



# Generalized quantum phase spaces for the $\kappa$ -deformed extended Snyder model

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## ABSTRACT

We describe, in an algebraic way, the  $\kappa$ -deformed extended Snyder models, that depend on three parameters  $\beta$ ,  $\kappa$  and  $\lambda$ , which in a suitable algebra basis are described by the de Sitter algebras  $o(1, N)$ . The commutation relations of the algebra contain a parameter  $\lambda$ , which is used for the calculations of perturbative expansions. For such  $\kappa$ -deformed extended Snyder models we consider the Heisenberg double with dual generalized momenta sector, and provide the respective generalized quantum phase space depending on three parameters mentioned above. Further, we study for these models an alternative Heisenberg double, with the algebra of functions on de Sitter group. In both cases we calculate the formulae for the cross commutation relations between generalized coordinate and momenta sectors, at linear order in  $\lambda$ . We demonstrate that in the commutators of quantum space-time coordinates and momenta of the quantum-deformed Heisenberg algebra the terms generated by  $\kappa$ -deformation are dominating over  $\beta$ -dependent ones for small values of  $\lambda$ .

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## 1. Introduction

The non-commutative (NC) quantum space-times, where the coordinates are elements of NC quantum space-time algebra, have been considered as a tool for the description of quantum gravity (QG) (see, e.g. [1], [2]), which unifies two basic theories in physics: general relativity and quantum mechanics. One of the first examples of quantum space-time was proposed by Snyder already in 1947 [3]. Other quantum space-time models were studied in the 1990's (e.g.  $\kappa$ -deformed relativistic theories [4–7]). Due to the increasing interest in QG, the framework of quantum symmetries described by the theory of quantum groups was developed. The NC structure of space-time at the Planck scale appeared as a way to describe QG at very short distances and led to the description of quantum-mechanical and quantum-deformed relativistic phase spaces. It appeared that these NC structures of quantum space-times and quantum relativistic symmetries are associated with the formalism of Hopf algebras and the case of quantum phase spaces has been generalized to the Hopf algebroids framework (see, e.g. [8], [9]).

In this paper we find the quantum-deformed relativistic phase space for the  $\kappa$ -deformed extended Snyder model [10]<sup>1</sup> which will be obtained by the Heisenberg double construction. The Heisenberg double, which constitutes a Hopf algebroid, is defined within the Hopf algebraic framework and can be considered as providing the way to define quantum-deformed phase spaces in the presence of QG effects.

The  $\kappa$ -deformed Snyder model, proposed firstly in [11], [12], unifies the two perhaps best known models which describe the NC relativistic space-time coordinates  $\hat{x}_i$ <sup>2</sup>:

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<sup>1</sup> We use “ $\kappa$ -extended Snyder model” as a short notation for the  $\lambda$ -dependent  $\kappa$ -deformed extended Snyder model.

<sup>2</sup> N-dimensional Latin indices include zero, which describes time-like dimension, i.e.  $i, j = 0, \dots, N - 1$ .

i) the Snyder model:

$$[\hat{x}_i, \hat{x}_j] = i\beta M_{ij} \tag{1}$$

where  $\beta$  is a parameter with dimension of length square and  $M_{ij}$  are the generators of Lorentz transformations;

ii)  $\kappa$ -Minkowski quantum space-time:

$$[\hat{x}_i, \hat{x}_j] = i(a_i \hat{x}_j - a_j \hat{x}_i) \tag{2}$$

where  $a_i$  is a constant vector multiplied by the parameter  $\kappa^{-1}$  with dimension of length. Both Snyder and  $\kappa$ -deformed Snyder models lead to non-associativity and non-coassociativity [13], [14]. In order to avoid problems with non-associativity, an alternative Snyder model, in which Snyder space is a subspace of a larger non-commutative space was proposed in [14] for the 3 dimensional Euclidean Snyder space, and its generalization to the extended Snyder model was proposed in [16].<sup>3</sup> If one uses Latin indices the algebra describing the  $\lambda$ -dependent extended Snyder model looks as follows [16]

$$[\hat{x}_i, \hat{x}_j] = i\lambda\beta\hat{x}_{ij}, \quad [\hat{x}_{ij}, \hat{x}_{kl}] = i\lambda(\eta_{ik}\hat{x}_{jl} - \eta_{il}\hat{x}_{jk} - \eta_{jk}\hat{x}_{il} + \eta_{jl}\hat{x}_{ik}), \tag{4}$$

$$[\hat{x}_{ij}, \hat{x}_k] = i\lambda(\eta_{ik}\hat{x}_j - \eta_{jk}\hat{x}_i), \tag{5}$$

where  $\hat{x}_i = \sqrt{\beta}\hat{x}_{iN}$ .

Unification of  $\kappa$ -Minkowski and extended Snyder space-times was proposed in [10] generalizing the extended Snyder model [16] by including the  $\kappa$ -Minkowski algebra terms (2) using orthogonal algebra with metric tensor  $g$  [17], [18]. The main step for achieving this unification was introducing particular modification of constant metric tensor instead of Minkowski metric.

In this paper we introduce the generalization of quantum-mechanical phase space corresponding to the  $\kappa$ -extended Snyder model. The phase spaces with NC space-time coordinates, have been already considered for Snyder model and for models covariant under the  $\kappa$ -deformed Poincaré algebra: the generalized phase spaces containing the  $\kappa$ -Minkowski NC space-time were considered in [19–24] and the phase space for the Snyder model in, e.g. [3], [25], [26] as well as for the extended Snyder model in [27]. Quantum space-times and deformed phase spaces for Snyder and Yang models were discussed in [28], [29] together with their extensions to supersymmetric models, see also [30].

Here we propose the description of the corresponding deformed phase spaces for the  $\kappa$ -extended Snyder model. The unification of the extended Snyder and  $\kappa$ -Minkowski models proposed in [10], [31] is realized in the framework of associative and coassociative Hopf algebra, what permits to apply the Heisenberg double construction that is considered in the present paper. The resulting generalized phase spaces offer interesting insights into the  $\kappa$ -extended Snyder model, which is described by the superposition of two quantum deformations with a further parameter  $\lambda$ .

## 2. $\kappa$ -deformed extended Snyder model

Let us recall the algebra corresponding to the  $\kappa$ -extended Snyder model, which was introduced in [10] as the extended unified  $\kappa$ -Minkowski Snyder model. We shall denote it as  $o(1, N; g)$ . It is defined by the following Lie-algebraic set of commutation relations:

$$[\hat{X}_{\mu\nu}, \hat{X}_{\rho\sigma}] = i\lambda(g_{\mu\rho}\hat{X}_{\nu\sigma} - g_{\nu\rho}\hat{X}_{\mu\sigma} + g_{\nu\sigma}\hat{X}_{\mu\rho} - g_{\mu\sigma}\hat{X}_{\nu\rho}). \tag{6}$$

The metric  $g \equiv (g_{\mu\nu})$  has the form:

$$g = \begin{pmatrix} -1 & 0 & \dots & 0 & g_0 \\ 0 & 1 & \dots & 0 & g_1 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 1 & g_{N-1} \\ g_0 & g_1 & \dots & g_{N-1} & g_N \end{pmatrix}$$

with  $\det g = -g_0^2 + \sum_{i=1}^{N-1} g_i^2 - g_N$ . If we rewrite the generators  $\hat{X}_{\mu\nu}$  as  $\hat{X}_{ij}$  and  $\hat{X}_{kN} = \kappa\hat{X}_k$  where  $\kappa$  is a new mass-like parameter, and rewrite the metric as follows

$$g_{ij} = \eta_{ij}, \quad g_{iN} = \kappa a_i, \quad g_{NN} = g_N = \kappa^2\beta$$

the algebra (6) splits to the following set of relations

$$[\hat{X}_i, \hat{X}_j] = i\lambda(a_i\hat{X}_j - a_j\hat{X}_i + \beta\hat{X}_{ij}), \tag{7}$$

$$[\hat{X}_{ij}, \hat{X}_k] = i\lambda(\eta_{ik}\hat{X}_j - \eta_{jk}\hat{X}_i + a_j\hat{X}_{ik} - a_i\hat{X}_{jk}), \tag{8}$$

$$[\hat{X}_{ij}, \hat{X}_{kl}] = i\lambda(\eta_{ik}\hat{X}_{jl} - \eta_{jk}\hat{X}_{il} + \eta_{jl}\hat{X}_{ik} - \eta_{il}\hat{X}_{jk}). \tag{9}$$

<sup>3</sup> The  $N$ -dimensional extended Snyder algebra [14], after introducing the  $\lambda$ -dependence [16], takes the form of an  $N$ -dimensional de Sitter algebra given by  $(\mu = 0, 1 \dots N)$

$$[\hat{x}_{\mu\nu}, \hat{x}_{\rho\sigma}] = i\lambda(\eta_{\mu\rho}\hat{x}_{\nu\sigma} - \eta_{\nu\rho}\hat{x}_{\mu\sigma} + \eta_{\nu\sigma}\hat{x}_{\mu\rho} - \eta_{\mu\sigma}\hat{x}_{\nu\rho}) \tag{3}$$

where  $\hat{x}_{\mu\nu}$  are the standard  $o(1, N)$  generators. A specific new feature of such formulation of extended Snyder algebra introduced in [16], is the appearance of a dimensionless parameter  $\lambda$  such that for  $\lambda \rightarrow 0$  the extended Snyder algebra reduces to the Abelian algebra and generators  $\hat{x}_{\mu\nu}$  become commutative. The extended Snyder algebra (3) contains the generators defining Snyder quantum space-time coordinates  $\hat{x}_i$  (for which the classical limit is obtained when  $\beta \rightarrow 0$ ) as well as tensorial coordinates  $\hat{x}_{ij}$  (see (4), (5)). For the interpretation of  $\hat{x}_{ij}$  we refer the reader to [15].

From the first commutator we see how the Snyder and  $\kappa$ -Minkowski space-time relations are unified. The  $\hat{X}_i$  are the NC space-time coordinates, and  $\hat{X}_{ij}$  can be interpreted as non-commutative tensorial coordinates.

Moreover, when  $g_N = 0$ , the relations (6) describe the  $\kappa$ -Minkowski space-time with Lorentz covariance algebra. Alternatively, when  $g_0 = \dots = g_{N-1} = 0$  and  $g_N = 1$ , then the relations (6) reduce to the algebra describing the extended Snyder model. Also note that if  $g_{\mu\nu} \rightarrow \eta_{\mu\nu}$  then the algebra (6) becomes a standard orthogonal algebra, which describes the  $N$ -dimensional suitably rescaled de Sitter algebra.

The coalgebra sector is a classical one (primitive), i.e.

$$\Delta(\hat{X}_{\mu\nu}) = \Delta_0(\hat{X}_{\mu\nu}), \tag{10}$$

$$\epsilon(\hat{X}_{\mu\nu}) = 0 \text{ and } S(\hat{X}_{\mu\nu}) = -\hat{X}_{\mu\nu} \tag{11}$$

and reduces accordingly to coproducts, counits and antipodes for  $\hat{X}_k$  and  $\hat{X}_{ik}$ .

If one defines the following change of coordinates:

$$\hat{X}_{\mu\nu} = (O\hat{x}O^T)_{\mu\nu}, \quad g_{\mu\nu} = (O\eta O^T)_{\mu\nu}, \tag{12}$$

with the choice of the matrix

$$O = \begin{pmatrix} 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 1 & 0 \\ -g_0 & g_1 & \dots & g_{N-1} & \rho \end{pmatrix} \tag{13}$$

where  $\rho = \sqrt{g_N - g_k g_k} = \sqrt{-\det g}$ , one can show that the algebra (6) reduces to (see also footnote 3)

$$[\hat{X}_{\mu\nu}, \hat{x}_{\rho\sigma}] = i\lambda(\eta_{\mu\rho}\hat{x}_{\nu\sigma} - \eta_{\nu\rho}\hat{x}_{\mu\sigma} + \eta_{\nu\sigma}\hat{x}_{\mu\rho} - \eta_{\mu\sigma}\hat{x}_{\nu\rho}), \tag{14}$$

where  $\eta_{\mu\nu} = (-1, 1, \dots, 1)$  and  $g_N = 1$ . This algebra has been discussed in [14], [16], as the extended Snyder algebra. We recall (see footnote 3 and (4), (5)) that the algebra (14) reduces to the  $N$ -dimensional Snyder space extended by the Lorentz algebra. We add that Heisenberg double for the extended Snyder model (14) was studied in [27].

Due to the form of the matrix  $O$  we have the following relations between two sets of coordinates:

$$\hat{X}_i = \rho\hat{x}_i + a_j\hat{x}_{ij}, \quad \hat{X}_{ij} = \hat{x}_{ij}. \tag{15}$$

Both sets of the coordinates  $\hat{X}_{\mu\nu}$  and  $\hat{x}_{\mu\nu}$  describe NC extended space-time and are related via relations (15).

In the classical limit ( $\lambda \rightarrow 0$ ) the generators of the algebras (6) and (14) reduce to the commutative (Abelian) ones [31]  $\hat{X}_{\mu\nu} \rightarrow X_{\mu\nu}$  with  $[X_{\mu\nu}, X_{\rho\sigma}] = 0$ ; ( $\hat{x}_{\mu\nu} \rightarrow x_{\mu\nu}$  with  $[x_{\mu\nu}, x_{\rho\sigma}] = 0$ ). These Abelian coordinates are related with each other via the matrix  $O$  given by (13)

$$X_{\mu\nu} = (OXO^T)_{\mu\nu},$$

i.e. explicitly:

$$X_i = \rho x_i + a_j x_{ij}, \quad X_{ij} = x_{ij},$$

where  $X_i = \frac{1}{\kappa} X_{iN}$  and  $x_i = \frac{1}{\kappa} x_{iN}$ . One can say that the algebra (6), with the generators  $\hat{X}_{\mu\nu}$  can be seen as the deformation of the underlying commutative space described by  $X_{\mu\nu}$ , with  $\lambda$  as the deformation parameter.

For the commutative extended spacetime coordinates  $X_{\mu\nu}$  and  $x_{\mu\nu}$  we can introduce the extended momenta ( $P^{\mu\nu}, p^{\mu\nu}$ ) which can be realized in a standard way as  $P^{\mu\nu} = -i\frac{\partial}{\partial X_{\mu\nu}}$  and  $p^{\mu\nu} = -i\frac{\partial}{\partial x_{\mu\nu}}$ . In this way one can introduce two copies of the generalized Heisenberg algebra as unital, associative algebras generated by the extended coordinates and momenta, with the tensorial coordinates antisymmetric under the exchange  $\mu \leftrightarrow \nu$ . The following commutation relations are valid (we put  $\hbar = 1$ )

$$[X_{\mu\nu}, X_{\alpha\beta}] = 0 = [P^{\mu\nu}, P^{\alpha\beta}], \quad [X_{\mu\nu}, P^{\rho\sigma}] = i(\delta_\mu^\rho \delta_\nu^\sigma - \delta_\mu^\sigma \delta_\nu^\rho), \tag{16}$$

and similarly

$$[x_{\mu\nu}, x_{\alpha\beta}] = 0 = [p^{\mu\nu}, p^{\alpha\beta}], \quad [x_{\mu\nu}, p^{\rho\sigma}] = i(\delta_\mu^\rho \delta_\nu^\sigma - \delta_\mu^\sigma \delta_\nu^\rho), \tag{17}$$

where Greek indices are raised and lowered by the metric  $g_{\mu\nu}$ .

One can introduce, as well, the formulae for the momenta with a single index (i.e.  $P^i = \kappa P^{iN}$ ,  $p^i = \kappa p^{iN}$ ) as dual to space-time coordinates  $X_i$  and  $x_i$ ; Latin indices are raised and lowered by the flat metric  $\eta_{ij}$ . We also have analogous relations between the two types of momenta (16) and (17)

$$p_i = \rho P_i, \quad p_{ij} = P_{ij} - a_i P_j + a_j P_i, \tag{18}$$

with the relation  $P^{\mu\nu} = ((O^{-1})^T p (O^{-1}))^{\mu\nu}$ . For more details about these coordinates and momenta we refer the reader to [10], [31] where this model was introduced and studied.

In order to discuss the phase spaces associated with the  $\kappa$ -deformed extended Snyder model (6) one can use the Heisenberg double construction method.

### 2.1. Commutative momenta for the $\kappa$ -extended Snyder model and their coproducts

One can introduce the Abelian momenta, dual to the coordinates  $\hat{X}_{\mu\nu}$  which describe the algebra  $o(1, N; g)$  (see (6)), by deforming the canonically conjugate Abelian momenta  $P^{\mu\nu}$ . The technique to calculate the coproducts  $\Delta P_i$  and  $\Delta P_{ij}$  was proposed in [12], [13]. The commutation relations for  $P^i = \kappa P^{iN}$ ,  $P^{ij}$  remain unchanged:

$$[P_i, P_j] = 0, \quad [P_{ik}, P_{jl}] = 0, \quad [P_i, P_{jl}] = 0, \tag{19}$$

while the coalgebraic sector of the momentum generators [31] looks as follows<sup>4</sup>:

$$\begin{aligned} \Delta P_i &= P_i \otimes 1 + 1 \otimes P_i + \lambda[-c_1(P_j \otimes P_{ij} + P_j \otimes P_i a_j) \\ &\quad + (1 - c_1)(P_{ij} \otimes P_j + P_i \otimes P_j a_j) + (2c_1 - 1)P_j \otimes P_j a_i] + O(\lambda^2) \end{aligned} \tag{20}$$

$$\begin{aligned} \Delta P_{ij} &= P_{ij} \otimes 1 + 1 \otimes P_{ij} + \frac{\lambda}{2}[-\beta P_i \otimes P_j - (P_{ik} \otimes P_{jk} + a_k(P_i \otimes P_{jk} - P_{jk} \otimes P_i)) \\ &\quad - (2c_1 - 1)(P_k \otimes P_{jk} + P_k \otimes P_j a_k + P_{jk} \otimes P_k + P_j \otimes P_k a_k) a_i - (i \leftrightarrow j)] + O(\lambda^2), \end{aligned} \tag{21}$$

$$S(P_i) = -P_i + \lambda(1 - 2c_1)(P_j P_{ij} + a_k P_k P_i - P_k P_k a_i), \tag{22}$$

$$S(P_{ij}) = -P_{ij} - \lambda(2c_1 - 1)(a_i P_{jk} P_k + a_i P_j a_k P_k - (i \leftrightarrow j)), \tag{23}$$

$$\epsilon(P_i) = \epsilon(P_{ij}) = 0, \tag{24}$$

where a new parameter  $c_1$  depends on the realization (see Sec. 3 in [31]), and coproducts are presented up to the first order in the deformation parameter  $\lambda$ . The coproducts (20), (21) correspond to the so-called generic realization [31], and define the momentum sector of the  $\kappa$ -extended Snyder model as parametrized by  $\lambda, \beta, a_i$  and  $c_1$ . Such parametrization occurs when the coproducts are written up to the first order in  $\lambda$ ; in higher orders of  $\lambda$  additional parameters appear (see Sec. 3 in [31] for more details). For the concrete Weyl realization [10], one should set  $c_1 = \frac{1}{2}$  and for the natural realization (i.e. with classical algebra basis)  $c_1 = 0$ .

We postulate the standard duality relation:

$$\langle P_j, \hat{X}_i \rangle = -i\eta_{ij}, \tag{25}$$

$$\langle P_k, \hat{X}_{ij} \rangle = 0, \tag{26}$$

$$\langle P_{kl}, \hat{X}_i \rangle = 0, \tag{27}$$

$$\langle P_{kl}, \hat{X}_{ij} \rangle = -i(\eta_{ik}\eta_{jl} - \eta_{jk}\eta_{il}). \tag{28}$$

One can check that from the duality relation between the products in the algebra generated by  $\hat{X}$  and the coproducts in the coalgebra generated by  $P$ , the following compatibility conditions hold

$$\langle b_{(1)}, a \rangle \langle b_{(2)}, a' \rangle = \langle b, a \cdot a' \rangle, \tag{29}$$

$$\langle b, a_{(1)} \rangle \langle b', a_{(2)} \rangle = \langle b \cdot b', a \rangle. \tag{30}$$

### 3. Generalized quantum phase space from the Heisenberg double

We introduce the left Hopf action  $\triangleright$  of momenta on coordinates defined by the formula  $P \triangleright \hat{X} = \langle P, \hat{X}_{(2)} \rangle \hat{X}_{(1)}$ , and recall that we use the Sweedler notation for the coproduct. From (25)-(28) it follows immediately that  $P_j \triangleright \hat{X}_i = -i\eta_{ij}$ ,  $P_k \triangleright \hat{X}_{ij} = 0$ ,  $P_{kl} \triangleright \hat{X}_i = 0$ ,  $P_{kl} \triangleright \hat{X}_{ij} = -i(\eta_{ik}\eta_{jl} - \eta_{jk}\eta_{il})$ . The corresponding Heisenberg double commutators follow from the cross product construction:

$$[P, \hat{X}] = \hat{X}_{(1)} \langle P_{(1)}, \hat{X}_{(2)} \rangle P_{(2)} - \hat{X}P, \tag{31}$$

written shortly without the indices. Doing the calculation explicitly and using the coproducts for momenta (20)-(21), case by case, we obtain:

$$[P_j, \hat{X}_i] = -i\eta_{ji} + i\lambda[c_1(P_{ji} + P_j a_i) - (1 - c_1)\eta_{ij} P_r a_r - (2c_1 - 1)P_i a_j] + O(\lambda^2), \tag{32}$$

$$[P_j, \hat{X}_{is}] = -i\lambda(1 - c_1)(\eta_{ij} P_s - \eta_{sj} P_i) + O(\lambda^2), \tag{33}$$

<sup>4</sup> Note that there was a typo in sign in term  $P_{jk} \otimes P_i a_k$  in the coproduct of  $\Delta P_{ij}$  in [31].

$$[P_{ij}, \hat{X}_k] = \frac{i\lambda}{2} \{ \beta(\eta_{ik}P_j - \eta_{jk}P_i) + (\eta_{ik}P_{jl} - \eta_{jk}P_{il})a_l + (2c_1 - 1)[a_i(P_{jk} + P_ja_k + \eta_{jk}P_la_i) - a_j(P_{ik} + P_ia_k + \eta_{ik}P_la_i)] \} + O(\lambda^2), \quad (34)$$

$$[P_{ij}, \hat{X}_{st}] = -i(\eta_{si}\eta_{tj} - \eta_{ti}\eta_{sj}) + \frac{i\lambda}{2} [\eta_{si}(P_{jt} + P_ja_t - (2c_1 - 1)P_t a_j) - \eta_{ti}(P_{js} + P_ja_s - (2c_1 - 1)P_s a_j) - \eta_{sj}(P_{it} + P_ia_t - (2c_1 - 1)P_t a_i) + \eta_{tj}(P_{is} + P_ia_s - (2c_1 - 1)P_s a_i)] + O(\lambda^2). \quad (35)$$

We add that all of the generalized phase space relations, including the tensorial coordinates and momenta, depend on the  $c_1$  parameter. In particular for  $c_1 = 0$ , we obtain in place of (32)

$$[P_k, \hat{X}_i] = -i\eta_{ki}(1 + \lambda a_j P_j) + i\lambda P_i a_k + O(\lambda^2). \quad (36)$$

Such commutator would correspond to the so-called classical basis of  $\kappa$ -Poincaré (named also the natural realization) [32], [33], [34]. For  $c_1 = \frac{1}{2}$  we get:

$$[P_k, \hat{X}_i] = -i\eta_{ki}(1 + \frac{1}{2}\lambda a_j P_j) - \frac{i}{2}\lambda(P_{ik} - P_k a_i) + O(\lambda^2) \quad (37)$$

which is the result corresponding to the so-called Weyl realization of  $\kappa$ -Poincaré algebra [32].

**i) Reduction to the  $\kappa$ -Minkowski and relation with the  $\kappa$ -de-Sitter case**

When  $g_N = 0$  the relations (6) describe the  $\kappa$ -Minkowski space-time with Lorentz covariance algebra as symmetry and the cross commutators obtained above (32)-(35) are reduced to the particular case of  $\kappa$ -Minkowski phase space relations. Since the expressions provided in (32), (33) and (35) are  $\beta$ -independent they will remain the same in the reduction to  $\kappa$ -Minkowski case ( $g_N = 0$ ) up to the linear order in  $\lambda$ . The relation (34) reduces to:

$$[P_{ij}, \hat{X}_k] = \frac{i\lambda}{2} [\eta_{ik}P_{jl}a_l + (2c_1 - 1)(P_{jk}a_i + P_ja_k a_i + \eta_{jk}P_la_i) - (i \leftrightarrow j)] + O(\lambda^2). \quad (38)$$

Basic quantum-deformed Heisenberg algebra relation is described by the cross commutation relation (32), which in linear order of  $\lambda$  does not depend on the  $\beta$  parameter and contains only the terms generated by  $\kappa$ -deformation. We know however from the standard Snyder model (see [3]) that the term linear in  $\beta$  is bilinear in the momenta. Because terms bilinear in  $P_i$  are as well bilinear in  $\lambda$ , the parameter  $\beta$  can contribute only at the second perturbative order in  $\lambda$ . One can state that for small  $\lambda$  the terms generated by  $\kappa$ -deformation dominate over the  $\beta$ -dependent terms, but for more precise statement the perturbative  $\lambda^2$  order terms should also be calculated for the relation (32).

Further one can compare these particular relations with the results obtained in the literature (see, e.g. in [22], [23]) where the Heisenberg double construction was investigated for the  $\kappa$ -Minkowski space-time and quantum symmetry described by the  $\kappa$ -Poincaré algebra (in Snyder model the quantum symmetry is linked with the de Sitter algebra).

**ii) Reduction to the extended Snyder model**

We can reduce the  $\kappa$ -dependent terms in commutation relations (32)-(35) and compare such results with the results obtained from the Heisenberg double of the extended Snyder model [27]. When  $g_0 = \dots = g_{N-1} = 0$  and  $g_N = 1$ , the algebra (6), after the change of variables (15), (18), provides the extended Snyder algebra. The phase space relations calculated in (32)-(35) are reduced to the following ones

$$[p_k, \hat{x}_i] = -i\eta_{ki} + i\lambda c_1 p_{ki} + O(\lambda^2), \quad (39)$$

$$[p_k, \hat{x}_{ij}] = -i\lambda(1 - c_1)(\eta_{ik}p_j - \eta_{jk}p_i) + O(\lambda^2), \quad (40)$$

$$[p_{ij}, \hat{x}_k] = i\frac{\lambda}{2}\beta(\eta_{ik}p_j - \eta_{jk}p_i) + O(\lambda^2), \quad (41)$$

$$[p_{ij}, \hat{x}_{st}] = -i(\eta_{si}\eta_{tj} - \eta_{ti}\eta_{sj}) + i\frac{\lambda}{2}[(\eta_{si}p_{jt} - \eta_{sj}p_{it}) - (\eta_{ti}p_{js} - \eta_{tj}p_{is})] + O(\lambda^2). \quad (42)$$

The cross commutation relations in the generalized phase space obtained by the Heisenberg double method for the extended Snyder model were calculated in [27] and resulted in the following:

$$[p_k, \hat{x}_i] = -i\eta_{ki} + i\frac{\lambda}{2}p_{ki} + O(\lambda^2), \quad (43)$$

$$[p_k, \hat{x}_{ij}] = -i\frac{\lambda}{2}(\eta_{ik}p_j - \eta_{jk}p_i) + O(\lambda^2), \quad (44)$$

$$[p_{ij}, \hat{x}_k] = i\frac{\lambda}{2}\beta(\eta_{ik}p_j - \eta_{jk}p_i) + O(\lambda^2), \quad (45)$$

$$[p_{ij}, \hat{x}_{st}] = -i(\eta_{si}\eta_{tj} - \eta_{ti}\eta_{sj}) + i\frac{\lambda}{2}[(\eta_{si}p_{jt} - \eta_{sj}p_{it}) - (\eta_{ti}p_{js} - \eta_{tj}p_{is})] + O(\lambda^2). \quad (46)$$

They do agree with the above results (39)-(42) for  $c_1 = 1/2$ .

#### 4. Another Heisenberg double for the $\kappa$ -extended Snyder model

Using the algebra (6) describing  $\kappa$ -extended Snyder model, also other Heisenberg double construction can be considered. If we introduce the algebra of functions generated by dual de Sitter group matrices  $\Lambda_{\alpha\beta} : \{\Lambda_{\alpha\beta} : [\Lambda_{\alpha\beta}, \Lambda_{\mu\nu}] = 0 : \Lambda^T g \Lambda = g\}$  we should postulate

$$\Delta(\Lambda_{\rho\sigma}) = \Lambda_{\rho\alpha} \otimes \Lambda_{\alpha\sigma}; \quad \epsilon(\Lambda_{\rho\sigma}) = g_{\rho\sigma}; \quad S(\Lambda_{\rho\sigma}) = (\Lambda^{-1})_{\rho\sigma} = \Lambda_{\sigma\rho}. \tag{47}$$

One can also introduce the matrices  $\tilde{\Lambda}_{\alpha\beta}$  which are related with the above group elements via the map  $\Lambda_{\rho\sigma} = \left(O \tilde{\Lambda} O^T\right)_{\rho\sigma}$ ,<sup>5</sup> ( $\alpha, \beta = 0, 1, \dots, N$ ). The basic duality relation is given by:

$$\langle \Lambda_{\rho\sigma}, \hat{X}_{\mu\nu} \rangle = -i\lambda(g_{\rho\mu}g_{\sigma\nu} - g_{\rho\nu}g_{\sigma\mu}). \tag{48}$$

We consider the following left Hopf action  $\triangleright$ :

$$\Lambda_{\rho\sigma} \triangleright \hat{X}_{\mu\nu} = \langle \Lambda_{\rho\sigma}, \hat{X}_{\mu\nu(2)} \rangle \hat{X}_{\mu\nu(1)} = g_{\rho\sigma} \hat{X}_{\mu\nu} - i\lambda(g_{\rho\mu}g_{\sigma\nu} - g_{\rho\nu}g_{\sigma\mu}) \tag{49}$$

and we obtain the cross commutation relations which are defined by the Heisenberg double method

$$\begin{aligned} [\Lambda_{\rho\sigma}, \hat{X}_{\mu\nu}] &= \hat{X}_{\mu\nu(1)} \langle \Lambda_{\rho\sigma(1)}, \hat{X}_{\mu\nu(2)} \rangle \Lambda_{\rho\sigma(2)} - \hat{X}_{\mu\nu} \Lambda_{\rho\sigma} \\ &= -i\lambda(g_{\rho\mu} \Lambda_{\nu\sigma} - g_{\rho\nu} \Lambda_{\mu\sigma}). \end{aligned} \tag{50}$$

If we recall the relations  $\hat{X}_{kN} = \kappa \hat{X}_k$  and  $g_{ij} = \eta_{ij}$ ,  $g_{iN} = \kappa a_i$ ,  $g_{NN} = g_N = \kappa^2 \beta$ , we get from (50) the cross commutation relations between the dual group elements and quantum algebra generators

$$[\Lambda_{jk}, \hat{X}_i] = -\frac{i\lambda}{\kappa}(g_{ji} \Lambda_{Nk} - g_{jN} \Lambda_{ik}) = -\frac{i\lambda}{\kappa}(\eta_{ji} \Lambda_{Nk} - \kappa a_j \Lambda_{ik}), \tag{51}$$

$$[\Lambda_{jN}, \hat{X}_i] = -\frac{i\lambda}{\kappa}(g_{ji} \Lambda_{NN} - g_{jN} \Lambda_{iN}) = -\frac{i\lambda}{\kappa}(\eta_{ji} \Lambda_{NN} - \kappa a_j \Lambda_{iN}), \tag{52}$$

$$[\Lambda_{Nk}, \hat{X}_i] = -i\lambda(a_i \Lambda_{Nk} - \kappa \beta \Lambda_{ik}), \tag{53}$$

$$[\Lambda_{NN}, \hat{X}_i] = -i\lambda(a_i \Lambda_{NN} - \kappa \beta \Lambda_{iN}). \tag{54}$$

For the tensorial coordinates  $\hat{X}_{ij}$  we obtain the cross relations:

$$[\Lambda_{lk}, \hat{X}_{ij}] = -i\lambda(\eta_{li} \Lambda_{jk} - \eta_{lj} \Lambda_{ik}), \tag{55}$$

$$[\Lambda_{lN}, \hat{X}_{ij}] = -i\lambda(\eta_{li} \Lambda_{jN} - \eta_{lj} \Lambda_{iN}), \tag{56}$$

$$[\Lambda_{Nk}, \hat{X}_{ij}] = -i\lambda\kappa(a_i \Lambda_{jk} - a_j \Lambda_{ik}), \tag{57}$$

$$[\Lambda_{NN}, \hat{X}_{ij}] = -i\lambda\kappa(a_i \Lambda_{jN} - a_j \Lambda_{iN}). \tag{58}$$

Note that  $\Lambda_{ij} = \tilde{\Lambda}_{ij}$ .<sup>6</sup> One can check explicitly that the duality  $\langle \Lambda_{\rho\sigma}, \hat{X}_{\mu\nu} \rangle$  given above reduces to

$$\langle \tilde{\Lambda}_{\rho\sigma}, \hat{x}_{\mu\nu} \rangle = -i\lambda(\eta_{\rho\mu}\eta_{\sigma\nu} - \eta_{\rho\nu}\eta_{\sigma\mu}) \tag{61}$$

for the extended Snyder model (i.e. when  $g_{Ni} = g_0 = \dots = g_{N-1} = 0$ ,  $g_{ij} = \eta_{ij}$ ,  $g_{NN} = g_N = \kappa^2 \beta$ ). Therefore, all the cross commutation relations (51)-(58) are the same as the ones obtained in [27] where the extended Snyder model was investigated.

In the reduction of the above results to the  $\kappa$ -Minkowski case ( $g_N = 0$ ) only two relations (53), (54) depend on  $\beta$  and we easily see the result of reduction. Further, focusing on the relation (51), after putting  $a_j = \frac{1}{\kappa} \delta_j^0$  (i.e. assuming a time-like  $\kappa$ -deformation) we obtain the following covariance relations

$$[\Lambda_{jk}, \hat{X}_i] = -\frac{i}{\kappa} \lambda \left( \eta_{ji} \Lambda_{Nk} - \delta_j^0 \Lambda_{ik} \right). \tag{62}$$

<sup>5</sup> In this case we have:  $\{\tilde{\Lambda}_{\alpha\beta} : [\tilde{\Lambda}_{\alpha\beta}, \tilde{\Lambda}_{\mu\nu}] = 0 : \tilde{\Lambda}^T \eta \tilde{\Lambda} = \eta\}$ ,

$$\Delta(\tilde{\Lambda}_{\rho\sigma}) = \tilde{\Lambda}_{\rho\alpha} \otimes \tilde{\Lambda}_{\alpha\sigma}; \quad \epsilon(\tilde{\Lambda}_{\rho\sigma}) = \eta_{\rho\sigma}; \quad S(\tilde{\Lambda}_{\rho\sigma}) = (\tilde{\Lambda}^{-1})_{\rho\sigma} = \tilde{\Lambda}_{\sigma\rho}.$$

Such algebra was also used in [27] in our studies of the Heisenberg double for the extended Snyder model.

<sup>6</sup> Footnote 5 can be supplemented by the following relations obtained from (46), (50):

$$[\tilde{\Lambda}_{jk}, \hat{X}_i] = -\frac{i\lambda}{\kappa}(\eta_{ji} \tilde{\Lambda}_{Nk} - \kappa a_j \tilde{\Lambda}_{ik}), \tag{59}$$

$$[\tilde{\Lambda}_{lk}, \hat{X}_{ij}] = -i\lambda(\eta_{li} \tilde{\Lambda}_{jk} - \eta_{lj} \tilde{\Lambda}_{ik}). \tag{60}$$

## 5. Discussion and outlook

For the  $\kappa$ -extended Snyder model introduced in [10] we constructed the generalized quantum phase space, which depends on three parameters,  $\beta$ ,  $\kappa$  and  $\lambda$ . The Hopf algebra describing the quantum symmetries of the model is coassociative, hence one can use the Heisenberg double construction for the correct description of the respective generalized quantum phase space. Since this model contains Snyder ( $\beta \neq 0$ ) and  $\kappa$ -Minkowski sectors ( $\kappa \neq 0$ ), one can discuss how the generalized quantum phase space can be reduced to these special cases. The algebraic relations determining the generalized phase space for the  $\kappa$ -extended Snyder model have been calculated to the first order in the  $\lambda$  parameter, and for most commutation relations to such order we get only the formulae modified by  $\kappa$ -deformation. Our algebra also includes the orthogonal (de Sitter) algebra generators, which are described by tensorial coordinates.

The quantum group describing quantum symmetries of the  $\kappa$ -extended Snyder model is built up from deformed coproducts of momenta [31] and undeformed Lorentz algebra relations. When  $\kappa \rightarrow \infty$  the  $\kappa$ -extended Snyder model reduces to the extended Snyder model which describes the noncommutative de-Sitter space-time, with  $\beta$  as the inverse square of constant curvature parameter  $R$  characterizing dual de-Sitter pseudosphere in momentum space. Considering the case, when  $\beta \rightarrow 0$  but keeping parameter  $\kappa$  we obtain the known  $\kappa$ -deformed Minkowski spacetime. Keeping both parameters, one gets an analogue of  $\kappa$ -deformation of noncommutative de-Sitter space related with the classical  $r$ -matrices of de-Sitter algebra [35,36]. Such classical  $r$ -matrices, which introduce two de-Sitter parameters, namely one related with de-Sitter curvature  $R$  and other related with  $\kappa$  deformation, can be considered and it would be interesting to investigate the connection between  $\kappa$ -deformed extended Snyder model and the deformations introducing the  $\kappa$ -deformations of de-Sitter geometry. Also, it would be of interest to see what realizations of  $\kappa$ -Poincaré, after the interpretation of  $\beta$  as the inverse square of de Sitter radius ( $\beta \sim R^{-2}$ ), could be obtained by the quantum Inonu-Wigner contraction procedure.

The Heisenberg double for the  $\kappa$ -Poincaré quantum group leads to the  $\kappa$ -deformed phase space that was investigated in different bases for the  $\kappa$ -Poincaré Hopf algebra [19–23,31]. In  $\kappa$ -extended Snyder model with different quantum symmetry group and different coalgebra sector, the comparison between these two cases and the present model involving tensorial coordinates should still be discussed.

Further, one of the tasks which would be interesting to investigate is the Hopf algebroid providing the generalized quantum phase spaces of the  $\kappa$ -extended Snyder model. Hopf algebroids describing the quantum phase spaces with NC space-time coordinates have been studied in the literature recently [20,37–41], and it should be recalled that Heisenberg double construction provides a natural example of the Hopf bialgebroid structure [8].

It can finally be added that the parameters  $\beta$ ,  $\kappa$  and  $\lambda$  are not treated here as genuine quantum deformation parameters introducing quantum deformation determined by classical  $r$ -matrices, which satisfy classical or modified Yang Baxter equations. These parameters appear in the procedure of changing the  $o(1, N)$  basis, which effectively leads to the modified de Sitter relations and noncanonical modified de Sitter metric  $g_{\mu\nu}$ . The parameters  $\beta$ ,  $\kappa$  and  $\lambda$  appear as determining the algebra basis and in this way we introduce physical parameters ( $\beta$  is related with de Sitter radius,  $\kappa$  usually is linked with Planck mass, and the parameter  $\lambda$  is related with the Planck constant). Thus, in order to describe the  $\kappa$ -extended Snyder model we only redefine the standard basis of classical de Sitter algebra, this mathematically rather trivial operation leads to results which might be significant in physical applications.

## Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

## Data availability

No data was used for the research described in the article.

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