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Keywords: Heat equation; Coupled PDE-ODE; Boundary Control; Sliding mode control; Dynamic actuator

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Sliding Mode Boundary Control for Heat Equations with Uncertain Dynamic Actuators

Alessandro Pilloni, Alessandro Pisano, Elio Usai, Yury Orlov

Abstract—The present study focuses on designing a boundary controller for an uncertain heat equation with a dynamic actuator, leading to a coupled PDE-ODE representation of the system’s dynamics. Our goal is to regulate the state of the PDE to a constant set-point, despite the presence of uncertainties in both the diffusivity parameter and the actuator time constant. The proposed design is based on sliding mode control and guarantees pointwise-in-space exponential stabilization of the closed-loop system in the second-order Sobolev space $H^2(0,1)$. A Lyapunov analysis is conducted to demonstrate the stability characteristics of the closed-loop system and numerical simulation results are provided to corroborate the effectiveness of the proposed controller. The simulations also confirm the rejection of smooth time-varying disturbances acting at the controlled boundary, specifically between the actuator process and the controlled PDE boundary.

Index Terms—Heat equation; Coupled PDE-ODE; Boundary Control; Sliding mode control; Dynamic actuator.

I. CONTROL OF PARABOLIC PDE-ODE SYSTEMS

Engineering studies concerning “parabolic” partial differential equations (PDEs) are popular because they are predominately used in describing fluid, thermal, and chemical processes, see [1, Section I.A] and reference therein.

From a control perspective, such interest is recently moved to the stabilization problem and the controller synthesis of parabolic PDEs coupled with ordinary differential equations (ODEs), thus considering finite-dimensional dynamics in the actuating and sensing paths of PDEs, see [1]–[8].

At the moment are few studies available, and most consider the backstepping boundary control problem of 1-D parabolic PDEs cascaded with either a dynamic actuator, or a dynamic sensor, or “sandwiched” between a dynamic actuator and a sensor, collocated at the two opposite boundaries. Among them it is worth mentioning [1], [2], [5], which propose “one step” backstepping designs for, respectively, a Burgher PDE coupled with an integrator at the plant input [2], a Heat PDE coupled with a dynamic sensor [5], and an ODE-Heat PDE-ODE system [1]. However, due to the coupled system’s structure, mapping the plant into the target system in “one step” leads to kernel equations of PDEs coupled to ODEs. Hence, it may be hard to determine the corresponding kernel.

Alternatively, in [3], the “successive backstepping approach”, firstly proposed in [4] for “hyperbolic” systems, is considered. In this case, the kernel equation of the PDE only have to be solved, thus simplifying the design. Indeed, the coupled kernel equations resulting from the one-step approach are now decomposed into several easily solvable equations. Finally, we mention [6], which in contrast to backstepping strategies, proposes a constructive LMI-based method to verify, through the use of a polynomial approximation of the infinite-dimensional state, the exponential stability of a heat PDE coupled with a dynamic sensor numerically.

Although the generalization of finite-dimensional control strategies dealing with external disturbances and/or parameter uncertainties to the PDE setting has received attention in recent years as practically motivated (see [7]–[11] and references therein), only few works consider the extension to parabolic PDEs coupled with ODEs. To the best of our knowledge one of the first attempts is [10]. There, with the aim to extended second-order sliding mode control (SMC) concepts to the infinite-dimensional setting, a twisting-based boundary SMC is proposed for the stabilization of an uncertain heat PDE subjected to unbounded matched disturbances. To achieve the goal, while making the process input smooth, a dynamic input extension as that used in [2] was applied, thus providing a heat PDE coupled with an integrator ODE. Then, in [7], the combination of a one-step backstepping controller with a relay-based first-order SMC term is proposed to stabilize a heat PDE coupled with a dynamic sensor, and in the presence of matched disturbances acting at the controlled boundary. Finally, it is worth mentioning [11], in which the strategy [10], combined with a backstepping term, is also effective for stabilizing an unstable reaction-diffusion process subjected to matched disturbances acting at the controlled boundary, and under mixed Robin-type boundary conditions.

Contribution: All the controllers above address the PDE’s boundary actuation while neglecting the dynamics of the actuator itself. However, when considering an uncertain PDE system coupled with uncertain ODE actuator dynamics, and potentially subjected to disturbances at the controlled boundary, the control problem becomes significantly more challenging. It is also worth noting that among the reviewed studies, only [1] considers ODE actuator dynamics, but neither uncertainties nor disturbances were addressed.

Thus motivated, this paper investigates the robust boundary control of an uncertain heat PDE coupled to an uncertain ODE representing the actuator dynamics. From a methodological viewpoint, we make some steps beyond the

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A. Pilloni and A. Pisano, and E. Usai are with Department of Electrical and Electronic Engineering, University of Cagliari, Italy. Y. Orlov is with the CICESE Research Center, Ensenada, Mexico. Email: alessandro.pilloni@unica.it, pisano@unica.it, elio.usai@unica.it, yorlov@cicese.mx
Alessandro Pilloni is the corresponding author.

existing literature in that the considered PDE-ODE system is subject to uncertainties in either the finite- and the infinite-dimensional part of the model, respectively, on the actuator time-constant, and heat diffusivity. In particular, we devised a boundary relay-based SMC strategy that, when combined with proportional and derivative (PD) control terms, can exponentially stabilize the resulting closed-loop system in higher-order spatial norms, achieving pointwise-in-space stabilization. This constitutes the main contribution of the paper.

Notice that the proposed control is discontinuous, thus we conjecture that the scheme could effectively reject unmatched smooth time-varying disturbances acting at the actuated PDE boundary, specifically between the actuator process and the controlled PDE boundary. Although this has been confirmed through simulations, further investigation will be conducted to extend the stability analysis, drawing inspiration from related finite-dimensional results, such as those in [12], [13], to derive a proper tuning rule for the relay control gain. This would represent further advancement beyond [7], [10], [11], which only consider PDEs affected by matched disturbances.

Organization: Preliminary results are introduced in Section II. Section III presents the class of uncertain coupled PDE-ODE Boundary Value problems (BVPs) and formulates the control problem. The proposed robust boundary controller is detailed in Section IV, where the pointwise exponential stabilization of the closed-loop system in the H^2 -norm is proved. Numerical simulations are provided in Section VI. Finally, Section VII offers some concluding remarks and suggests directions for further related investigations.

II. PRELIMINARY RESULTS

The *Sobolev space* of absolutely continuous scalar functions $y(\zeta)$ with square-integrable derivatives $y^{(\ell)}(\zeta)$ on $(0, 1)$ up to the ℓ -th order is denoted by $H^\ell(0, 1)$. Specifically, $H^0(0, 1)$ corresponds to the $L^2(0, 1)$ *Hilbert space*. Finally, the H^ℓ -norm of $y(\zeta)$ is

$$\|y(\cdot)\|_{H^\ell} = \left(\int_0^1 \sum_{i=0}^{\ell} \left[y^{(i)}(\zeta) \right]^2 d\zeta \right)^{\frac{1}{2}}. \quad (1)$$

For $a \in \mathbb{R}$, $|a|$ denotes its absolute value, while $\text{sign}(a)$ represents the set-valued sign operator, which yields 1 if $a > 0$, -1 if $a < 0$, and $[-1, 1]$ if $a = 0$.

Lemma 1. *Let $a, b \in \mathbb{R}$. The next inequalities hold:*

a) Hölder's Inequality:

$$ab < |ab| < |a| |b|.$$

b) Extended triangle Inequality:

$$-\left(\frac{a^2}{2\gamma} + \frac{\gamma b^2}{2} \right) \leq ab \leq \left(\frac{a^2}{2\gamma} + \frac{\gamma b^2}{2} \right), \quad \forall \gamma > 0.$$

Lemma 2 (see [14]). *Consider $y(\zeta) \in H^1(0, 1)$, then*

$$\|y(\cdot)\|_{H^0}^2 \leq 2(y^2(i) + \|y_\zeta(\cdot)\|_{H^0}^2), \quad i = 0, 1.$$

Lemma 3. *A symmetric matrix $M \in \mathbb{R}^{2 \times 2}$ is positive definite ($M \succ 0$) if and only if the next constraints holds true*

$$M = \begin{bmatrix} a & b \\ b & c \end{bmatrix} \succ 0 \iff a > 0, c > 0, ac > b^2.$$

Proof. Let $y = [y_1, y_2]^\top$. The quadratic form

$$y^\top M y = a \left(y_1 + \frac{b}{a} y_2 \right)^2 + y_2^2 \left(c - \frac{b^2}{a} \right) > 0 \quad (2)$$

is strictly positive if $a > 0$ and $ac > b^2$. Thus, also $c > 0$. $\square$

III. PROBLEM FORMULATION

Consider the scalar field $q(\xi, t)$ evolving in the Hilbert space $H^0(0, 1)$, with the spatial variable $\xi \in [0, 1]$ and the time variable $t \in [0, \infty)$. Let $q(\xi, t)$ be governed by the parabolic PDE known as the *Heat Equation*

$$q_t(\xi, t) = \theta q_{\xi\xi}(\xi, t), \quad (3)$$

where q_t and $q_{\xi\xi}$ denote the temporal and second-order spatial derivatives of the scalar field $q(\xi, t)$, and $\theta > 0$ is a positive unknown coefficient, conventionally known as *diffusivity*. In this work, we consider *Neumann-type* boundary conditions (BCs) of the form

$$q_\xi(1, t) = u(t), \quad q_\xi(0, t) = 0, \quad (4)$$

where $u(t) \in \mathbb{R}$ denotes the output provided by an actuator modeled by the following first-order uncertain ODE

$$\tau \dot{u}(t) + u(t) = v(t), \quad (5)$$

where $\tau > 0$ is the uncertain time constant of the actuator. Here $v(t)$ is the manipulable control variable to be designed. At time $t = 0$ the next spatial initial condition (IC) is

$$q(\xi, 0) = q^0(\xi) \in H^0(0, 1). \quad (6)$$

Let q^r be an a-priori given time-independent and spatially-invariant reference profile, the **control objective** is to design $v(t)$ to exponentially steer to zero the H^2 -norm of the deviation field

$$x(\xi, t) = q(\xi, t) - q^r \quad (7)$$

where

$$\|x(\cdot, t)\|_{H^2}^2 = \|x(\cdot, t)\|_{H^0}^2 + \|x_\xi(\cdot, t)\|_{H^0}^2 + \|x_{\xi\xi}(\cdot, t)\|_{H^0}^2. \quad (8)$$

Steering (8) to zero implies that the $\|x(\xi, t)\|_{H^1}$ goes to zero, indicating the deviation field vanishes pointwise in space.

A. Assumption and preliminary considerations

A-priori information requirements on the plant uncertainties are specified in the following assumption.

Assumption 1. *There exist a-priori known positive constants $\theta_m, \theta_M, \tau_m, \tau_M$ such that $\theta \in [\theta_m, \theta_M]$, and $\tau \in [\tau_m, \tau_M]$.*

Let us now make some preliminary considerations on the considered control objective. Our goal is to enforce, through

$v(t)$, a feedback mechanism ensuring the H^2 -norm of the solution $x(\xi, t)$ of the combined PDE-ODE BVP

$$x_t(\xi, t) = \theta x_{\xi\xi}(\xi, t), \quad (9)$$

$$x_\xi(0, t) = 0, \quad (10)$$

$$x_\xi(1, t) = u(t), \quad (11)$$

$$\tau \dot{u}(t) = v(t) - u(t), \quad (12)$$

$$x(\xi, 0) = q^0(\xi) - q^r \in H^2(0, 1), \quad (13)$$

exponentially decay, as time progresses. From (8) and (9) this further implies the stabilization of the augmented state

$$z(\xi, t) = x_t(\xi, t) \propto x_{\xi\xi}(\xi, t)$$

in $H^0(0, 1)$. For later use, we introduce the auxiliary BVP

$$z_t(\xi, t) = \theta z_{\xi\xi}(\xi, t), \quad (14)$$

$$z_\xi(0, t) = 0, \quad (15)$$

$$z_\xi(1, t) = \dot{u}(t), \quad (16)$$

$$z(\xi, 0) = z^0(\xi) = \theta x_{\xi\xi}^0(\xi) \in H^0(0, 1), \quad (17)$$

which is obtained by differentiating (9)-(13) in the time variable whereas the IC (17) is derived from (9), and (13).

Unfortunately, following [15, Theorem 3.3.3], the existence of classical solutions for the BVP (14)-(17) is not guaranteed. This is because $\dot{u}(0)$ do not necessarily verify the resulting compatibility condition $x_{\xi\xi}^0(1) = \dot{u}(0)$. Because of that, in the remainder, the meaning of the auxiliary BVP (14)-(17) will be viewed in the mild sense [15]. Moreover, in accordance with [16], let us further note that the mild solutions of (14)-(17) coincide with the corresponding weak solutions of the so-called standardizing PDE in distributions

$$\begin{aligned} z_t(\xi, t) &= \theta z_{\xi\xi}(\xi, t) + \theta \dot{u}(t) \delta(\xi - 1) \\ &= \theta z_{\xi\xi}(\xi, t) + \theta \left[\frac{v(t) - x_\xi(1, t)}{\tau} \right] \delta(\xi - 1) \end{aligned} \quad (18)$$

where $u(t)$ is derived from (11), under the homogeneous BCs

$$z_\xi(1, t) = 0, \quad z_\xi(0, t) = 0, \quad (19)$$

and to the same IC (17). Notice that since (18) contains the Dirac distribution $\delta(\xi - 1)$, the solutions to the BVP (18)-(19) are defined indirectly in the weak sense [17, pag. 258].

Definition 1. A function $z(\cdot, t) \in H^0(0, 1)$, satisfying the homogenous BCs (19) is called weak solution of the BVP (18)-(19) on $t \in [0, t']$ if for every $\varphi(\xi) \in H^0(0, 1)$, the quantity $\int_0^1 z(\xi, t) \varphi(\xi) d\xi$ is absolutely continuous on $[0, t']$ and, for almost all $t \in [0, t']$, its results that

$$\begin{aligned} \frac{d}{dt} \int_0^1 z(\xi, t) \varphi(\xi) d\xi &= -\theta \int_0^1 z_{\xi\xi}(\xi, t) \varphi_\xi(\xi) d\xi \\ &+ \theta \left[\frac{v(t) - x_\xi(1, t)}{\tau} \right] \varphi(1). \end{aligned} \quad (20)$$

The concept of weak solution in (20) relies on the property $\int_0^1 \varphi(\xi) \delta(\xi - 1) d\xi = \varphi(1)$ of the shifted Dirac distribution

$\delta(\xi - 1)$, on an arbitrary test function $\varphi(\xi) \in H^1(0, 1)$, and it is based on the integration-by-parts property

$$\int_0^1 z_{\xi\xi}(\xi, t) \varphi(\xi) d\xi = - \int_0^1 z_\xi(\xi, t) \varphi_\xi(\xi) d\xi \quad (21)$$

of the Sobolev derivatives of the $H^1(0, 1)$ -valued functions under the given homogeneous BCs in (19).

IV. CONTROLLER SYNTHESIS

To achieve the control goal, the following non-smooth boundary controller is proposed

$$v(t) = -\beta_1 x(1, t) - \beta_2 x_t(1, t) - W \text{sign}(x(1, t)), \quad (22)$$

where β_1 , β_2 and W are positive gains to be determined. This controller can be viewed as a PD controller with an additional relay term and explicitly utilizes the measurement $x(\xi, t)$ and its time derivative $x_t(\xi, t)$ at the boundary $\xi = 1$.

Remark 1. Explicit state differentiation is typically not allowed in control synthesis because it can lead to algebraic loops. However, in this case, it is acceptable due to the presence of the dynamic actuator.

Since (22) is discontinuous, along the manifold $x(1, t) = 0$ the right-hand side of (12) is differentiable almost everywhere. Thus, by *Filippov Theory* [18], its solution is absolutely continuous and corresponds to the solution of the associated *differential inclusion*. Nonetheless, following [19], such a generalized *Filippov solution* is equivalent to the *weak solution* of the BVP (18)-(19) with the multi-valued switched feedback (22). The uniqueness of the Filippov solution justifies that, in the remainder, the solution of the closed-loop BVP will be studied, according to Definition 1, in the weak sense.

Notice that as for the finite-dimensional system, any motion along a discontinuity manifold is a *sliding motion*.

A. Main result

Theorem 1. Consider the PDE-ODE BVP (9)-(13) (or equivalently (18)-(19)) with $v(t)$ as in (22). Let Assumption 1 be in force. If the controller gains satisfy

$$\begin{aligned} \beta_1 &> 0, \\ \beta_2 &> \frac{2}{\tau_m \theta_m^2} + \frac{\tau_M}{4} + \frac{2\tau_M}{\theta_m \left(2\tau_m + \frac{8}{3\theta_M} \right)}, \\ W &> 0, \end{aligned} \quad (23)$$

then the solutions $x(\cdot, t)$ of the resulting closed-loop BVP are globally exponentially stable in the space $H^2(0, 1)$.

Its proof is preceded by the next instrumental lemma.

Lemma 4. Let Assumption 1 be satisfied, the functional

$$\begin{aligned} V(t) &= \frac{1}{2} \int_0^1 [x(\zeta, t) \quad x_t(\zeta, t)] \begin{bmatrix} \frac{1}{\tau} & 1 \\ 1 & (2\tau + \frac{8}{3\theta}) \end{bmatrix} \begin{bmatrix} x(\zeta, t) \\ x_t(\zeta, t) \end{bmatrix} d\zeta \\ &+ \frac{\theta}{2} \int_0^1 x_\xi^2(\zeta, t) d\zeta + \left(2\tau + \frac{8}{3\theta} \right) \theta \frac{W}{\tau} |x(1, t)| \\ &+ \frac{1}{2} \left(\left(2\tau + \frac{8}{3\theta} \right) \frac{\theta \beta_1}{\tau} + \frac{\theta \beta_2}{\tau} \right) x^2(1, t), \end{aligned} \quad (24)$$

computed along the solutions $x(\cdot, t)$ of the BVP (9)-(13) is non-negative, and equivalent to the H^2 -norm of $x(\cdot, t)$, i.e.,

$$\exists \rho_1, \rho_2 > 0 : 0 \leq \rho_1 \|x(\cdot, t)\|_{H^2}^2 \leq V(t) \leq \rho_2 \|x(\cdot, t)\|_{H^2}^2. \quad (25)$$

Proof of Lemma 4 Following Lemma 3, and because $\frac{1}{\tau}(2\tau + \frac{8}{3\theta}) > 1$, one can conclude the first integral term in (24) is non-negative. Therefore, it is easy to conclude that for all $W > 0$, $\tau > 0$, it yields $V(t)$ is positive definite.

Let us now focus on the proof of (25). Let $\gamma_1 > 0$, and by means of left inequality of (2), the mixed product within the first integral of (24) satisfies

$$-\left(\frac{1}{2\gamma_1}(2x)^2 + \frac{\gamma_1}{2}x_t^2\right) \leq 2xx_t \leq \left(\frac{1}{2\gamma_1}(2x)^2 + \frac{\gamma_1}{2}x_t^2\right). \quad (26)$$

Then, by noticing that $\|x_t(\cdot, t)\|_{H^0}^2 = \theta^2 \|x_{\xi\xi}(\cdot, t)\|_{H^0}^2$, and by considering the first to terms only of (24), it yields

$$V(t) \geq \left(\frac{1}{2\tau} - \frac{1}{\gamma_1}\right) \|x(\cdot, t)\|_{H^0}^2 + \frac{\theta}{2} \|x_{\xi}(\cdot, t)\|_{H^0}^2 + \frac{1}{2} \left(2\tau + \frac{8}{3\theta} - \frac{\gamma_1}{2}\right) \frac{1}{\theta^2} \|x_{\xi\xi}(\cdot, t)\|_{H^0}^2. \quad (27)$$

Now, let $\gamma_1 = 2(\tau + \varepsilon^2)$ and $\varepsilon^2 < (\tau + \frac{8}{3\theta})$, and because of (8), the left-hand side of (25) is verified, where

$$\rho_1 = \min \left\{ \left(\frac{\varepsilon^2}{\tau(2\tau + 2\varepsilon^2)} \right), \frac{\theta}{2}, \frac{1}{2\theta^2} \left(\tau + \frac{8}{3\theta} - \varepsilon^2 \right) \right\} > 0. \quad (28)$$

To reproduce the right inequality of (25), notice that by virtue of the BC (10) and (1), thus it results that

$$(x_{\xi}(\xi, t))^2 = \left(\int_0^{\xi} x_{\xi\xi}(\zeta, t) d\zeta \right)^2 \leq \|x_{\xi\xi}(\cdot, t)\|_{H^0}^2 \quad (29)$$

holds $\forall \xi \in [0, 1]$ and $t \geq 0$. Now, notice that by virtue of (1) and (29), it holds that

$$\begin{aligned} x^2(1, t) &= \int_0^1 [x(1, t) - x(\zeta, t) + x(\zeta, t)]^2 d\zeta \\ &\leq \frac{1}{2} \left\{ \int_0^1 [x(1, t) - x(\zeta, t)]^2 d\zeta + \int_0^1 (x(\zeta, t))^2 d\zeta \right\} \\ &\leq \frac{1}{2} \left\{ \int_0^1 \max_{\zeta \in [0, 1]} \{ (x_{\xi}(\zeta, t))^2 \} d\zeta + \|x(\cdot, t)\|_{H^0}^2 \right\} \\ &\leq \frac{1}{2} \left\{ \|x_{\xi\xi}(\cdot, t)\|_{H^0}^2 + \|x(\cdot, t)\|_{H^0}^2 \right\} \leq \frac{1}{2} \|x(\cdot, t)\|_{H^2}^2. \end{aligned} \quad (30)$$

To complete the proof, note that by virtue of (28) and (30), and because of (26) holds, the right-hand side of (25) is guaranteed for all $\theta > 0$, $\tau > 0$ and $\gamma_1 > 0$. $\square$

B. Proof of Theorem 1

Consider the functional in (24) where β_1 , β_2 and W are selected in accordance with (23). By Lemma 4, we conclude $V(t) \succ 0$. Let us now differentiate $V(t)$ along the solutions

of the BVP (14)-(17), as follows

$$\begin{aligned} \dot{V}(t) &= \frac{1}{\tau} \int_0^1 x(\zeta, t) x_t(\zeta, t) d\zeta + \int_0^1 x(\zeta, t) z_t(\zeta, t) d\zeta \\ &\quad + \int_0^1 x_t(\zeta, t) z(\zeta, t) d\zeta \\ &\quad + (2\tau + \frac{8}{3\theta}) \int_0^1 z(\zeta, t) z_t(\zeta, t) d\zeta \\ &\quad + \theta \int_0^1 x_{\xi}(\zeta, t) x_{\xi t}(\zeta, t) d\zeta \\ &\quad + (2\tau + \frac{8}{3\theta}) \theta \frac{W}{\tau} z(1, t) \text{sign}(x(1, t)) \\ &\quad + \left((2\tau + \frac{8}{3\theta}) \frac{\beta_1}{\tau} + \frac{\beta_2}{\tau} \right) \theta x(1, t) z(1, t). \end{aligned} \quad (31)$$

Certain terms in the right-hand side of (31) are now manipulated according to the BVP (9)-(13), and the BVP (18)-(19) in distributions. In particular, by substituting (9) into the first term of (31), and integrating it by parts, and due to the BC (10), one obtains

$$\begin{aligned} \frac{1}{\tau} \int_0^1 x(\zeta, t) x_t(\zeta, t) d\zeta &= \frac{\theta}{\tau} \int_0^1 x(\zeta, t) x_{\xi\xi}(\zeta, t) d\zeta \\ &= \frac{\theta}{\tau} x(1, t) x_{\xi}(1, t) - \frac{\theta}{\tau} \|x_{\xi}(\cdot, t)\|_{H^0}^2. \end{aligned} \quad (32)$$

Consider now the second term of (31). Since $z_t(\xi, t)$ is differentiable almost everywhere, following Definition 1, and (21), and the homogeneous BCs (19), one derives

$$\begin{aligned} \int_0^1 x(\zeta, t) z_t(\zeta, t) d\zeta &= -\theta \int_0^1 x_{\xi}(v, t) z_{\xi}(v, t) dv \\ &\quad + \theta x(1, t) \left[\frac{v(t) - x_{\xi}(1, t)}{\tau} \right]. \end{aligned} \quad (33)$$

The third and fifth terms of (31) are instead rewritten as

$$\begin{aligned} \int_0^1 x_t(\zeta, t) z(\zeta, t) d\zeta &= \int_0^1 z(\zeta, t) z(\zeta, t) d\zeta = \|z(\cdot, t)\|_{H^0}^2, \\ \theta \int_0^1 x_{\xi}(\zeta, t) x_{\xi t}(\zeta, t) d\zeta &= \theta \int_0^1 x_{\xi}(\zeta, t) z_{\xi}(\zeta, t) d\zeta. \end{aligned} \quad (34)$$

Finally, the fourth term of (31), following Definition 1, and (21), and the homogeneous BCs (19), is rewritten as

$$\begin{aligned} (2\tau + \frac{8}{3\theta}) \int_0^1 z(\zeta, t) z_t(\zeta, t) d\zeta &= \\ (2\tau + \frac{8}{3\theta}) \left[-\theta \|z_{\xi}(\cdot, t)\|_{H^0}^2 + \theta z(1, t) \left(\frac{v(t) - x_{\xi}(1, t)}{\tau} \right) \right]. \end{aligned} \quad (35)$$

Substituting (32)-(35) into (31), and (22) to $v(t)$, and by applying simplifications and reordering, one derives

$$\begin{aligned} \dot{V}(t) &= -\frac{\theta}{\tau} \|x_{\xi}(\cdot, t)\|_{H^0}^2 - (2\tau + \frac{8}{3\theta}) \theta \|z_{\xi}(\cdot, t)\|_{H^0}^2 \\ &\quad + \|z^2(\zeta, t)\|_{H^0}^2 + \theta x(1, t) \left[-\frac{\beta_1}{\tau} x(1, t) - \frac{W}{\tau} \text{sign}(x(1, t)) \right] \\ &\quad + (2\tau + \frac{8}{3\theta}) \theta z(1, t) \left[-\frac{\beta_2}{\tau} z(1, t) - \frac{1}{\tau} x_{\xi}(1, t) \right]. \end{aligned}$$

Now, by applying Lemma 2 to the third positive term, $\dot{V}(t)$ can be upper-estimated as next

$$\begin{aligned} \dot{V}(t) &\leq -\frac{\lambda'_1 > 0}{\theta \beta_1} x^2(1, t) - \frac{\lambda'_2 > 0}{\theta} \|x_{\xi}(\cdot, t)\|_{H^0}^2 - \frac{\lambda'_3 > 0}{\theta W} |x(1, t)| \\ &\quad - (2\tau + \frac{8}{3\tau}) \frac{\theta \beta_2}{\tau} z^2(1, t) - (2\tau + \frac{8}{3\theta}) \theta \|z_{\xi}(\cdot, t)\|_{H^0}^2 \\ &\quad + 2 \left(z^2(1, t) + \|z_{\xi}(\cdot, t)\|_{H^0}^2 \right) + (2\tau + \frac{8}{3\theta}) \frac{\theta}{\tau} \left| z(1, t) x_{\xi}(1, t) \right|. \end{aligned} \quad (36)$$

Consider now the last non-negative term of (36). Using Lemma 1.b, such term can further be upper-estimated as

$$|z(1,t) x_\xi(1,t)| \leq \left(\frac{1}{2\gamma_2} z^2(1,t) + \frac{\gamma_2}{2} x_\xi^2(1,t) \right), \quad (37)$$

where, because of $x_\xi(0,t) = 0$, and Lemma 2, one derives

$$\begin{aligned} x_\xi^2(1,t) &= (x_\xi(1,t) - x_\xi(0,t))^2 = \left(\int_0^1 x_{\xi\xi}(\zeta,t) d\zeta \right)^2 \\ &\leq \int_0^1 x_{\xi\xi}^2(\zeta,t) d\zeta = \|x_{\xi\xi}(\cdot,t)\|_{H^0}^2 \\ &= \frac{1}{\theta^2} \|z(\cdot,t)\|_{H^0}^2 \leq \frac{2}{\theta^2} \left(z^2(1,t) + \|z_\xi(\cdot,t)\|_{H^0}^2 \right). \end{aligned}$$

Now, by substituting it into (37), as follows

$$|z(1,t) x_\xi(1,t)| \leq \left(\frac{1}{2\gamma_2} + \frac{\gamma_2}{\theta^2} \right) z^2(1,t) + \frac{\gamma_2}{\theta^2} \|z_\xi(\cdot,t)\|_{H^0}^2,$$

and by letting $\gamma_2 = \tau\theta^2/4 > 0$ one further obtains

$$|z(1,t)| |x_\xi(1,t)| \leq \left(\frac{2}{\tau\theta^2} + \frac{\tau}{4} \right) z^2(1,t) + \frac{\tau}{4} \|z_\xi(\cdot,t)\|_{H^0}^2. \quad (38)$$

Now, replacing (38) to the last term of (36), and neglecting its third negative term, $\dot{V}(t)$ is further upper-estimated as next

$$\begin{aligned} \dot{V}(t) &\leq -\lambda_1' x^2(1,t) - \lambda_2' \|x_\xi(\cdot,t)\|_{H^0}^2 \\ &\quad - \overbrace{\left(2\tau + \frac{8}{3\theta} \right) \frac{\theta}{\tau} \left[\beta_2 - \left(\frac{2}{\tau\theta^2} + \frac{\tau}{4} \right) - \frac{2\tau}{\theta \left(2\tau + \frac{8}{3\theta} \right)} \right]}^{\lambda_1'' > 0 \text{ (due to (23))}} z^2(1,t) \\ &\quad - \overbrace{\left[\left(2\tau + \frac{8}{3\theta} \right) \theta - \left(2\tau + \frac{8}{3\theta} \right) \frac{\theta}{\tau} \frac{\tau}{4} - 2 \right]}^{\lambda_2''} \|z_\xi(\cdot,t)\|_{H^0}^2. \end{aligned}$$

Finally, by simply noticing that $\forall \theta > 0, \tau > 0$, it yields

$$\lambda_2'' = \left[2\theta\tau - \frac{\tau\theta}{2} + \frac{8}{3} + \frac{2}{3} - 2 \right] = \left[\frac{3\tau\theta}{2} + \frac{4}{3} \right] > 0,$$

one can easily conclude that

$$\begin{aligned} \dot{V}(t) &\leq -\lambda_1' x^2(1,t) - \frac{1}{2} \lambda_2' \|x_\xi(\cdot,t)\|_{H^0}^2 - \frac{1}{2} \lambda_2' \|x_\xi(\cdot,t)\|_{H^0}^2 \\ &\quad - \lambda_1'' z^2(1,t) - \lambda_2'' \|z_\xi(\cdot,t)\|_{H^0}^2 < 0. \end{aligned} \quad (39)$$

Following Lyapunov's Theorem, the control (22), with gains selected as in (23), asymptotically stabilize the heat process (9)-(13) despite the diffusivity parameter uncertainty, and the uncertain actuator dynamics.

Now, by invoking Lemma 2, and the identity $\|z(\xi,t)\|_{H^0}^2 = \|x_t(\cdot,t)\|_{H^0}^2 = \theta^2 \|x_{\xi\xi}(\xi,t)\|_{H^0}^2$, it results

$$\lambda_1' x^2(1,t) + \frac{1}{2} \lambda_2' \|x_\xi(\cdot,t)\|_{H^0}^2 \geq \rho_3' \|x(\cdot,t)\|_{H^0}^2, \quad (40)$$

$$\lambda_1'' z^2(1,t) + \lambda_2'' \|z_\xi(\cdot,t)\|_{H^0}^2 \geq \rho_3'' \|x_{\xi\xi}(\cdot,t)\|_{H^0}^2$$

with $\rho_3' = \frac{1}{2} \min \{ \lambda_1', \frac{1}{2} \lambda_2' \}$ and $\rho_3'' = \frac{1}{2\theta^2} \min \{ \lambda_1'', \lambda_2'' \}$. Thus, by substituting (40) into (39), one concludes that

$$\dot{V}(t) \leq -\rho_3 \|x(\cdot,t)\|_2^2 \quad \text{with} \quad \rho_3 = \min \left\{ \rho_3', \frac{1}{2} \lambda_2', \rho_3'' \right\}.$$

From that, and thanks to (25), the point-wise exponential stability of the closed-loop process in the H^2 -norm is proved.

V. REJECTION OF DISTURBANCES AT THE BOUNDARY

Since (22) is discontinuous, we conjecture that it could effectively reject smooth time-varying disturbances acting at the actuated PDE boundary $\xi = 1$, as described next:

$$q_\xi(1,t) = d(t) + u(t). \quad (41)$$

These disturbances occur between the actuator and the controlled PDE boundary, making them unmatched with respect to the control $v(t)$ provided to the actuator (5). Indeed, by exploiting the same state-augmentation used in (14)-(17), then (16) would be replaced by:

$$z_\xi(1,t) = \dot{u}(t) + \dot{d}(t), \quad (42)$$

while (18) would become

$$z_t(\xi,t) = \theta z_{\xi\xi}(\xi,t) + \theta \left[\frac{1}{\tau} v(t) + \frac{d(t) - x_\xi(1,t)}{\tau} + \dot{d}(t) \right] \delta(\xi - 1).$$

From this, we have reason to believe (22) with a large enough $W > 0$ can completely reject the effect of smooth disturbances, satisfying:

$$\exists D_0, D_1 : \|d(t)\|_\infty \leq D_0, \|\dot{d}(t)\|_\infty \leq D_1 \quad \forall t \geq 0. \quad (43)$$

At present, this has been confirmed through simulations. Thus further investigations are needed to extend our stability analysis to this case and to derive a proper tuning rule for the relay control gain. This would be beyond also [7], [10], [11] which only considers PDEs affected by matched disturbances.

VI. SIMULATIONS

In this section, we present the results of simulations where θ and τ in (3) and (5) were randomly selected within the domains $\theta \in [\theta_m, \theta_M] = [8, 12]$ and $\tau \in [\tau_m, \tau_M] = [0.5, 1]$. We applied Neumann-type boundary conditions as in (4), with $u(0) = v(0) = 0$, and set the initial conditions (ICs) to $q^0(\xi) = 10 + 5 \cos(4\pi\xi)$.

To numerically solve the closed-loop combined PDE-ODE boundary value problem (BVP) (9)-(13), we employed the finite-difference approximation method. The spatial domain $\xi \in [0, 1]$ was discretized into $N = 30$ uniformly spaced nodes $\xi_i = ih$, where $h = 1/(N-1)$ and $i = 0, 1, 2, \dots, N-1$. The resulting N -th order system was then solved using the Euler-Forward method with a time step of 10^{-5} .

For comparison, two pairs of tests were conducted. In all tests, $q^r = 20$. The first pair of tests did not account for the presence of disturbances. The left plot of Fig. 1 shows the behavior of the deviation field when only the proportional control component of $v(t)$ was active, specifically with $\beta_2 = W = 0$. In the right plot, following the tuning rules in (23), the control was implemented with $\beta_1 = 1$, $\beta_2 = 0.6$, and $W = 1.1$. It follows the proposed PD-plus-Relay control achieved much better settling time and reduced oscillations.

The second pair of tests considered the effect of an unmatched disturbance as in (41), where $d(t) = \sin(2\pi t)$, with $|d(t)| \leq D_0 = 1$ and $|\dot{d}(t)| \leq D_1 = 2\pi$. The left plot of Fig. 2 shows the behavior of the deviation field when $\beta_1 = 1$, $\beta_2 = 0.6$, and $W = 1.1 > D_0 = 1$, as in the previous

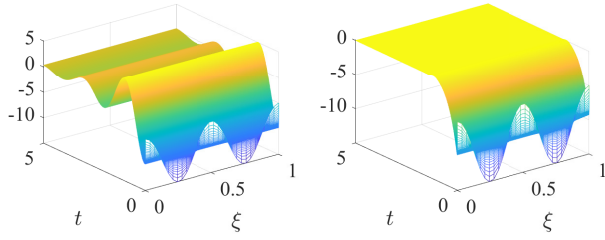


Fig. 1: Profiles of $x(\xi, t)$ when $d(t) = 0$ and only $\beta_1 = 1$ is non-zero (left), and with $\beta_1 = 1$, $\beta_2 = 0.6$, and $W = 1.1$ satisfying (23) (right).

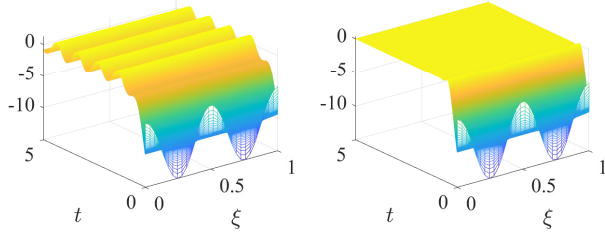


Fig. 2: Profiles of $x(\xi, t)$ when $d(t) = \sin(2\pi t)$ and $\beta_1 = 1$, $\beta_2 = 0.6$, and $W = 1.1 > D_0 = 1$ (left), and with $\beta_1 = 1$, $\beta_2 = 0.6$, and $W = 7.3 > D_0 + D_1 = 7.2$ (right).

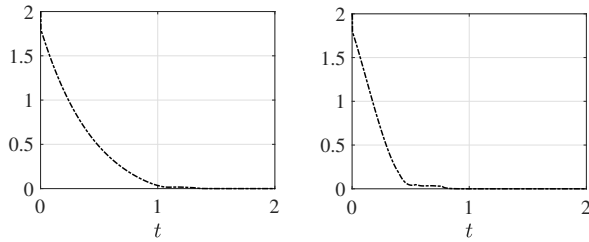


Fig. 3: Evolution of $\|x(\cdot, t)\|_{H^2}$ when $d(t) = 0$ and $\beta_1 = 1$, $\beta_2 = 0.6$, and $W = 1.1$ (left) and with $d(t) = \sin(2\pi t)$ and $\beta_1 = 1$, $\beta_2 = 0.6$, and $W = 7.3 > D_0 + D_1 = 7.2$ (right).

test. The right plot considers a larger, discontinuous gain with $W = 7.3 > D_0 + D_1 = 7.2$. These simulations confirm our conjecture that W should dominate, following (V), the sum of the upper bound of the disturbance and that of its first derivative.

Finally, the left plot of Fig. 3 confirms the results summarized by Theorem 1 and Lemma 4, regarding the decay to zero of the H^2 -norm of the deviation field (7). It is noteworthy that a sufficiently large discontinuous gain W guarantees the same convergence characteristics, despite the perturbation effects of the unmatched disturbance.

The attainment of the prescribed set-point confirms the expected robustness to the presence of a non-instantaneous actuation, the uncertainties on θ and τ , and the unmatched non-vanishing disturbance of the proposed controller.

VII. CONCLUSIONS

The boundary control problem of an uncertain heat equation coupled with an uncertain first-order dynamic actuator

was addressed by proposing a PD-plus-Relay control algorithm that relies solely on boundary actuation and sensing. A formal analysis confirmed the pointwise stabilization of the deviation field in the $H^2(0, 1)$ space.

Simulations further demonstrated the control algorithm's ability to manage unmatched perturbations at the controlled boundary. Future work will focus on developing a formal Lyapunov analysis to derive tuning rules for the control gains in this context. Additionally, extending this approach to higher-order uncertain actuator dynamics and distributed in-space actuation with “ N ” dynamic actuators would be of interest.

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