \geqslant N+1$. Therefore, $x \preceq x'$. Finally, by lemma 9, one has that $d_{\mathrm{tr}}(x,x') \leqslant \delta$. □

It is also interesting to note that, as we demonstrate in the following proposition, the newly introduced approximation scheme preserves the majorization order (see figure 1(a)).

Proposition 11. *Let $x, y \in \Delta^{\downarrow}_{\infty}$ be such that $x \preceq y$ and let $\delta > 0$. Then there exist $x', y' \in \Delta'^{\downarrow}_{\infty}$ such that $d_{\mathrm{tr}}(x,x') \leqslant \delta$ and $d_{\mathrm{tr}}(y,y') \leqslant \delta$, and $x' \preceq y'$.*

Proof. Given $\delta > 0$, there exists y' such that $y \preceq y'$ and $d_{\mathrm{tr}}(y,y') \leqslant \delta$, by proposition 10. Let $K \in \mathbb{N}$ be such that $y'_n = 0$ for all $n \geqslant K$. For this K and for the given $\delta > 0$, there exists x' such that $x \preceq x'$ and $d_{\mathrm{tr}}(x,x') \leqslant \delta$, by proposition 10. Let us see that $x' \preceq y'$. By construction, we have $s_k(x') \leqslant s_k(y')$ for $1 \leqslant k \leqslant K$. For all $k \geqslant K$, $s_k(x') \geqslant 1 = s_k(y')$. □

In addition, we have the converse result.

Proposition 12. *Let $x, y \in \Delta^{\downarrow}_{\infty}$. If for all $\delta > 0$, there exists x', y' with finite support such that $d_{\mathrm{tr}}(x,x') < \delta$, $d_{\mathrm{tr}}(y,y') < \delta$ and $x' \preceq y'$. Then, $x \preceq y$.*

Proof. By hypothesis we can construct sequences $\{x'_m\}_{m=1}^{\infty}$ and $\{y'_m\}_{m=1}^{\infty}$ such that $d_{\mathrm{tr}}(x,x'_m) < 1/m$ and $d_{\mathrm{tr}}(y,y'_m) < 1/m$ for all $m \in \mathbb{N}$. Notice that the first k coordinates of x'_m (resp. y'_m) converge to the first k coordinates of x (resp. y). Indeed, let $\overline{x} = (x_1, \ldots, x_k)$ and $\overline{x'_m} = (x'_{m1}, \ldots, x'_{mk})$. Then,

$$\begin{aligned}
\|\sqrt{\overline{x}} - \sqrt{\overline{x'_m}}\|_2^2 &= \sum_{n=1}^{k} \left(\sqrt{x_n} - \sqrt{x'_{mn}}\right)^2 \\
&\leqslant \sum_{n=1}^{\infty} \left(\sqrt{x_n} - \sqrt{x'_{mn}}\right)^2 \\
&= 2 - 2\sum_{n=1}^{\infty} \sqrt{x_n}\sqrt{x'_{mn}} \\
&= 2 - 2\left(x \cdot x'_m\right) \\
&\leqslant 2 - 2\left(x \cdot x'_m\right)^2 \\
&= 2\, d_{\mathrm{tr}}\left(x, x'_m\right)^2 \\
&< \frac{2}{m^2}.
\end{aligned}$$

Given that all norms are equivalent in $\mathbb{R}^k$, we get that $\overline{x'_m}$ converges to $\overline{x}$ and in particular, being s_k a continuous function, $s_k(\overline{x'_m})$ converges to $s_k(\overline{x})$. But we have the equalities $s_k(\overline{x'_m}) = s_k(x'_m)$ and $s_k(\overline{x}) = s_k(x)$ (same for $s_k(y'_m)$ and $s_k(y)$). Then, taking limit in m to the relation $s_k(x'_m) \leqslant s_k(y'_m)$ it follows that $s_k(x) \leqslant s_k(y)$. □

Finally, from proposition 11 and appealing to the properties of the lattice, the following observation about the infimum and supremum elements for infinite sequences and their finite-support counterparts follows (see figures 1(b) and (c)).

Observation 13. Consider $x, y \in \Delta^{\downarrow}_{\infty}$ with x' and y' representing their respective approximated finite-support sequences. Then, one has $x \wedge y \preceq x' \wedge y'$. Similarly, for the supremum, one can establish $x \vee y \preceq x' \vee y'$.

3. LOCC convertibility for infinite-dimensional systems

In this section, we present a new definition of LOCC convertibility for infinite-dimensional systems. Later on, we will prove that our definition coincides with the one already defined by Owari *et al* [2], which is known as ϵ-LOCC convertibility.

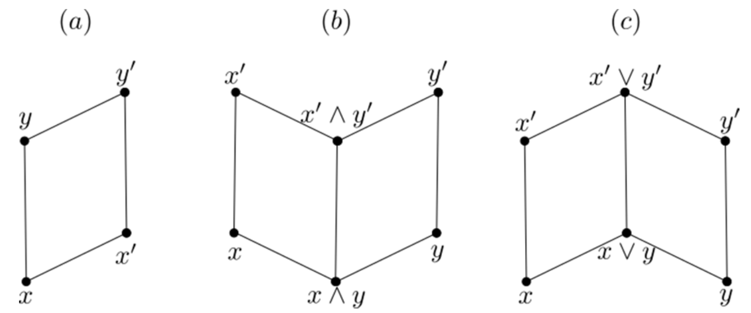

Figure 1. Hasse diagrams showing the majorization relations proved in: (a) proposition 11; (b) and (c) observation 13.

In what follows, we consider composite systems that consist of two parties, A and B, such that the Hilbert space of the joint system is $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$, with $\dim \mathcal{H}_A = \infty$ and $\dim \mathcal{H}_B = \infty$.

3.1. ϵ-LOCC convertibility

First, let us recall the Schmidt decomposition of bipartite pure states in the infinite-dimensional case [2].

Theorem 14 ([2, theorem 4]). *For any $|\psi\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B$, there exist orthonormal sets (but not necessarily basis sets) $\{|a_n\rangle\}_{n\in\mathbb{N}}$ and $\{|b_n\rangle\}_{n\in\mathbb{N}}$ of $\mathcal{H}_A$, and $\mathcal{H}_B$, respectively, such that*

$$|\psi\rangle = \sum_{n=1}^{\infty} \sqrt{\psi_n} |a_n\rangle |b_n\rangle, \tag{6}$$

where $\psi = (\psi_i)_{i\in\mathbb{N}} \in \Delta_\infty^\downarrow$.

Second, we review the notion of ϵ-LOCC convertibility introduced in [2], defined in terms of the trace distance $\| |\psi\rangle\langle\psi| - |\phi\rangle\langle\phi| \|_{\mathrm{tr}} = \sqrt{1 - |\langle\psi|\phi\rangle|^2}$ between pure states.

Definition 15 ([2, definition 1]). $|\psi\rangle$ is ϵ-convertible to $|\phi\rangle$ by LOCC, denoted as $|\psi\rangle \underset{\epsilon-\mathrm{LOCC}}{\longrightarrow} |\phi\rangle$, if for any $\epsilon > 0$, there exists a LOCC operation Λ_ϵ such that $\|\Lambda_\epsilon(|\psi\rangle\langle\psi|) - |\phi\rangle\langle\phi|\|_{\mathrm{tr}} < \epsilon$.

Finally, we recall the following theorem stating the equivalence between ϵ-LOCC and majorization of the squared Schmidt coefficients, which can be viewed as the infinite-dimensional version of Nielsen's theorem [4].

Theorem 16 ([2, theorem 1]). *Let $|\psi\rangle$ and $|\phi\rangle$ bipartite pure states belonging to $\mathcal{H}_A \otimes \mathcal{H}_B$. Then, $|\psi\rangle \underset{\epsilon-\mathrm{LOCC}}{\longrightarrow} |\phi\rangle$ if and only if $\psi \preceq \phi$, where ψ and ϕ are the sequences formed by the squared Schmidt coefficients of $|\psi\rangle$ and $|\phi\rangle$, respectively.*

It is worth mentioning that in the work [2], the authors not only give an infinite-dimensional extension of the deterministic conversion protocol via LOCC, but they also study the probabilistic case via stochastic-LOCC. Here, we focus on the deterministic scenario and leave the probabilistic case for future work.

3.2. δ-LOCC convertibility

The following two observations inspire our definition of LOCC convertibility:

- The trace distance between any two bipartite pure states belonging to an infinite-dimensional Hilbert space is always minimized when they share the same Schmidt orthonormal sets.
- For any bipartite pure state $|\psi\rangle$ belonging to $\mathcal{H}_A \otimes \mathcal{H}_B$, there exists another pure state $|\psi'\rangle$ with same orthonormal Schmidt sets than $|\psi\rangle$ and finite support, such that $\| |\psi\rangle\langle\psi| - |\psi'\rangle\langle\psi'| \|_{\mathrm{tr}} = d_{\mathrm{tr}}(\psi, \psi')$.

The latter is a direct consequence of proposition 10. Indeed, for any $\delta \in (0,1)$, proposition 10 tell us how to choose the Schmidt coefficients of $|\psi'\rangle$ in order to have $\| |\psi\rangle\langle\psi| - |\psi'\rangle\langle\psi'| \|_{\mathrm{tr}} = d_{\mathrm{tr}}(\psi, \psi') < \delta$.

Regarding the first observation, we can prove the following proposition that resembles the finite-dimensional case (see [17, lemma 1]).

Proposition 17. *Let $|\psi\rangle$ and $|\phi\rangle$ bipartite pure states belonging to $\mathcal{H}_A \otimes \mathcal{H}_B$ with the same Schmidt orthonormal sets. Then,*

$$\| |\psi\rangle\langle\psi| - |\phi\rangle\langle\phi| \|_{\mathrm{tr}} \leqslant \| |\psi_{UV}\rangle\langle\psi_{UV}| - |\phi\rangle\langle\phi| \|_{\mathrm{tr}}, \tag{7}$$

where $|\psi_{UV}\rangle = U \otimes V|\psi\rangle$, with U and V being isometries. In particular, we have $\| |\psi\rangle\langle\psi| - |\phi\rangle\langle\phi| \|_{\mathrm{tr}} = d_{\mathrm{tr}}(\psi, \phi)$, where ψ and ϕ denotes the corresponding sequences formed by the squared Schmidt coefficients of the states.

Proof. We will extend the second proof in [17, lemma 1] to the infinite-dimensional case. Let us prove

$$|\langle\psi|U \otimes V|\phi\rangle|^2 \leqslant |\langle\psi|\phi\rangle|^2.$$

Let σ^ϕ and σ^ψ be diagonal matrices of infinite dimensions constructed from the ordered Schmidt coefficients of $|\phi\rangle$ and $|\psi\rangle$.

Then, it follows that

$$\begin{aligned} |\langle\psi|U \otimes V|\phi\rangle|^2 &= |\operatorname{tr}\left(\sigma^\phi V \sigma^\psi U^t\right)|^2 \\ &= |\operatorname{tr}\left(\sqrt{\sigma^\phi} V \sqrt{\sigma^\psi}\sqrt{\sigma^\psi} U^t \sqrt{\sigma^\phi}\right)|^2 \\ &\leqslant |\operatorname{tr}\left(\sigma^\psi V^\dagger \sigma^\phi V\right)| |\operatorname{tr}\left(\sigma^\phi \overline{U} \sigma^\psi U^t\right)|. \end{aligned}$$

Since σ^ψ is diagonal, $\operatorname{tr}(\sigma^\psi C) = \operatorname{tr}(\sigma^\psi \operatorname{diag}(C))$. Also, given that $C = V^\dagger \sigma^\phi V$ is congruent to σ^ϕ, it follows from [15, corollary 6.1 S(iii'is)] that c, the diagonal of C, is majorized by ϕ, where ϕ is the diagonal of σ^ϕ,

$$c \preceq \phi.$$

Furthermore, from (the proof of) [18, theorem 4.2] it follows that for every $\epsilon > 0$, there exists a convex combination of permutations such that $\|c - \sum_{i=1}^{s} p_i^\epsilon P_i^\epsilon \phi\|_1 < \epsilon$. In particular, there exists a sequence $\{\sum_{i=1}^{s_n} p_i^n P_i^n \phi\}_{n\geqslant 0}$ such that

$$\lim_{n\to\infty} \sum_{i=1}^{s_n} p_i^n P_i^n \phi = c.$$

Then, by the continuity of the map $\mu_\psi : \ell^1 \to \mathbb{R}$ given by $\mu_\psi(x) = \sum_i \psi_i x_i$ it follows

$$\begin{aligned} \operatorname{tr}\left(\sigma^\psi C\right) &= \mu_\psi(c) \\ &= \lim_{n\to\infty} \mu_\psi\left(\sum_{i=1}^{s_n} p_i^n P_i^n \phi\right) \\ &= \lim_{n\to\infty} \sum_{i=1}^{s_n} p_i^n \mu_\psi\left(P_i^n \phi\right) \\ &= \lim_{n\to\infty} \sum_{i=1}^{s_n} p_i^n \operatorname{tr}\left(\sigma^\psi P_i^n \sigma^\phi P_i^{n\dagger}\right). \end{aligned}$$

Now, from lemma 27, $\operatorname{tr}(\sigma^\psi P \sigma^\phi P^\dagger) \leqslant \operatorname{tr}(\sigma^\psi \sigma^\phi)$ and then, for $n \geqslant 0$,

$$\sum_{i=1}^{s_n} p_i^n \operatorname{tr}\left(\sigma^\psi P_i^n \sigma^\phi P_i^{n\dagger}\right) \leqslant \sum_i p_i^n \operatorname{tr}\left(\sigma^\psi \sigma^\phi\right) = \operatorname{tr}\left(\sigma^\psi \sigma^\phi\right).$$

Taking limit, $\operatorname{tr}(\sigma^\psi C) \leqslant \operatorname{tr}(\sigma^\psi \sigma^\phi)$, that is,

$$\operatorname{tr}\left(\sigma^\psi V^\dagger \sigma^\phi V\right) \leqslant \operatorname{tr}\left(\sigma^\psi \sigma^\phi\right).$$

Similarly, it is possible to prove the result,

$$\operatorname{tr}\left(\sigma^\phi \overline{U} \sigma^\psi U^t\right) \leqslant \operatorname{tr}\left(\sigma^\psi \sigma^\phi\right).$$

Therefore,

$$|\langle\psi|U \otimes V|\phi\rangle|^2 \leqslant |\operatorname{tr}\left(\sigma^\psi V^\dagger \sigma^\phi V\right)| |\operatorname{tr}\left(\sigma^\phi \overline{U} \sigma^\psi U^t\right)| \leqslant |\operatorname{tr}\left(\sigma^\psi \sigma^\phi\right)|^2 = |\langle\psi|\phi\rangle|^2. \tag{8}$$

□

Building on the previous results, we can now provide our definition of δ-convertibility.

Definition 18. $|\psi\rangle$ is δ-convertible to $|\phi\rangle$ by LOCC, denoted as $|\psi\rangle \underset{\delta-\mathrm{LOCC}}{\longrightarrow} |\phi\rangle$, if for any $\delta > 0$, there exist states $|\psi_\delta\rangle$ and $|\phi_\delta\rangle$, both with sequences of Schmidt coefficients with finite support, such that $\||\psi\rangle\langle\psi| - |\psi_\delta\rangle\langle\psi_\delta|\|_{\mathrm{tr}} < \delta$, $\||\phi\rangle\langle\phi| - |\phi_\delta\rangle\langle\phi_\delta|\|_{\mathrm{tr}} < \delta$ and there exists a LOCC operation Λ_δ such that $|\phi_\delta\rangle\langle\phi_\delta| = \Lambda_\delta(|\psi_\delta\rangle\langle\psi_\delta|)$.

From this definition, we can state the following version of Nielsen's theorem in the context of infinite-dimensional systems.

Proposition 19. *Let $|\psi\rangle$ and $|\phi\rangle$ be bipartite pure states belonging to $\mathcal{H}_A \otimes \mathcal{H}_B$. Then, $|\psi\rangle \underset{\delta-\mathrm{LOCC}}{\longrightarrow} |\phi\rangle$ if and only if $\psi \preceq \phi$, where ψ and ϕ are the sequences formed by the squared Schmidt coefficients of $|\psi\rangle$ and $|\phi\rangle$, respectively.*

Proof. $(\Longrightarrow)$
Based on the hypothesis that $|\psi\rangle \underset{\delta-\mathrm{LOCC}}{\longrightarrow} |\phi\rangle$, for any $\delta > 0$, there exist states $|\psi_\delta\rangle$ and $|\phi_\delta\rangle$ with sequences of Schmidt coefficients of finite support, such that $\||\psi\rangle\langle\psi| - |\psi_\delta\rangle\langle\psi_\delta|\|_{\mathrm{tr}} < \delta$ and $\||\phi\rangle\langle\phi| - |\phi_\delta\rangle\langle\phi_\delta|\|_{\mathrm{tr}} < \delta$ and there exists a LOCC operation Λ_δ such that $|\phi_\delta\rangle\langle\phi_\delta| = \Lambda_\delta(|\psi_\delta\rangle\langle\psi_\delta|)$.

Furthermore, there exist states $|\psi'_\delta\rangle$ and $|\phi'_\delta\rangle$ that share the same Schmidt coefficients as $|\psi_\delta\rangle$ and $|\phi_\delta\rangle$, respectively, but have the same Schmidt orthonormal basis as $|\psi\rangle$ and $|\phi\rangle$, respectively.

Let $\psi, \phi \in \Delta^{\downarrow}_{\infty}$ be the squared Schmidt coefficients of $|\psi\rangle$ and $|\phi\rangle$, respectively, and $\psi', \phi' \in \Delta'^{\downarrow}_{\infty}$ be the squared Schmidt coefficients of $|\psi_\delta\rangle$ (or $|\psi'_\delta\rangle$) and $|\phi_\delta\rangle$ (or $|\phi'_\delta\rangle$), respectively.

Then, by proposition 17, we have that

$$d_{\mathrm{tr}}(\psi, \psi') \leqslant \||\psi\rangle\langle\psi| - |\psi\rangle\langle\psi|_\delta\|_{\mathrm{tr}} < \delta, \quad d_{\mathrm{tr}}(\phi, \phi') \leqslant \||\phi\rangle\langle\phi| - |\phi\rangle\langle\phi|_\delta\|_{\mathrm{tr}} < \delta$$

and, by Nielsen's Theorem, $\psi' \preceq \phi'$. Finally, by proposition 12, we obtain $\psi \preceq \phi$.

$(\Longleftarrow)$ Let $|\psi\rangle$ and $|\phi\rangle$ such that $\psi \preceq \phi$. Then, from proposition 11, for any $\delta > 0$, we can obtain states $|\psi'\rangle$ and $|\phi'\rangle$ with $\psi', \phi' \in \Delta'^{\downarrow}_{\infty}$ such that $\psi' \preceq \phi'$ (hence $|\psi'\rangle \underset{\mathrm{LOCC}}{\rightarrow} |\phi'\rangle$ by Nielsen's theorem), and $\||\psi\rangle\langle\psi| - |\psi'\rangle\langle\psi'|\|_{\mathrm{tr}} = d_{\mathrm{tr}}(\psi, \psi') < \delta$ and $\||\phi\rangle\langle\phi| - |\phi'\rangle\langle\phi'|\|_{\mathrm{tr}} = d_{\mathrm{tr}}(\phi, \phi') < \delta$. Therefore, $|\psi\rangle \underset{\delta-\mathrm{LOCC}}{\longrightarrow} |\phi\rangle$. □

As a corollary, we obtain that ϵ-LOCC and δ-LOCC are equivalent notions:

Corollary 20. *Given two bipartite pure states $|\psi\rangle$ and $|\phi\rangle$, belonging to $\mathcal{H}_A \otimes \mathcal{H}_B$, the following three statements are equivalent:*

- $|\psi\rangle \underset{\delta-\mathrm{LOCC}}{\longrightarrow} |\phi\rangle$
- $|\psi\rangle \underset{\epsilon-\mathrm{LOCC}}{\longrightarrow} |\phi\rangle$
- $\psi \preceq \phi$

where ψ and ϕ are the sequences formed by the squared Schmidt coefficients of $|\psi\rangle$ and $|\phi\rangle$, respectively.

4. Optimal common resource and optimal common product

We have already studied the convertibility between infinite-dimensional entangled states via LOCC, and we have seen how this operation is subject to a majorization relationship between the sequences of Schmidt coefficients. We now introduce the notions of optimal common resource and optimal common product. Both concepts are related to the completeness of the majorization lattice, i.e. on the ability to define supremum and infimum elements for any subset of sequences. We will formulate these definitions in terms of δ-LOCC convertibility, but they can be formulated in an equivalent way in the ϵ-LOCC setting.

4.1. Optimal common resource

First, let us introduce the definitions of common resource and optimal common resource.

Definition 21. Let $\mathcal{P}$ be an arbitrary set of bipartite pure states in $\mathcal{H}_A \otimes \mathcal{H}_B$. The state $|\psi^{\mathrm{cr}}\rangle$ is said to be a common resource of $\mathcal{P}$, if

$$|\psi^{\mathrm{cr}}\rangle \underset{\delta-\mathrm{LOCC}}{\longrightarrow} |\phi\rangle \quad \forall |\phi\rangle \in \mathcal{P}. \tag{9}$$

Moreover, the state $|\psi^{\mathrm{ocr}}\rangle$ is said to be an optimal common resource of $\mathcal{P}$, if $|\psi^{\mathrm{ocr}}\rangle$ is a common resource and for any other common resource $|\psi^{\mathrm{cr}}\rangle$, one has

$$|\psi^{\mathrm{cr}}\rangle \underset{\delta-\mathrm{LOCC}}{\longrightarrow} |\psi^{\mathrm{ocr}}\rangle. \tag{10}$$

For finite-dimensional $\mathcal{P}$, there always exists an optimal common resource. Unlike the finite-dimensional case, the existence of an optimal common resource for infinite-dimensional systems is conditioned to the existence of a common resource. At the same time, this is subject to the completeness of the lattice of sequences discussed in proposition 6. More precisely,

Proposition 22. *Let $\mathcal{P}$ be an arbitrary set of bipartite pure states in $\mathcal{H}_A \otimes \mathcal{H}_B$. Then, if there exists a common resource $|\psi^{\mathrm{cr}}\rangle$ of $\mathcal{P}$, there also exists an optimal common resource $|\psi^{\mathrm{ocr}}\rangle$ of $\mathcal{P}$.*

Proof. Let $|\psi^{\mathrm{cr}}\rangle$ be a common resource for the set $\mathcal{P}$, as in definition 21. In that case, proposition 19 says that $\psi^{\mathrm{cr}} \preceq \phi \forall |\phi\rangle \in \mathcal{P}$, with ψ^{cr}, ϕ the corresponding sequences of Schmidt coefficients. Thus, ψ^{cr} is a lower bound for all the considered sequences and, by lemma 5, there exists an infimum. That infimum gives us the sequence of Schmidt coefficients associated with the optimal common resource $|\psi^{\mathrm{ocr}}\rangle$. □

Let us see two examples, where in the first an optimal common resource does not exit whereas in the second it does. In particular, the next example was introduced in [13] in the context of the Gaussian channel minimum entropy conjecture. Here, we use this example to illustrate two sets of bipartite pure states that do not admit an optimal common resource.

Example 23. Let a two-mode squeezer of parameter r, that is, $U(r) = \exp\left[r(ab - a^\dagger b^\dagger)\right]$ where $a, b, a^\dagger$ and $b^\dagger$ are the creation and annihilation operator of the inputs modes, respectively. The action of the two-mode squeezer over the input state $|k\rangle|0\rangle$ can be expressed in the Schmidt decomposition as $|\psi_\lambda^{(k)}\rangle = U(r)|k\rangle|0\rangle = \sum_{n=0}^{\infty} \sqrt{x_n^{(k)}(\lambda)}|n+k\rangle|n\rangle$ with $x_n^{(k)}(\lambda)$ given by equation (2) and $\lambda = \tanh r$ [13]. Let consider the sets $\left\{|\psi_\lambda^{(k)}\rangle\right\}_{k\in\mathbb{N}_0}$ and $\left\{|\psi_\lambda^{(k)}\rangle\right\}_{\lambda\in(0,1)}$, which have the peculiarities that $|\psi_\lambda^{(k+1)}\rangle \underset{\delta-\mathrm{LOCC}}{\longrightarrow} |\psi_\lambda^{(k)}\rangle$ and $|\psi_\lambda^{(k)}\rangle \underset{\delta-\mathrm{LOCC}}{\longrightarrow} |\psi_{\lambda'}^{(k)}\rangle$ for $\lambda' \leqslant \lambda$. The corresponding sets of sequences of Schmidt coefficients were studied in example 7, in which we showed they do not have infima. Then, optimal common resources for these sets do not exist.

The following example was introduced in [19] in order to show that the entropy of entanglement for infinite-dimensional quantum systems is not necessarily continuous in the trace-norm. We use this example in order to illustrate the case of a set of bipartite pure sates having an optimal common resource.

Example 24. Let consider the set of bipartite pure sates $\left\{|\psi^{(k)}\rangle\right\}_{k\in\mathbb{N}\geqslant 3}$, where $|\psi^{(k)}\rangle = \sum_{n=1}^{k+1}\sqrt{x_n^{(k)}}|a_n\rangle|b_n\rangle$ with $x^{(k)}$ given by equation (3). In particular, we have that all the states are not LOCC convertible to each other, that is, $|\psi^{(k)}\rangle \underset{\delta-\mathrm{LOCC}}{\not\longleftrightarrow} |\psi^{(k')}\rangle$ for all $k \neq k'$. However, an optimal common resource of the set $\left\{|\psi^{(k)}\rangle\right\}_{k\in\mathbb{N}\geqslant 3}$ exists and its Schmidt coefficients can be computed algorithmically as shown in example 8.

4.2. Optimal common product

We introduce now the notion of a common product and the optimal common product of a set of states.

Definition 25. Let $\mathcal{P}$ an arbitrary set of bipartite pure states in $\mathcal{H}_A \otimes \mathcal{H}_B$. The state $|\psi^{\mathrm{cp}}\rangle$ is said to be a common product of $\mathcal{P}$, if

$$|\phi\rangle \underset{\delta-\mathrm{LOCC}}{\longrightarrow} |\psi^{\mathrm{cp}}\rangle \ \forall |\phi\rangle \in \mathcal{P}. \tag{11}$$

Moreover, the state $|\psi^{\mathrm{ocp}}\rangle$ is said to be an optimal common product of $\mathcal{P}$, if $|\psi^{\mathrm{ocp}}\rangle$ is a common product and for any other common product $|\psi^{\mathrm{cp}}\rangle$, one has

$$|\psi^{\mathrm{ocp}}\rangle \underset{\delta-\mathrm{LOCC}}{\longrightarrow} |\psi^{\mathrm{cp}}\rangle. \tag{12}$$

Just as the common resource problem is associated with the existence of lower bounds in the space of Schmidt sequences, the common product problem is linked to the existence of upper bounds. In that sense, given that the majorization lattice is $\bigvee$-complete, there always exists an optimal common product.

Proposition 26. *Let $\mathcal{P}$ an arbitrary set of bipartite pure states in $\mathcal{H}_A \otimes \mathcal{H}_B$. Then, there exists an optimal common product $|\psi^{\mathrm{ocp}}\rangle$ of $\mathcal{P}$.*

Proof. It follows directly from proposition 6, noting that there always exists a supremum for the corresponding set of sequences of squared Schmidt coefficients. □

Reviewing the examples 23 and 24 just discussed, it is evident that in both cases there exist the so-called optimal common products, whose Schmidt coefficients are determined by the suprema outlined in examples 7 and 8.

5. Concluding remarks

In conclusion, this article delves into the intricacies of infinite-dimensional systems, specifically focusing on the convertibility of entangled states through LOCC. In particular, we have introduced a new definition of LOCC convertibility for infinite-dimensional systems, termed δ-LOCC *convertibility*, which is fully characterized by a majorization relation between sequences of squared Schmidt coefficients and proves to be equivalent to ϵ-LOCC convertibility. Notably, this definition offers the mathematical advantage of dealing with finitely supported sequences.

Moreover, we have explored the LOCC convertibility problem in practical situations involving two parties aiming to perform various quantum information tasks using a single entangled state. In these scenarios, the concepts of optimal common resource and optimal common product for a given set of infinite-dimensional target states naturally arise. While the existence of an optimal common product is always guaranteed, an optimal common resource is conditionally dependent to the existence of a common resource, highlighting a novel difference in the entanglement properties between finite and infinite-dimensional systems.

We have leveraged the majorization lattice characterization for infinite sequences to establish these results. This not only contributes to the understanding of the LOCC paradigm in the infinite-dimensional case, but also presents mathematical insights with broader applicability beyond the specific scope of the addressed problem. Moreover, our results can be applied to other majorization-based resource theories. Overall, the exploration of these issues for infinite-dimensional systems enhances our comprehension of the fundamental properties of entanglement and its role as a quantum resource.

Data availability statement

No new data were created or analysed in this study.

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Appendix. Rearrangement inequality

Let us prove an infinite version of the rearrangement inequality (see e.g. [20, 10.2]).

Lemma 27. *Let $\psi, \phi \in \Delta_\infty^\downarrow$ and let $\tau : \mathbb{N} \to \mathbb{N}$ be a bijection. Then,*

$$\sum_{i=1}^{\infty} \psi_i \phi_{\tau(i)} \leqslant \sum_{i=1}^{\infty} \psi_i \phi_i. \tag{A.1}$$

Proof. Let us prove the given inequality for any $N > 0$ and any bijection $\tau : \mathbb{N} \to \mathbb{N}$,

$$\sum_{i=1}^{N} \psi_i \phi_{\tau(i)} \leqslant \sum_{i=1}^{\infty} \psi_i \phi_i.$$

First, we define a bijection τ' on the set $\{0, 1, \ldots, N\}$ such that $\tau'(i) = \tau(i)$ if $\tau(i) \leqslant N$, and otherwise $\tau'(i)$ takes any unused value from the set. From the finite case [20, 10.2], we have

$$\sum_{i=1}^{N} \psi_i \phi_{\tau'(i)} \leqslant \sum_{i=1}^{N} \psi_i \phi_i. \tag{A.2}$$

By construction, $\phi_{\tau(i)} \leqslant \phi_{\tau'(i)}$ since $\tau(i) \geqslant \tau'(i)$ and ϕ is a non-increasing sequence.

Therefore, combining both equations gives us

$$\sum_{i=1}^{N} \psi_i \phi_{\tau(i)} \leqslant \sum_{i=1}^{N} \psi_i \phi_{\tau'(i)} \leqslant \sum_{i=1}^{N} \psi_i \phi_i \leqslant \sum_{i=1}^{\infty} \psi_i \phi_i. \tag{A.3}$$

□

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