Globally indeterminate growth paths in the Lucas model of endogenous growth

Giovanni Bella

Department of Economics and Business, University of Cagliari, Italy.

Tel: +390706753340. Fax: +39070660929. Email: bella@unica.it.

Paolo Mattana

Department of Economics and Business, University of Cagliari, Italy.

Tel: +390706753312. Fax: +39070660929. Email: mattana@unica.it.

Beatrice Venturi

Department of Economics and Business, University of Cagliari, Italy.

Tel: +390706753333. Fax: +39070660929. Email: venturi@unica.it.

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Corresponding Author: Giovanni Bella, Department of Economics and Business, University of Cagliari, Via S. Ignazio 17, 09123 Cagliari (Italy). Tel: +390706753340. Fax: +39070660929. Email: bella@unica.it.

Abstract

This paper shows that global indeterminacy may characterize the three-dimensional vector field implied by the Lucas (1988) endogenous growth model. To achieve this result, we demonstrate the emergence of a family of homoclinic orbits connecting the steady state to itself in backward and forward time, when the stable and unstable manifolds are locally governed by real eigenvalues. In this situation, we prove that if the saddle quantity is negative, and other genericity conditions are fulfilled, a stable limit cycle bifurcates from the homoclinic orbit. Orbits originating in a tubular neighborhood of the homoclinic orbit are then bound to converge to this limit cycle, creating the conditions for the onset of global indeterminacy. Some economic intuitions related to this phenomenon are finally explored.

Keywords: Lucas model; Homoclinic orbit to real saddle; Global indeterminacy.

1 Introduction

In the field of the two-sector, continuous-time, endogenous growth model with infinitely-lived agents, an influential literature has clearly established that in presence of non-competitive elements (often an externality factor), the "determinacy" of the perfect foresight equilibrium path, namely the uniqueness of the collection of agents' actions corresponding to given fundamentals, is not warranted (Del Ray and Lopez-Garcia, 2017). This is of course of great theoretical interest: it implies that given the aggregate capital stock, for given preferences and technology, there are multiple future equilibrium growth paths, each of which depending on the initial values of the jump variables of the system.

The phenomenon is commonly investigated by means of the instruments of the local analysis: if the number of stable roots of the linearization matrix (often evaluated at a hyperbolic equilibrium point corresponding to a balanced growth path) is greater than the number of state or predetermined variables, then there exists a *continuum* of solution trajectories, each of which departing from the initial choice of the jump variables. In this case, the equilibrium is said to be *locally indeterminate*.[1]

To obtain information on the behavior of indeterminate economies over a wider range of initial conditions, and for longer periods of time, a recent liter-

[1] There is a huge bulk of literature investigating the local indeterminacy properties of the equilibrium for the general case of the two-sector continuous-time endogenous growth model. Significant contributions are Chamley (1993), Benhabib and Farmer (1996), Benhabib and Perli (1994), Benhabib *et al.* (1994, 2000), Garnier *et al.* (2013), Minea and Villieu (2013).

ature is seeking conditions for indeterminacy to emerge when the dynamics is not constrained to be linear. This is the idea of *global indeterminacy*, which dates back, at least, to the seminal works of Krugman (1991) and Matsuyama (1991). Global indeterminacy is more difficult to detect since it requires more complicate mathematical techniques. For the case of the two-sector, continuous-time, endogenous growth model with infinitely-lived agents, the property of global indeterminacy has been proved in connection with: *i*) the existence of a Hopf cycle around a unique rest point in a well-located bi-dimensional manifold (Nishimura and Shigoka, 2006); *ii*) the existence of a homoclinic orbit connecting a saddle point to itself and enclosing a source or a sink, again in a well-located bi-dimensional manifold (Mino, 2004; Mattana *et al.*, 2009; Bella and Mattana, 2014); *iii*) different ω-limit sets (Brito and Venditti, 2010; Antoci *et al.*, 2014); *iv*) a chaotic attractor which appears after the rupture of a Shilnikov homoclinic orbit doubly asymptotic to a saddle-focus in $\mathbb{R}^3$ (Bella *et al.*, 2017).

We show that the system of dynamic laws characterizing the competitive solution of the standard Lucas (1988) model gives rise to global indeterminacy independently of the onset of a chaotic attractor (Barnett *et al.*, 2015). The mathematical theory behind this result exploits the existence of a homoclinic (or connecting) orbit that converges in both forward and backward time to a saddle equilibrium whose linearization matrix admits three real eigenvalues, two negative and one positive. If such homoclinic orbit is broken, by varying a bifurcation parameter, then a single periodic orbit, either stable, saddle or unstable, bifurcates in suitable cross sections near the equilibrium. In the full $\mathbb{R}^3$ dimension, this implies the existence of a tubular neighborhood of the homoclinic orbit. Moreover, given any value of the aggregate capital stock belonging to this tubular neighborhood, there exists a *continuum* of trajectories, each of which starting from different initial values of the control variables, which are bound to stay inside this tubular neighborhood.[2]

We can now present the plan of the paper. The second section recalls the three-dimensional system of first-order differential equations characterizing the competitive solution of the Lucas (1988) model, as obtained by Benhabib and Perli (1994). Extant indeterminacy results are also discussed. In the third section, we prove the existence of a homoclinic orbitand propose the relevant mathematical theorem. The fourth section exploits the conditions for the emergence of a stable limit cycle bifurcation from the homoclinic orbit and the role of global indeterminacy. The conclusion reassesses the main findings of the paper and suggests some policy implications.

2 The model

2.1 The dynamic system and the steady state

As shown by Benhabib and Perli (1994), the three-dimensional vector field

$$\begin{aligned}
\dot{x} &= Ax^{\beta}u^{1-\beta} + \tfrac{\delta(1-\beta+\gamma)}{\beta-1}(1-u)x - qx \\
\dot{u} &= \tfrac{\delta(\beta-\gamma)}{\beta}u^2 + \tfrac{\delta(1-\beta+\gamma)}{\beta}u - qu \qquad (\mathcal{S}) \\
\dot{q} &= q^2 + A\tfrac{\beta-\sigma}{\sigma}x^{\beta-1}u^{1-\beta}q - \tfrac{\rho}{\sigma}q
\end{aligned}$$

[2] To simplify discussion throughout the paper, we neglect equilibrium dynamics outside the tubular neighborhood in the case the boundary of the set is attractive from outside.

characterizes the equilibrium paths of the Lucas (1988) model. Specifically, $x = kh^{-\frac{1-\beta+\gamma}{1-\beta}}$ and $q = \frac{c}{k}$ denote stationarizing transformations of the physical capital (k), human capital (h) and consumption (c) variables. u is the fraction of total time allocated to goods production (y). Since total time is normalized to one, $(1-u)$ is the fraction of total time devoted to the sector of education. By construction, x is the predetermined variable, whereas u and q are the jump variables of the system. Parameters are as follows. The pair $(A, \delta) \in \mathbb{R}^2_{++}$ measures the technological levels in the physical capital and human capital sectors, respectively; $\beta \in (0, 1)$ is the share of physical capital in the goods sector; $\rho \in \mathbb{R}_{++}$ is the time preference rate; $\sigma \in \mathbb{R}_+ - \{1\}$ is the inverse of the intertemporal elasticity of substitution. Finally, $\gamma \in \mathbb{R}_{++}$ is the externality parameter in the production of human capital. Therefore, the set of parameters $\theta \equiv (A, \beta, \gamma, \delta, \rho, \sigma)$ lives inside $\Theta = \mathbb{R}_{++} \times (0, 1) \times \mathbb{R}^3_{++} \times \mathbb{R}_+ - \{1\}$.

Furthermore, the triplet

$$x^* = \left[\frac{\beta(\rho/\sigma)-\delta(1-\beta+\gamma)-\delta(\beta-\gamma)u^*}{A\beta(\beta-\sigma)/\sigma}\right]^{1/(\beta-1)} u^* \quad (1.a)$$

$$u^* = 1 - \frac{(1-\beta)(\rho-\delta)}{\delta[\gamma-\sigma(1-\beta+\gamma)]} \quad (1.b)$$

$$q^* = \frac{\delta(1-\beta+\gamma)}{\beta} + \frac{\delta(\beta-\gamma)}{\beta}u^* \quad (1.c)$$

corresponds to values of (x, u, q) such that $\dot{x} = \dot{u} = \dot{q} = 0$. Therefore $P^* \equiv (x^*, u^*, q^*)$ is the (interior) steady state of system $\mathcal{S}$. When θ is located in either one of the following two subsets of Θ

$$\Theta_1 = \left\{\theta \in \Theta : \rho \in (0, \delta); \sigma > 1 - \frac{\rho(1-\beta)}{\delta(1-\beta+\gamma)}\right\} \quad (2)$$

$$\Theta_2 = \left\{ \theta \in \Theta : \rho \in (\delta, \delta + \tfrac{\delta\gamma}{1-\beta}), \sigma \in (0, 1 - \tfrac{\rho(1-\beta)}{\delta(1-\beta+\gamma)}) \right\} \qquad (3)$$

P^* is unique, and $0 < u^* < 1$. When $(x, u, q) = (x^*, u^*, q^*)$, the transversality condition (TVC) is satisfied, and the growth rate of the economy is

$$g^* = \tfrac{\delta(1-\beta+\gamma)}{\beta}(1 - u^*)$$

Let $\mathbf{J}$ denote the Jacobian matrix of system $\mathcal{S}$, evaluated at the steady state, P^*. The eigenvalues of $\mathbf{J}$ are the solutions of its characteristic equation

$$\det(\lambda\mathbf{I}-\mathbf{J}) = \lambda^3 - \mathrm{Tr}(\mathbf{J})\lambda^2 + \mathrm{B}(\mathbf{J})\lambda - \mathrm{Det}(\mathbf{J}) \qquad (4)$$

where $\mathbf{I}$ is the identity matrix. $\mathrm{Tr}(\mathbf{J})$ and $\mathrm{Det}(\mathbf{J})$ are Trace and Determinant of $\mathbf{J}$, respectively. $\mathrm{B}(\mathbf{J})$ is the sum of principal minors of order 2. Formally,

$$\mathrm{Tr}(\mathbf{J}) = \tfrac{2\beta-\gamma}{\beta}\delta u^*; \;\; \mathrm{Det}(\mathbf{J}) = j_{11}^*\delta u^* q^* \tfrac{\gamma-\sigma(1-\beta+\gamma)}{\sigma(\beta-1)}; \;\; \mathrm{B}(\mathbf{J}) = j_{11}^* q^* + \tfrac{\beta-\gamma}{\beta}(\delta u^*)^2$$

By applying the Routh-Hurwitz criterion, Benhabib and Perli (1994) obtain the following stability result.

Proposition 1 (*Local indeterminacy*) *Let $\theta \in \Theta_1$. Then $\mathbf{J}$ has one negative eigenvalue and two eigenvalues with positive real parts. Let now $\theta \in \Theta_2$. Then i) if $\gamma > \beta$, $\mathbf{J}$ has one positive eigenvalue and two eigenvalues with negative real parts. The equilibrium is locally indeterminate; ii) if $0 < \gamma \leq \beta$, there exist two subsets, Θ_2^A and Θ_2^B, such that if $\theta \in \Theta_2^A$, there is again a continuum of equilibria: $\mathbf{J}$ has one positive eigenvalue and two eigenvalues with negative real parts, whereas if $\theta \in \Theta_2^B$, there are no equilibria converging* *to the steady state:* **J** *has three eigenvalues with positive real parts.*

Therefore, if $\theta \in \Theta_1$ the equilibrium is locally unique, whereas, if $\theta \in \Theta_2$ the equilibrium is either locally indeterminate or totally unstable.[3]

Other researchers have complemented Proposition 1 by using the mathematics of the global analysis. In particular, Nishimura and Shigoka (2006) prove conditions for closed orbits to emerge at the boundary between Θ_2^A and Θ_2^B subsets. If this happens, there is a *continuum* of equilibria departing from a given aggregate capital stock; namely, P^* is globally indeterminate. Additionally, Bella *et al.* (2017) have recently shown that, in the local determinacy region of the parameter space, Θ_1, when the steady state is a saddle-focus, perfect-foresight chaotic solution trajectories may emerge in an open neighborhood $U \subset \mathbb{R}^3$ of a homoclinic orbit. Again, given the initial value of the predetermined variable, there exists a *continuum* of initial values of the jump variables such that each path starting in U continues to stay in this neighborhood. Since this *continuum* of equilibria is outside the small neighborhood relevant for the local analysis, the equilibrium is therefore globally indeterminate.

[3] Xie (1994), under the parametric restriction $\sigma = \beta$, shows that, if $\gamma > \beta$, there exists a *continuum* of equilibria converging to the balanced growth path (BGP) in the original $\mathbb{R}^4$ dimension.

3 A limit cycle in the neighborhood of a homoclinic orbit

3.1 Propedeutical definitions and Theorem

Consider a system in $\mathbb{R}^3$ with a homoclinic orbit Γ_0 to a saddle x_0. Assume that dim $W^u = 1$, where W^u is the unstable manifold, and introduce a two-dimensional cross-section Σ with coordinates (ξ, η). Suppose that $\xi = 0$ corresponds to the intersection of Σ with the stable manifold W^s of x_0. Let conversely the point with coordinates (ξ^u, η^u) corresponds to the intersection of W^u with Σ. The following definition is useful.

Definition 1 *A split function can be defined as $\varphi = \xi^u$. Its zero ($\varphi = 0$) gives a condition for the homoclinic bifurcation in $\mathbb{R}^3$.*

We can now propose the following result (Kuznetsov, 2004, p. 214).

Theorem 1 *Consider a three-dimensional system*

$$\dot{x} = f(x, \alpha), \qquad x \in \mathbb{R}^3, \qquad \alpha \in \mathbb{R}^1 \tag{5}$$

with smooth f, having at $\alpha = 0$ a saddle equilibrium point $x_0 = 0$ with real eigenvalues $\lambda_1(0) > 0 > \lambda_2(0) \geq \lambda_3(0)$ and a homoclinic orbit Γ_0. Assume the following genericity conditions also hold:

$(H.1)$ $s = \lambda_1(0) + \lambda_2(0) < 0$;

$(H.2)$ $\lambda_2(0) \neq \lambda_3(0)$;

$(H.3)$ Γ_0 *returns to $x_0 = 0$ along the leading eigenspace*;

$(H.4)$ $\varphi'(0) \neq 0$, *where $\varphi(\alpha)$ is the split function in definition* 1.

Then, system (5) *has a unique and stable limit cycle* L_φ *in a neighborhood* U_0 *of* $\Gamma_0 \cup x_0$ *for all sufficiently small* $\varphi > 0$.

In the following subsections, we show that Theorem 1 applies to system $\mathcal{S}$ in specific regions of the parameter space.

3.2 Parametric conditions for P^* to be a real saddle

Consider the following statement.

Lemma 1 *Let* $\theta \in \Theta_2$. *Then* $\bar{\Theta}_2 = \{\theta \in \Theta_2 : \mathbf{J}$ *has three real eigenvalues with* $s < 0\} \neq \varnothing$.

Proof. Solving (4) by Cardano's formula, we obtain the real roots

$$\begin{aligned}
\lambda_1 &= \tfrac{\mathrm{Tr}(\mathbf{J})}{3} + 2\sqrt{-\tfrac{m}{3}}\cos\tfrac{\theta}{3} \\
\lambda_2 &= \tfrac{\mathrm{Tr}(\mathbf{J})}{3} + 2\sqrt{-\tfrac{m}{3}}\cos\tfrac{\theta+2\pi}{3} \\
\lambda_3 &= \tfrac{\mathrm{Tr}(\mathbf{J})}{3} + 2\sqrt{-\tfrac{m}{3}}\cos\tfrac{\theta+4\pi}{3}
\end{aligned}$$

where $\tan\theta = -\frac{2\sqrt{-\Delta}}{n}$. Moreover, $m = \frac{3B(\mathbf{J})-Tr(\mathbf{J})^2}{3}$, $n = -\mathrm{Det}(\mathbf{J}) - 2\frac{\mathrm{Tr}(\mathbf{J})^3}{27} + \frac{\mathrm{Tr}(\mathbf{J})\mathrm{B}(\mathbf{J})}{3}$, and $\Delta = \left(\frac{m}{3}\right)^3 + \left(\frac{n}{2}\right)^2$ is the discriminant. ∎

Consider the following illustrative example.

Example 1 *Set* $(A, \beta, \gamma, \delta, \rho, \sigma) = (0.05, 0.25, 0.65, 0.021, 0.022, 0.4) \in \Theta_2$. *This economy has* $P^* \equiv (x^*, u^*, q^*) \simeq (0.204, 0.601, 0.097)$ *and a long-run growth rate* $g^* \simeq 1.56\%$. *The computation of the eigenvalues of* $\mathbf{J}$ *leads to* $\lambda_1 \simeq 0.09032$, $\lambda_2 \simeq -0.09424$ *and* $\lambda_3 \simeq -0.00368$. *The saddle quantity is negative,* $s = -0.00392 < 0$. *Using* σ *as a bifurcation parameter, we observe that parameters remain inside* $\bar{\Theta}_2$ *for* $0.37 \lesssim \sigma \lesssim 0.43$.

3.3 Existence of homoclinic orbits

We apply now the procedure developed by Shang and Han (2005), and show that a homoclinic loop emerges as a solution trajectory of system $\mathcal{S}$ for parameter values belonging to the $\bar{\Theta}_2 \subset \Theta_2$ subset. The application of the method leads to the following relationship

$$\varphi \equiv \eta - \frac{F_{3d}}{\lambda_3(0) + 2\lambda_1(0)}\xi^2 + \frac{1}{\lambda_3(0)} \frac{F_{2d}\left[\lambda_2(0) + 2\lambda_3(0)\right]}{F_{2f}\ \left[\lambda_2(0) - 2\lambda_1(0)\right]} F_{3f}\xi^2 = 0 \qquad (6)$$

which needs to hold for the emergence of a homoclinic orbit connecting P^* to itself.[4] In (6), $\eta \in (0,1)$ and $\xi \in (0,1)$ are arbitrary constants, and the $F_{i,j}$ coefficients, $i = 2, 3$ and $j = d, f$, are intricate combinations of the original parameters of the model and of three scaling factors (ψ_1, ψ_2, ψ_3) associated with the choice of the eigenvectors. Finally, $\lambda_1(0) > 0 > \lambda_2(0) > \lambda_3(0)$ are the eigenvalues of the linearization matrix $\mathbf{J}$.

Consider the following example.

Example 2 *Set* $(\beta, \gamma, \delta, \rho, \sigma) = (0.25, 0.65, 0.021, 0.022, 0.4)$ *as in Example* 1*. Figure* 1 *represents equation* (6) *in the remaining* (A, η, ξ) *parameters.Let us now take* A *as the bifurcation parameter. Let also* $\hat{A}$ *denote the critical*

[4] The application of this method requires the accomplishment of a number of steps. We are first required to translate system $\mathcal{S}$ to the origin and operate a coordinate change using the associated eigenbasis. Then, we need to obtain an explicit polynomial approximation of the analytical expressions of both the two-dimensional stable manifold associated with λ_2 and λ_3, and of the one-dimensional unstable manifold associated with λ_1. This is obtained by means of the undetermined coefficient method. The procedure is fully detailed in Bella *et al.* (2017) for a very similar problem. Due to space constraint, we do not report these computations, but they are available upon request.

value of A such that the condition (6) *is satisfied, i.e.,* $\varphi(\hat{A}) = 0$. *As it appears clear from Figure* 1, *there is a minimum value at* $A \simeq 0.33$, *at which condition* (6) *is satisfied. Therefore, if we set* $A = 0.05$ *as in Example* 1, *no homoclinic orbit can emerge as a solution trajectory of system* $\mathcal{S}$. *Set* $(\xi, \eta) = (0.05, 0.5)$. *Then,* $\hat{A} = 0.343$ *is the critical value of the bifurcation parameter such that* (6) *is satisfied. Hence, if* $A \equiv \hat{A} = 0.343$, *a homoclinic orbit emerges as a solution of system* $\mathcal{S}$.

Figure 1

Therefore, the following is implied.

Lemma 2 (*Existence of homoclinic orbits to* P^*) *Let* $\theta \in \bar{\Theta}_2$. *Then*

$$\bar{\Theta}_2^H = \left\{ \theta \in \bar{\Theta}_2 : \mathcal{S} \text{ possesses a homoclinic orbit} \right\} \neq \varnothing$$

To derive economic insights into from an economy undergoing the homoclinic bifurcation in Theorem 1, it is useful to locate the boundaries of $\bar{\Theta}_2^H$ in the parameter space. We already know that the structure of the eigenvalues which satisfies the requirements in Theorem 1 can be achieved in the Lucas model only if the degree of risk aversion has to be lower than unity, i.e., $\sigma < 1 - \frac{\rho(1-\beta)}{\delta(1-\beta+\gamma)} < 1$, and the discount rate is higher than the productivity in the schooling sector, i.e., $\rho > \delta$.

Furthermore, extensive numerical simulations also allow us to propose the following result.

Remark 1 *Inside $\bar{\Theta}_2^H$, γ and A are higher than established in standard empirical estimates. The contrary applies for β.*[5]

We are now ready to prove the following statement.

Proposition 2 (*Existence of a limit cycle in the neighborhood of the homoclinic orbit*). *Let $\theta \in \bar{\Theta}_2^H$. Then $\hat{A}$ is the bifurcation parameter such that a homoclinic orbit emerges as a solution trajectory of system $\mathcal{S}$. Since there exist values of A sufficiently close to $\hat{A}$ such that $\varphi(A) > 0$ and $\varphi'(A) \neq 0$, a unique stable limit cycle around the homoclinic orbit emerges.*

Proof. To prove the statement, we need to show that conditions in Theorem 1 are satisfied. By Lemma 2, we already know that since $\bar{\Theta}_2^H \neq \varnothing$, there exist regions in the parameters space such that P^* is a real saddle with $s < 0$, and there is a critical value of the parameter $A = \hat{A}$ at which an orbit connecting P^* to itself in backward and forward time emerges as a solution trajectory. Our examples also show that, generically, $\lambda_2(0) \neq \lambda_3(0)$ so that (H.2) is also satisfied. Furthermore, by construction, the homoclinic orbit returns to P^* along the leading eigenspace. Finally equation (6) represents the split function of the homoclinic orbit, as stated in Definition 1. Since (6) is smooth in A and $\varphi(A) > 0$, Theorem 1 can by applied to system $\mathcal{S}$. ■

[5] For instance, Greiner *et al.* (2005) estimate a triplet $(A, \beta, \gamma) \simeq (0.06, 0.44, 0)$ for both US and Germany. Moreover, assuming the economy behaves as described by the Lucas model, Altăr *et al.* (2008) take $(A, \beta, \gamma) \simeq (0.1, 0.37, 0)$ to forecast the trajectory of the Romanian economy. However, Lucas (1988), Benhabib and Perli (1994), and Xie (1994) assume values $(\beta, \gamma) = (0.25, 0.417)$ more in line with our simulations. Notice also that in these contributions A is normalized to 1 to simplify the analysis.

4 Global indeterminacy in the Lucas model of endogenous growth

As discussed in the Introduction, the study of global indeterminacy of the equilibrium requires the proof that, given an initial condition in terms of the predetermined variable, there are regions of the parameter space at which multiple equilibria, lying outside the “small” neighborhood relevant for the local analysis, exist.

Proposition 2 implies that, when $\theta \in \bar{\Theta}_2^H$, system $\mathcal{S}$ has a bounded region enclosed by a tubular neighborhood of the homoclinic orbit, given by the basin of attraction of the stable limit cycle emerging after the rupture of the homoclinic orbit (see, Aguirre *et al.*, 2013, for a technical discussion on this issue). Let $\mathcal{T}_A \subset \mathbb{R}^3$ denote this tubular neighborhood and let also Int$\mathcal{T}_A$ and Bd$\mathcal{T}_A$ denote the set of all interior points on this tubular neighborhood and its boundary, respectively. Let finally

$$\mathcal{E}_{_A} = \{(x, u, q) \in \mathbb{R}^3 : (x, u, q) \in \text{Int}\mathcal{T}_A\}$$

be a three-dimensional manifold containing the set of all possible paths starting on Int$\mathcal{T}_A$. Then, by construction, all paths starting on the (compact) set $\mathcal{E}_{_A}$ are bound to stay in $\mathcal{E}_{_A}$. Consider therefore the following.

Proposition 3 (*Global indeterminacy of the equilibrium*). *Let $\theta \in \bar{\Theta}_2^H$ and recall Proposition* 2. *Assume the bifurcation parameter A sufficiently close to its critical value $\hat{A}$. Consider $x(0) \in \mathcal{E}_{_A}$. Then, Lucas* (1988) *model exhibits global indeterminacy of the equilibrium. If we appropriately choose* *the scaling constants (ψ_1, ψ_2, ψ_3), the equilibrium path satisfies the TVC.*

Proof. In Proposition 2 we show that there exist regions in the parameter space such that a stable limit cycle bifurcates from a homoclinic orbit connecting P^* to itself. We have also discussed that this implies the existence of a tubular neighborhood of the homoclinic orbit $\mathcal{T}_A \subset \mathbb{R}^3$. Let therefore $x(0) \in \mathcal{E}_A = \{(x, u, q) \in \mathbb{R}^3 : (x, u, q) \in \text{Int}\mathcal{T}_A\}$. Then, there is a *continuum* of values of the jump variables giving rise to recurrent orbits, namely equilibrium trajectories which are bound to stay in $\mathcal{E}_A$ for all times. Benhabib and Perli (1994) show that, at the steady state, the TVC is satisfied when $0 < u^* < 1$. Bella *et al.* (2017) show that if the scaling factors (ψ_1, ψ_2, ψ_3) are appropriately chosen then, along the homoclinic orbit, $u \in (0, 1)$ and the TVC still holds. Finally, according to Theorem 1, if we choose a sufficiently small deviation of the bifurcation parameter A from its neighborhood along the homoclinic orbit, then the TVC is also satisfied. ■

Consider now the following example.

Figure 2

Example 3 *Set* $(\beta, \gamma, \delta, \rho, \sigma) = (0.25, 0.65, 0.021, 0.022, 0.4)$ *and* $(\xi, \psi) = (0.05, 0.5)$ *as in Example* 2*. Then, we know that if* $A \equiv \hat{A} = 0.343$ *there is a homoclinic loop connecting* P^* *to itself. Moreover, since* $\varphi\left(\hat{A}\right) \simeq 0.036$ *and* $\varphi'\left(\hat{A}\right) \simeq 0.045$*, Theorem* 1 *applies to system* $\mathcal{S}$*. Let us now assume* $A \neq \hat{A}$*, and consider the case of* $A = 0.35$*. Then, by Theorem* 1*, a stable limit cycle bifurcates from the homoclinic orbit. At* $A = 0.35$*,* $P^* \equiv (x^*, u^*, q^*) \simeq (2.728, 0.603, 0.097)$*. Consider Figure* 2*, where we plot in* $\mathbb{R}^3$ *two equilib-*

rium trajectories, starting at $P_1(0) = (2.728, 0.626, 0.095)$ (*blue curve*) *and* $P_2(0) = (2.728, 0.642, 0.094)$ (*red curve*)*, converging to the cycle* (*thick black line*)*. Notice that, to better characterize the implications of global indeterminacy, we choose the same value of the aggregate capital stock.*

The two patterns of convergence to the limit cycle are interesting to compare. The dynamics is initially dramatically different: the equilibrium starting at $P_1(0) = (2.728, 0.626, 0.095)$ implies a decrease in the state variable x and an increase in both u and q; the contrary happens with the equilibrium starting at $P_2(0) = (2.728, 0.642, 0.094)$. However, when the equilibrium trajectories enter a neighborhood of the steady state, the dynamics tend to become virtually indistinguishable as they approach the unique limit cycle.

5 Conclusions and policy implications

In this paper we show that the equilibrium in the Lucas (1988) two-sector endogenous growth model may be globally indeterminate in the region of the parameter space characterized by a low convenience of agents for consumption smoothing and a high discount rate with regard to the productivity in the schooling sector. The phenomenon is likely to occur for economies where high technological levels in the goods sector and a strong externality factor combine with a low share of physical capital.

The mathematical underpinnings behind our results exploit the onset of a stable limit cycle appearing in a three dimensional ambient space after the rupture of a homoclinic orbit connecting the unique steady state to itself. As clearly pointed out in the literature (cf., *inter al.*, Aguirre *et al.*, 2013), this implies the existence of a tubular neighborhood of the original homoclinic orbit, such that any initial condition starting inside this tubular neighborhood gives rise to a perfect-foresight equilibrium which is bound to converge to the limit cycle. Since the result is valid beyond the small neighborhood relevant for the local analysis, we have global indeterminacy of the equilibrium.

Results are important. Beyond the theoretical relevance of global indeterminacy in the Lucas (1988) model, there are a number of policy implications which is worth discussing. First of all, we find that, when the linear analysis is put aside, another ω-limit set (besides the steady state) emerges. Furthermore, since all eigenvalues are real, it is likely that convergence to the steady state occurs smoothly or even monotonically; conversely, as should be also clear, if initial conditions are chosen so that the economy converges to the limit cycle, the economic system may wander quite wildly before approaching a oscillating pattern in the long-run. Of course, since the equilibrium is also locally indeterminate, this statement works if we neglect the possibility of extrinsic uncertainty affecting agents' expectations and causing sunspot fluctuations (cf. Mino, 2004, for an interesting discussion on these elements).

How should a government act in order to make economic growth more predictable, or follow a more desirable/controllable pattern? A realistic move would be, *ceteris paribus*, to raise the parameter measuring the productivity in the educational sector, δ, above the discount rate, ρ. If this happens, and σ is below unity, parameters start to belong to the Θ_1 subset in the main text. First of all, from a global perspective, this implies that

Theorem 1 cannot find application anymore. The possibility of limit cycles is then excluded and a saddle-type steady state results as the only long run attractor. Secondly, in the local perspective, when parameters belong to the Θ_1 subset, uniqueness of the equilibrium is achieved and, as shown in Bella *et al.* (2017), if $\sigma < 1$ all eigenvalues are real, and the economy converges to the rest point along a predictable and smooth perfect foresight equilibrium.

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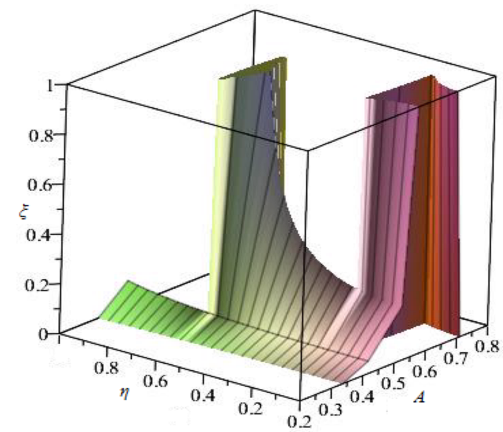

Figure 1: The parametric surface $\varphi = 0$

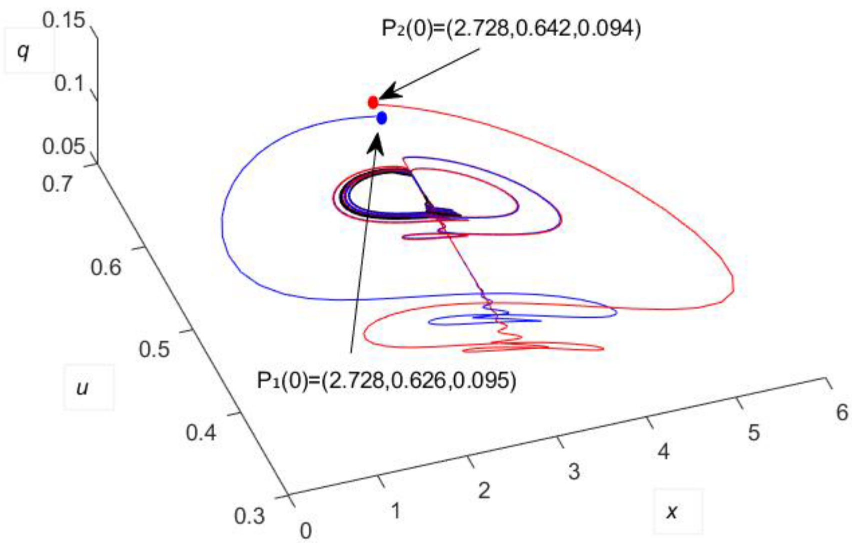

Figure 2: Equilibrium trajectories converging to the unique limit cycle