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Scale-free estimation of an average state in large-scale systems

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***Abstract*— This paper provides a computationally tractable necessary and sufficient condition for the existence of an average state observer for large-scale linear time-invariant (LTI) systems. Two design procedures, each with its own significance, are also devised. When the necessary and sufficient condition is not satisfied, a methodology is proposed to obtain the best asymptotic estimate of the average state. In particular, the estimation problem is addressed by aggregating the unmeasured states of the original system and obtaining a projected system of reduced dimension. Thus, the dimension of the proposed observer equals the dimension of the projected system. Moreover, one obtains an observer of dimension equal to one under the second design procedure irrespective of the system dimension. Therefore, the estimation is 'scale-free' since the complexity of the estimation task doesn't scale with the size of the system.**

***Index Terms*— Large-scale systems, state aggregation, average state observer, scale-free estimation.**

I. Introduction

Network processes such as transportation [1], epidemic spread [2], building thermal [3], and power grid [4] are examples of large-scale systems. Such systems require tremendous amount of computational and sensing resources for monitoring purposes. Therefore, it is crucial to develop scalable state estimation algorithms.

Estimating the whole state of a system is not only computationally expensive but sometimes impossible due to limited number of available sensors, which may render the system unobservable. Nevertheless, for control and monitoring, the knowledge of an aggregated quantity of the state is often sufficient. For instance, in positive systems [5], the average of the state vector provides a suitable estimate for the state norm, which finds applications in feedback stabilization [6]. Thus we study the estimation of an average state for LTI systems in this paper.

Average state estimation has been recently studied in [7], where clustered model reduction [8] is used to reduce the complexity of the estimation task. However, the proposed methodology is only suitable for stable LTI systems and the clustering algorithm provided doesn't minimize the model approximation error, which in turn impairs the estimation.

We follow the approach of functional observers, [9], but by addressing the problem from a lower-order projection of the original system. In particular, our goal is to estimate the average of unmeasured part of the state vector, which is aggregated to obtain a system with dimension equal to the number of measured states plus one. This handles the issue of complexity in large-scale network systems and enables us to design an observer whose dimension does not scale with the dimension of the original system, hence the term "scale-free estimation," similar to which is introduced in [10].

The notion of average observability is studied in [11], which presents the observability analysis and provides a necessary and sufficient condition for reconstructing the average state from the output of the system. This paper, on the other hand, provides the design procedures for average state observers. We remark that the necessary and sufficient condition for the existence of such an observer is only sufficient for average observability, and not necessary.

To summarize, our contribution in this paper is as follows: 1) We provide a necessary and sufficient condition for the existence of an average state observer; 2) We provide two design procedures for the average state observer; 3) We prove the boundedness of the estimation error under suitable assumptions when the average state observer does not exist; 4) We devise an optimal methodology to minimize the asymptotic value of the estimation error.

II. Notation

The real and complex numbers in the interval a are denoted as $\mathbb{R}_a$ and $\mathbb{C}_a$, respectively. Matrices are denoted by uppercase letters and vectors by boldface lowercase letters. A vector of ones and an identity matrix are denoted as $\mathbf{1}_n$ and I_n, respectively, whereas $0_{n\times m}$ and 0_n denote a matrix and a vector of zeros, respectively, with subscripts indicating their dimensions. The subscripts are sometimes omitted for brevity. For a matrix $A \in \mathbb{R}^{n\times m}$, we denote its transpose and pseudo-inverse by A^T and A^+, respectively; and $\text{eig}(A)$ denotes the set of its eigenvalues when $n = m$. A (block) diagonal matrix is written as $\text{diag}[A_1, \cdots, A_n]$ with matrices $A_1, \cdots, A_n$ at its diagonal. The absolute value of a scalar is denoted by $|\cdot|$ and a generic p-norm for vectors by $\|\cdot\|$. With each vector norm $\|\cdot\|$ on $\mathbb{R}^n$, we associate a matrix norm $|||\cdot|||$ that is induced by $\|\cdot\|$ on $\mathbb{R}^{n\times n}$.

III. Problem definition

Linear time-invariant (LTI) systems of the form

$$\begin{aligned} \dot{\mathbf{x}}(t) &= A\mathbf{x}(t) + B\mathbf{u}(t) \\ \mathbf{y}(t) &= C\mathbf{x}(t) \end{aligned} \qquad (\boldsymbol{\Sigma})$$

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are considered, where $\mathbf{x}(t) \in \mathbb{R}^n$, $\mathbf{u}(t) \in \mathbb{R}^p$, and $\mathbf{y}(t) \in \mathbb{R}^m$ are the state, input, and output vectors, respectively, and $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times p}$, $C \in \mathbb{R}^{m\times n}$ are constant matrices. Without loss of generality, we assume the partition of the state vector $\mathbf{x}(t) = [\mathbf{x}_1^T(t) \;\; \mathbf{x}_2^T(t)]^T$, where $\mathbf{x}_1(t) \in \mathbb{R}^k$ and $\mathbf{x}_2(t) = \mathbf{y}(t) \in \mathbb{R}^m$ with $n = k + m$. Thus, the system matrices are partitioned as

$$\begin{aligned} A &= \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \qquad B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}, \qquad (1) \\ C &= \begin{bmatrix} 0_{m\times k} & I_m \end{bmatrix}; \end{aligned}$$

where the dimensions of the matrix blocks are in accordance with the state partition.

Referring to $\mathbf{x}_1(t)$ and $\mathbf{x}_2(t)$ as unmeasured and measured states, respectively, we aim to estimate the average of the unmeasured state, i.e.

$$z_1(t) = \frac{1}{k}\mathbf{1}_k^T \mathbf{x}_1(t).$$

We obtain a lower-order system by aggregating the unmeasured state $\mathbf{x}_1(t)$ as

$$\begin{aligned} \dot{\mathbf{z}}(t) &= E\mathbf{z}(t) + F\boldsymbol{\sigma}(t) + G\mathbf{u}(t) \\ \mathbf{y}(t) &= H\mathbf{z}(t) \end{aligned} \quad , \qquad (\boldsymbol{\Sigma}_r)$$

where the state vector $\mathbf{z}(t) = \begin{bmatrix} z_1(t) & \mathbf{x}_2^T(t) \end{bmatrix}^T \in \mathbb{R}^{m+1}$ and the matrices are given as

$$\begin{aligned} E &= \begin{bmatrix} \frac{1}{k}\mathbf{1}^T A_{11}\mathbf{1} & \frac{1}{k}\mathbf{1}^T A_{12} \\ A_{21}\mathbf{1} & A_{22} \end{bmatrix}, \; F = \begin{bmatrix} \frac{1}{k}\mathbf{1}^T A_{11} \\ A_{21} \end{bmatrix} \\ H &= \begin{bmatrix} 0_m & I_m \end{bmatrix}, \qquad G = \begin{bmatrix} \frac{1}{k}\mathbf{1}^T B_1 \\ B_2 \end{bmatrix}. \qquad (2) \end{aligned}$$

The average deviation vector $\boldsymbol{\sigma}(t) \in \mathbb{R}^k$ is given by

$$\boldsymbol{\sigma}(t) := (I - \frac{1}{k}\mathbf{1}_k\mathbf{1}_k^T)\mathbf{x}_1(t), \qquad (3)$$

where we note that

$$\mathbf{1}_k^T \boldsymbol{\sigma}(t) = 0 \qquad (4)$$

for all $t \in \mathbb{R}_{\geq 0}$.

The goal is to devise an observer whose estimate $\hat{z}_1(t)$ converges to the true value $z_1(t)$ asymptotically as $t \to \infty$. This necessarily involves filtering $\boldsymbol{\sigma}(t)$ from $\boldsymbol{\Sigma}_r$. Thus the problem considered here is similar to the problem of observer design for systems with unknown inputs, [12], [13], where $\boldsymbol{\sigma}(t)$ is assumed to be completely arbitrary. However, in our case, $\boldsymbol{\sigma}(t)$ is not arbitrary but satisfies (4). It is quite intuitive that the possibility to filter out the effect of $\boldsymbol{\sigma}(t)$ is equivalent to the existence of an average state observer. This consideration directs the study in two directions:

1) Find a necessary and sufficient condition under which the effect of $\boldsymbol{\sigma}(t)$ can be filtered out by an average state observer and devise the design procedures;
2) If the average state observer doesn't exist, devise a methodology to obtain a best possible estimate $\hat{z}_1(t)$ as $t \to \infty$.

IV. Asymptotic estimation of average state

The following form of the observer is considered

$$\begin{aligned} \dot{\mathbf{w}}(t) &= M\mathbf{w}(t) + K\mathbf{y}(t) + NG\mathbf{u}(t) \\ \hat{\mathbf{z}}(t) &= \mathbf{w}(t) + L\mathbf{y}(t) \end{aligned} \quad , \qquad (\widehat{\boldsymbol{\Sigma}}_r)$$

where $\mathbf{w}(t), \hat{\mathbf{z}}(t) \in \mathbb{R}^{m+1}$ are the state and output of $\widehat{\boldsymbol{\Sigma}}_r$, respectively, and M, K, N, L are matrices of appropriate dimensions. Let the estimation error be $\mathbf{e}(t) := \mathbf{z}(t) - \hat{\mathbf{z}}(t)$, which satisfies

$$\dot{\mathbf{e}}(t) = M\mathbf{e}(t) + NF\boldsymbol{\sigma}(t)\,, \qquad (5)$$

where

$$\begin{aligned} N = I - LH, &\quad M = NE - K_1 H, \qquad (6) \\ K_2 = ML, &\qquad K = K_1 + K_2. \end{aligned}$$

If $\|\mathbf{e}(t)\| \to 0$ as $t \to \infty$, then we say that the observer $\widehat{\boldsymbol{\Sigma}}_r$ estimates $\mathbf{z}(t)$ asymptotically.

In what follows, we abide by the following assumption.

Assumption IV.1. The pair (H, E) is observable, i.e. $\operatorname{rank} \mathcal{O}_{H,E} = m + 1$, where H, E are given in (2) and $\mathcal{O}_{H,E} = \begin{bmatrix} H^T & (HE)^T & \cdots & (HE^m)^T \end{bmatrix}^T$.

Observability of the pair (H, E) is a necessary condition for the observability of $\boldsymbol{\Sigma}_r$, [11], which is necessary for the existence of an observer $\widehat{\boldsymbol{\Sigma}}_r$ such that $\|\mathbf{e}(t)\| \to 0$ as $t \to \infty$ at an arbitrary rate, [14]. To show that it is a very mild assumption, we present the following lemma.

Lemma IV.1. The pair (H, E) is observable if and only if $A_{21}\mathbf{1}_k \neq 0_m$, where H, E are given in (2) and A_{21} in (1).

Proof. For necessity, suppose $A_{21}\mathbf{1}_k = 0_m$, then the pair (H, E) is in an observable canonical form, [15], hence not observable. For sufficiency, consider

$$\mathcal{O}_{H,E} = \begin{bmatrix} 0_m & I_m \\ A_{21}\mathbf{1}_k & A_{22} \\ * & * \end{bmatrix},$$

where $*$ denotes the remaining terms. Here, notice that if $A_{21}\mathbf{1}_k \neq 0_m$, then $\operatorname{rank} \mathcal{O}_{H,E} = m + 1$. $\square$

A. Necessary and sufficient condition

We provide a necessary and sufficient condition that is easy to check computationally since it doesn't require, unlike [16], to construct several observability matrices, which is computationally not feasible for large-scale systems.

Theorem IV.1. Given a system $\boldsymbol{\Sigma}$, consider its lower-order projection $\boldsymbol{\Sigma}_r$ and an observer $\widehat{\boldsymbol{\Sigma}}_r$. Suppose that $\boldsymbol{\sigma}(t) \neq 0_k$ for some interval in $t \in \mathbb{R}_{\geq 0}$. Then, the estimation error $\mathbf{e}(t) := \mathbf{z}(t) - \hat{\mathbf{z}}(t)$ converges to 0 as $t \to \infty$ at an arbitrary rate if and only if

$$\operatorname{rank} \begin{bmatrix} \mathbf{1}_k^T \\ \mathbf{1}_k^T A_{11} \\ A_{21} \end{bmatrix} = \operatorname{rank} A_{21}, \qquad (7)$$

where $A_{11} \in \mathbb{R}^{k\times k}$ and $A_{21} \in \mathbb{R}^{m\times k}$ are given in (1).

Proof. Recall the error dynamics in (5) and the property (4). Then, it is easy to see that $NF\boldsymbol{\sigma}(t) = 0$ if and only if

$NF = \mathbf{v}\mathbf{1}^T$ for some $\mathbf{v} \in \mathbb{R}^{m+1}$, where F is given in (2) and N in (6). The equation $NF = \mathbf{v}\mathbf{1}^T$ with $NF = (I - LH)F$ can be written as

$$\begin{bmatrix} 1 & -\boldsymbol{\ell}_1^T \\ 0_m & I_m - L_2 \end{bmatrix} \begin{bmatrix} \frac{1}{k}\mathbf{1}^T A_{11} \\ A_{21} \end{bmatrix} = \begin{bmatrix} v_1 \mathbf{1}^T \\ \mathbf{v}_2 \mathbf{1}^T \end{bmatrix}, \tag{8}$$

where $L = \begin{bmatrix} \boldsymbol{\ell}_1^T \\ L_2 \end{bmatrix} \in \mathbb{R}^{(m+1)\times m}$ and $\mathbf{v} = \begin{bmatrix} v_1 \\ \mathbf{v}_2 \end{bmatrix} \in \mathbb{R}^{m+1}$. To satisfy the upper part of (8), we must have

$$\operatorname{rank} \begin{bmatrix} \frac{1}{k}\mathbf{1}^T A_{11} - v_1 \mathbf{1}^T \\ A_{21} \end{bmatrix} = \operatorname{rank} A_{21}, \tag{9}$$

for some $v_1 \in \mathbb{R}$. The lower part of (8) is satisfied either if $L_2 \neq I_m$ and

$$\operatorname{rank} \begin{bmatrix} \mathbf{1}^T \\ A_{21} \end{bmatrix} = \operatorname{rank} A_{21}, \tag{10}$$

or if $L_2 = I_m$. We prove the necessity and sufficiency of the theorem as follows.

Sufficiency: If $L_2 \neq I_m$, then $N = I - LH$ is such that the pair (H, NE) is observable (see Theorem 8.M3 in [15]). Therefore, $\operatorname{eig}(M) \subset \mathbb{C}_{<0}$ can be assigned arbitrarily by the algorithm presented in [17]. Thus $\|\mathbf{e}(t)\| \to 0$ as $t \to \infty$ at an arbitrary rate determined by $\operatorname{eig}(M) \subset \mathbb{C}_{<0}$. To satisfy (8) for $L_2 \neq I_m$, the condition (10) must hold. Since (7) implies (9) and (10), the sufficiency of (7) is proved.

Necessity: If $L_2 = I_m$, then $N = I - LH$ is such that (H, NE) is not an observable pair. However, if we choose $K = [\mathbf{k}_{11} \;\; K_{12}^T]^T$ such that $\mathbf{k}_{11}^T = \frac{1}{k}\mathbf{1}^T A_{12} - \boldsymbol{\ell}_1^T A_{22}$ and $K_{12} = -\operatorname{diag}[\lambda_2, \cdots, \lambda_{m+1}]$, then we have

$$M = \operatorname{diag}[\lambda_1(v_1), \lambda_2, \cdots, \lambda_{m+1}], \tag{11}$$

where $\lambda_1(v_1) = \frac{1}{k}\mathbf{1}^T A_{11}(I - A_{21}^+ A_{21})\mathbf{1} + v_1 \mathbf{1}^T A_{21}^+ A_{21}\mathbf{1}$ and $\lambda_i \in \mathbb{R}_{<0}$ for $i = 2, \cdots, m+1$. For M to be a Hurwitz matrix, it is necessary that

$$v_1 < \frac{\mathbf{1}^T A_{11}(A_{21}^+ A_{21} - I)\mathbf{1}}{k\mathbf{1}^T A_{21}^+ A_{21}\mathbf{1}}. \tag{12}$$

Furthermore, to have $\lambda_1(v_1) \in \mathbb{R}_{<0}$ arbitrary, i.e. $\|\mathbf{e}(t)\| \to 0$ as $t \to \infty$ at an arbitrary rate determined by $\operatorname{eig}(M) \subset \mathbb{R}_{<0}$, it must hold that (9) is satisfied for all $v_1 \in \mathbb{R}$ satisfying (12). In other words, the rows of A_{21} must span the plane formed by $\operatorname{span}\{\mathbf{1}^T A_{11}, \mathbf{1}^T\}$ excluding the line $\operatorname{span}\{\mathbf{1}^T\}$. That is, for general $\boldsymbol{\Sigma}$, (9) must hold for all $v_1 \in \mathbb{R}$, which proves the necessity of (7). □

B. Design procedures for average state observer

As a consequence of the proof of Theorem IV.1, we develop two design procedures for $\widehat{\boldsymbol{\Sigma}}_r$. The first design provides the matrix L such that the pair (H, NE) is observable, where $N = I - LH$. Hence, $\operatorname{eig}(M) \subset \mathbb{C}_{<0}$ can be arbitrarily assigned by finding the matrix K_1 by the pole-placement algorithm [17]. The second design provides the choice of L and K such that M is a diagonal matrix with diagonal entries in $\mathbb{R}_{<0}$. This design yields an average state observer of dimension one, whose convergence is tuned by choosing appropriate value of v_1 satisfying (12).

Proposition IV.2 (Design 1). Let the system $\boldsymbol{\Sigma}$ be such that (7) is satisfied. Consider the observer $\widehat{\boldsymbol{\Sigma}}_r$ such that

$$\begin{aligned} L &= \begin{bmatrix} \frac{1}{k}\mathbf{1}^T A_{11} - v_1 \mathbf{1}^T \\ A_{21} - \mathbf{v}_2 \mathbf{1}^T \end{bmatrix} A_{21}^+, \\ K_1 &= \operatorname{place}(NE, H, [\lambda_1, \ldots, \lambda_{m+1}]), \end{aligned} \tag{13}$$

where $\lambda_i \in \mathbb{C}_{<0}$, for $i = 1, \cdots, m+1$, are the desired eigenvalues of M, "place" is the pole-placement algorithm provided in [17], $v_1 \in \mathbb{R}$ and $\mathbf{v}_2 \in \mathbb{R}^m \setminus \{0_m\}$ are arbitrary, and the matrices M, N, K are chosen according to (6). Then, the estimation error $\mathbf{e}(t)$ converges to 0 as $t \to \infty$ at a rate determined by $\operatorname{eig}(M) = \{\lambda_1, \cdots, \lambda_{m+1}\}$.

Proof. Since (7) is satisfied, L in (13) is the solution satisfying (8) for some arbitrary values of $v_1 \in \mathbb{R}$ and $\mathbf{v}_2 \in \mathbb{R}^m$, see [18]. Therefore, $N = I - LH$ is such that $NF = \mathbf{v}\mathbf{1}^T$, hence $NF\boldsymbol{\sigma}(t) = 0$ for all $t \in \mathbb{R}_{\geq 0}$. Moreover, N is such that the pair (H, NE) is observable (refer to the sufficiency part of the proof of Theorem IV.1). Therefore, K_1 obtained by the pole placement algorithm [17] gives $M = NE - K_1 H$ with $\operatorname{eig}(M) = \{\lambda_1, \cdots, \lambda_{m+1}\}$, where $\lambda_i \in \mathbb{C}_{<0}$. □

The purpose of the second design procedure is to obtain a diagonal M as in (11), with its first entry depending on $v_1 \in \mathbb{R}$, where v_1 must satisfy (12) to make M a Hurwitz matrix.

Proposition IV.3 (Design 2). Let the system $\boldsymbol{\Sigma}$ be such that (7) is satisfied. Consider the observer $\widehat{\boldsymbol{\Sigma}}_r$ such that

$$\begin{aligned} L &= \begin{bmatrix} \boldsymbol{\ell}_1^T \\ L_2 \end{bmatrix} = \begin{bmatrix} \left(\frac{1}{k}\mathbf{1}^T A_{11} - v_1 \mathbf{1}^T\right) A_{21}^+ \\ I_m \end{bmatrix}, \\ K_1 &= \begin{bmatrix} \mathbf{k}_{11}^T \\ K_{12} \end{bmatrix} = \begin{bmatrix} \frac{1}{k}\mathbf{1}^T A_{12} - \boldsymbol{\ell}_1^T A_{22} \\ -\operatorname{diag}[\lambda_2, \cdots, \lambda_{m+1}] \end{bmatrix}, \end{aligned} \tag{14}$$

where $\lambda_i \in \mathbb{R}_{<0}$, for $i = 2, \cdots, m+1$, and $v_1 \in \mathbb{R}$ such that (12) is satisfied, and the matrices M, N, K chosen according to (6). Then, the estimation error $\mathbf{e}(t)$ converges to 0 as $t \to \infty$ at a rate determined by $\operatorname{eig}(M) = \{\lambda_1(v_1), \cdots, \lambda_{m+1}\}$, where M is given in (11) and

$$\lambda_1(v_1) = \frac{1}{k}\mathbf{1}^T A_{11}(I - A_{21}^+ A_{21})\mathbf{1} + v_1 \mathbf{1}^T A_{21}^+ A_{21}\mathbf{1}. \tag{15}$$

Proof. The choice of L is such that $NF\boldsymbol{\sigma}(t) = 0$ for all $t \in \mathbb{R}_{\geq 0}$. Moreover, the choice of matrices in (14) yields a diagonal M, as in (11), where v_1 is chosen such that it satisfies (12). Therefore, $\operatorname{eig}(M) \subset \mathbb{R}_{<0}$ are assigned arbitrarily and $\|\mathbf{e}(t)\| \to 0$ as $t \to \infty$. □

In design 2, the eigenvalues can only be chosen in $\mathbb{R}_{<0}$ and not in $\mathbb{C}_{<0}$. However, this design yields a reduced-order average observer of dimension equal to 1.

Proposition IV.4 (Reduced-order average observer). Consider the observer $\widehat{\boldsymbol{\Sigma}}_r$. Then, the choice of design matrices in (14) yields a reduced order average observer

$$\begin{aligned} \dot{w}_1(t) &= \lambda_1(v_1) w_1(t) + \mathbf{k}_{11}^T \mathbf{y}(t) + \mathbf{g}^T \mathbf{u}(t) \\ \hat{z}_1(t) &= w_1(t) + \boldsymbol{\ell}_1^T \mathbf{y}(t) \end{aligned} \tag{$\widehat{\boldsymbol{\Sigma}}_r^1$}$$

such that $z_1(t) - \hat{z}_1(t) \to 0$ as $t \to \infty$, where $w_1(t) \in \mathbb{R}$, ℓ_1^T and $\mathbf{k}_{11}$ given in (14), $\lambda_1(v_1)$ given in (15), and

$$\mathbf{g}^T = \left(\frac{1}{k}\mathbf{1}^T B_1 - \boldsymbol{\ell}_1^T B_2\right) G$$

with $B_1 \in \mathbb{R}^{k\times p}$ and $B_2 \in \mathbb{R}^{m\times p}$ given in (1).

Proof. We can obtain $\widehat{\boldsymbol{\Sigma}}_r^1$ from $\widehat{\boldsymbol{\Sigma}}_r$ since all states in $\widehat{\boldsymbol{\Sigma}}_r$ under (14) are decoupled and are stable. □

The significance of both design procedures are summarized as follows. In design 1, $\mathrm{eig}(M)$ can be arbitrarily assigned in $\mathbb{C}_{<0}$, which gives an extra control over the estimation performance in the transient phase. Design 2, on the other hand, enables us to obtain a reduced-order average observer $\widehat{\boldsymbol{\Sigma}}_r^1$ of dimension equal to 1, which makes the estimation problem scale-free.

V. Approximate estimation of average state

If a system $\boldsymbol{\Sigma}$ doesn't satisfy the necessary and sufficient condition in Theorem IV.1, the estimation error doesn't converge to zero. In this section, we devise a methodology to obtain a best estimate asymptotically. In other words, we minimize $\lim_{t\to\infty} \|\mathbf{e}(t)\|$, where $\lim$ is considered to be supremum limit if the steady state is not achieved.

A. Boundedness of estimation error

We prove under suitable assumptions that the estimation error is bounded, i.e. $\lim_{t\to\infty} \|\mathbf{e}(t)\| < \infty$. The result in this section is a gateway to error minimization considered in the next section.

Theorem V.1. Given a system $\boldsymbol{\Sigma}$, consider its lower-order projection $\boldsymbol{\Sigma}_r$ and an observer $\widehat{\boldsymbol{\Sigma}}_r$ such that $\mathbf{e}(t)$ satisfies (5). Assume one of the following holds:

(i) $\mathrm{eig}(A) \subset \mathbb{C}_{\le 0}$ and $\int_0^\infty \|\mathbf{u}(t)\| dt < \infty$.
(ii) $\mathrm{eig}(A) \subset \mathbb{C}_{<0}$ and $\|\mathbf{u}(t)\| < \infty$ for all $t \in \mathbb{R}_{\ge 0}$.

Then, it holds that $\lim_{t\to\infty} \|\mathbf{e}(t)\| < \infty$.

Proof. Consider (5), where the error trajectory satisfies $\|\mathbf{e}(t)\| \le \left\|e^{Mt}\mathbf{e}(0)\right\| + \left\|\int_0^t e^{M(t-\tau)} NF\boldsymbol{\sigma}(\tau) d\tau\right\|$. Note that M can be chosen to be Hurwitz (i.e. all eigenvalues have negative real parts). Thus, $\lim_{t\to\infty} \left|\left|\left|e^{Mt}\right|\right|\right| = 0$ and

$$\begin{aligned}\lim_{t\to\infty} \|\mathbf{e}(t)\| &\le \lim_{t\to\infty} \left\| \int_0^t e^{M(t-\tau)} NF\boldsymbol{\sigma}(\tau) \right\| d\tau \\ &\le \lim_{t\to\infty} \int_0^t \left\| e^{M(t-\tau)} NF\boldsymbol{\sigma}(\tau) \right\| d\tau \\ &\le \lim_{t\to\infty} \int_0^t \left|\left|\left| e^{M(t-\tau)} \right|\right|\right| \|NF\boldsymbol{\sigma}(\tau)\| \, d\tau.\end{aligned}$$

Since the two functions in the integral are positive by definition of the norm, it holds (see [19])

$$\begin{aligned}\lim_{t\to\infty} \|\mathbf{e}(t)\| &\le \left[\max_{t\ge 0} \|NF\boldsymbol{\sigma}(\boldsymbol{t})\|\right] \left[\lim_{t\to\infty} \int_0^t \left|\left|\left| e^{M(t-\tau)} \right|\right|\right| d\tau\right] \\ &\le \frac{|||V^{-1}||| |||V||| |||NFJ|||}{\lambda^*} \max_{t\ge 0} \|\mathbf{x}_1(t)\|, \qquad (16)\end{aligned}$$

where $J = I - \frac{1}{k}\mathbf{1}\mathbf{1}^T$, $\lambda^* = \min_{\lambda\in \mathrm{eig}(M)} |\mathrm{Re}\{\lambda\}| > 0$, and V is the matrix of eigenvectors in eigenvalue decomposition of M. The inequality (16) is obtained since

$$\begin{aligned}\lim_{t\to\infty} \int_0^t \left|\left|\left| e^{M(t-\tau)} \right|\right|\right| d\tau &= \int_0^\infty \left|\left|\left| e^{M\tau} \right|\right|\right| d\tau \\ &\le \frac{|||V^{-1}||| |||V|||}{\lambda^*}.\end{aligned}$$

Note that, if either of the assumptions (i) and (ii) hold, then we have $\|\mathbf{x}_1(t)\| < \infty$ for all $t \in \mathbb{R}_{\ge 0}$, which completes the proof. □

The bound on the error (16) depends on the condition number of V, which is defined as $\mathrm{cond}(V) := |||V^{-1}||| |||V|||$. An upper bound on the condition number of matrices is widely studied, see [20], however, to our knowledge, there is no upper bound on the condition number of eigenvector matrix V. Therefore, in general, to minimize $|||NFJ|||$ by a suitable choice of N doesn't ensure the minimization of the bound (16) in the case of design 1 (Proposition IV.2). However, in design 2 (Proposition IV.3), the matrix M is diagonal, therefore $V = I$ and $\mathrm{cond}(V) = 1$.

B. Minimizing estimation error at $t \to \infty$

We devise a methodology based on design 2 (Proposition IV.3) to obtain a best average state estimate at $t \to \infty$. Since M is diagonal, (16) is given by

$$\lim_{t\to\infty} \|\mathbf{e}(t)\| \le \frac{|||N(v_1)FJ|||}{\lambda^*} \max_{t\ge 0} \|\mathbf{x}_1(t)\|. \qquad (17)$$

The eigenvalues $\lambda_2, \cdots, \lambda_{m+1}$ of M can be chosen freely, therefore we assume that $\lambda^* = |\lambda_1(v_1)|$, where $\lambda_1(v_1)$ is given in (15).

Proposition V.2. Consider the observer $\widehat{\boldsymbol{\Sigma}}_r$ with design (14). Assume the conditions of Theorem V.1 hold such that (17) is bounded. Then, the bound (17) is minimized by $v_1 \in \mathbb{R}$, which is the solution to

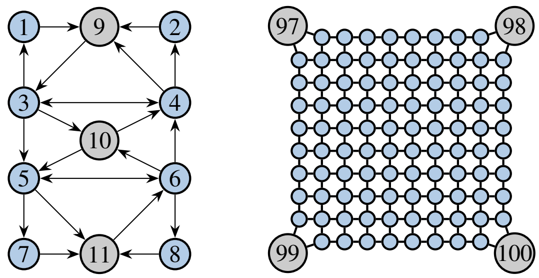
$$\begin{aligned}&\min_{v_1\in\mathbb{R}} \quad \frac{\left\|v_1\mathbf{p}^T - \mathbf{q}^T\right\|}{|v_1\alpha - \beta|} \\ &\text{subject to} \quad v_1\alpha - \beta < 0\,,\end{aligned} \qquad (18)$$

where $\mathbf{p}^T = \mathbf{1}^T(A_{21}^+ A_{21} - I)$, $\mathbf{q}^T = \frac{1}{k}\mathbf{1}^T A_{11}(A_{21}^+ A_{21} - I)$, $\alpha = \mathbf{p}^T\mathbf{1} + k$, $\beta = \mathbf{q}^T\mathbf{1}$, and $\lambda^* = |\alpha v_1 - \beta|$. Here, we assume that $\alpha \neq 0$ and $\beta \neq 0$.

Proof. Consider (17). If (7) doesn't hold, then $|||NFJ||| \neq 0$, where F, N, and J are given in (2), (6), and (16), respectively. Note that $NFJ = 0$ if and only if $NF = \mathbf{v}\mathbf{1}^T$ since $\mathrm{rank}\, J = k-1$, where $\mathbf{v} = [v_1 \;\; \mathbf{v}_2^T]^T \in \mathbb{R}^{m+1}$. Thus minimizing $|||NFJ|||$ is equivalent to minimizing $\left|\left|\left|NF - \mathbf{v}\mathbf{1}^T\right|\right|\right|$, where $\mathbf{v} \in \mathbb{R}^{m+1}$ is a free parameter. By explicitly computing the objective function, we obtain

$$NF - \mathbf{v}\mathbf{1}^T = \begin{bmatrix} \frac{1}{k}\mathbf{1}^T A_{11} - \boldsymbol{\ell}_1^T A_{21} - v_1\mathbf{1}^T \\ (I_m - L_2)\, A_{21} - \mathbf{v}_2\mathbf{1}^T \end{bmatrix}.$$

For a given $\mathbf{v}$, the solution to this minimization problem is given by the method of least squares of the corresponding linear equations, see [18], which gives an analytic solution



(a) Example 1 (b) Example 2

Fig. 1: Graph structures

of L in terms of $\mathbf{v}$ as given in (13). However, notice that $\mathbf{v}_2 = 0_m$ is necessary for a minimizing solution, which gives a matrix L as given in (14) in terms of $v_1 \in \mathbb{R}$. Therefore, if $\lambda^* = |\alpha v_1 - \beta|$, the bound on the steady state estimation error (16) is minimized by minimizing the following

$$\min_{L(v_1)} \frac{\|NFJ\|}{\lambda^*} = \min_{v_1} \frac{\left\|(\frac{1}{k}\mathbf{1}^T A_{11} - v_1 \mathbf{1}^T)(I - A_{21}^+ A_{21})\right\|}{|\alpha v_1 - \beta|},$$

where α, β are given in (18). $\square$

The optimization problem (18) can be solved by an appropriate optimization technique. However, if the norm function in (18) is chosen to be differentiable, one obtains an analytic solution for optimal v_1^*.

Corollary V.2.1. Consider the minimization problem (18) with a Euclidean norm, then the optimal solution is given by

$$v_1^* = \frac{\beta}{\alpha} - \sqrt{\left(\frac{\beta}{\alpha}\right)^2 + \left(\frac{\mathbf{p}^T\mathbf{p}}{\mathbf{q}^T\mathbf{q}}\right) - 2\left(\frac{\beta}{\alpha}\right)\left(\frac{\mathbf{p}^T\mathbf{q}}{\mathbf{q}^T\mathbf{q}}\right)}, \tag{19}$$

where $\alpha, \beta, \mathbf{p}, \mathbf{q}$ are given in (18). Here, we assume that $\alpha \neq 0$, $\beta \neq 0$, and $\mathbf{q}^T\mathbf{q} \neq 0$.

VI. Numerical Simulations

In this section, we present some examples to illustrate the design procedures presented in the preceding sections. For large-scale systems, we represent the state matrix A by a graph, where the nodes represent the states of the system and the edges represent the non-zero entries of A.

Example VI.1. A linear compartmental system [21] with a structure as shown in Fig. 1(a) is considered, where blue nodes represent the unmeasured states and grey nodes the measured states. The state at each node i satisfies $\dot{x}_i(t) = \sum_{j=1}^n a_{ij}x_j(t) - \sum_{h=1}^n a_{hi}x_i(t)$, where $a_{ij} = [A]_{ij}$ with $[A]_{ij} = 1$ if there is edge (i,j), for $i \neq j$, and 0 otherwise. By constructing the matrix A, it can be verified that (7) is satisfied. We generate $\mathbf{x}(0)$ uniformly random in $\mathbb{R}^n_{(-2,2)}$ with $n = 11$. The output matrix $C = [0_{3\times 8} \quad I_3]$ and the input matrix $B = C^T$ with $\mathbf{u}(t) = 10[\sin t \quad \sin 10t \quad \sin 20t]^T$. Fig. 2 shows the estimation results of the two design procedures presented in this paper. The eigenvalues of M are chosen to be same in both procedures to present a comparison.

Design 1. Choose $v_1 = 1$ and $\mathbf{v}_2 = \mathbf{1}$. Compute L by (13). The desired $\operatorname{eig}(M) = \{-0.75, -1, -2, -3\}$, for which we

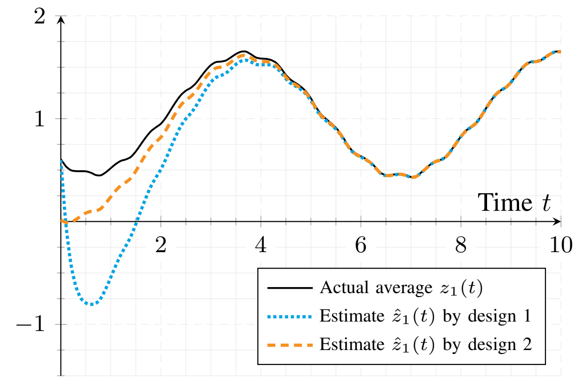


Fig. 2: Average state estimation for Example 1.

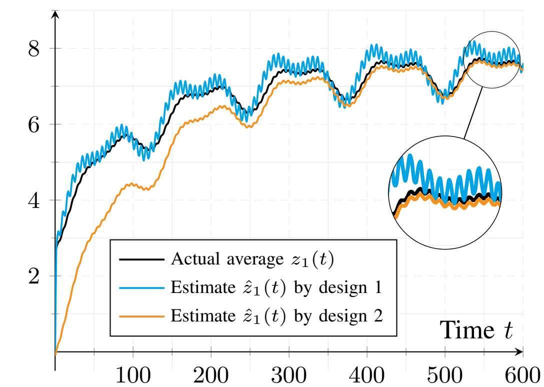


Fig. 3: Average state estimation for Example 2.

obtain by [17]

$$K_1 = \begin{bmatrix} 3.3199 & 2.0276 & 3.0276 \\ 4.4858 & 1.2582 & 2.2582 \\ 2.4849 & 2.6321 & 2.8821 \\ 2.4849 & 1.8821 & 3.6321 \end{bmatrix},$$

where $N = I - LH$ and $K = K_1 + K_2$ with $K_2 = ML$.

Design 2. Choose $\mathbf{v}_2 = 0$ and $v_1 = -0.0938$ such that $\lambda_1(v_1) = \alpha v_1 - \beta = -0.75$, where α, β are given in (18). Choose $\{\lambda_2, \lambda_3, \lambda_4\} = \{-1, -2, -3\}$. Then, the matrices L and K_1 are computed by (14) and M, N, and K by (6). Obtain an average state observer $\widehat{\mathbf{\Sigma}}_r^1$ with $\lambda_1 = -0.75$ and $\boldsymbol{\ell}_1^T = [-0.0312, 0.2187, -0.0312]$. Fig.2 shows better estimation performance for design 2, however the estimation for both designs converges at the same time due to same eigenvalues.

Example VI.2. For network systems to satisfy (7), it is necessary that every unmeasured node is connected to at least one measured node. It is because (7) requires the rows of A_{12} to span the vector $\mathbf{1}^T$. Thus this condition in general requires a large number of measured nodes, which is not feasible due to limited number of available sensors. The purpose of this example is to show that even if (7) doesn't hold, the observers $\widehat{\mathbf{\Sigma}}_r$ and $\widehat{\mathbf{\Sigma}}_1$ estimate the average state with a satisfactory performance if the conditions of Theorem V.1 hold.

Consider a reaction-diffusion system over a grid network shown in Fig.1(b), where the state at each node satisfies

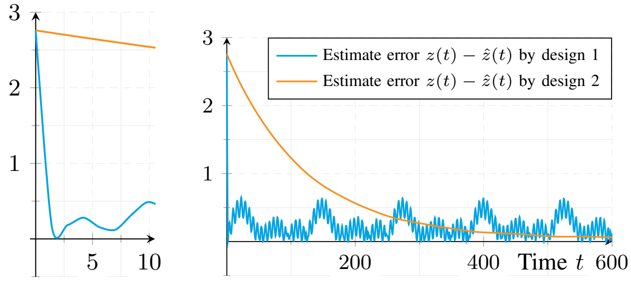


Fig. 4: Estimation error for Example 2.

$\dot{x}_i(t) = -r_i x_i(t) + \sum_{j=1}^{n} a_{ij}[x_j(t) - x_i(t)]$, where $r_i = 0.2$ is the reaction rate and $a_{ij} = a_{ji}$ is the diffusion rate, [8]. The term $a_{ij} = 1$ if the nodes i and j are connected and 0 otherwise, and the output matrix $C = [0_{4\times 96} \quad I_4]$. The inputs are such that $u_1(t) = \sin(0.05t)$ applied at nodes 97 and 98, $u_2(t) = \sin(t)$ applied at nodes 99 and 100, and $u_3(t) = 0.01$ applied at the remaining boundary nodes of the grid (Fig.1(b)). For design 2, We obtain $\widehat{\boldsymbol{\Sigma}}_1$ by finding $v_1^* = -0.0021$ by (19), and $\lambda_1(v_1^*) = -0.0237$. Let $\{\lambda_2, \cdots, \lambda_5\} = \{-1, \cdots, -4\}$. For design 1, we choose $\mathbf{v}_2 = 0.1\,\mathbf{1}_m$ and a faster eigenvalue $\lambda_1 = -0.5$ while other eigenvalues same as in design 2.

Design 1 gives better performance in the transient phase with faster response, however design 2 achieves smaller estimation error at the final state as shown in Fig. 3 and 4. Thus, this example verifies the minimization setup (18) and the solution (19) that minimizes asymptotic value of the estimation error.

Finally, we observed in this example that even though only 8 unmeasured nodes out of 96 are connected to the measured nodes, which signals a huge distance from the necessary and sufficient condition (7), we are still able to obtain a decent estimate of the average state as shown in Fig. 3. This might be the case due to the symmetric structure of the grid, which may have rendered the average deviation vector to have smaller values. However, to present a detailed analysis on this type of behavior is beyond the scope of this paper and is postponed for the future work.

VII. Concluding Remarks

We provided a necessary and sufficient condition for the existence of an average state observer and devised two design procedures. When the necessary and sufficient condition is not satisfied, an error minimization methodology is devised based on the proposed design procedures. We observed that the first design is suitable for achieving better transient behavior, whereas the second design yields a simple observer of dimension equal to 1 and achieves a minimal asymptotic error.

The future prospects of this work include an extension to the estimation of averages of multiple clusters or zones in physical networks. Moreover, estimating nonlinear functionals of the state is also under consideration. For instance, the norm of average deviation vector provides a spread of states around average, which has applications in output control of systems.

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