A Keller–Segel type taxis model with ecological interpretation and boundedness due to gradient nonlinearities

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ABSTRACT. We introduce a novel gradient-based damping term into a Keller–Segel type taxis model with motivation from ecology and consider the following system equipped with homogeneous Neumann-boundary conditions:

$$(\diamond) \quad \begin{cases} u_t = \Delta u - \chi \nabla \cdot (u \nabla v) + a u^\alpha - b u^\beta - c|\nabla u|^\gamma & \text{in } \Omega \times (0, T_{max}), \\ \tau v_t = \Delta v - v + u & \text{in } \Omega \times (0, T_{max}). \end{cases}$$

The problem is formulated in a bounded and smooth domain Ω of $\mathbb{R}^N$, with $N \geq 2$, for some positive numbers $a, b, c, \chi > 0$, $\tau \in \{0, 1\}$, $\gamma \geq 1$, $\beta > \alpha \geq 1$ and with $T_{max} \in (0, \infty]$. As far as we know, Keller–Segel models with gradient-dependent sources are new in the literature and, accordingly, beyond giving a reasonable ecological interpretation the objective of the paper is twofold:

 i) to provide a rigorous analysis concerning the local existence and exensibility criterion for a class of models generalizing problem $(\diamond)$, obtained by replacing $au^\alpha - bu^\beta - c|\nabla u|^\gamma$ with $f(u) - g(\nabla u)$;

 ii) to establish sufficient conditions on the data of problem $(\diamond)$ itself, such that it admits a unique classical solution (u, v), for $T_{max} = \infty$ and with both u and v bounded.

We handle i) whenever appropriately regular initial distributions $u(x, 0) = u_0(x) \geq 0$, $\tau v(x, 0) = \tau v_0(x) \geq 0$ are considered and f and g obey some regularity properties and, moreover, some growth restrictions. Further, as to ii), for the same initial data considered in the previous case, global boundedness of solutions is proven for any $\tau \in \{0, 1\}$, provided that $\frac{2N}{N+1} < \gamma \leq 2$.

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1. Introduction, main claims and organization of the paper

1.1. **Taxis terms in ecological models.** The original Keller–Segel model

$$(1) \qquad\qquad u_t = \Delta u - \nabla \cdot (u\nabla v), \qquad v_t = \Delta v - v + u$$

was introduced to describe the self-organized aggregation in a starving slime mould colony (see [20, 4]), where individual cells (density u) direct their otherwise random movement toward higher concentrations of a chemical signal substance they produce (concentration v). On much larger scales yet, this taxis system and its relatives find their use, for example in modelling the motion of predator populations in response to the concentration of their prey (or vice versa), see e.g. [18].

But also if the second species is not explicitly modelled (that is, also if the second equation remains of the form in (1) and does not incorporate Lotka–Volterra type interactions), and even in the absence of a chemical signal, it is possible to understand the second component in (1) as portrayal of the tendency of animals to have their motion influenced by their spatial memory [7].

If attraction to (or, with a different sign, repulsion from) previously visited sites is incorporated in a heat equation model of random motion by means of a so-called weak memory kernel, it is precisely (1) that results, [31].

1.2. **Interaction terms in equations of population dynamics.** Especially in a setting involved with population dynamics, that is, u denoting the density of a biological species occupying a certain domain, the combination of (1) with terms describing population growth or decay is very natural, for example logistic growth [35] (in the context of chemotaxis models, see [1, 27]). In this case, the temporal evolution of the distribution of the species is described as interplay between displacements (in the form of diffusion and directed motion), reproduction and death, with the death rate increasing for large populations, so that the zero-order terms in $u_t = \Delta u - \nabla \cdot (u\nabla v) + f(u)$ may have the form $f(u) = au^\alpha - bu^\beta$, where $\alpha = 1$ if the birth rate is constant (a) and where $\beta > 1$.

Nevertheless, not only the current size of a population has an influence on its growth; beyond such density-mediated effects, also its behaviour can have a significant impact. Since it is not unusual for predators to react to motion (e.g. [3] or [26]), for the prey population, motility can increase the risk to be decimated by attacks; see [11] for supporting evidence in the case of katydids, or [37, 2] for studies of zooplankton, whose mortality rate depends on the swimming velocity.

In a macroscopic model, such effects appear as additional decay terms depending on the size of the gradient of the solution, i.e. of the gradient of the population density, so that we obtain $u_t = \Delta u - \nabla \cdot (u\nabla v) + f(u) - g(\nabla u)$, where g is an increasing function of $|\nabla u|$. (Indeed, this coincides with the interpretation of gradient terms in scalar parabolic equations as 'accidental deaths' suggested by Souplet in [32].)

1.3. **The model.** In conclusion, we aim to study the signal-production chemotaxis model

$$
(2) \quad \begin{cases}
u_t = \Delta u - \chi \nabla \cdot (u \nabla v) + a u^\alpha - b u^\beta - c|\nabla u|^\gamma & \text{in } \Omega \times (0, T_{max}), \\
\tau v_t = \Delta v - v + u & \text{in } \Omega \times (0, T_{max}), \\
\partial_\nu u = \partial_\nu v = 0 & \text{on } \partial\Omega \times (0, T_{max}), \\
u(x,0) = u_0(x), \; \tau v(x,0) = \tau v_0(x) & x \in \bar{\Omega},
\end{cases}
$$

where $a, b, c, \chi > 0$, $1 \le \alpha < \beta$, $\gamma \ge 1$, $\tau \in \{0,1\}$, and where the problem is posed in a bounded and smooth domain Ω of $\mathbb{R}^N$, with $N \ge 2$. Moreover, sufficiently regular initial data $u_0 \ge 0$, $\tau v_0 \ge 0$ are given, ∂_ν indicates the outward normal derivative on $\partial\Omega$, whereas $T_{max} \in (0, \infty]$ is the maximal time up to which solutions to the system are defined. (For a local existence result and a characterization of the maximal existence time for a class of models generalizing (2) see Theorem 1.1 below.)

The parameter $\tau \in \{0,1\}$ distinguishes between the 'fully parabolic' chemotaxis system ($\tau = 1$) and its 'parabolic-elliptic' counterpart ($\tau = 0$). The latter is a frequent simplification of the model, which is helpful or necessary in certain proofs, and from a biological perspective can be justified whenever the spread of v occurs on a much faster time scale than that of u.

1.4. **Analysis of the Keller–Segel system.** Mathematically, (1) is known for its ability to admit global bounded solutions and unbounded solutions blowing up in finite time, depending on the total mass of initial data and the specific initial configurations. More precisely, for $N = 1$ all solutions are uniformly bounded in time, whereas for $N \ge 3$ and any positive mass m, initial data with this mass $m = \int_\Omega u_0$ that lead to blow-up within finite time have been detected. On the other hand, when $N = 2$, the total mass m separates the case where diffusion overcomes self-attraction (if $m < 4\pi$, solutions are global in time) from the case where self-attraction dominates and initial data giving rise to finite-time blow-up exist. An overview can be found in the surveys [23, 4], and we mention [14, 28, 38] as some notable results in this context.

1.5. **The Keller–Segel system with logistic source terms.** While mathematical intuition indicates that the presence of logistic terms (and thus of a superlinear damping effect) should benefit the boundedness of solutions (paralleling the boundedness of solutions to the ODE $u' = au - bu^\beta$ for $\beta > 1$) and prevent the extreme structure formation in the sense of finite-time blow-up found in (1), this has only been shown for large values of b (if $\beta = 2$, see [34], [39]; and for weak solutions featuring eventual smoothness see [22]), whereas the possibility of blow-up was established for certain values of $\beta > 1$, first for dimension 5 or higher [40], when the equation for v reads $0 = \Delta v - \frac{1}{|\Omega|} \int_\Omega u(x,t)dx + u$ (see also [10] for a recent improvement of [40]), but later also in 3-dimensional domains (with $0 = \Delta v - v + u$ and $\beta < \frac{7}{6}$), [41], with extensions to more general systems with nonlinear diffusion and sensitivity functions, [5, 33].

1.6. **Gradient nonlinearities.** The study of gradient terms in parabolic equations, and their impact on the possibility of blow-up is a classical topic (see, e.g. [6, 19, 8], where the

relations between exponents γ and α were investigated that can prevent or admit blow-up in the Dirichlet problem for a semilinear heat equation of the form $u_t = \Delta u - c|\nabla u|^\gamma + u^\alpha$, or [29, Sec. IV]). Additionally, a possible biological interpretation of these terms was given in the above-mentioned [32]. Nevertheless, in chemotaxis models, gradient nonlinearities of this kind seem to have been treated only in [36], where a lower bound of the maximal existence time of solutions was derived.

1.7. **Presentation of the main results.** A general existence theory or any result on boundedness of solutions, however, seem to be lacking for Keller-Segel type taxis models with gradient terms. In particular motivated by the observation that some generalized logistic sources still permit blow-up (see above: [40, 41, 5, 33, 10]) in such models, we aim at deriving sufficient conditions such that the combined actions of the reproductive growth $(+au^\alpha)$ and the two degradation mechanisms $(-bu^\beta - c|\nabla u|^\gamma)$ may guarantee their global existence. It is our goal to pose the condition on the strength of the 'accidental death', i.e. the gradient term, and not the (zero-order) decay by 'natural death' that was part of earlier works.

A natural question we want to deal with thus is:

$\mathcal{Q}_1$: For $\tau = c = 0$, $\alpha = 1$ and $\beta < 2$ as in [41], let u_0 be such that the related solution to (2) blows at finite time. Do some conditions on $\gamma \geq 1$ exist so to ensure that, for $c > 0$ only, such blow-up scenario is prevented?

In order to engage with this question, we first require a rigorous analysis concerning the local existence of model (2), which was not available in the literature. As consequence, we firstly address questions connected to existence of (local) classical solutions and related extensibility criteria applicable to a wider context than that specifically formulated in system (2). In particular, once Ω, $\chi > 0$, $\tau \in \{0, 1\}$ are given as indicated above, we will deal with this problem:

$$
(3) \quad \begin{cases}
u_t = \Delta u - \chi \nabla \cdot (u \nabla v) + f(u) - g(\nabla u) & \text{in } \Omega \times (0, T_{max}), \\
\tau v_t = \Delta v - v + u & \text{in } \Omega \times (0, T_{max}), \\
\partial_\nu u = \partial_\nu v = 0 & \text{on } \partial\Omega \times (0, T_{max}), \\
u(x,0) = u_0(x), \ \tau v(x,0) = \tau v_0(x) & x \in \bar{\Omega}.
\end{cases}
$$

Here f and g will suitably generalize the terms of (2) and, more precisely may have to fulfill these assumptions:

$$
\begin{cases}
(4a) \quad f \colon \mathbb{R} \to \mathbb{R} \text{ is locally Lipschitz continuous, with } f(0) \geq 0; \\
(4b) \quad \text{there is } C_f > 0 \text{ such that } f(s) \leq C_f \quad \text{for all } s \geq 0; \\
(4c) \quad g \colon \mathbb{R}^N \to [0, \infty) \text{ is locally Lipschitz continuous, with } g(0) = 0; \\
(4d) \quad \text{there is } C_g > 0 \text{ such that } g(z) \leq C_g(1 + |z|^2) \text{ for all } z \in \mathbb{R}^N.
\end{cases}
$$

In these positions, we first prove the following

Theorem 1.1 (Local existences and extensibility criterion)**.** *Let $N \in \mathbb{N}$, $\Omega \subset \mathbb{R}^N$ be a bounded domain with smooth boundary, $\chi \in \mathbb{R}$, $\delta > 0$, $\tau \in \{0, 1\}$, let f and g comply with*

(4). *For every* $u_0 \in C^{1+\delta}(\overline{\Omega})$, $\tau v_0 \in C^{2+\delta}(\overline{\Omega})$ *satisfying*

$$\partial_\nu u_0|_{\partial\Omega} = 0, \tau \partial_\nu v_0|_{\partial\Omega} = 0 \ and \ u_0 \geq 0, \tau v_0 \geq 0,$$

problem (3) *has a unique and nonnegative classical solution*

$$(u,v) \in \left(C^{1,\frac{1}{2}}(\overline{\Omega}\times[0,T_{max})) \cap C^{2,1}(\overline{\Omega}\times(0,T_{max}))\right) \times C^{2,1}(\overline{\Omega}\times[0,T_{max}))$$

for some maximal $T_{max} \in (0,\infty]$ *which is such that for every* $p > \frac{N}{2}$

(5) $$either \ T_{max} = \infty \quad or \quad \limsup_{t \nearrow T_{max}} \|u(\cdot,t)\|_{L^p(\Omega)} = \infty.$$

With the aid of the above result, we can obtain some more information on solutions to model (3) if more restrictive assumptions on some data are required. Precisely, we have this further

Theorem 1.2. *Let* $N \geq 2$, $\Omega \subset \mathbb{R}^N$ *be a bounded domain with smooth boundary, f and g comply with* (4), $\chi > 0$ *and* $\tau \in \{0,1\}$. *Then, for every* $u_0 \in C^{1+\delta}(\overline{\Omega})$, $\tau v_0 \in C^{2+\delta}(\overline{\Omega})$ *satisfying*

$$\partial_\nu u_0|_{\partial\Omega} = 0, \tau \partial_\nu v_0|_{\partial\Omega} = 0 \ and \ u_0 \geq 0, \ \tau v_0 \geq 0,$$

whenever

(6) $$\frac{2N}{N+1} < \gamma \leq 2,$$

and there are $C_1, C_2, C_3 > 0$ *such that f and g also comply with*

(7) $$f(s) \leq C_1 - C_2 s \ for \ all \ s \geq 0 \quad and \quad g(z) \geq C_3|z|^\gamma \quad for \ all \ z \in \mathbb{R}^N,$$

problem (3) *has a unique and nonnegative classical solution*

$$(u,v) \in \left(C^{1,\frac{1}{2}}(\overline{\Omega}\times[0,\infty)) \cap C^{2,1}(\overline{\Omega}\times(0,\infty)) \cap L^\infty(\Omega\times(0,\infty))\right)$$
$$\times \left(C^{2,1}(\overline{\Omega}\times[0,\infty)) \cap L^\infty(\Omega\times(0,\infty))\right),$$

which, in particular, is bounded in the sense that there exists $C > 0$ *such that*

$$\|u(\cdot,t)\|_{L^\infty(\Omega)} + \|v(\cdot,t)\|_{W^{1,\infty}(\Omega)} \leq C \quad for \ all \ t \geq 0.$$

As a by-product, we can specialize Theorem 1.2 to situations where f and g are external sources exactly reading as in model (2), and thus answer $\mathcal{Q}_1$.

Corollary 1.3. *Let* $N \geq 2$, $\Omega \subset \mathbb{R}^N$ *be a bounded domain with smooth boundary, $a,b,c,\chi,\delta > 0$, $\tau \in \{0,1\}$ and $1 \leq \alpha < \beta$. Additionally, let relation* (6) *hold. Then, for every* $u_0 \in C^{1+\delta}(\overline{\Omega})$, $\tau v_0 \in C^{2+\delta}(\overline{\Omega})$ *satisfying*

$$\partial_\nu u_0|_{\partial\Omega} = 0, \tau \partial_\nu v_0|_{\partial\Omega} = 0 \ and \ u_0 \geq 0, \tau v_0 \geq 0,$$

problem (2) *has a unique and nonnegative classical solution, which is bounded.*

Remark 1 (Open question)**.** *While the extensibility criterion (5) gives some indication how blow-up would occur (and, in particular, excludes pure gradient blow-up), and while there are conditions on f that prevent blow-up even for $g \equiv 0$, we here have to leave open the question whether g alone can prevent unboundedness for small exponents, i.e. whether for some $\gamma \in \left[1, \frac{2N}{N+1}\right]$ there may exist unbounded solutions to problem (2) emanating from certain data.*

2. Existence of local-in-time solutions and extensibility criterion. Proof of Theorem 1.1

While C_f, C_g, C_1, C_2, C_3 will always refer to the constants in (4) and (7), other constants, and in particular the small-letter constants c_i, are local to each proof.

The proof of Theorem 1.1 is consequence of this series of lemmas. Until we confirm – in Lemma 2.4 – the $C^{2,1}$-regularity of solutions, showing that they are, indeed, classical, "solution of (3)" occasionally must be understood as solution of the weak form of (3). The current regularity of (u, v) will always be indicated.

Lemma 2.1. *Let $N \in \mathbb{N}$, $\Omega \subset \mathbb{R}^N$ be a bounded domain with smooth boundary, $\delta > 0$, $\chi \in \mathbb{R}$, $\tau \in \{0, 1\}$, let $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R}^N \to [0, \infty)$ be continuous. For every $M_1 > 0$ and $M_2 > 0$ there is $T > 0$ such that system (3) for every $u_0 \in C^{1+\delta}(\overline{\Omega})$, $\tau v_0 \in C^{2+\delta}(\overline{\Omega})$ satisfying*

$$(8) \qquad \|u_0\|_{C^{1+\delta}(\overline{\Omega})} \le M_1, \|\tau v_0\|_{C^{1+\delta}(\overline{\Omega})} \le M_2 \text{ and } \partial_\nu u_0|_{\partial\Omega} = 0, \tau \partial_\nu v_0|_{\partial\Omega} = 0,$$

has a solution

$$(u, v) \in C^{1+\delta, \frac{1+\delta}{2}}(\overline{\Omega} \times [0, T]) \times C^{2+\delta, 1+\frac{\delta}{2}}(\overline{\Omega} \times [0, T]).$$

Proof. We fix $T_0 > 0$. For $R > M_1 + 1$ and $T \in (0, T_0]$ (to be specified later) we let

$$X_{R,T} = \{u \in C^1(\overline{\Omega} \times [0, T]) \mid \sup_{t \in [0,T]} \|u(\cdot, t)\|_{C^1(\overline{\Omega})} \le R\}$$

which is a closed bounded convex subset of $C^1(\overline{\Omega} \times [0, T])$. For $\overline{u} \in X_{R,T}$ we let v be the solution of

$$(9) \qquad \tau v_t = \Delta v - v + \overline{u}, \ \ \partial_\nu v = 0, \ \ \tau v(\cdot, 0) = \tau v_0$$

(whose existence as classical solution and uniqueness follow from, e.g., [21, IV.5.3] if $\tau = 1$ and from [13, 6.31] if $\tau = 0$), and define $\Phi(\overline{u}) = u \in L^2((0, T); W^{1,2}(\Omega)) \cap C([0, T]; L^2(\Omega))$ to be the weak solution of

$$(10) \qquad u_t = \Delta u - \chi \nabla \cdot (u \nabla v) + f(\overline{u}) - g(\nabla \overline{u}), \ \ \partial_\nu u = 0 \ \ u(\cdot, 0) = u_0$$

(whose existence follows by [21, III, §5], uniqueness due to [21, III.3.1]).

There is $c_1 = c_1(M_2, R) > 0$ such that every solution of (9) (for any $\overline{u} \in X_{R,T}$ and v_0 with (8)) satisfies

$$\sup_{t \in [0,T]} \|v(\cdot, t)\|_{C^{1+\delta}(\overline{\Omega})} \le c_1$$

(cf. [25, Thm. 1.2] if $\tau = 1$ or [13, Thm 6.30] if $\tau = 0$) and there is $c_2 = c_2(M_1, R, c_1, f, T_0) > 0$ such that (for any u_0 as in (8) and any $\overline{u} \in X_{R,T}$) the solution u of (10) satisfies

$$\|u\|_{L^\infty(\Omega \times (0, T_0))} \leq c_2,$$

because from

$$u_t \leq \Delta u - \chi \nabla \cdot (uh) + \sup_{s \in [-R, R]} f(s) \quad \text{and} \quad u_t \geq \Delta u - \chi \nabla \cdot (uh) + \inf_{s \in [-R, R]} f(s) - \sup_{z \in B_R(0)} g(z)$$

with the bounded function $h = \nabla v$, we obtain upper and lower bounds for u from semigroup estimates like [38, Lemma 1.3(iv)]. Having ensured boundedness of u, we can now employ [25, Thm. 1.2] to see that there is $c_3 > 0$ such that (for any u_0 as in (8) and any $\overline{u} \in X_{R,T}$) the solution u of (10) satisfies

$$\|u\|_{C^{1+\delta, \frac{1+\delta}{2}}(\overline{\Omega} \times [0, T])} \leq c_3.$$

Consequently,

$$\|u(\cdot, t)\|_{C^1(\overline{\Omega})} \leq \|u_0\|_{C^1(\overline{\Omega})} + \|u(\cdot, t) - u_0\|_{C^1(\overline{\Omega})} \leq M_1 + c_3 T^\delta,$$

which shows that for $T < c_3^{-\frac{1}{\delta}}$, Φ maps $X_{R,T}$ to $X_{R,T}$. Moreover, $\Phi(X_{R,T})$ is bounded in $C^{1+\delta, \frac{1+\delta}{2}}(\overline{\Omega} \times [0, T])$, hence compact in $C^1(\overline{\Omega} \times [0, T])$, and Φ is continuous. Schauder's fixed point theorem asserts the existence of a fixed point $u \in \Phi(X_{R,T})$ of Φ. $\square$

Lemma 2.2. *In addition to the assumptions of Lemma 2.1, assume $u_0 \geq 0$ and let f and g fulfill* (4a) *and* (4c). *Then $u \geq 0$.*

Proof. Since 0 is a subsolution due to $f(0) \geq 0$ and $g(0) = 0$, this follows from a classical comparison principle ([29, 52.6]), applied to the first equation of (3) for fixed (and $C^2(\overline{\Omega})$-bounded) v. $\square$

Lemma 2.3. *Let Ω, τ, χ, u_0, τv_0, f, g be given as in Lemma 2.1, and additionally let f and g be locally Lipschitz continuous. Let $(u_1, v_1), (u_2, v_2) \in C^0(\overline{\Omega} \times [0, T])$ satisfy $\nabla u_1, \nabla u_2, \nabla v_1, \nabla v_2 \in C^0(\overline{\Omega} \times [0, T])$, $(u_1)_t, (u_2)_t \in C^0(\overline{\Omega} \times [0, T])$ (and, if $\tau = 1$, also $(v_1)_t, (v_2)_t \in C^0(\overline{\Omega} \times [0, T])$) and be solutions of* (3). *Then $(u_1, v_1) = (u_2, v_2)$ in $\overline{\Omega} \times [0, T]$.*

Proof. By means of a Grönwall-type argument this is obtained from a study of

$$\frac{d}{dt} \left(\int_\Omega (u_1 - u_2)^2 + \tau \int_\Omega (v_1 - v_2)^2 \right) \quad \text{for all } t \in [0, T]. \qquad \square$$

Lemma 2.4. *Let $N \in \mathbb{N}$, $\Omega \subset \mathbb{R}^N$ be a bounded domain with smooth boundary, $\delta > 0$, $\chi \in \mathbb{R}$, $\tau \in \{0, 1\}$, and let f and g fulfill assumptions* (4a) *and* (4c). *For $u_0 \in C^{1+\delta}(\overline{\Omega})$, $\tau v_0 \in C^{2+\delta}(\overline{\Omega})$ satisfying*

$$(11) \qquad \partial_\nu u_0|_{\partial\Omega} = 0, \tau \partial_\nu v_0|_{\partial\Omega} = 0 \text{ and } u_0 \geq 0, \tau v_0 \geq 0$$

there is $T_{max} \in (0, \infty]$ such that (3) *has a unique solution*

$$(u, v) \in \left(C^{1, \frac{1}{2}}(\overline{\Omega} \times [0, T_{max})) \cap C^{2,1}(\overline{\Omega} \times (0, T_{max})) \right) \times C^{2,1}(\overline{\Omega} \times [0, T_{max})),$$

which is nonnegative, and such that

$$either \ T_{max} = \infty \quad or \quad \lim_{t \nearrow T_{max}} \left(\|u(\cdot,t)\|_{C^{1+\delta}(\overline{\Omega})} + \|v(\cdot,t)\|_{C^{1+\delta}(\overline{\Omega})} \right) = \infty.$$

Proof. Repeated application of the local existence result of Lemma 2.1, aided by the uniqueness statement of Lemma 2.3, ensures the existence of a solution $(u,v) \in C^{1+\delta,\frac{1+\delta}{2}}(\overline{\Omega} \times [0,T_{max})) \times C^{2+\delta,1+\frac{\delta}{2}}(\overline{\Omega} \times [0,T_{max}))$ with some maximally chosen $T_{max} > 0$. If T_{max} were finite and we did not have $\lim_{t \nearrow T_{max}} \left(\|u(\cdot,t)\|_{C^{1+\delta}(\overline{\Omega})} + \|v(\cdot,t)\|_{C^{1+\delta}(\overline{\Omega})} \right) = \infty$, we could find $M > 0$ and a sequence $t_k \nearrow T_{max}$ such that $\left(\|u(\cdot,t_k)\|_{C^{1+\delta}(\overline{\Omega})} + \|v(\cdot,t_k)\|_{C^{1+\delta}(\overline{\Omega})} \right) \leq M$ for all $k \in \mathbb{N}$, which would enable us to extend the solution on $(t_k, t_k + T(M))$ with $T(M) := T(M,M)$ taken from Lemma 2.1. As $t_k + T(M) > T_{max}$ for large k, this would contradict maximality of T_{max}.

Further application of classical regularity theory (cf. the Schauder estimates contained in [21, IV.5.3]) ensures that also $u \in C^{2,1}(\overline{\Omega} \times (0,T_{max}))$.

Finally, we observe that from Lemma 2.2 we have that $u \geq 0$ and, in turn, $v \geq 0$ is consequence of the elliptic or parabolic maximum principle applied to the second equation of (3). $\qquad\square$

The presence of additional gradient terms becomes noticable in the next step. In order to rely on classical regularity theory, we require some restriction of their growth, so as to preclude gradient blow-up.

Lemma 2.5. *Let Ω, χ, τ, u_0, v_0, f and g be as in Lemma 2.4, and let g satisfy assumption (4d). Then there is $T_{max} \in (0,\infty]$ such that (3) has a solution*

$$(u,v) \in \left(C^{1,\frac{1}{2}}(\overline{\Omega} \times [0,T_{max})) \cap C^{2,1}(\overline{\Omega} \times (0,T_{max})) \right) \times C^{2,1}(\overline{\Omega} \times [0,T_{max}))$$

and such that

$$either \ T_{max} = \infty \quad or \quad \limsup_{t \nearrow T_{max}} \|u(\cdot,t)\|_{L^{\infty}(\Omega)} = \infty$$

Proof. Assuming that the maximal existence time T_{max} (cf. Lemma 2.4) were finite, by the extensibility criterion in Lemma 2.4 it is sufficient to show that boundedness of $\|u(\cdot,t)\|_{L^{\infty}(\Omega)}$ for $t \in [0,T_{max}]$ entails boundedness of $\|u(\cdot,t)\|_{C^{1+\delta}(\overline{\Omega})} + \|v(\cdot,t)\|_{C^{1+\delta}(\overline{\Omega})}$ for $t \in (\frac{T_{max}}{2}, T_{max})$. Boundedness of $\|v(\cdot,t)\|_{C^{1+\delta}(\overline{\Omega})}$ results from that of $\|u(\cdot,t)\|_{L^{\infty}(\Omega)}$ by means of [25, Thm. 1.2] (if $\tau = 1$) or, e.g. from [9, Thm. I.19.1] (if $\tau = 0$). According to the nomenclature in [25] letting $A(x,t,z,p) = p - z\nabla v(x,t)$ and $B(x,t,z,p) = f(z) - g(p)$, we conclude from the fact that u is a bounded solution to $u_t = \nabla \cdot (A(x,t,u,\nabla u)) + B(x,t,u,\nabla u)$ and [25, Thm. 1.2] that $\sup_{t \in (\frac{3T_{max}}{4}, T_{max})} \|u(\cdot,t)\|_{C^{1+\delta}(\overline{\Omega})}$ is finite. $\qquad\square$

Lemma 2.6. *Let $N \in \mathbb{N}$, $\Omega \subset \mathbb{R}^N$ be a bounded domain with smooth boundary, $\delta > 0$, $\chi \in \mathbb{R}$, $\tau \in \{0,1\}$, let f and g comply with (4a), (4b) and (4c). For every $p > \frac{N}{2}$ and $K > 0$, there are $q > N$, $L > 0$ and $M > 0$ such that if (u,v) is a solution to (3) for any $u_0 \in C^{1+\delta}(\overline{\Omega})$, $\tau v_0 \in C^{2+\delta}(\overline{\Omega})$ satisfying (11) and*

$$\|u(\cdot,t)\|_{L^p(\Omega)} \leq K \quad \text{for all } t \in [0,T_{max}), \quad \|u_0\|_{L^{\infty}(\Omega)} \leq K \text{ and } \|\nabla v_0\|_{L^q(\Omega)} \leq K,$$

then

$$\|\nabla v(\cdot,t)\|_{L^q(\Omega)} \leq L \ for \ all \ t \in (0, T_{max}) \tag{12}$$

and

$$\|u(\cdot,t)\|_{L^\infty(\Omega)} \leq M \ for \ all \ t \in (0, T_{max}).$$

Proof. From $\|u(\cdot,t)\|_{L^p(\Omega)} \leq K$ and $p > \frac{N}{2}$, which implies that $\frac{Np}{(N-p)_+} > N$, by elliptic regularity theory [9, I.19.1] (if $\tau = 0$) or (if $\tau = 1$) semigroup estimates ([38, Lemma 1.3(ii)] applied in $\nabla v(t) = \nabla e^{t(\Delta-1)}v_0 + \int_0^t \nabla e^{(t-s)(\Delta-1)}u(s)ds$) we find $L > 0$ and $q > N$ such that (12) holds. We let $c_1 > 0$ and $c_2 > 0$ be the constants from the semigroup estimates in [38, Lemma 1.3(i)] and [38, Lemma 1.3(iv)], and use that $f(s) \leq C_f$ for all $s \in \mathbb{R}$. Choosing $\tilde{p} > p$ so large that $(\frac{1}{q} + \frac{1}{\tilde{p}})^{-1} > N$, we find that

$$0 \leq u(t) \leq e^{t\Delta}u_0 - \chi \int_0^t e^{(t-s)\Delta}\nabla \cdot (u\nabla v)ds + \int_0^t e^{(t-s)\Delta}f(u(s))ds$$

$$\leq \|u_0\|_{L^\infty(\Omega)} + c_2|\chi| \int_0^t (1 + (t-s)^{-\alpha})\|u(\cdot,s)\|_{L^p(\Omega)}^{\frac{p}{\tilde{p}}}\|u(\cdot,s)\|_{L^\infty(\Omega)}^{1-\frac{p}{\tilde{p}}}\|\nabla v(\cdot,s)\|_{L^q(\Omega)}ds + C_f$$

for $t \leq \min\{1, T_{max}\}$ and $\alpha = \frac{1}{2} + \frac{N}{2}(\frac{1}{q} + \frac{1}{\tilde{p}}) < 1$ and that with $\tau = t - 1 + s$

$$0 \leq u(\cdot,t) \leq e^{1\Delta}u(\cdot,t-1) - \chi \int_0^1 e^{(1-s)\Delta}\nabla \cdot (u(\cdot,\tau)\nabla v(\cdot,\tau))ds + \int_0^1 e^{(1-s)\Delta}f(u(\cdot,\tau))ds$$

$$\leq 2c_1\|u(\cdot,t-1)\|_{L^p(\Omega)}$$

$$+ c_2|\chi| \int_0^1 (1 + (1-s)^{-\alpha})\|u(\cdot,\tau)\|_{L^p(\Omega)}^{\frac{p}{\tilde{p}}}\|u(\cdot,\tau)\|_{L^\infty(\Omega)}^{1-\frac{p}{\tilde{p}}}\|\nabla v(\cdot,\tau)\|_{L^q(\Omega)}ds + C_f$$

for $t \in (1, T_{max})$, so that upon defining $M(t) = \sup_{\tau \in (0,t)} \|u(\cdot,\tau)\|_{L^\infty(\Omega)}$, we obtain

$$M(t) \leq \underbrace{C_f + \max\{\|u_0\|_{L^\infty(\Omega)}, 2c_1 K\}}_{=:c_3} + \underbrace{c_2|\chi| \int_0^1 (1 + \tau^{-\frac{1}{2} - \frac{N}{2}(\frac{1}{q} + \frac{1}{\tilde{p}})})d\tau K^{\frac{p}{\tilde{p}}}L}_{=:c_4}(M(t))^{1-\frac{p}{\tilde{p}}},$$

for any $t \in (0, T_{max})$, which shows that

$$M(t) \leq \sup\{x \mid x \leq c_3 + c_4 x^{1-\frac{p}{\tilde{p}}}\} =: M < \infty. \qquad \square$$

Proof of Theorem 1.1. The existence and the nonnegativity follow from Lemma 2.4; this form of the extensibility criterion is obtained from Lemma 2.6 in combination with Lemma 2.5. $\qquad \square$

3. Some a priori estimates. Proofs of Theorem 1.2 and Corollary 1.3

The proof of Theorem 1.2 goes through some steps, which altogether ensure uniform boundedness of u in some L^p-norm.

3.1. **A priori estimates.** From now to §3.4, we assume that $u_0, \tau v_0$ are given as required by Theorem 1.2 and, for $\chi > 0$, by (u, v) we will indicate the local solution to model (3) provided by Theorem 1.1; moreover, with the aim of exploiting the related extensibility criterion (5), let us focus on the derivation of some *a priori* L^p-estimates for the u-component, by specifying how each of the constraints in Theorem 1.2 takes part in our computations. In a first step, let us derive a control on the total mass of u from nonnegativity of g and the behaviour of f.

Lemma 3.1. *Let f and g satisfy* (7). *Then, for some $m_0 > 0$ we have that u is such that*

$$(13) \qquad \int_\Omega u(x,t)dx \le m_0 \quad \text{for all } t \in (0, T_{max}).$$

Proof. An integration over Ω of the first equation in (3) due to the estimates in (7) shows that

$$\frac{d}{dt} \int_\Omega u = \int_\Omega (f(u) - g(\nabla u)) \le C_1 |\Omega| - C_2 \int_\Omega u \quad \text{in } (0, T_{max}),$$

so that we arrive at $\int_\Omega u(\cdot, t) \le m_0 = \max\left\{\int_\Omega u_0, \frac{C_1|\Omega|}{C_2}\right\}$ for all $t \in (0, T_{max})$. $\qquad\square$

The term $\int_\Omega |\nabla u^{\frac{p}{2}}|^2$ in any dimension controls $\int_\Omega u^p$:

Lemma 3.2. *Let $p > 1$. Then there is $k > 0$ such that*

$$\int_\Omega u^p \le 4\frac{p-1}{p} \int_\Omega |\nabla u^{\frac{p}{2}}|^2 + k \quad \text{on } (0, T_{max}).$$

Proof. We let

$$\check{\theta} = \frac{\frac{p}{2} - \frac{1}{2}}{\frac{p}{2} - \frac{1}{2} + \frac{1}{N}} \in (0, 1),$$

and apply the Gagliardo-Nirenberg inequality (possibly used in the less common version with exponents below one, given in, e.g. [24, Lemma 2.3]) and

$$(14) \qquad (A + B)^l \le \max\{1, 2^{l-1}\}(A^l + B^l), \quad l > 0, A, B \ge 0,$$

which together with (13) yields $\check{C}_{GN} > 0$ and $k > 0$ such that

$$\int_\Omega u^p = \|u^{\frac{p}{2}}\|^2_{L^2(\Omega)} \le \check{C}_{GN}\left(\|\nabla u^{\frac{p}{2}}\|^{2\check{\theta}}_{L^2(\Omega)}\|u^{\frac{p}{2}}\|^{2(1-\check{\theta})}_{L^{\frac{2}{p}}(\Omega)} + \|u^{\frac{p}{2}}\|^2_{L^{\frac{2}{p}}(\Omega)}\right) \le 4\frac{p-1}{p} \int_\Omega |\nabla u^{\frac{p}{2}}|^2 + k$$

on $(0, T_{max})$. $\qquad\square$

Lemma 3.3. *Let f and g satisfy* (7) *and let $p > 1$. There is $K > 0$ such that*

$$\frac{d}{dt} \int_\Omega u^p + \int_\Omega u^p$$

$$\le -\chi(p-1) \int_\Omega u^p \Delta v + \chi(p-1) \int_\Omega u^{p+1} + K - C_3 p\left(\frac{p-1+\gamma}{\gamma}\right)^{-\gamma} \int_\Omega |\nabla u^{\frac{p-1+\gamma}{\gamma}}|^\gamma$$

on $(0, T_{max})$.

Proof. Standard testing procedures, together with the estimates on f and g from (7), give

$$\frac{d}{dt}\int_\Omega u^p = p\int_\Omega u^{p-1}\left(\Delta u - \chi\nabla\cdot(u\nabla v) + f(u) - g(\nabla u)\right)$$

$$\le -4\frac{p-1}{p}\int_\Omega |\nabla u^{\frac{p}{2}}|^2 - \chi(p-1)\int_\Omega u^p\Delta v + C_1 p\int_\Omega u^{p-1} - C_3 p\int_\Omega u^{p-1}|\nabla u|^\gamma$$

on $(0,T_{max})$, and straightforward manipulation of the term $-C_3 pu^{p-1}|\nabla u|^\gamma$ results in the claim, if we take into account that by Young's inequality with some $c_1 > 0$ we have

$$C_1 p\int_\Omega u^{p-1} \le \chi(p-1)\int_\Omega u^{p+1} + c_1 \quad \text{on }\ (0,T_{max}),$$

and employ Lemma 3.2. $\qquad\qquad\square$

Suitably estimating $\int_\Omega u^{p+1}$ will use the largeness assumption on γ given in (6):

Lemma 3.4. *Let* $\frac{2N}{N+1} < \gamma \le 2$. *Then there is* $p > \frac{N}{2}$ *such that for any* $K > 0$ *there is* $c = c(K,p) > 0$ *such that*

$$K\int_\Omega u^{p+1} \le C_3 p\left(\frac{p-1+\gamma}{\gamma}\right)^{-\gamma}\int_\Omega |\nabla u^{\frac{p-1+\gamma}{\gamma}}|^\gamma + c \quad on\ \ (0,T_{max}).$$

Proof. Due to the assumption on γ, it is possible to choose $p > \frac{N}{2}$ such that

$$(15)\qquad\qquad \theta = \frac{\frac{p-1+\gamma}{\gamma}(1-\frac{1}{p+1})}{\frac{1}{N} - \frac{1}{\gamma} + \frac{p-1+\gamma}{\gamma}} \in (0,1) \quad \text{and}\quad \frac{\theta(p+1)}{p-1+\gamma} \in (0,1).$$

By (15), the Gagliardo–Nirenberg and Young inequalities as in Lemma 3.2, in conjunction with (13) for any $K > 0$, provide also some $C_{GN}, c > 0$ such that

$$K\int_\Omega u^{p+1} = K\|u^{\frac{p-1+\gamma}{\gamma}}\|_{L^{\frac{\gamma(p+1)}{p-1+\gamma}}(\Omega)}^{\frac{\gamma(p+1)}{p-1+\gamma}}$$

$$\le KC_{GN}\left(\|\nabla u^{\frac{p-1+\gamma}{\gamma}}\|_{L^\gamma(\Omega)}^{\frac{\theta\gamma(p+1)}{p-1+\gamma}}\|u^{\frac{p-1+\gamma}{\gamma}}\|_{L^{\frac{\gamma}{p-1+\gamma}}(\Omega)}^{\frac{(1-\theta)\gamma(p+1)}{p-1+\gamma}} + \|u^{\frac{p-1+\gamma}{\gamma}}\|_{L^{\frac{\gamma}{p-1+\gamma}}(\Omega)}^{\frac{\gamma(p+1)}{p-1+\gamma}}\right)$$

$$\le C_3 p\left(\frac{p-1+\gamma}{\gamma}\right)^{-\gamma}\int_\Omega |\nabla u^{\frac{p-1+\gamma}{\gamma}}|^\gamma + c \quad \text{on }\ (0,T_{max}). \qquad\square$$

For the remaining study toward the uniform boundedness of $\int_\Omega u^p$, we will distinguish the cases $\tau = 0$ and $\tau = 1$.

3.2. Analysis of the parabolic-elliptic case: $\tau = 0$.

Lemma 3.5. *Let f and g satisfy (7) and let* $\frac{2N}{N+1} < \gamma \le 2$. *Then there exists* $L_0 > 0$ *such that for some* $p > \frac{N}{2}$

$$\int_\Omega u^p \le L_0 \quad for\ all\ \ t \in (0,T_{max}).$$

Proof. We take p from Lemma 3.4. By explicitly using that $-u^p \Delta v = u^p(u-v) \le u^{p+1}$ on $\Omega \times (0, T_{max})$, from Lemma 3.3 we find that

$$(16) \qquad \frac{d}{dt} \int_\Omega u^p + \int_\Omega u^p \le 2\chi(p-1) \int_\Omega u^{p+1} - C_3 p \left(\frac{p-1+\gamma}{\gamma} \right)^{-\gamma} \int_\Omega |\nabla u^{\frac{p-1+\gamma}{\gamma}}|^\gamma + c_1$$

on $(0, T_{max})$. Finally by taking into account relation (16) and Lemma 3.4 with $K = 2\chi(p-1)$, we are given some $c_2 > 0$ fulfilling

$$\frac{d}{dt} \int_\Omega u^p + \int_\Omega u^p \le c_2 \quad \text{on} \quad (0, T_{max}),$$

which combined with $(\int_\Omega u^p)_{|t=0} = \int_\Omega u_0^p$ concludes the proof through comparison reasoning for ordinary differential inequalities. $\qquad \square$

3.3. **Analysis of the parabolic-parabolic case:** $\tau = 1$. In the fully parabolic case, it is no longer possible to replace Δv easily by inserting $u - v$; in the spirit of [17, 16, 30] we will therefore rely on Maximal Sobolev regularity for the second equation, which will enable us to properly deal with integral terms arising in testing procedures.

As the essential tool, we thus first recall the following consequence of Maximal Sobolev regularity results (like [15] or [12, Thm. 2.3]):

Lemma 3.6. *Let $N \in \mathbb{N}$, $\Omega \subset \mathbb{R}^N$ be a bounded domain with smooth boundary, $q \in (1, \infty)$, $\mu > \frac{1}{q}$ and $K > 1$. Then there is $C_{MR} > 0$ such that the following holds: Whenever $T \in (0, \infty]$, $I = [0, T)$, $f \in L^q(I; L^q(\Omega))$ and $v_0 \in W^{2,q}(\Omega)$, with $\partial_\nu v_0 = 0$ on $\partial\Omega$, is such that $\|v_0\|_{W^{2,q}(\Omega)} \le K$, every solution $v \in W_{loc}^{1,q}(I; L^q(\Omega)) \cap L_{loc}^q(I; W^{2,q}(\Omega))$ of*

$$v_t = \Delta v - \mu v + f \quad in \quad \Omega \times (0,T); \quad \partial_\nu v = 0 \quad on \quad \partial\Omega \times (0,T); \quad v(\cdot, 0) = v_0 \quad on \quad \Omega$$

satisfies

$$\int_0^t e^s \left(\int_\Omega |\Delta v(\cdot, s)|^q \right) ds \le C_{MR} \left[1 + \int_0^t e^s \left(\int_\Omega |f(\cdot, s)|^q \right) ds \right] \quad for \ all \ t \in (0,T).$$

Proof. For $(x, t) \in \Omega \times (0, T)$, let us set $z(x,t) := e^{\frac{t}{q}} v(x,t)$. Then easy computations establish that z solves

$$\begin{cases} z_t = \Delta z - \left(\mu - \frac{1}{q} \right) z + e^{\frac{t}{q}} f & \text{in } \Omega \times (0,T), \\ \partial_\nu z = 0 & \text{on } \partial\Omega \times (0,T), \\ z(x,0) = v_0(x) & x \in \Omega. \end{cases}$$

Subsequently, let us apply Maximal Sobolev regularity ([15, (3.8)], [12, Thm. 2.3]) to $A = \Delta - (\mu - \frac{1}{q})$, $X = L^q(\Omega)$ and $X_1 = D(A) = W_{\frac{\partial}{\partial\nu}}^{2,q}(\Omega) = \{w \in W^{2,q}(\Omega) : \frac{\partial w}{\partial\nu} = 0 \text{ on } \partial\Omega\}$, which asserts that with some $c_1 > 0$ we have for every $t \in (0, T)$ that

$$\|\Delta z\|_{L^q([0,t];L^q(\Omega))} + \|z_t\|_{L^q([0,t];L^q(\Omega))} \le c_1 \left(\|v_0\|_{1-\frac{1}{q},q} + \left(\int_0^t \|e^{\frac{s}{q}} f(\cdot, s)\|_{L^q(\Omega)}^q \, ds \right)^{\frac{1}{q}} \right),$$

where $\|\cdot\|_{1-\frac{1}{q},q}$ represents the norm in the interpolation space $(X, X_1)_{1-\frac{1}{q},q}$, which can be estimated by K, according to the assumption on v_0. In turn, we have that for $C_{MR} = c_1^q 2^{q-1} K^q$ (cf. (14)) and all $t \in (0, T)$

$$(17) \qquad \int_0^t \Big(\int_\Omega |\Delta z(\cdot, s)|^q \Big) ds \le c_1^q \bigg[K + \Big(\int_0^t \|e^{\frac{s}{q}} f(\cdot, s)\|_{L^q(\Omega)}^q \, ds \Big)^{\frac{1}{q}} \bigg]^q$$

$$\le C_{MR} \bigg[1 + \Big(\int_0^t e^s \Big(\int_\Omega |f(\cdot, s)|^q \Big) ds \Big) \bigg].$$

We can finally obtain the claim by re-substituting $z(\cdot, t) := e^{\frac{t}{q}} v(\cdot, t)$ into relation (17). $\qquad\square$

With the aid of the above lemma, let us continue with the following L^p bound.

Lemma 3.7. *Let f and g satisfy (7) and let $\frac{2N}{N+1} < \gamma \le 2$. Then there exists $L_1 > 0$ such that for some $p > \frac{N}{2}$*

$$\int_\Omega u^p \le L_1 \quad \text{for all} \quad t \in (0, T_{max}).$$

Proof. Again we let $p > \frac{N}{2}$ be as in Lemma 3.4. From Lemma 3.3 we have that

$$\frac{d}{dt} \int_\Omega u^p + \int_\Omega u^p \le \chi(p-1) \int_\Omega |u^p \Delta v| + \chi(p-1) \int_\Omega u^{p+1} + c_1 - C_3 p \Big(\frac{p-1+\gamma}{\gamma} \Big)^{-\gamma} \int_\Omega |\nabla u^{\frac{p-1+\gamma}{\gamma}}|^\gamma$$

on $(0, T_{max})$. By invoking Young's inequality, we infer that on $(0, T_{max})$,

$$\frac{d}{dt} \int_\Omega u^p + \int_\Omega u^p \le \chi \frac{(p-1)(2p+1)}{p+1} \int_\Omega u^{p+1} + \chi \frac{p-1}{p+1} \int_\Omega |\Delta v|^{p+1} + c_1 - C_3 p \Big(\frac{p-1+\gamma}{\gamma} \Big)^{-\gamma} \int_\Omega |\nabla u^{\frac{p-1+\gamma}{\gamma}}|^\gamma.$$

Successively, solving the above ordinary differential inequality for $\int_\Omega u(\cdot, t)^p$, we can derive for all $t \in (0, T_{max})$,

$$e^t \int_\Omega u^p \le \|u_0\|_{L^p(\Omega)}^p$$

$$+ \int_0^t e^s \bigg[\chi \frac{(p-1)(2p+1)}{p+1} \int_\Omega u^{p+1} + \chi \frac{p-1}{p+1} \int_\Omega |\Delta v|^{p+1} - C_3 p \Big(\frac{p-1+\gamma}{\gamma} \Big)^{-\gamma} \int_\Omega |\nabla u^{\frac{p-1+\gamma}{\gamma}}|^\gamma + c_1 \bigg] ds.$$

In estimating the integral containing Δv, let us now exploit Lemma 3.6 with $f = u$, $q = p+1$, $\mu = 1$ to obtain $C_{MR} > 0$ such that for all $t \in (0, T_{max})$

$$e^t \int_\Omega u^p \le c_2 + \int_0^t e^s \bigg[c_3 \int_\Omega u^{p+1} - C_3 p \Big(\frac{p-1+\gamma}{\gamma} \Big)^{-\gamma} \int_\Omega |\nabla u^{\frac{p-1+\gamma}{\gamma}}|^\gamma \bigg] ds + c_4(e^t - 1),$$

where $c_2 = \|u_0\|_{L^p(\Omega)}^p$, $c_3 = \chi \frac{(2p+1)(p-1)}{p+1} + \chi \frac{p-1}{p+1} C_{MR}$ and $c_4 = c_1 + \frac{p-1}{p+1} \chi C_{MR}$. In this way, we again invoke Lemma 3.4, applied with $K = c_3$, so as to rephrase the previous inequality as

$$e^t \int_\Omega u^p \le c_2 + c_4(e^t - 1) \quad \text{for all } t \in (0, T_{max}),$$

from which we obtain the assertion with $L_1 := c_2 + c_4$. $\qquad\square$

Finally we are in a position to conclude.

3.4. **Final proofs of boundedness and global existence.**

Proof of Theorem 1.2. By Lemma 2.6 and Theorem 1.1, whenever the first component of the local solution (u, v) to problem (3) belongs to $L^\infty((0, T_{max}); L^p(\Omega))$ for some $p > \frac{N}{2}$, then actually $u \in L^\infty(\Omega \times (0, \infty))$; evidently, by using the equation for v, if u lies in $L^\infty(\Omega \times (0, \infty))$, so does v. Subsequently, the claim follows for $\gamma > \frac{2N}{N+1}$ and $\tau = 0$ and $\tau = 1$, by invoking Lemmas 3.5 and 3.7, respectively.   □

Proof of Corollary 1.3. Let us observe that for any $a, b, C_2 > 0$ and $\beta > \alpha \geq 1$, the function defined by $f(s) = as^\alpha - bs^\beta$ is continuously differentiable, so that (4a) holds, and satisfies $\lim_{s \to \infty}(f(s) + C_2 s) = -\infty$, so that there is $C_1 > 0$ such that $f(s) + C_2 s \leq C_1$ for every $s \geq 0$. In particular, f satisfies (4b) and (7). Similarly, for any $\gamma \in [1, 2]$ and $c > 0$, $g: \mathbb{R}^N \to \mathbb{R}$, $g(z) = c|z|^\gamma$ obeys (4c), (4d) and (7). We therefore can apply Theorem 1.2.   □

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