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# Effective Young's modulus of a unidirectional fiber reinforced composite: a mean Green operator interpretation

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**Abstract** Experimental measurements of the effective Young's modulus of an unidirectional composite obtained by embedding 3D-printed PLA–CF fibers in a cold-curing polyurethane elastomeric matrix are presented. Specimens with a controlled number of aligned fibers were tested in uniaxial compression along the direction of alignment. The force-displacement records were processed with a routine that identifies the linear loading regime, converts it to stress-strain through the specimen geometry, and returns the longitudinal modulus together with a combined uncertainty built from the repeatability of multiple tests and from the scatter of the experimental curve. The measured modulus increases monotonically, on average, with the fiber volume fraction, yet remains several orders of magnitude below the modulus of the fiber material. Finally, the data are interpreted within a mean Green-operator (mGo) based Ponte-Castañeda and Willis homogenization scheme, which provides the full anisotropic effective moduli.

**Keywords** Composite materials · Fiber reinforced matrix · Homogenization · Elasticity

## 1 Introduction

The mechanics of composite media constitutes one of the most consolidated fields of the wider panorama of microstructured continua. Moreover, it has progressively migrated toward different domains, from architected media [1–4] to biological tissues [5,6]. Generally, composites are constituted by stiff and slender inclusions, i.e. glass, carbon or aramid fibers, immersed in a matrix which, albeit more compliant, possesses a rigidity of comparable order of magnitude. In such a situation the longitudinal response is described with excellent approximation by the rule of mixtures, whereas the transverse and the shear responses are furnished by the classical micromechanical estimates of Eshelby [7], Hashin-Shtrikman [8], Mori-Tanaka [9] or self-consistent

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type. The predictive capacity of these estimates in the regime of moderate contrast is, by now, out of discussion, and it has conferred upon the volume fraction the role of principal descriptor of the microstructure.

Considerably less attention has been dedicated to the antithetic regime, in which very stiff fibers are immersed in an extremely compliant, rubber-like matrix. Here the contrast between the elastic moduli of the two constituents attains three or four orders of magnitude; the matrix, quasi-incompressible and of exiguous shear modulus, offers to the reinforcement a scarce lateral confinement; and phenomena that in a stiff matrix are legitimately neglected become fundamental in the mechanical response: the instability of the slender inclusions, the finite amplitude of the strains attained already under moderate loads and, above all, the circumstance that the effective moduli are dictated by the geometrical disposition of the reinforcement, by its architecture, no less than by its volumetric abundance [10–12]. In this regime the volume fraction  $f$ , taken in isolation, forfeits its status of sufficient descriptor: two composites possessing the same  $f$ , but a different geometric arrangement of the very same fibers, may exhibit effective responses differing by a conspicuous factor. This aspect, which is natural in the field of metamaterials, where architecture (or microstructure) is conceived *ad hoc* in order to obtain a macroscopic behavior that no natural material exhibits [13–15], becomes relevant also in the field of composites such that we could even talk of *metacomposites*.

In these media the desideratum is not the maximal stiffness, but a deformability which is tailored, frequently strongly anisotropic and, in the most fortunate cases, programmable [16, 17]. In this field the classical Cauchy continuum manifests its insufficiency: whenever the microstructure possesses an intrinsic length which is not negligible with respect to the dimension of the specimen, or whenever the interaction among the microscopic constituents is transmitted over distances commensurate with them, the homogenized description must be enriched, and one is naturally conducted toward generalized continua and, in particular, toward second-gradient models [18–22], as it was already clear to P. Germain [23, 24].

From the experimental point of view, the advent of fused-deposition additive manufacturing has rendered accessible the fabrication of fibrous architectures of controlled geometry, spacing and number, and therefore the realization of model metacomposites in which the content of reinforcement may be varied in a systematic and reproducible manner [25–29]. This opens the way to a clean experimental exploration of the manner in which the effective elastic response evolves with the fiber content and, above all, to a direct confrontation with the predictions of micromechanical homogenization.

Among the theoretical instruments available for such a confrontation we have elected the one founded upon the Green operators of the infinite elastic medium [30–33]. The Green tensor, and the modified (symmetrized) operator deduced from it, permit to express the perturbation of the strain field induced by an inhomogeneity, and their mean over the domain occupied by the inclusion furnishes the mean Green operator (mGo), which governs the totality of the mean-field estimates [34–36]. The interest of this formulation, in the version elaborated for patterns of inclusions, resides in a multiplicative dissociation: once transported into Fourier space, the operator factorizes into a part depending exclusively upon the elasticity of the matrix and a part depending exclusively upon the morphology of the pattern [37]. The interactions among the inclusions belonging to one and the same pattern are consequently retained exactly, and the restriction to dilute concentrations, which afflicts the classical mean-field schemes, is thereby removed. When the operator so constructed is introduced into the estimate of Ponte Castañeda–Willis [38], which descends, let us recall it, from a variational principle of Hashin–Shtrikman type, one obtains an effective stiffness sensitive not only to the volume fraction, but to the entire architecture of the reinforcement. In this way, the planar alignment of circular cylinders which is studied in the present work may be carried to its term analytically [39].

In the present work, we manufacture unidirectional specimens by embedding different numbers  $N$  of parallel, cylindrical fibers, disposed along a single plane, in a very compliant cold-curing elastomeric matrix, and we measure, by means of uniaxial compression tests performed along the in-plane direction  $x_3$  of the alignment (orthogonal to the fiber axis  $x_2$ ) the effective Young modulus  $E_3$  as a function of the fiber volume fraction  $f$ . From the point of view of the homogenization, the medium is biphasic: the printed fibers enter the estimate as homogeneous and isotropic constituent. The constituent materials, their provenance and the fabrication protocol are detailed in Sect. 2. Every specimen has been subjected to ten repeated loading cycles, and their elastic modulus has been identified by an automatic procedure which individuates the linear branch of the response and evaluates the associated uncertainty.

The paper is organized as follows. Section 2 describes the specimens, the materials and the methodology of measurement and of analysis of the data; Sect. 3 presents the measured moduli and the onset of the instability; Sect. 4 develops the homogenization of the planar fiber alignment by means of the mean Green operator and of the Ponte Castañeda–Willis estimate; Sect. 5 confronts the analytical prediction with the experimental data; Sect. 6 draws the conclusions and delineates the perspectives that the present investigation opens.

## 2 Materials and methods

We adopt a Cartesian frame  $(x_1, x_2, x_3)$  attached to the specimen, sketched in Fig. 1. The fibers are straight circular cylinders of radius  $r = 1$  mm and length  $\ell = 50$  mm, with their axis aligned along  $x_2$ . They are arranged parallel to one another and equally spaced along  $x_3$ , so that all fiber axes lie in a single  $x_2$ – $x_3$  plane; the  $x_1$  axis is orthogonal to that plane. The fibers are embedded in a rectangular parallelepiped of matrix of dimensions  $6$  mm ( $x_1$ )  $\times$   $50$  mm ( $x_2$ )  $\times$   $50$  mm ( $x_3$ ), i.e. a thin plate of thickness  $6$  mm normal to  $x_1$ .

Because the reinforcement is confined to a single in-plane layer, of dimensions  $2R$  ( $x_1$ )  $\times$   $\ell$  ( $x_2$ )  $\times$   $\ell$  ( $x_3$ ), we define the fiber volume fraction  $f$  as the ratio of the total fiber volume to the volume of the *reinforced strip* that contains them, and not to the volume of the whole specimen. With  $N$  fibers of cross-sectional area  $\pi R^2$  and length  $\ell$ , and a reinforced strip of thickness equal to the fiber diameter  $2R$  and width equal to  $\ell$ ,

$$f = \frac{N \pi R^2 \ell}{(2R) \ell^2} = \frac{N \pi R}{2\ell}. \quad (1)$$

For  $R = 1$  mm (that is, a fiber diameter  $d = 2R = 2$  mm) and  $\ell = 50$  mm this reduces to the working definition used throughout the paper,

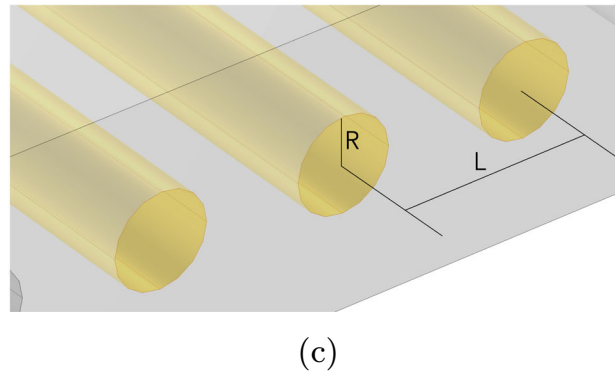
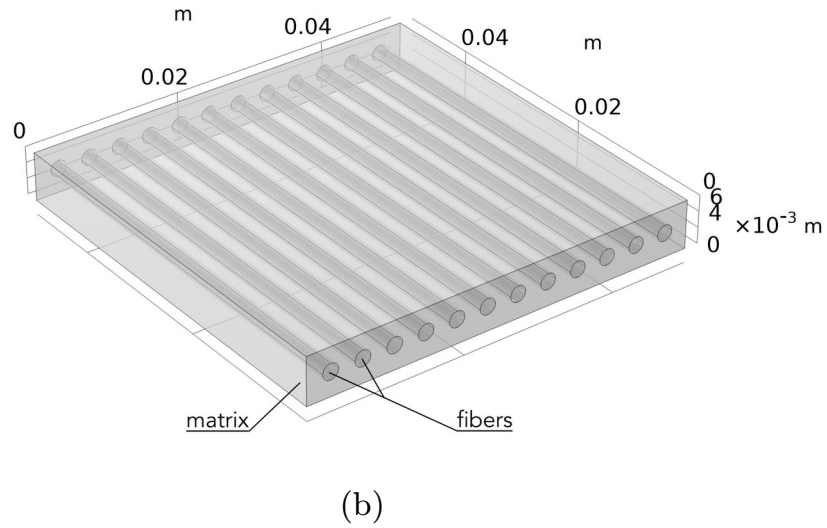
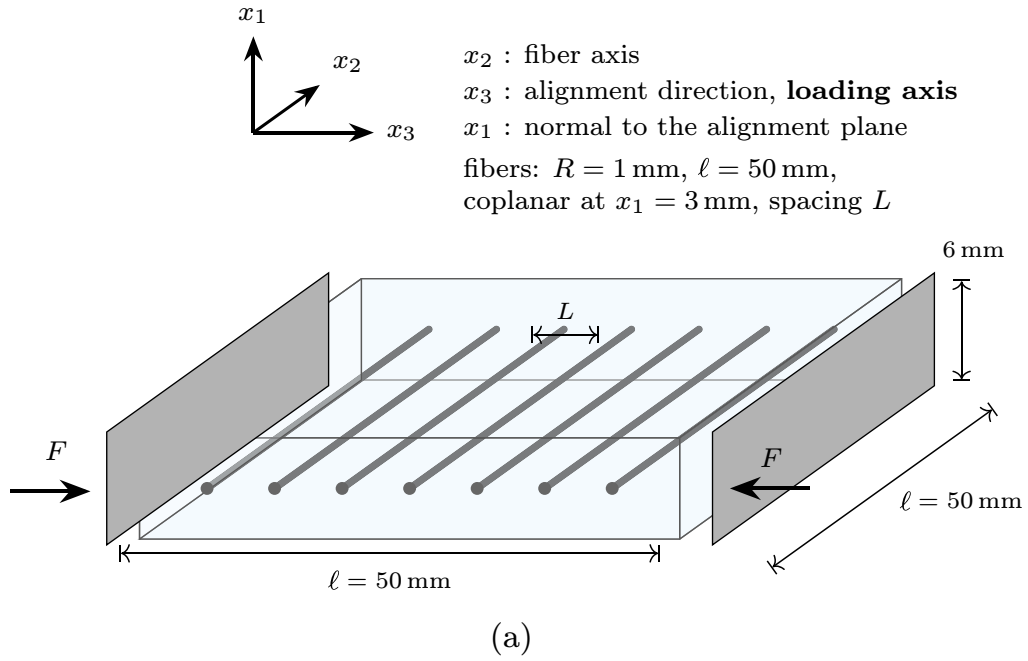
$$f = \frac{\pi N}{100}, \quad (2)$$

so that, e.g.,  $N = 8$  gives  $f \simeq 0.2513$  and  $N = 21$  gives  $f \simeq 0.6597$ . If the same fiber volume had been referred to the entire cross-section of the specimen,  $A = 6$  mm  $\times$   $50$  mm  $= 300$  mm<sup>2</sup>, one would have obtained instead  $f_{\text{tot}} = N \pi R^2 / A = \pi N / 300$ , that is exactly one third of (2), the ratio being the quotient  $6$  mm/ $2R = 3$  between the thickness of the specimen and that of the reinforced layer. The two conventions are therefore interchangeable,  $f = 3f_{\text{tot}}$ , and we adopt the former for a mechanical reason: the medium described by the homogenization estimate of Sect. 4 is the reinforced layer, whereas the two fiber-free skins of  $2$  mm which flank it are an appendage of the casting, mechanically in series with that layer and containing no reinforcement whatsoever. Including them would divide  $f$  by three without altering the physics, thus translating the prediction along an abscissa which does not govern the response. This point is resumed, and its consequences on the identification of the constituent moduli examined, in Sect. 5.1.

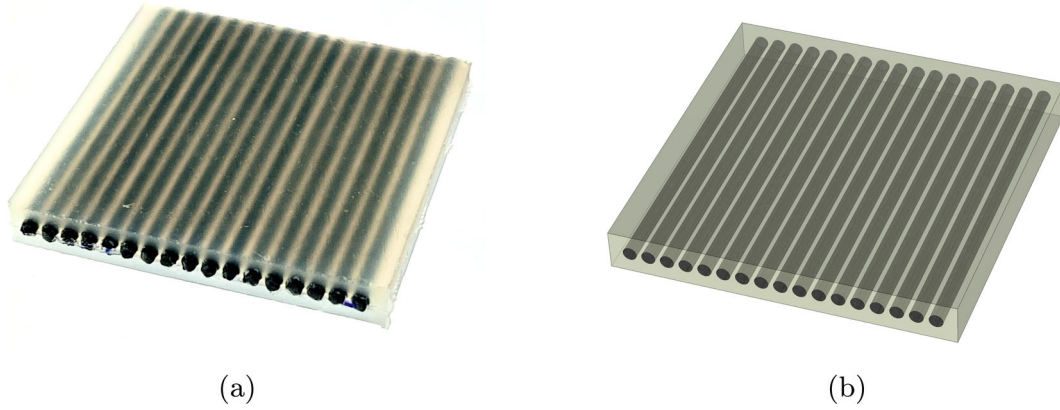
**Materials and fabrication** The fibers were produced by fused-deposition modelling from a carbon-fiber-reinforced polylactic-acid filament (PLA–CF, Bambu Lab, nominal filament diameter  $1.75$  mm), extruded on a Bambu Lab X1 Carbon machine. The rods were printed with an infill density of  $100\%$ , solidity being here a requirement, and with their axis orthogonal to the build plate. The matrix material is SINTAGOM 703/30 (Prochim), a two-component cold-curing polyurethane resin which polymerizes at room temperature to give a soft, rubber-like elastomer, dimensionally stable and curing essentially without shrinkage. Specimens were obtained by arranging the printed rods in the desired parallel configuration within a mould and casting the liquid resin around them, demoulding after cure; see Fig. 2. The number of fibers was varied from  $N = 0$ , the unreinforced matrix used as a reference, up to  $N = 21$ . The fibers are straight circular cylinders of radius  $R = 1$  mm, that is of diameter  $d = 2R = 2$  mm, and of length  $\ell = 50$  mm.

**Mechanical testing** Each specimen was subjected to uniaxial *compression* applied along the in-plane alignment axis  $x_3$ , orthogonal to the fiber axis  $x_2$ , under displacement control, the reaction force and the prescribed displacement being recorded synchronously. The loaded cross-section is the  $x_1$ – $x_2$  face, of nominal area  $A = 6$  mm  $\times$   $50$  mm  $= 300$  mm<sup>2</sup>, and the gauge length is  $\ell = 50$  mm. Tests were performed on a custom-built electromechanical testing machine, designed and assembled in-house.<sup>1</sup> The moving crosshead is driven by an SFU1204 ball screw (nominal diameter  $12$  mm, pitch  $p = 4$  mm/rev, useful travel  $300$  mm) mounted on twin parallel linear guides, and actuated by a NEMA 23 closed-loop stepper motor (STEPPERONLINE, holding torque  $1.2$  Nm) equipped with an integrated  $1000$ -line optical incremental encoder ( $4000$  counts per revolution in quadrature). The motor is driven by a STEPPERONLINE CL57T (V4.1) closed-loop stepper controller configured at  $1600$  pulses per revolution: the crosshead displacement is therefore prescribed with a nominal resolution of  $2.5$   $\mu$ m, the closed-loop control guaranteeing that the motor shaft tracks the commanded position within one encoder count. The reaction force was measured by a  $50$  kg S-beam strain-gauge load cell (nominal sensitivity  $2.0$  mV/V), digitized by a  $24$ -bit HX711  $\Sigma\Delta$  analog-to-digital converter at its intrinsic sampling rate of  $10$  Hz. Prior to the test campaign the load cell was calibrated in situ against a reference

<sup>1</sup> The testing machine was built by the authors GS and MS.



**Fig. 1** Reference frame and geometry of the specimen (a): the fiber axis is  $x_2$ , the fibers are coplanar and equally spaced along  $x_3$ , which is also the direction of the applied compression, and  $x_1$  is normal to the alignment plane. Geometry of the sample (b) and a particular of it (c), where the interdistance between the fibers is shown



**Fig. 2** One of the realized specimens (a), where  $N = 17$ , and its implementation in CAD (b) for successive numerical analyses

mass weighed on a precision balance with a resolution of 0.01 g. Motion control and synchronous acquisition of force and prescribed displacement are handled by an ESP32 microcontroller running dedicated firmware, which streams the samples to a host PC at 20 Hz over USB serial for storage and post-processing. The specimen being extremely compliant, its axial stiffness  $K = E_3 A / \ell$  ranges from about 6 N/mm for the unreinforced matrix to about 24 N/mm at  $N = 21$ , against a frame stiffness of the order of  $10^4$  N/mm: the error committed in identifying the crosshead displacement with the shortening of the specimen is thus below 0.1 %, two orders of magnitude smaller than the uncertainty of Eq. (5), and no compliance correction has been applied.

For each value of  $N$  a single specimen was subjected to ten successive loading cycles, every cycle being confined to the branch preceding the load peak and therefore to the recoverable range, the specimen returning to its initial configuration between one cycle and the next. The ten repetitions serve to assess the repeatability of the measurement; the absence of a monotone drift in the ten identified moduli excludes a progressive degradation of the specimen. The results of the tests are reported in Fig. 3.

**Data analysis and identification of the modulus** The force vs displacement curves were processed with a dedicated, fully automatic routine written in Wolfram Mathematica in the linear regime. The routine denoises the signal and evaluates the slope along the curve. The slope is obtained by least-squares fitting in the stress-strain plane, where the engineering stress and strain are:

$$\sigma = \frac{F}{A}, \quad \varepsilon = \frac{\delta}{\ell}, \quad (3)$$

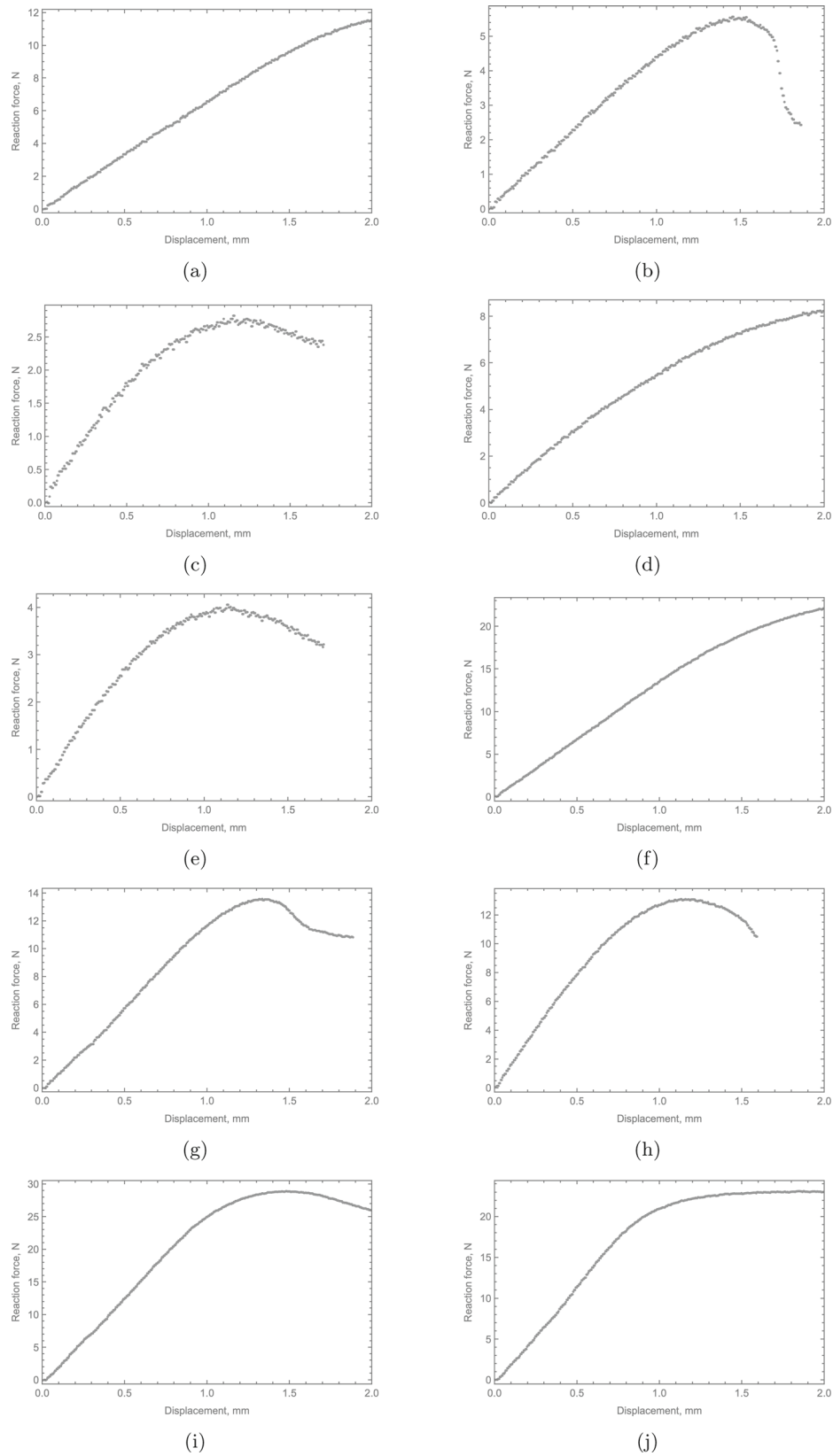
where  $F$  is the measured reaction force and  $\delta$  the value of the prescribed displacement, so that the Young's modulus of a branch is the slope  $E = d\sigma/d\varepsilon = (\ell/A) dF/d\delta$ .

The modulus reported for a specimen is the mean of its  $m$  repeated tests,  $\bar{E} = \frac{1}{m} \sum_{k=1}^m E^{(k)}$ , and its standard uncertainty combines two independent contributions. The first is a *repeatability* (between-tests) term, quantifying the specimen-to-specimen scatter of nominally identical tests. It is the standard error of the mean,

$$u_{\text{rep}} = \frac{s}{\sqrt{m}}, \quad s = \sqrt{\frac{1}{m-1} \sum_{k=1}^m (E^{(k)} - \bar{E})^2}, \quad (4)$$

where  $s$  is the sample standard deviation of the  $m$  measured moduli. This is a first evaluation of uncertainty: for  $m = 2$  it reduces to  $u_{\text{rep}} = |E^{(1)} - E^{(2)}|/2$ , while for the present  $m = 10$  tests it rests on a far more reliable estimate of the dispersion and, through the  $1/\sqrt{m}$  factor, yields a correspondingly tighter uncertainty on the mean. The second contribution is a *curve-noise* (within-test) term arising from the scatter of the data points about the fitted line. For each test it is the standard error of the fitted slope,  $\text{SE}(E) = \sigma_{\text{res}} / \sqrt{S_{\varepsilon\varepsilon}}$ , with  $\sigma_{\text{res}} = \sqrt{\text{RSS}/(n-2)}$  the residual standard deviation over the  $n$  points of the branch and  $S_{\varepsilon\varepsilon} = \sum_i (\varepsilon_i - \bar{\varepsilon})^2$ ; the per-test values are propagated to the mean as  $u_{\text{noise}} = \frac{1}{m} \sqrt{\sum_{k=1}^m \text{SE}(E^{(k)})^2}$ . As the two contributions are statistically independent (one capturing between-test variability, the other within-test regression scatter) they are combined in quadrature,

$$u(E) = \sqrt{u_{\text{rep}}^2 + u_{\text{noise}}^2}, \quad (5)$$



**Fig. 3** Measured reaction force vs. prescribed displacement plots for the following specimens defined by the number of embedded fibers: **a** no fibers; **b**  $N = 2$  fibers; **c**  $N = 4$  fibers; **d**  $N = 5$  fibers; **e**  $N = 8$  fibers; **f**  $N = 10$  fibers; **g**  $N = 12$  fibers; **h**  $N = 17$  fibers; **i**  $N = 19$  fibers; **j**  $N = 21$  fibers

**Table 1** Representative effective longitudinal modulus  $E_3$  versus fiber number  $N$  and volume fraction  $f = \pi N/100$ . Uncertainties from Eq. (5)

| $N$ | $f$   | $E_3$ [MPa]       |
|-----|-------|-------------------|
| 0   | 0.00  | $1.055 \pm 0.015$ |
| 2   | 0.063 | $0.715 \pm 0.011$ |
| 4   | 0.126 | $0.537 \pm 0.011$ |
| 5   | 0.157 | $0.997 \pm 0.035$ |
| 8   | 0.251 | $0.853 \pm 0.034$ |
| 10  | 0.314 | $2.18 \pm 0.04$   |
| 12  | 0.377 | $1.958 \pm 0.009$ |
| 17  | 0.534 | $2.49 \pm 0.05$   |
| 19  | 0.597 | $4.44 \pm 0.04$   |
| 21  | 0.660 | $4.061 \pm 0.027$ |

and the modulus is reported as  $\bar{E} \pm u(E)$ .

### 3 Results

#### 3.1 Effective longitudinal modulus

Table 1 reports representative values of the effective longitudinal modulus  $E_3$  as a function of the fiber volume fraction  $f = \pi N/100$ , together with the associated uncertainty; the full set of ten specimens follows the same processing. Two observations stand out.

First, the modulus increases, on average, with the fiber content: across the investigated range the trend is clearly positive (with the steepest rise at the higher fiber counts), while at low fiber counts the values scatter around the unreinforced matrix modulus, of the order of 1 MPa. Adding fibers therefore stiffens the composite in the loading direction, as expected.

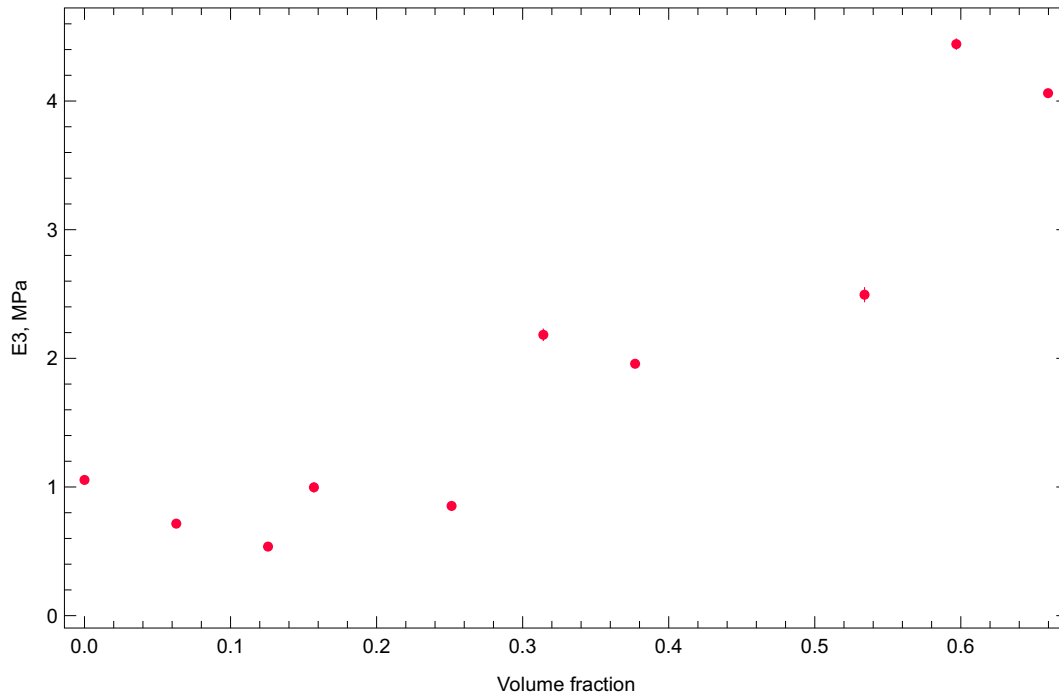
Second, the measured modulus is of the order of a few megapascals, that is of the order of the modulus of the matrix, and remains several orders of magnitude below the modulus of the fiber material (PLA–CF, of the order of  $2160 \pm 90$  MPa according to the manufacturer's data sheet). In fact, being the loading axis  $x_3$  the direction of the alignment and orthogonal to the fiber axis  $x_2$ , the modulus  $E_3$  is therefore a transverse modulus, and the axial rule of mixtures, which would indeed predict a stiffness dominated by the fibers, does not apply to it. The effective transverse modulus consequently remains of the order of the matrix modulus and grows slowly, and convexly, with  $f$ . The interest of the measurement resides not in the magnitude of  $E_3$  but in the manner of its growth, which, as Sect. 4 will show, is dictated by the architecture of the alignment and not by the volume fraction alone.

A third observation concerns the four specimens with  $N = 2, 4, 5, 8$ , whose modulus falls below that of the unreinforced matrix. The depression attains 49 % at  $N = 4$  (0.537 MPa against 1.055 MPa) and 32 % at  $N = 2$ , whereas the uncertainty of Eq. (5) does not exceed a few per cent. This incongruence could be due to different reasons, as, for example the way the specimens are fabricated.

#### 3.2 Onset of buckling

Because the loading is compressive and the specimens are slender in the loading direction, all curves terminate in a load peak followed by softening. We do *not* identify the onset of buckling with the load peak: at the peak the instability is already well developed. Instead we define the buckling onset as the displacement at which the curve first departs from the linear engaged branch, i.e. the end of the stiff linear regime, which precedes the peak. On the settling-free strain scale this yields the critical buckling strain  $\varepsilon_{\text{buck}}$ , and, evaluated on the engaged branch, an estimate of the critical stress  $\sigma_{\text{cr}} \simeq E_3 \varepsilon_{\text{buck}}$ .

The data indicate that the critical stress  $\sigma_{\text{cr}}$  increases with the fiber content, whereas the critical strain  $\varepsilon_{\text{buck}}$  tends to decrease: adding fibers raises the load-carrying capacity at instability but, because the stiffness grows faster, the composite reaches the critical state at a smaller strain (see Fig. 3). This behavior is consistent with the picture of a buckling-dominated soft-matrix composite.



**Fig. 4** Experimentally evaluated elastic modulus  $E_3$  plotted versus volume fraction  $f$ . Remark that error bar is almost of the same dimension of the experimental points and so it is hardly visible

#### 4 Homogenization of a planar alignment of cylindrical fibers

We want now to use the experimental data which have been presented in the previous Sections to validate a model of composite material based on Green operators [30,31,33,34] and on a homogenization scheme similar to that developed by Ponte-Castañeda and Willis [38].

**Hypotheses of the model and their expected influence.** The estimate developed in this Section rests upon the following hypotheses, which we state at the outset together with the sense in which each of them biases the comparison with the measurements of Sect. 3.

- (H1) *Infinitely long cylinders.* The manufactured fibers have  $\ell/2R = 25$ , large but finite; over an end region of the order of the fiber diameter the assumed field is not transmitted.
- (H2) *Infinite planar alignment.* The pair interaction decays as  $i^{-2}$  and  $i^{-4}$ , so that the influence zone extends over some ten to twenty fiber radii and the infinite-alignment operator is an excellent approximation for any longer alignment. The fibers occupying the extremities of the row, however, lack a neighbour on one side, and their relative weight is  $2/N$ : unity at  $N = 2$ , one half at  $N = 4$ , less than one tenth at  $N = 21$ . The finite-size bias is therefore severe at low fiber counts and negligible at high ones.
- (H3) *Perfect bonding at the fiber-matrix interface.*
- (H4) *Circular cross-section.*
- (H5) *Linear elasticity and infinitesimal strains.*
- (H6) *Cauchy continuum.*

Hypotheses (H2) and (H3) are the two which dominate the residual dispersion between prediction and measurement, and they act in the same direction at low  $N$ , where the population of fibers is essentially a population of edges.

##### 4.1 Geometry and material symmetry

We consider an infinite fiber planar alignment (FPA): a set of identical, parallel, infinitely long circular cylinders of radius  $R$  embedded in an isotropic elastic matrix of shear modulus  $\mu_m$  and Poisson ratio  $\nu_m$ . Obviously,

as we have seen in the previous Sections, the real specimens which have been tested are not composed by infinitely long fibers and the alignment itself is not infinitely extended. These assumptions are fundamental for developing the model in a simple way, but cannot be obtained in experiments: one should take into account these discrepancies when interpreting the experimental results.

The orientation convention adopted throughout is the following:

- i. the fiber axis is aligned with  $x_2$ ;
- ii. the fibers are coplanar and regularly spaced along  $x_3$ , with axis-to-axis inter-distance  $L$ ;
- iii. the alignment plane is therefore the  $(x_2, x_3)$  plane, whose unit normal is  $\mathbf{x}_1$ .

The single dimensionless parameter controlling the microstructure is the ratio

$$\rho_0 = \frac{R}{L} \in (0, 1], \quad (6)$$

with  $\rho_0 \rightarrow 0$  the dilute (well-separated) limit and  $\rho_0 = 1$  the contact configuration. The transverse fiber volume fraction is proportional to  $\rho_0$  through the packing geometry.

Because the three coordinate planes  $\{x_1 = 0\}$ ,  $\{x_2 = 0\}$  and  $\{x_3 = 0\}$  are mirror-symmetry planes of the microstructure, the homogenized medium is *orthotropic* with principal material axes  $(x_1, x_2, x_3)$ . The three axes play distinct mechanical roles:  $x_2$  is the (stiff) fiber direction;  $x_3$  is the in-plane transverse direction, along which the fibers interact most strongly (and along which we have tested the specimens); and  $x_1$  is the out-of-plane direction normal to the alignment. This orthotropy fixes, once and for all, which tensor components may be non-zero and, ultimately, that the medium is described by nine independent elastic constants.

#### 4.2 Mean Green operator: separation of matrix and geometry

We briefly recall (see [39] for all the details) the construction of the mean Green operator (mGo) within an infinite homogeneous elastic medium, characterized by the fourth-order elasticity tensor  $\mathbf{C}$ .

Let  $\mathbf{u}(\mathbf{r}) = (u_1, u_2, u_3)$  denote the displacement field at point  $\mathbf{r}$ , induced by a point force  $\mathbf{f}(\mathbf{r}')$  applied at  $\mathbf{r}'$ . The displacement field is expressed through the Green tensor  $G_{pj}$  as

$$u_p(\mathbf{r}) = G_{pj}(\mathbf{r} - \mathbf{r}') f_j(\mathbf{r}'). \quad (7)$$

Since the medium is infinite, the Green tensor depends only on  $\mathbf{r} - \mathbf{r}'$ . Introducing the stress equilibrium equation

$$\sigma_{mn,n}(\mathbf{r}) = \frac{\partial \sigma_{mn}(\mathbf{r})}{\partial x_n} = 0, \quad (8)$$

and using the constitutive relation  $\sigma_{mn} = C_{mnpq} u_{p,q}$ ,<sup>2</sup> one obtains the governing equation for the Green tensor:

$$C_{mnpq} G_{pj,qn}(\mathbf{r} - \mathbf{r}') + \delta(\mathbf{r} - \mathbf{r}') \delta_{mj} = 0. \quad (9)$$

The modified Green operator is defined as the symmetrized second derivative of the Green tensor:

$$\Gamma_{pqjn}(\mathbf{r} - \mathbf{r}') = - \left( \frac{\partial^2 G_{pj}(\mathbf{r} - \mathbf{r}')}{\partial x_q \partial x_n} \right)_{\text{sym}}, \quad (10)$$

where the symmetrization is performed over the index pairs  $(p, q)$  and  $(j, n)$ . The operator associated with an inclusion domain  $V$  is the average

$$t_{pqjn}^V(\mathbf{r}) = \int_V \Gamma_{pqjn}(\mathbf{r} - \mathbf{r}') d\mathbf{r}'. \quad (11)$$

<sup>2</sup> The displacement gradient, rather than the infinitesimal strain, is employed here for convenience. The two writings are equivalent by virtue of the minor symmetry  $C_{mnpq} = C_{mqnp}$ , which we assume throughout and which entails that the antisymmetric part of  $u_{p,q}$ , that is the infinitesimal rotation, does not contribute to the stress. We stress that this places the present construction squarely within the classical Cauchy setting: in a micropolar or Cosserat medium the minor symmetry fails and the antisymmetric part of the gradient acquires a constitutive role.

Passing to Fourier space and introducing the acoustic tensor  $M_{mp}(\boldsymbol{\omega}) = C_{mnpq} \omega_q \omega_n$ , with  $\boldsymbol{\omega} = \mathbf{k}/|\mathbf{k}|$ , the operator factorizes into a purely material part and a purely geometric part,

$$t_{pqjn}^V(\mathbf{r}) = \int_{\Omega} \underbrace{M_{pj}^{-1}(\boldsymbol{\omega}) \omega_q \omega_n}_{t_{pqjn}^e(\boldsymbol{\omega})} \psi_V(\boldsymbol{\omega}, \mathbf{r}) d\boldsymbol{\omega}, \quad (12)$$

where  $t^e(\boldsymbol{\omega})$  is the *elementary operator*, depending only on the matrix elasticity, and  $\psi_V$  is a *shape function* depending only on the geometry of  $V$ . This multiplicative dissociation is the central computational advantage of the Radon-transform formulation: once  $t^e(\boldsymbol{\omega})$  is tabulated for a given matrix symmetry, it is reused for any inclusion pattern by updating  $\psi_V$  alone.

**Matrix part** For an *isotropic* matrix the angular integration of  $t^e(\boldsymbol{\omega})$  over the unit sphere collapses onto two scalar constants,

$$B = \frac{1}{\mu_m}, \quad A = -\frac{B}{2(1-\nu_m)} = -\frac{1}{2\mu_m(1-\nu_m)}, \quad (13)$$

in terms of which every non-vanishing component of the mean Green operator can be written. Thus  $A$  and  $B$  carry *all* the matrix-dependence of the operator.

**Geometric part** The mean shape function is obtained from the sectional area  $s_V(z, \boldsymbol{\omega})$ , i.e. the area of the intersection of  $V$  with the plane  $\boldsymbol{\omega} \cdot \mathbf{r} = z$ ,

$$\bar{\psi}_V(\boldsymbol{\omega}) = \frac{1}{8\pi^2 v} \int_{D_V^-(\boldsymbol{\omega})}^{D_V^+(\boldsymbol{\omega})} \left( \partial_z s_V(z, \boldsymbol{\omega}) \right)^2 dz, \quad (14)$$

with  $v$  the volume of  $V$  and  $D_V^{\pm}$  the support of  $V$  along  $\boldsymbol{\omega}$ . The computation of the shape function  $\bar{\psi}_V(\boldsymbol{\omega})$  is a central point in the determination of the mGo components for different kind of inclusion domains. In this work we are considering an alignment of cylindric fibers and the components on mGo have been already computed.

For a pattern made of several sub-domains  $V = \bigcup_i V_i$ , the mean operator splits into an interior term and a pairwise-interaction term,

$$\bar{\mathbf{t}}^V = \sum_i \frac{v_i}{v} \bar{\mathbf{t}}^{V_i} + \sum_i \sum_{j>i} \frac{v_i + v_j}{v} \bar{\mathbf{t}}^{V_i, V_j}, \quad (15)$$

in which only the interaction term depends on the relative positions of the inclusions.

#### 4.3 The complete operator of the infinite alignment

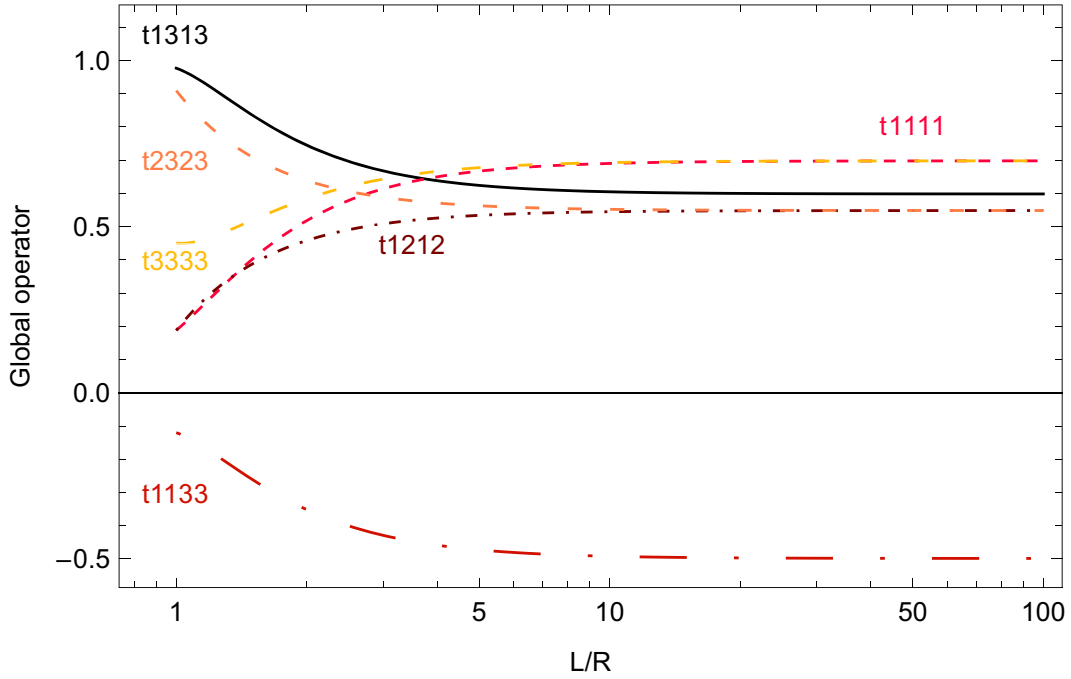
Applying (15) to  $n$  identical parallel cylinders (same shape and orientation, spaced along  $x_3$ ) and using the fact that the pair interaction between two circular cylinders depends on their spacing through even powers of  $\rho_0$  only, the mean operator of the finite alignment reads [39]

$$\bar{\mathbf{t}}_{\rho_0}^{ncyl} = \bar{\mathbf{t}}^{cyl} + 2 \sum_{i=1}^{n-1} \frac{n-i}{n} \left( \frac{\mathbf{v}_0}{i^2} \rho_0^2 + \frac{\mathbf{w}_0}{i^4} \rho_0^4 \right), \quad (16)$$

where  $\bar{\mathbf{t}}^{cyl}$  is the interior operator of a single isolated cylinder, transversely isotropic about the fiber axis  $\mathbf{x}_2$  and independent of  $\rho_0$ , while the constant fourth-order tensors  $\mathbf{v}_0, \mathbf{w}_0$  collect the non-zero components of the cylinder-pair interaction. The  $i^{-2}$  and  $i^{-4}$  weights are the signature of the circular cross-section entering the inverse Radon transform.

In the infinite-alignment limit  $n \rightarrow \infty$  the two lattice sums converge to Riemann zeta values,  $\sum i^{-2} = \zeta(2) = \pi^2/6$  and  $\sum i^{-4} = \zeta(4) = \pi^4/90$ , so that the operator admits the closed form

$$\bar{\mathbf{t}}_{\rho_0}^{\infty cyl} = \bar{\mathbf{t}}^{cyl} + 2 \left[ \zeta(2) \mathbf{v}_0 \rho_0^2 + \zeta(4) \mathbf{w}_0 \rho_0^4 \right] \quad (17)$$



**Fig. 5** Non-zero components of the complete operator vs the inverse of  $\rho_0$ . Remark that when the interactions between the fibers become negligible the components “collapse” to simply 3 values

which we take as the *complete Green operator of the FPA*,  $\bar{\mathbf{t}}^V \equiv \bar{\mathbf{t}}_{\rho_0}^{\infty \text{cyl}}$ , to be inserted into the homogenization estimate of Sect. 4.4.

The expressions of the non-zero components of the global, interaction and complete operator for cylindrical fibers are reported in [37, 39] and here we report their plot versus the inverse of parameter  $\rho_0$  in Fig. 5.

As it can be observed in Fig. 5, Eq. (17) implies the existence of a finite influence zone around each fiber. Because the pair contribution decays as  $i^{-2}$  and  $i^{-4}$ , it becomes negligible beyond roughly ten to twenty fiber radii. This results in the fact that the infinite-alignment operator is an excellent approximation for any finite alignment longer than this critical length, irrespective of the actual fiber count. Moreover, since all interactions inside the influence zone are exactly retained in the mean operator, the dilute concentration restriction typical of mean-field schemes is effectively lifted: the description remains valid up to the maximal admissible fraction set by the no-overlap condition  $\rho_0 \leq 1$ .

It is worth making explicit the consequence of the behavior shown in Fig. 5. When the interaction contribution becomes negligible one has  $t_{3333} = t_{1111}$  and  $t_{1212} = t_{2323}$ : the complete operator has reduced to the interior operator of the isolated cylinder, which is transversely isotropic about the fiber axis  $x_2$ . The homogenized medium then degenerates from orthotropic, with nine independent constants, to transversely isotropic, with five, so that  $E_1 = E_3$  and  $\mu_{1212} = \mu_{2323}$ . In this regime the planar alignment is indistinguishable, as far as the effective moduli are concerned, from any transversely isotropic arrangement of the same volume fraction: the architecture ceases to be a design variable and the estimate reduces to a classical mean-field one.

The threshold admits an immediate experimental translation. For the present geometry the spacing is  $L = \ell/N$ , whence  $\rho_0 = R/L = NR/\ell$ , and comparison with (2) yields the exact relation

$$f = \frac{\pi}{2} \rho_0. \quad (18)$$

The condition  $L/R = 10$ , that is  $\rho_0 = 0.1$ , therefore corresponds to  $f \simeq 0.157$  and to  $N = 5$  exactly. All the specimens with  $N \leq 5$  lie in the non-interacting regime; the interaction terms  $\zeta(2)\rho_0^2$  and  $\zeta(4)\rho_0^4$  become progressively decisive from  $N = 8$  onwards ( $\rho_0 = 0.16$ ) and attain  $\rho_0 = 0.42$  at  $N = 21$ , where the quadratic contribution is some eighteen times its value at  $N = 5$ . It is thus the upper half of the experimental campaign which interrogates the architecture of the reinforcement, the lower half testing in substance the dilute limit, a circumstance which should be borne in mind when reading Fig. 8, and which is not unrelated to the anomalous specimens discussed in Sect. 3.1.

#### 4.4 Effective stiffness: the Ponte Castañeda–Willis estimate

The macroscopic stiffness is obtained by inserting the FPA operator  $\bar{\mathbf{t}}^V = \bar{\mathbf{t}}_{\rho_0}^{\infty \text{cyl}}$  into the Ponte Castañeda–Willis (PCW) estimate [38,39]. For a single embedded phase (the fibers) of stiffness  $\mathbf{C}_f$  and volume fraction  $f_f$ ,

$$\mathbf{C}_{\text{eff}} = \mathbf{C}_M - f_f \left[ (\mathbf{C}_M - \mathbf{C}_f)^{-1} - \bar{\mathbf{t}}_{\mathbf{C}_M}^V + f_f \mathbf{t}_{\mathbf{C}_M}^{\text{SDist}} \right]^{-1}. \quad (19)$$

The estimate involves two distinct geometric operators, and this separation is what distinguishes PCW from the Mori–Tanaka scheme. In fact, (i)  $\bar{\mathbf{t}}_{\mathbf{C}_M}^V$  encodes the shape of the reinforcement pattern (here, the planar fiber alignment), while (ii)  $\mathbf{t}_{\mathbf{C}_M}^{\text{SDist}}$  encodes the shape of the spatial distribution of the patterns within the matrix, i.e. the Hill operator of the ellipsoid describing how the fibers are statistically arranged. A statistically isotropic distribution corresponds to the spherical Green operator; a distribution inheriting the FPA symmetry corresponds to a cylindrical one.

#### 4.5 Engineering elastic and shear moduli

Once  $\mathbf{C}_{\text{eff}}$  is known, the effective compliance is its tensor inverse,

$$\mathbf{S}_{\text{eff}} = \mathbf{C}_{\text{eff}}^{-1}. \quad (20)$$

For computing the effective properties of the composite, the values of the shear modulus and Poisson ratio of both matrix and fiber materials have been assumed as the theoretical ones reported on the data sheet of Bambulab PLA-CF for the fibers and of Prochima SINTAGOM 703/30 for the matrix. More precisely, it has been assumed:  $\mu_F = 830$  MPa,  $\nu_F = 0.35$ ;  $\mu_M = 0.206$  MPa,  $\nu_M = 0.45$ . Since the medium is orthotropic with axes  $(x_1, x_2, x_3)$ , all nine constants are in general distinct and are read directly off the diagonal of  $\mathbf{S}_{\text{eff}}$ . In full fourth-order (tensor) notation,

$$E_i = \frac{1}{S_{iiii}^{\text{eff}}} \quad (\text{no sum}), \quad \mu_{ijij} = \frac{1}{4 S_{ijij}^{\text{eff}}} \quad (i \neq j), \quad (21)$$

so that, explicitly, the three Young moduli are

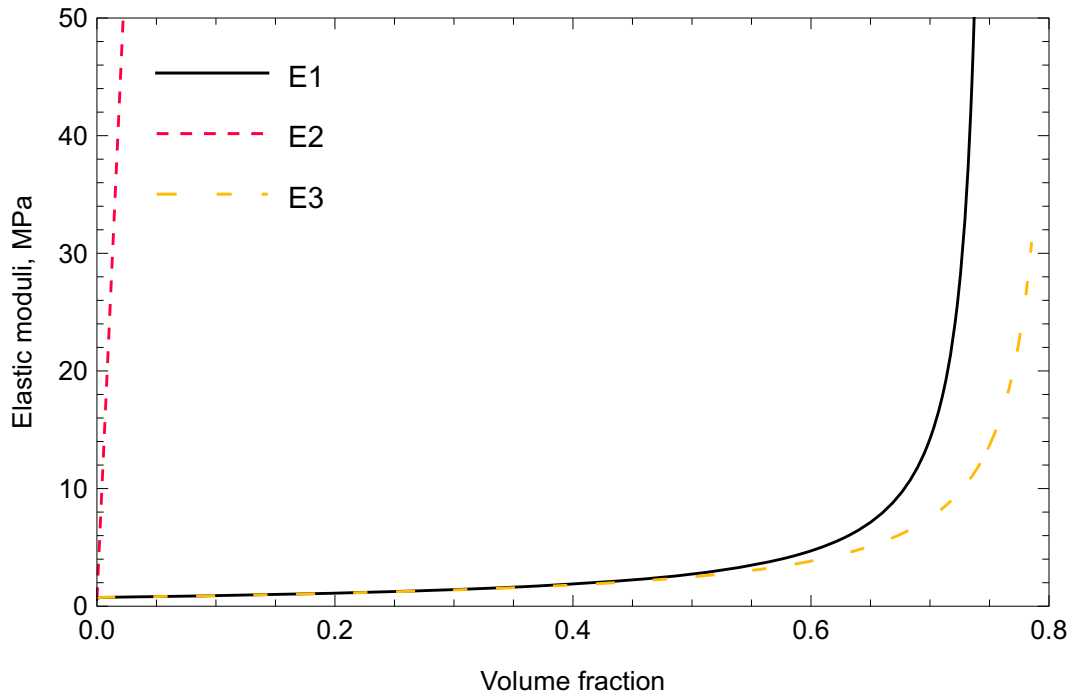
$$E_1 = \frac{1}{S_{1111}^{\text{eff}}}, \quad E_2 = \frac{1}{S_{2222}^{\text{eff}}}, \quad E_3 = \frac{1}{S_{3333}^{\text{eff}}}, \quad (22)$$

with  $E_2$  the fiber-direction (stiff) modulus and  $E_1, E_3$  the two transverse moduli. Their plot versus the fiber volume fraction is reported in Fig. 6. We observe that, as it is expected, the longitudinal (with respect to the fiber direction) elastic modulus  $E_2$  is much stiffer than the other two.

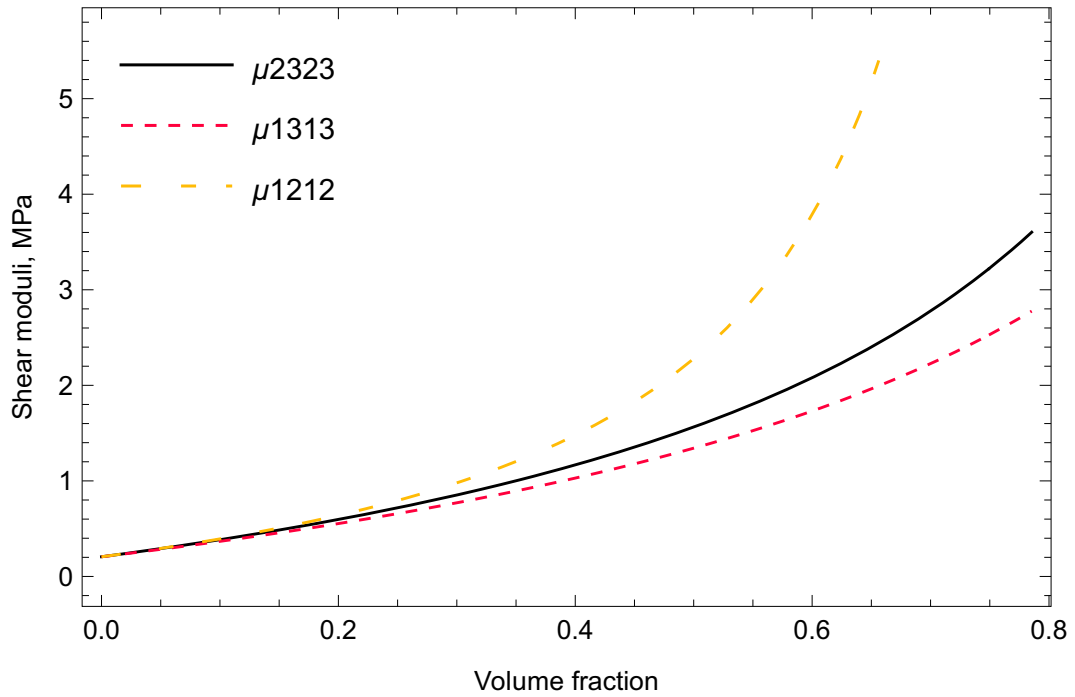
The three shear moduli are

$$\mu_{1212} = \frac{1}{4 S_{1212}^{\text{eff}}}, \quad \mu_{2323} = \frac{1}{4 S_{2323}^{\text{eff}}}, \quad \mu_{1313} = \frac{1}{4 S_{1313}^{\text{eff}}}. \quad (23)$$

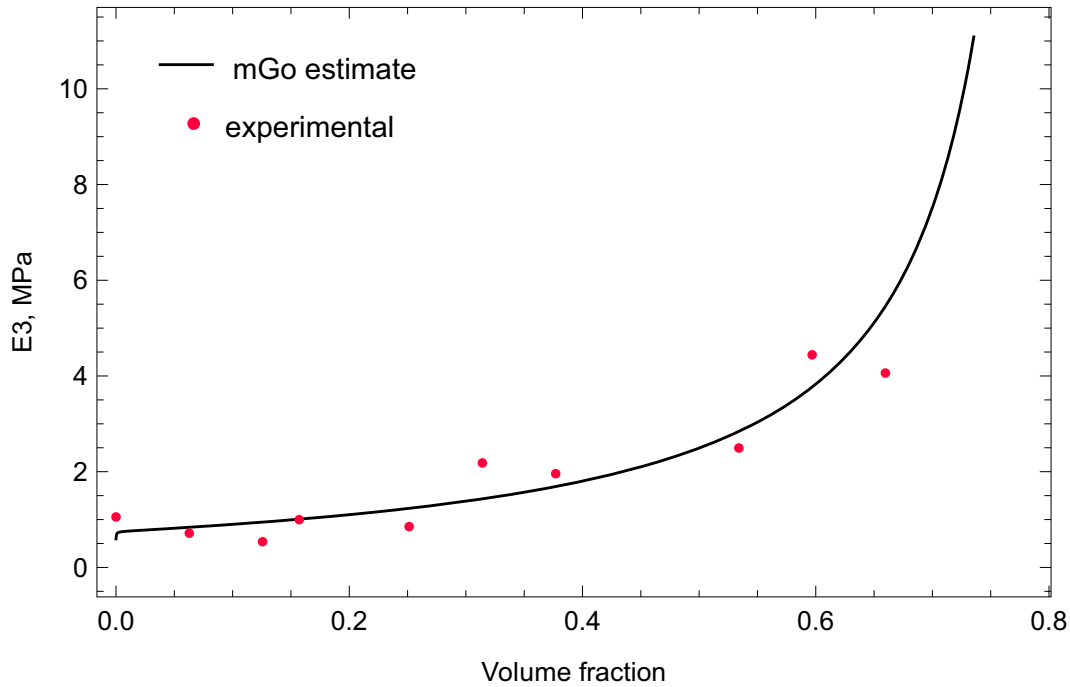
Here  $\mu_{1212}$  and  $\mu_{2323}$  are the two *longitudinal* (fiber-plane) shear moduli, distinct precisely because the interaction breaks the transverse isotropy between  $x_1$  and  $x_3$ , while  $\mu_{1313}$  is the transverse shear modulus in the plane perpendicular to the fibers. The plot of these shear moduli versus the fiber volume fraction is shown in Fig. 7. The Poisson ratios follow from the off-diagonal compliances,  $\nu_{ij} = -S_{ijij}^{\text{eff}}/S_{iiii}^{\text{eff}}$ .



**Fig. 6** Plots of the three elastic moduli  $E_1$ ,  $E_2$  (stiff, in fiber direction) and  $E_3$  versus the fiber volume fraction



**Fig. 7** Plots of the three shear moduli  $\mu_{1212}$ ,  $\mu_{2323}$  and  $\mu_{1313}$  versus the fiber volume fraction



**Fig. 8** Comparison between mGo estimate (black line) and experimentally evaluated elastic modulus  $E_3$  (red points), plotted versus volume fraction  $f$

## 5 Validation of the mGO computed effective modulus

The results presented in the previous two Sections can be compared relatively to elastic modulus  $E_3$ , by superimposing the experimental points shown in Fig. 4 and the analytical curve of Fig. 6 for  $E_3$ . The result is shown in Fig. 8 and it is clear that experimental measures are in good accord with the prevision based on mGo technique.

Figure 8 reports the effective Young's modulus  $E_3$  of the composite as a function of the fiber volume fraction. The solid line is the analytical prediction obtained by inserting the mean Green operator (mGO) of the fiber alignment into the Ponte-Castañeda Willis (PCW) homogenization estimate, whereas the markers are the moduli extracted from the compression tests on the manufactured specimens.

It is worth insisting that the uncertainty of Eq. (5) is a repeatability and not a reproducibility: it measures the fidelity of the measurement performed upon a given specimen, not the variability of the fabrication from one specimen to another. The dispersion of the experimental points about the analytical curve in Fig. 8 is of the latter kind and is consequently not represented by the error bars, which explains why it exceeds them by so wide a margin.

Over the entire range of volume fractions investigated, the analytical estimate follows the experimental data closely: it reproduces both the magnitude of the fiber-induced stiffening and its monotonic growth with volume fraction. This agreement confirms that the mGO/PCW scheme correctly captures the elastic interaction between the reinforcement and the compliant matrix even in the high-contrast regime that characterizes this class of elastomer-based composites, where the fiber-to-matrix stiffness ratio is large. The model, however, rests on two idealizations that also delimit its range of validity. First, the reinforcement is described as a single planar fiber alignment: all fibers lie in one plane and are spaced along  $x_3$ , so the microstructure is only one fiber thick in the out-of-plane direction  $x_1$ . Second, the mean Green operator is evaluated in the infinite-alignment limit, while the physical specimens contain only a finite number of fibers. As a result, the fibers near the ends of the row lack neighbors on one side and, in a thin finite specimen, these boundary regions carry a disproportionate weight. This finite-size, edge-dominated character is the most plausible origin of the residual scatter observed between prediction and measurement. Both limitations are expected to be removed, or, at least, reduced, by replacing the single planar alignment with a fiber bundle, i.e. a genuinely three-dimensional packing of several stacked alignments. In such a configuration the interior fibers are surrounded by matrix and by neighboring fibers on all sides, so that their local environment approaches the representative bulk assumed

by the homogenization estimate. The relative fraction of boundary fibers decreases accordingly, edge effects are attenuated, and a wider and more physically meaningful range of volume fractions becomes accessible, bringing the experimental arrangement closer to the idealized medium underlying the mGO/PCW prediction.

### 5.1 Limitations of the analysis

The interpretation of the experimental results by means of the proposed analytical model is subject to intrinsic limitations associated with the geometry of the manufactured specimens. More specifically, the specimens are characterized by a comparatively high proportion of polyurethane matrix surrounding the reinforcing fibers. In particular, the upper and lower matrix layers each possess a thickness of approximately 2mm, which is equal to the diameter of the reinforcing fibers. Consequently, a significant fraction of the polymeric material is located in regions that do not contain any reinforcement and therefore does not contribute to the fiber-matrix interaction in the same manner as the matrix within the reinforced domain.

From a mechanistic point of view, it would not be appropriate to evaluate the fiber volume fraction by considering the total volume of the polymeric matrix, including these fiber-free outer layers. Such an approach would artificially reduce the estimated fibre volume fraction, yielding a value that is not representative of the actual reinforcement ratio within the mechanically active region of the composite. Since the constitutive model is formulated under the assumption of a homogeneous composite medium, an inaccurate estimation of the fiber volume fraction would propagate directly into the determination of the effective material parameters, thereby compromising the interpretation of the experimental observations.

For this reason, the upper and lower matrix layers were deliberately excluded from the calculation of the fiber volume fraction. This simplification leads to an approximate estimate of the effective reinforcement content, which, although not exact from a purely geometrical perspective, is considered to provide a more faithful representation of the composite architecture governing the mechanical response. The adopted procedure therefore constitutes a rational approximation aimed at reducing the discrepancy between the idealized assumptions of the model and the actual morphology of the specimens.

As a direct consequence of this approximation, it is not appropriate to assume the nominal elastic moduli of either the reinforcing fibers nor the polyurethane matrix as constitutive input parameters. Indeed, the effective stiffness inferred from the experimental response reflects not only the intrinsic mechanical properties of the constituent materials but also the geometric correction introduced through the modified evaluation of the fibre volume fraction. Accordingly, corrected values of the elastic moduli have been adopted, which have been declared in subsection 4.5.

It is worth emphasizing that this approximation does not invalidate the predictive capability of the proposed model. On the contrary, it enables a physically consistent interpretation of the experimental data while preserving the simplicity of the analytical framework. Nevertheless, the present approach highlights the sensitivity of the model to the geometrical definition of the representative volume element and to the precise evaluation of the reinforcement content.

Further investigations will therefore be undertaken to establish a more rigorous methodology for determining the effective fiber volume fraction in specimens characterized by substantial fiber-free matrix regions. Such developments are expected to improve the physical interpretation of the effective constitutive parameters, reduce the reliance on empirical corrections, and ultimately enhance both the predictive accuracy and the generality of the proposed modeling framework.

## 6 Conclusions

We have manufactured, tested and modeled a family of model composites constituted by a single planar alignment of stiff, 3D printed PLA-CF cylinders immersed in a very compliant polyurethane elastomer, and we have followed the evolution of the effective Young modulus  $E_3$ , measured along the direction of the alignment and orthogonal to the fiber axis, as a function of the number  $N$  of fibers, i.e. of the volume fraction  $f = \pi N/100$ , from the unreinforced matrix up to  $f \simeq 0.66$ .

The effective modulus grows with the fiber content, passing from the 1.055(15) MPa of the pure matrix to the 4.06(3) MPa measured at  $f \simeq 0.66$  (Table 1 and Fig. 4), i.e. by a factor of about four. The growth is monotone in the mean but not point by point: the specimens containing few fibers scatter around the modulus of the unreinforced matrix, and some of them are even more compliant than it, an occurrence that

no homogenization scheme is able to predict and which we ascribe to the technology of fabrication (residual porosity of the casting, imperfect adhesion at the fiber, matrix interface, slight misalignment of the printed rods) rather than to the metrology, the uncertainty of Eq. (5) being of the order of the percent. It is worth remarking that, notwithstanding a contrast of stiffness of nearly three orders and a half between the two phases, the composite remains matrix-dominated in the direction of loading.

This effect is correctly described by the mean Green operator. The comparison presented in Fig. 8 shows that the analytic curve of  $E_3$  follows the experimental points over the whole investigated range, reproducing both the order of magnitude and the convex growth with the volume fraction. This accord corroborates the mGo/PCW scheme in a regime of high contrast which lies far outside the one for which the mean-field schemes were originally conceived.

A second, and less obvious, indication comes from the behavior in compression of the specimens. In fact, every curve terminates in a peak of the load followed by softening, and the departure from the linear branch, which we have adopted as the onset of the instability, in preference to the peak, where the phenomenon is by then fully developed, occurs at a critical stress that grows with the fiber content and at a critical strain that, on the contrary, tends to diminish. This shows that the reinforcement augments the load that the specimen sustains before losing its stability, but it simultaneously restricts the window of strain within which the material may be described as linearly elastic. For the conception of soft architected media this datum is at least as important as the value of the modulus itself: the architecture of the reinforcement dictates not only the effective stiffness, but also the boundary of the domain in which such a stiffness is a meaningful notion.

The model, on the other hand, rests upon idealizations which delimit its range of validity and which it is honest to enumerate. The reinforcement is described as a single planar alignment, one fiber thick in the direction  $x_1$ : the specimen is therefore dominated by its own edges, the fibers occupying the extremities of the row lacking a neighbor on one side, and for exiguous  $N$  these boundary fibers constitute the majority of the population. The mean Green operator is moreover evaluated in the limit of the infinite alignment, whereas the manufactured specimens contain from two to twenty-one fibers. To this it must be added that the printed rods possess a section which is a polygon dictated by the deposition path rather than a perfect circle, that their adhesion to the matrix is postulated as perfect, and that the elastic constants of the two phases have been assumed from the technical data sheets: in particular the modulus of the matrix so deduced ( $E_m \simeq 0.6$  MPa, from  $\mu_m = 0.206$  MPa and  $\nu_m = 0.45$ ) is appreciably inferior to the one that we have measured on the unreinforced specimen (1.055(15) MPa). An identification of the matrix constants performed directly upon the specimen with  $N = 0$  would translate the analytic curve upwards and would presumably ameliorate the accord: we consider this the first refinement to be effected, and the residual dispersion between prediction and measurement is to be imputed, in our opinion, to the confluence of this datum with the finite-size, edge-dominated character of the microstructure.

Several perspectives are opened by the present investigation. The most immediate consists in substituting for the single planar alignment a bundle of fibers, i.e. a genuinely three-dimensional packing of several superposed alignments: the interior fibers would then be surrounded by matrix and by neighbors on every side, their local environment approaching the bulk representative environment postulated by the homogenization, the relative weight of the boundary fibers diminishing and a wider range of volume fractions becoming accessible. The machinery of the mean Green operator extends to this configuration without conceptual modification, the pattern being simply redefined. In parallel, the CAD reproduction of the specimens (Fig. 2b) will permit finite element analyses of the actual geometry: these will allow the estimate to be validated outside the range accessible to the experiment, the imperfections of fabrication (defective adhesion, misalignment, porosity) to be quantified, and the critical load to be computed as a bifurcation of the equilibrium, thus closing the circle between model, simulation and measurement. Other numerical approaches can be employed to perform simulations in this respect and obtaining results that can be compared to experiments. For this reason, we may use already developed numerical frameworks as the one presented in [40–44].

A more ambitious perspective concerns the nature itself of the homogenized description. The pair interaction internal to the alignment decays as  $i^{-2}$  and  $i^{-4}$  and therefore possesses a finite, but not negligible, range: an influence zone of some ten fiber radii, which endows the microstructure with an intrinsic length. Whenever the ratio between this length and the macroscopic dimension of the specimen ceases to be exiguous, as it happens in the specimens with few fibers, the first-gradient continuum of Cauchy is no longer an adequate receptacle for the homogenized behavior, and a second-gradient, or more generally a generalized, continuum must be substituted for it, its supplementary constitutive coefficients being inherited from the microstructure [45–48].

Furthermore, future investigations will benefit from the rigorous determination of the extremal values of the effective elastic moduli, as these provide fundamental bounds on the macroscopic mechanical response

of heterogeneous materials. Such information is expected to constitute an essential theoretical basis for the validation and refinement of the proposed model, particularly when extending the analysis to different fiber distributions and material symmetry classes. The significance of these extremal elastic properties has been extensively demonstrated in the following contributions, which establish the theoretical framework for the estimation of effective constitutive parameters [49,50].

The analysis presented in this work can be considered also for applications to the broader panorama of the mechanical characterization of fibrous composites. The experimental protocol here adopted prolongs a consolidated line of investigation, of which the recent study on the flexural performance of flax-fiber laminates constitutes a pertinent exemplum [51]. The conceptual nucleus of our work, moreover, reappears in construction materials as rammed earth reinforced with vegetal fibers, whose mechanical characteristics are dictated by similar mechanisms as between a slender reinforcement and a yielding matrix [52].

We conclude by observing that the number of the fibers, their spacing, their orientation and the superposition of several alignments are, all of them, design variables through which the entire tensor of the effective medium may be programmed. The composite examined here already renders manifest that in this class of materials the architecture is the constitutive parameter. This is precisely the epistemic nucleus of the paradigm of metamaterials [53,54]. The confluence of additive manufacturing, of an analytic homogenization founded upon Green operators and of the variational theory of generalized continua appears to us the most promising route toward the conception of soft architected media.

**Author contributions** G.S. designed and produced the specimens. M.S. and G.S. performed experimental measure. M.S., S.B. and F.S. interpreted results on the basis of the model. M.S. wrote the main manuscript text. F.S. revised the manuscript.

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**Data availability** All data supporting the findings of this study are available within the paper (in the form of the presented plots) and are available upon request to the authors.

#### Declarations

**Conflict of interest** The authors declare no conflict of interest.

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