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Homogenization and dimensional reduction for nonlinear multilayered plates including biological growth when the plate thickness and the size of the heterogeneities are not of the same order of magnitude

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E. Pruchnicki
Univ. Lille, ULR 7512
UML - Unité de Mécanique de Lille, Joseph Boussinesq,
59000 Lille, France
erick.pruchnicki@gmail.com

X. Chen
Bejiing Normal-Hong Kong Baptist University, Zhuhai 519087, China;
Guandong Provincial Key Laboratory of Interciplinary research
and Applications for Data Science.
xiaoyichen@bnbu.edu.cn

Y. Gao
Institute of Mechanics, Chinese Academy of Sciences,Beijing, 100190,
University of Chinese Academy of Sciences, Beijing 100049,
China
gaoyahe@imech.ac.cn

V. Eremeyev,
via Marengo 2, Edificio A, 09123 Cagliari, Italie
victor.eremeev@unica.it

Abstract

In this work, we address the important problem of the homogenization and the dimensional reduction for nonlinear plates including biological growth effect when the plate thickness and the size of the heterogeneities are not of the same order of magnitude. The theory when the thickness of the plate and the in-plane heterogeneities are of the same order of magnitude has been previously addressed by the first author. In our reduction method, the thickness of the plate is a small but does not go to zero, however the homogenization method adopted is a standard asymptotic analysis since the size of the heterogenities goes to zero. For the sake of simplicity, the distribution of material heterogeneities is assumed to be repetitive periodic. For the case when the period is very much smaller than the thickness of the plate, we have to consider first the limit when the size of the heterogeneities goes to zero. We obtain a multilayered plate with homogeneous layers for which we propose a two-dimensional plate model. Then we consider the case when the period is very much bigger than the thickness of the plate. We have to consider first homogenization in plane parallel to the mid-plane of the plate. This method is only meaningful when the geometry of the heterogeneities does not depend on the thickness direction of the plate. Then, we can obtain a plate model from multilayered plate in which each layer is homogeneous. Possible applications for future numerical works are given through work references and photographs.

Keywords: Homogenization, Heterogeneous plate, nonlinear elasticity, Multilayered-plate, Growth theory, two-dimensional plate model.

1 Introduction

This work gives new theoretical model for composite multilayered plates with a repetitive microscopic structure (*i.e.* periodic microstructure) valid for both manufactured industry products and plant structure in nonlinear elasticity including growth theory. The size of fine-scale details is typically much smaller compared to the characteristic in-plane dimension of the plate (denoted by l), and then direct numerical analyses are much too expensive. For the sake of simplicity, we suppose that each layer (numbered k=1, ..., n) of the multilayered plate is made up of different types of materials and can be generated by an assembly of many building blocks of the heterogeneous material called unit cell (denoted by $\mathbf{Y}^k$ for k=1,..., n). If the size of the unit cell (denoted by f) is much smaller than the size of the mid-plane of the plate (i.e., $\mu = f/l \ll 1$). We can consider that there exists a separation between the macroscopic scale and the microscopic scale. Indeed, the mechanical behaviour of a homogenized material associated to the heterogeneous one can be deduced by solving a micromechanical problem on the unit cell.

The specific geometry of the plate is such that its thickness is much smaller than the in-plane dimensions (l) i.e. $\varepsilon = h/l \ll 1$. For homogeneous plate, we have to consider only the deduction of an approximate two-dimensional model from the true three-dimensional equations which is called the reduction plate modelling. The rigorous limit of a two-dimensional variational problem when the thickness vanishes can be achieved through the $\Gamma-$convergence method ([1], [2]). This lead to a hierarchy of models depending on the order of magnitude of the applied load. Unfortunately, it is impossible to get a unique model independent of the order of magnitude of the applied external loads. To avoid this problem, it is necessary to consider different kind of ideas.

Indeed, Schneider *et al.*[3], Kienzler and Schneider [4], Steigmann [5] and Pruchnicki [6] introduce the new idea of the truncation of the elastic potential. While Dai and Song [7] and Wang *et al.* [8, 9] work directly on the static equations and the boundary conditions which reduce to a system of differential equations with boundary conditions.

Dai and Song [7] and Wang *et al.* [8, 9] consider a Taylor-Young expansion of the displacemment field up to fifth order with respect to the thickness of the plate and then deduce the static equations, the constituve law and the boundary conditions for a new kind of bidimensional model for homogeneous plate. A specific expansion of the displacement field including warping is proposed by Polizzoto [10].

An attempt to extend these methods to general geometry (non stratified) highly heterogeneous structures leads to intractable developments. So other ideas should be used.

When both the size of the microstructure heterogeneity and the thickness of the plate are of the same order of magnitude ($\mu \approx \varepsilon$), the homogenization and reduction processes can be performed simultaneously. The asymptotic expansion method applied by Caillerie [11] and Kohn and Vogelius [12] on a periodic linear elastic plate leads to a Kirchhoff–Love plate model. This result generalizes the known results established by Ciarlet [13] for homogeneous plates. However the transverse shearing effect can only be captured by considering up to second-order terms in the asymptotic expansion of the displacement field ([14]). A new idea is given in Sab and Lebée [15] and Lebée and Sab [16], these authors extend their bending gradient theory for homogeneous plate ([17]). So they propose a new homogenization procedure in order to include transverse shearing effect in the homogenization theory for elastic linear plate. Nevertheless, it seems to be very complicated to extend this approach in the fully nonlinear setting.

For the nonlinear hyperelastic case, by using the new idea of Pantz method of minimizing the potential energy with asymptotic expansion method proposed by Pantz for homogeneous plates [18-19], Pruchnicki [20-21] has obtained a membrane model and a bending model for heterogeneous plates. While Kalamkarov *et al.* [22] introduce directly the formal asymptotic expansion method in the strong formulation of the mechanical problem. They can justify the nonlinear von Karman's plate model, however the theory seems to be limited to material of which the mechanical behaviour is characterized by a linear expression of the second Piola-Kirchhoff tensor in terms of the Green-Lagrange strain tensor and many nonlinear terms are neglected.

The asymptotic expansion method is mathematically elegant but leads to intractable computation in nonlinear setting. As a remedy to these shortcomings, Berdichevsky [23] introduces the variational asymptotic method. Thus, by considering small strains, large rotations and displacements, Lee and Yu [24] and Lee *et al.* [25] obtain a new type of homogenized model for heterogeneous plates. Extensions of this idea is proposed by Pruchnicki [26-27] in the fully nonlinear setting including growth theory.

When both the order of magnitude of the heterogeneities of the microstructure and the thickness of the plate are not of the same order of magnitude ($\mu \ll \varepsilon$ or $\varepsilon \ll \mu$), the reduction and homogenization processes can be decoupled as explained in detail in Caillerie [11] and Bourgeois [28]. The two cases are summarized in Table 1 and detailed in the following sections. When there are many unit cells along the thickness direction (i.e., $\mu \ll \varepsilon$, see Figure 1), we can perform homogenization first to obtain homogenized effective properties of the heterogeneous material, then a dimensional reduction will give a plate model for structural analysis. Caillerie [11] has obtained classical results in linear elasticity. Another rigorous convergence result has been obtained by Hornung et *al.* [29] and Bufford *et al.* [30], they obtain by considering an appropriate energy scaling and by using Γ–convergence method, a nonlinear Kirchhoff-Love model valid for pure bending of plates.

As pointed out by Caillerie [11] when $\varepsilon \ll \mu$, the order of the aforementioned two-step approach is reversed. New convergence results have been obtained by Rohan and Miara [31] for the dynamical response of heterogeneous linear plates when the geometry of the heterogeneity will be assumed of cylindrical shape (i.e. independent from the transverse coordinate in thickness direction). They assume that the macroscopic behaviour is either of Kirchhoff-Love or of Reissner-Mindlin types. For the same hypothesis on the geometry of the heterogeneities and for linear Kirchhoff-Love plates made of metamaterials (man-made materials), Faraci *et al.* [32] study the transversal vibration of thin periodic elastic plates through asymptotic homogenization method.

The aim of this work is to give new general theoretical results for both the homogenization and the dimensional reduction for nonlinear elastic plates including biological growth effect when both the plate thickness and the size of the heterogeneities are not of the same order of magnitude. To our knowledge, this major research theme was never addressed in the literature. So this work is the continuation of the theoretical work of Pruchnicki [26-27] in which homogenization of heterogeneous nonlinear plates has been adressed when the thickness and the size of the heterogeneities are of the same order of magnitude.

In the reduction method adopted, the thickness of the plate is a constant and then does not go to zero i.e. asymptotic analysis is not considered. However, the homogenization results are obtained through a standard asymptotic analysis method i.e. the size of the heterogenities goes to zero. For the sake of simplicity, the distribution of material heterogeneities is assumed to be repetitive periodic. In Section 2, the notation and the problem considered are given. In Section 3, the general non linear quasi incompressible constitutive law including growth effect is described. In Section 4, the equation of non linear mechanical problem with boundary conditions is explicited. Section 5 is devoted to the case when the period is very much smaller than the thickness of the plate, then we have to consider first the limit when the size of the heterogeneities goes to zero. We obtain a multilayered plate with homogeneous layers (stratified plate) for which we are able to propose a two dimensional plate model. In Section 6, we consider the case when the period is very much bigger than the thickness of the plate.

So we study the successive methods in the inverse order to that considered in Section 5. We also explain that this method is only meaningful when the geometry of the heterogeneities does not depend on the thickness direction of the plate. It is the case for the well-known honeycomb structure with two skins. In Section 7, possible applications are also given.

2 Preliminary

2.1 Two scales description of heterogeneous plates

Let us denote by ($\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$) the basis of $\mathbb{R}^3$. Then we denote by $\mathbf{x} = (x_1, x_2, x_3)$,the Cartesian coordinate of a point of the reference position of the plate which also defines the macro-scale of the plate. In the plane (O, $\mathbf{e}_1$, $\mathbf{e}_2$), the domain $\boldsymbol{\omega} \subset \mathbb{R}^2$ defines the mid-plane of the plate. $\mathbf{x}^{'} = (x_1, x_2)$ represents the two orthogonal coordinates in the directions of the mid-plane of the plate. x_3 is the coordinate associated with the direction normal to mid-plane of the plate (thickness direction). This plate is composed of n heterogeneous layers with the initial thickness h^1, h^2, ... h^n, so the total thickness of the plate is $h = \sum_{i=1}^{k} h^k$. For the k-th layer of the plate a local orthonormal coordinate system (O_k, $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$) is established on the bottom surface of each layer. Within this local coordinate system, the reference configuration of the k-th layer occupies the region $\Omega^k = \boldsymbol{\omega}\times[0, \text{h}^k](k = 1, 2, \ldots n)$. The macroscopic coordinate in the k-th layer is then denoted by $\mathbf{x}$.

For the sake of simplicity of the notation, we do not introduce the superscript k for the cartesian coordinate in the k-th layer.

To describe the rapid change in the material characteristics, we introduce the local (microscopic) coordinates defined in the k-th layer by $\boldsymbol{y} = (y_1, y_2, y_3)$ $=\frac{\mathbf{x}}{\mu}$ and the global (macroscopic) counterpart coordinates is $\boldsymbol{x}$. We define $\boldsymbol{y}^{'} = (y_1, y_2)$. For the sake of simplicity, we suppose that the entire heterogeneous structure is periodic and then can be generated by an assembly of many building blocks of the heterogeneous material. Then the microstructure in each layer k can be characterized by the tridimensional connected unit cell denoted by $\mathbf{Y}^k$=$[0, a^k] \times [0, b^k] \times [0, c^k]$.

The displacement field of a material point in Ω^k is denoted as $\mathbf{u}_\mu^k$ (the subscript μ is added because the displacement field depend on the rapidly varying heterogeneities $\mu\mathbf{Y}^k$ and the superscript k is linked to the layer k) . The lateral boundary of each layer of the plate ($\partial\boldsymbol{\omega} \times [0, h^k]$) is split into two parts, the first one $\partial\boldsymbol{\omega}_{\overline{\mathbf{t}}} \times [0, h^k]$ is subjected to the surface force $\overline{\mathbf{t}}^k$ (in the initial configuration) and on the second one, denoted by $\partial\boldsymbol{\omega}_{\overline{\mathbf{u}}} \times [0, h^k]$, a prescribed displacement field $\overline{\mathbf{u}}^k$ is imposed. We denote by $\mathbf{n}$ the outward unit normal to the lateral boundary of the plate.

The load acting in the initial configuration on both the upper (denoted by $\overline{\mathbf{t}}^{\text{upp}}(\text{x}^{'})$ and the lower (denoted by $\overline{\mathbf{t}}^{\text{low}}(\text{x}^{'})$) boundaries of the plate and the body forces per unit mass (denoted by $\mathbf{f}_\mu^k(\mathbf{x}^k)$). Then the body force per unit volume in the reference configuration is $\mathbf{f}_\mu^{kv}(\mathbf{x}^k) = \rho_{0\mu}(\mathbf{x}^k)\mathbf{f}_\mu^k(\mathbf{x}^k)$ with $\rho_{0\mu}$

isthe mass density in the initial undeformed configuration.

2.2 Notation

To ensure clarity and consistency, we present the notations that will be used throughout the subsequent analysis in Table 1.

Symbol	Description
$\mathbf{x}$, $\mathbf{x}^{'}$, x_3	Macroscopic Cartesian coordinates (global, in-plane, thickness).
$\mathbf{y}$, $\mathbf{y}\prime$	Microscopic coordinates ($\mathbf{y}$= $\mathbf{x}$/µ).
k, n	Index of a layer and total number of layers.
i, j, α, β	Indices (Latin: 1 to 3; Greek: 1 to 2).
l	Characteristic in-plane dimension of the plate.
h, h^k	Total thickness of the plate and thickness of the k-th layer.
f	Characteristic size of the unit cell (heterogeneity).
$\varepsilon= h/l$	Small parameter representing the plate's aspect ratio.
$\mu= f/l$	Small parameter for the ratio of microscopic to macroscopic length scales.
ω	Domain of the mid-plane of the plate.
Ω^k	Reference configuration of the k-th layer.
$\mathbf{Y}^k$	Tridimensional unit cell of the k-th layer.
$\mathbf{u}_\mu^k$	Displacement field in the k-th layer.
$\mathbf{F}_\mu^k$	Total deformation gradient tensor in the k-th layer.
$\mathbf{A}_\mu^k$	Elastic deformation tensor in the k-th layer.
$\mathbf{G}_\mu^k$	Growth tensor in the k-th layer.

Table 1: The notations used for the multiscale model.

Frequent reference will be made to the definitions presented in Table 1 in the sections that follow. The small parameters ε and μ, for instance, are fundamental to the asymptotic expansion and homogenization procedures central to our model. As will be shown, the relationship between these two parameters dictates the specific form of the resulting governing equations.

We adopt the Einstein convention summation for repeated indices. The vector product (respectively the tensorial product) of vectors $\mathbf{a}$ and $\mathbf{b}$ are denoted by $\mathbf{a} \times \mathbf{b}$ (respectively $\mathbf{a} \otimes \mathbf{b}$). $\mathbf{I}$ is the unit second order tensor. The simple (respectively double) dot product of two tensors $\mathbf{A}$ and $\mathbf{B}$ is denoted by $\mathbf{A}.\mathbf{B}$ (respectively $\mathbf{A}$: $\mathbf{B}$). Det ($\mathbf{A}$) and

Cof ($\mathbf{A}$) are the determinant and the cofactor matrix of a second order tensor $\mathbf{A}$.

$\mathbf{v}_{,\alpha}$ (respectively $\mathbf{v}_{;\alpha}$) denotes the partial derivative of $\mathbf{v}$ with respect to x_α^k (respectively y_α^k), the *i-th* derivative with respect to x_3 is denoted by $(.)^{(i)}=\frac{\partial^i(.)}{\partial x_3^i}$ and obviously $(.)^{(0)}=(.)$,

$\mathrm{div}_{\mathbf{x}}$ and $\mathrm{grad}_{\mathbf{x}}$ denote divergence and gradient operators with respect to the variable specified in index, for example for a second order tensor, we have $\mathrm{div}_{\mathbf{x}'}\, \mathbf{B} = B_{i\alpha,\alpha}\ \mathbf{e}_i$.

The average over the unit cell $\mathbf{Y}^k$ of a tensor $\mathbf{Z}^k$ is $\langle \mathbf{Z}^k \rangle = \frac{1}{|\mathbf{Y}^k|} \int_{\mathbf{Y}^k} \mathbf{Z}^k d\mathbf{Y}^k$ and $|\mathbf{Y}^k|$ is the volume of $\mathbf{Y}^k$ when we average over an other domaine D (bidimensional one for example), we will adopt the notation $\langle \mathbf{Z}^k \rangle_{\mathbf{D}} = \frac{1}{|\mathbf{D}|} \int_{\mathbf{D}} \mathbf{Z}^k d\mathbf{D}$.

3 The three dimensional growth constitutive law

The total deformation gradient in the *k-th* layer can be given in terms of the displacement field denoted by $\mathbf{u}^k_\mu$

$$\mathbf{F}^k_\mu = \mathbf{I} + \mathrm{grad}_{\mathbf{x}}\, \mathbf{u}^k_\mu.$$

The total deformation of the plate comes from both the growth process and the elastic deformation. Following the basic assumption of elastic growth theory ([33-35]), the total deformation gradient tensor $\mathbf{F}^k_\mu$ can be decomposed into

$$\mathbf{F}^k_\mu = \mathbf{A}^k_\mu . \mathbf{G}^k_\mu, \tag{1}$$

in which $\mathbf{A}^k_\mu$ is the elastic deformation tensor and $\mathbf{G}^k_\mu$ is the growth tensor which is assumed to dependent of $\mathbf{x}'$ i.e. independent of x_3.

We assume that each layer can grow independently. We assume that each material constituting each layer of the plate is both nonlinear and isotropic. The stored energy function in the layer k is denoted by W^k. We assume that the constitutive law is isotropic and the material is quasi-incompressible (which is satisfied for most soft biological material) then the expression of the nominal stress tensor (the transpose of the first Piola-Kirchhoff tensor, see [7]) is given by ([33])

$$\boldsymbol{\pi}^k_\mu = J_{\mathbf{G}^k_\mu} \left(\mathbf{G}^k_\mu\right)^{-1} \boldsymbol{\pi}^{el\ k}_\mu, \tag{2}$$

where $J_{\mathbf{G}^k_\mu} = \det(\mathbf{G}^k_\mu)$ and $\boldsymbol{\pi}^{el\ k}_\mu = \left(\frac{\partial W^k_\mu(\mathbf{A}^k_\mu)}{\partial \mathbf{A}^k_\mu}\right)^T$.

To overcome the numerical difficulty linked to both the nearly incompressibility and the large deformation. The deformation gradient should be decoupled into the dilatational (volume changing) and the deviatoric parts ($\overline{\mathbf{A}}^k_\mu$ such that $\det(\overline{\mathbf{A}}^k_\mu)$=1)

$$\mathbf{A}^k_\mu = J^{\frac{1}{3}}_{\mathbf{A}^k_\mu} \overline{\mathbf{A}}^k_\mu.$$

Indeed, the strain-energy function is decoupled into a distortional part (W^k_{dis}) and a dilatational part (W^k_{vol}):

$$W^k_\mu(\mathbf{A}^k_\mu) = W^k_{dis\ \mu}(\bar{I}^k_{1\mu}, \bar{I}^k_{2\mu}) + W^k_{vol\ \mu}\left(J_{\mathbf{A}^k_\mu}\right).$$

where $\bar{I}^k_{1\mu}$ and $\bar{I}^k_{2\mu}$ are the two first invariants of $\mathbf{A}^k_\mu$.

For nearly incompressible material, we can choose

$$W_{vol\ \mu}^{k}(J_{\mathbf{A}_{\mu}^{k}}) = \frac{(J_{\mathbf{A}_{\mu}^{k}} - 1)^2}{2\alpha_{\mu}^{k}},$$

and α^k denotes the compressibility coefficient.

Then, we obtain ([36], p 19)

$$\boldsymbol{\pi}_{\mu}^{el\ k}\left(\mathbf{A}_{\mu}^{k}\right) = \left(\frac{\partial W_{dis\ \mu}^{k}(\overline{\mathbf{A}}_{\mu}^{k})}{\partial \overline{\mathbf{A}}_{\mu}^{k}} : \mathbf{P}_{\overline{\mathbf{A}}_{\mu}^{k}}\right)^{T} + \frac{\left(J_{\mathbf{A}_{\mu}^{k}} - 1\right)}{\alpha_{\mu}^{k}} \operatorname{Cof}\left(\mathbf{A}_{\mu}^{k}\right)^{T}, \tag{3}$$

where $\mathbf{P}_{\overline{\mathbf{A}}_{\mu}^{k}} = J_{\mathbf{A}_{\mu}^{k}}^{-\frac{1}{3}}\left[\mathbf{I}^4 - \frac{1}{3}\mathbf{A}_{\mu}^{k} \otimes \mathbf{A}_{\mu}^{k-T}\right]$ and $\mathbf{I}^4$ is the identity fourth order tensor.

So from expressions (2) and (3), the general form of the constitutive law for the quasi incompressible material is

$$\boldsymbol{\pi}_{\mu}^{k}\left(\mathbf{G}_{\mu}^{k}, \mathbf{A}_{\mu}^{k}\right) = \mathbf{CL}^{k}\left(\mathbf{G}_{\mu}^{k}, \mathbf{A}_{\mu}^{k}\right) = J_{\mathbf{G}_{\mu}^{k}}\left(\mathbf{G}_{\mu}^{k}\right)^{-1}\left(\left(\frac{\partial W_{dis\mu}^{k}(\overline{\mathbf{A}}_{\mu}^{k})}{\partial \overline{\mathbf{A}}_{\mu}^{k}} : \mathbf{P}_{\overline{\mathbf{A}}_{\mu}^{k}}\right)^{T} + \frac{\left(J_{\mathbf{A}_{\mu}^{k}} - 1\right)}{\alpha_{\mu}^{k}} \operatorname{Cof}\left(\mathbf{A}_{\mu}^{k}\right)^{T}\right). \tag{4}$$

4 Mechanical two-scale boundary problem

We assume that the different layers in the plate can grow independently.

The nonlinear mechanical problem (P) for the entire plate is obtained by considering successively the equilibrium equation, the nonlinear constitutive law defined by the tensorial function (4), the decomposition of the deformation gradient into elastic and growth parts (1) and the Dirichlet and the Neumann boundary conditions on the lateral boundary of the plate for each k-th layer (k = 1,2,..., n) and on the Neumann boundary conditions on both the upper and the lower faces of the plate.

$$\operatorname{div}_{\mathbf{x}} \boldsymbol{\pi}_{\mu}^{kT} + \mathbf{f}_{\mu}^{\ k\nu} = 0 \qquad \text{in} \qquad \Omega^{k}, \tag{5}$$

$$\boldsymbol{\pi}_{\mu}^{el\ k} = \mathbf{CL}^{k}\left(\mathbf{G}_{\mu}^{k}, \mathbf{A}_{\mu}^{k}\right) \qquad \text{in} \qquad \Omega^{k}, \tag{6}$$

$$\mathbf{A}_{\mu}^{k} = \mathbf{F}_{\mu}^{k}.\left(\mathbf{G}_{\mu}^{k}\right)^{-1} \qquad \text{in} \qquad \Omega^{k}, \tag{7}$$

$$\boldsymbol{\pi}_{\mu}^{\text{upp}\,T}.\mathbf{e}_3 = \bar{\mathrm{t}}^{\text{upp}} \qquad \text{on} \qquad \omega \times \{h^n\}, \tag{8}$$

$$\boldsymbol{\pi}_{\mu}^{\text{lowT}}.\mathbf{e}_3 = \bar{\mathrm{t}}^{\text{low}} \qquad \text{on} \qquad \omega \times \{0\}, \tag{9}$$

$$\boldsymbol{\pi}_{\mu}^{kT}.\mathbf{n} = \bar{\mathrm{t}}^{k} \qquad \text{on} \qquad \partial\boldsymbol{\omega}_{\bar{\mathbf{t}}} \times [0, h^k], \tag{10}$$

$$\mathbf{u}_\mu^k.\mathbf{n} = \overline{\mathrm{u}}^k \qquad \text{on} \qquad \partial\boldsymbol{\omega}_{\overline{\mathbf{u}}} \times [0, h^k]. \tag{11}$$

where $\mathbf{u}_\mu^k$, $\mathbf{G}_\mu^k$ and $\boldsymbol{\pi}_\mu^k$ are $\mu\mathbf{Y}^k$ periodic in $\mathbf{y}_k$.

On the interfaces between the different layers, we propose the following displacement and stress vector continuity conditions for k=1,..., n-1.

$$\mathbf{u}_\mu^k\left(h^k\right) = \mathbf{u}_\mu^{k+1}(0), \tag{12}$$

$$\boldsymbol{\pi}_\mu^{k\ T}\left(h^k\right).\mathbf{e}_3 = \boldsymbol{\pi}_\mu^{k+1\ T}(0).\mathbf{e}_3. \tag{13}$$

The original mechanical problem (P) defined on both the microcopic and macroscopic scales is then given by equations (5)-(13).

5 Plate theory when $\mu \ll \varepsilon$

In this section, we assume that the thickness of the plate is much greater than the size heterogeneity. As in Caillerie [11] for linear setting, we have to consider first the limiting behaviour of each layer of the plate when the size of the heterogeneities μ goes to zero (homogenized behaviour). We will show in the next subsection that the limiting homogenized behaviour of the k-th layer is obtained by solving a local microscopic problem on the three-dimensional unit cell $\mathbf{Y}^k$. This process gives a stratified plate with homogeneous layer for which we can derive a plate theory by using the new idea of Dai and Song [7].

We begin by considering the problem (P) (defined in the previous Section 4) which is valid for the initial heterogeneous structure.

This type of heterogeneous plate can be found in industrial plate made of wood. The structure of the local heterogeneities is described in [37]. If this case (the three dimensions are of the same order of magnitude) and in linear elastic setting, homogenization theories are given in [38-39].

5.1 Homogenization

This method can only be applied when the geometry of the heterogeneities (size μ) in each layer of the plate is very small compared with the thickness of the plate (size $\mu \ll \varepsilon$) as shown in Figure 1.

Figure 1: Plate with microstructure smaller than the thickness

So the homogenized tridimensional behaviour of each layer of the plate is obtained by a tridimensional asymptotic analysis. This type of homogenization approach was already considered by Marcellini [40], when the strain energy is a convex function of the deformation gradient. The homogenized behavior is given by the homogenized strain energy and this function is computed on a single unit cell. However, if the hyperelastic strain energy is not convex and the problem cannot be studied on a single unit cell ([41]). For the sake of simplicity, we assume that there is no instability in the range of loads studied and the homogenized strain energy can be computed in a single unit cell. Dumontet [42], [43], Geymonat *et al.* [44] and Pruchnicki [45] have studied the one dimensional case for stratified (each layer is homogeneous) three dimensional structure materials. For a a general non-stratified material, nonlinear homogenization of hyperelastic materials is numerically implemented and adapted to topology optimization ([46]). Other authors have chosen to linearize the initial problem by using an incremental method ([47-48]). Ponte Castaneda and Suquet [49], Talbot and Willis [50] and Ponte Castaneda [51] studied Voigt and Reuss and Hashin–Shtrikman bounds for hyperelastic materials. Ponte Castaneda and Tiberio [52] and for incompressible materials Lahellec *et al.* [53] applied the second-order theory, proposed by Ponte Castaneda [54], to hyperelastic composites. We will formally establish this homogenization theory by considering asymptotic expansion method, with the small adimensional ratio μ. The homogenized nonlinear law is theoretically the limit behaviour when μ goes to zero. The homogenization theory is valid in the region far enough from boundaries of the structure and so the homogenization theory is not satisfactory in the neighbourhood of these boundaries. In the neighbourhood of the lateral boundaries, we need to consider a specific boundary layer theory for correcting the homogenization theory ([43–45]).

Basically in the asymptotic method with double variables ([55-56]), we look for an asymptotic expansion of the displacement field $\mathbf{u}_\mu^k$ and the growth tensor $\mathbf{G}_\mu^k$ in terms of μ power which is solution of the problem (P)

$$\mathbf{u}_\mu^k(\mathbf{x}) = \mathbf{u}_0^k(\mathbf{x}) + \sum_{i=1}^{\infty} \mu^i \mathbf{u}_i^k\left(\mathbf{x}, \mathbf{y}\right), \tag{14}$$

$$\mathbf{G}_\mu^k(\mathbf{x}^{'}) = \mathbf{G}_0^k\left(\mathbf{x}^{'}, \mathbf{y}\right) + \sum_{i=1}^{\infty} \mu^i \mathbf{G}_i^k\left(\mathbf{x}^{'}, \mathbf{y}\right). \tag{15}$$

where functions $\mathbf{u}_i^k$ (for $i \geq 1$) and $\mathbf{G}_i^k$ are $\mathbf{Y}^k$-periodic in $\mathbf{y}$.

As in Du *et al.* [57-59], Wang *et al.* [60], we suppose that the growth is a given function which does not depend on the macroscopic thickness direction. For the body force, we assume that

$$\mathbf{f}\,_\mu^{k\nu}(\mathbf{x}) = \mathbf{f}\,^{k\nu}(\mathbf{x}, \mathbf{y}),$$

in which $\mathbf{f}\,^{k\nu}$ is $\mathbf{Y}^k$ periodic in $\mathbf{y}$.

The homogenization process consists in introducing the asymptotic expansion of the displacement field (14) and the growth tensor (15) into the problem (P). The same μ power terms are identified and the obtained problems are solved in series. Considering $\mathbf{x}$ and y as independent variables, by using the chain rule for derivative, we obtain for any terms of the form $\psi\left(\mathbf{x}, \mathbf{y} = \frac{\mathbf{x}}{\mu}\right)$.

$$\frac{\partial \psi\left(\mathbf{x}, \frac{\mathbf{x}}{\mu}\right)}{\partial x_i} = \frac{\partial \psi\left(\mathbf{x}, \mathbf{y}\right)}{\partial x_i} + \frac{1}{\mu}\frac{\partial \psi\left(\mathbf{x}, \mathbf{y}\right)}{\partial y_i}.$$

Thus from asymptotic expansion of the growth tensor (15) and the displacement field (14), we deduce the asymptotic expansion of both the deformation gradient and the nominal stress tensor

$$\mathbf{F}_{\mu}^{k}(\mathbf{x}) = \mathbf{F}_{0}^{k}(\mathbf{x},\mathbf{y}) + \sum_{i=1}^{\infty}\mu^{i}\mathbf{F}_{i}^{k}(\mathbf{x},\mathbf{y}),$$

$$\pi_{\mu}^{k}(\mathbf{x}) = \boldsymbol{\pi}_{0}^{k}(\mathbf{x},\mathbf{y}) + \sum_{i=1}^{\infty}\mu^{i}\boldsymbol{\pi}_{i}^{k}(\mathbf{x},\mathbf{y}). \tag{16}$$

in which $\mathbf{F}_{0}^{k}(\mathbf{x},\mathbf{y}) = \mathbf{F}^{Mk}(\mathbf{x})+\text{grad}_{\mathbf{y}}\mathbf{u}_{1}^{k}(\mathbf{x},\mathbf{y})$ with $\mathbf{F}^{Mk}$=I+$\text{grad}_{\mathbf{x}}\mathbf{u}_{0}^{k}(\mathbf{x})$,

$\mathbf{F}_{i}^{\ k}(\mathbf{x},\mathbf{y})$ =$\text{grad}_{\mathbf{x}}\mathbf{u}_{i}^{k}(\mathbf{x},\mathbf{y})$+$\text{grad}_{\mathbf{y}}\mathbf{u}_{i+1}^{k}(\mathbf{x},\mathbf{y})$ for $i \geq 1$, $\boldsymbol{\pi}_{0}^{k} = \mathbf{CL}^{k}\left(\mathbf{G}_{0}^{k}, \mathbf{A}_{0}^{k} = \mathbf{F}_{0}^{k}.\mathbf{G}_{0}^{k-1}\right)$.

We can remark that we do not need the expression $\boldsymbol{\pi}_{i}^{k}(\mathbf{x},\mathbf{y})$ $(i \neq 0)$ which is necessary for higher order homogenization theory.

By substituting the asymptotic expansion of the nominal stress tensor (16) into the equilibrium equation (5) and equating to zero terms of μ^{-1} and μ powers, we obtain:

$$\text{div}_{\mathbf{y}}\,\boldsymbol{\pi}_{0}^{kT} = 0 \qquad \text{in} \qquad \Omega^{k}\times\mathbf{Y}^{k}, \tag{17}$$

$$\text{div}_{\mathbf{x}}\,\boldsymbol{\pi}_{0}^{kT} + \text{div}_{\mathbf{y}}\,\boldsymbol{\pi}_{1}^{kT} + \mathbf{f}^{\ k\nu} = 0 \qquad \text{in} \qquad \Omega^{k}\times\mathbf{Y}^{k}. \tag{18}$$

5.1.1 Microscopic problem at first order

For each point $\mathbf{x}\in\Omega^{k}$ (associated with the macroscopic scale), we have to determine the microscopic displacement field $\mathbf{u}_{1}^{k}$ satisfying the microscopic problem (P_m) (Pruchnicki [26]) obtained from the local equilibrium equation (17) and the microscopic (or local) constitutive law (6) written at zero order

$$\text{div}_{\mathbf{y}}\,\boldsymbol{\pi}_{0}^{kT} = 0 \qquad \text{in} \qquad \mathbf{Y}^{k},$$

$$\boldsymbol{\pi}_{0}^{k} = \mathbf{CL}^{k}\left(\mathbf{G}_{0}^{k}, \mathbf{A}_{0}^{k}\right) \qquad \text{in} \qquad \mathbf{Y}^{k}, \tag{19}$$

$$\langle \mathbf{u}_{1}^{k}\rangle = 0, \tag{20}$$

where the functions $\mathbf{u}_{1}^{k}$ and $\boldsymbol{\pi}_{0}^{k}$ (for $i \geq 1$) are $\mathbf{Y}^{k}$-periodic in y.

The condition (20) eliminates the rigid translation indermination on the microscopic displacement field $\mathbf{u}_{1}^{k}$. The microscopic problem (P_m) shows that the microscopic displacement field $\mathbf{u}_{1}^{k}$ is a function of $\mathbf{F}^{Mk}$ and $\mathbf{G}_{0}^{k}$.

5.1.2 Macroscopic problem at zero order

Now we establish the macroscopic problem satisfied by the macroscopic displacement field $\mathbf{u}_{0}^{k}$. By averaging the microscopic equilibrium relation (18) over the unit cell $\mathbf{Y}^{k}$, we get the macroscopic equilibrium equation for each point $\mathbf{x}\in\Omega^{k}$

$$\text{div}_{\mathbf{x}}\,\boldsymbol{\Pi}_{0}^{k\,T} + \langle\mathbf{f}^{\ k\nu}\rangle = 0 \qquad \text{in} \qquad \Omega^{k}. \tag{21}$$

in which the expression of the macroscopic nominal stress tensor $\mathbf{\Pi}_0^k$ in terms of $\mathbf{F}^{Mk}$ (macroscopic constitutive law) is obtained by both averaging over the basic cell $\mathbf{Y}^k$ and taking into account that $\mathbf{u}_1^k$ is a function of $\mathbf{F}^{Mk}$ and $\mathbf{G}_0^k$ through the microsopic problem (P_m). By averaging the microscopic constitutive law (19) over the unit cell $\mathbf{Y}^k$, we obtain the macroscopic constitutive law

$$\mathbf{\Pi}_0^k(\mathbf{F}^{Mk}) = \left\langle \boldsymbol{\pi}_0^k\left(\mathbf{G}_0^k, \mathbf{A}_0^k\right)\right\rangle = \left\langle \mathbf{CL}^k\left(\mathbf{G}_0^k, \mathbf{A}_0^k\right)\right\rangle \quad \text{in} \quad \Omega^k, \tag{22}$$

Then, it will be interesting to comment the macroscopic constitutive law (22) with the earlier similar works. Due to the presence of the term $J_{\mathbf{G}^k}$ in the microscopic constitutive law (2), this law is not hyperelastic. Indeed, the macroscopic constitutive law is not too. Thus, it is not possible to show by using the Hill-Mandel macro-homogeneity condition at finite strain ([61]) that the macroscopic nonlinear consitutive law is hyperelastic ([45]).

We deduce static macroscopic problem (P_M) by considering for each layer k (k=1,2,.. n) the macroscopic equilibrium equation (21), the macroscopic constitutive law (22), the Dirichlet lateral boundary condition at zero-*th* order (11) and by averaging over the unit cell $\mathbf{Y}^k$ of the *k-th* layer at zero-*th* order the Neumann boundary conditions:

a) on both the upper (8) and the lower (9) faces of the plate,

b) on the lateral boundary (10) for k=1,2,..., n.

Finally we consider both the displacement and the averaged over the unit cell stress vector continuity conditions at zero-*th* order on the interfaces between the different layers for k=1,2,..., n–1.

$$\mathrm{div}_{\mathbf{x}}\,\mathbf{\Pi}_0^{kT} + \mathbf{F}^{k\nu} = 0 \qquad \text{in} \qquad \Omega^k, \tag{23}$$

$$\mathbf{\Pi}_0^k(\mathbf{F}^{Mk}) = \left\langle \mathbf{CL}^k\left(\mathbf{G}_0^k, \mathbf{A}_0^k\right)\right\rangle_{\mathbf{Y}^k} \qquad \text{in} \qquad \Omega^k,$$

$$\mathbf{u}_0^k.\mathbf{n} = \overline{\mathrm{u}}^k \qquad \text{on} \qquad \partial\boldsymbol{\omega}_{\overline{\mathbf{u}}} \times [0, h^k] \,\text{for} \quad k = 1, 2, \ldots .n,$$

$$\mathbf{\Pi}_0^{n\,T}.\mathbf{e}_3 = {}^{-}\mathbf{t}^{\mathrm{upp}} \qquad \text{on} \qquad \omega \times \{h^n\}, \tag{24}$$

$$\mathbf{\Pi}_0^{1\,T}.\mathbf{e}_3 = \overline{\mathbf{t}}^{\mathrm{low}} \qquad \text{on} \qquad \omega \times \{0\}, \tag{25}$$

$$\mathbf{\Pi}_0^{k\,T}.\mathbf{n} = \overline{\mathrm{t}}^k \qquad \text{on} \qquad \partial\boldsymbol{\omega}_{\overline{\mathbf{t}}} \times [0, h^k] \quad \text{for} \quad k = 1, 2, \ldots .n,$$

$$\mathbf{u}_0^k\left(h^k\right) = \mathbf{u}_0^{k+1}(0), \quad \text{for} \quad k = 1, 2, \ldots .n-1, \tag{26}$$

$$\mathbf{\Pi}_0^{k\,T}\left(h^k\right).\mathbf{e}_3 = \mathbf{\Pi}_0^{(k+1)\,T}(0).\mathbf{e}_3. \quad \text{for} \quad k = 1, 2, \ldots .n-1, \tag{27}$$

in which $\mathbf{F}_\nu^k = \langle \mathbf{f}\,{}_\nu^k \rangle$.

The macroscopic problem (P_M) shows that we obtain a nonlinear multilayered plate model in which each layer is characterized by a non analytical (*i.e.* numerical) expression of the macroscopic nomical stress tensor in terms of the homogeneous macroscopic displacement gradient $\mathbf{F}^{Mk}$. Due to growth, this macroscopic constitutive law is not hyperelastic.

5.2 Plate model for the homogenized problem (P_M)

In this section, the homogeneous plate model is derived from the homogenized problem (P_M). A series expansion-truncation approach will be adopted to derive the bidimensional vector plate equation ([7], [62-64]) from the tridimensional macroscopic problem (P_M). For that purpose, the displacement vector $\mathbf{u}^k$ in each layer is expanded in terms of the thickness variable x_3 (recall we suppose for the sake of simplicity that growth tensor is independent of the macroscopic thickness variable x_3)

$$\mathbf{u}_0^k(\mathbf{x}) = \sum_{i=0}^{4} \frac{x_3^i \mathbf{u}_0^{k(i)}(\mathbf{x}', 0)}{i!} + O((h^k)^5). \tag{28}$$

So in the following derivation, the vector unknows are $\mathbf{u}_0^k$, $\mathbf{u}_0^{k(1)}$, ... $\mathbf{u}_0^{k(4)}$ (5 vector unknowns for each layer k).

According to the series expansion (28), the deformation gradient tensor $\mathbf{F}^{Mk}$ in the homogeneous layer k, can also be expanded as

$$\mathbf{F}^{Mk}(\mathbf{x}) = \sum_{i=0}^{3} \frac{x_3^i \mathbf{F}^{Mk[i]}(\mathbf{x}', 0)}{i!} + O((h^k)^4). \tag{29}$$

in which $\mathbf{F}^{Mk[0]} = \mathbf{I} + \mathrm{grad}_{\mathbf{x}'}\ \mathbf{u}_0^k + \mathbf{u}_0^{k(1)} \otimes \mathbf{e}_3$ and for $i \geqslant 1$, $\mathbf{F}^{Mk[i]} = \mathrm{grad}_{\mathbf{x}'}\ \mathbf{u}_0^{k(i)} + \mathbf{u}_0^{k(i+1)} \otimes \mathbf{e}_3$.

We can note that the growth tensor is assumed to be independent of macroscopic scale in the thickness direction (see equation (15)) and then a Taylor-Young expansion of the growth tensor has not to be considered.

From the Taylor-Young expansion of $\mathbf{F}^{Mk}$ (29) and the constitutive law for the macroscopic nominal stress tensor (22), we obtain the Taylor-Young expansion of the macroscopic nominal stress tensor

$$\mathbf{\Pi}_0^k(\mathbf{x}) = \sum_{i=0}^{3} \frac{x_3^i \mathbf{\Pi}_0^{k[i]}(\mathbf{x}', 0)}{i!} + O((h^k)^4). \tag{30}$$

in which $\mathbf{\Pi}_0^{k[0]} = \mathbf{\Pi}_0^k\left(\mathbf{F}^{Mk[0]}\right)$, $\mathbf{\Pi}_0^{k[1]} = \frac{\partial \mathbf{\Pi}_0^k}{\partial \mathbf{F}^{Mk}} :\ \mathbf{F}^{Mk[1]}$, $\mathbf{\Pi}_0^{k[2]} = \frac{\partial \mathbf{\Pi}_0^k}{\partial \mathbf{F}^{Mk}} :\ \mathbf{F}^{Mk[2]} + (\frac{\partial^2 \mathbf{\Pi}_0^k}{\partial \mathbf{F}^{Mk\ 2}} :\ \mathbf{F}^{Mk[1]}) : \mathbf{F}^{Mk[1]}$,

$\mathbf{\Pi}_0^{k[3]} = \frac{\partial \mathbf{\Pi}_0^k}{\partial \mathbf{F}^{Mk}} :\ \mathbf{F}^{Mk[3]} + 3(\frac{\partial^2 \mathbf{\Pi}_0^k}{\partial \mathbf{F}^{Mk\ 2}} :\ \mathbf{F}^{Mk[1]}) : \mathbf{F}^{Mk[2]} + ((\frac{\partial^3 \mathbf{\Pi}_0^k}{\partial \mathbf{F}^{Mk\ 3}} :\ \mathbf{F}^{Mk[1]}) : \mathbf{F}^{Mk[1]}) : \mathbf{F}^{Mk[1]}$.

We can note that the analytical expression of the macroscopic nominal stress tensor ($\mathbf{\Pi}_0^k$) since in terms of the macroscopic gradient deformation ($\mathbf{F}^{Mk}$), which is the macroscopic constitutive law, is only known numerically. Then the partial derivative which appears in the expressions of $\mathbf{\Pi}_0^{k[i]}$ (for(i) i=1, 2, 3) can be only computed numerically.

Finally, we have to give the Taylor-Expansion of the macroscopic volumic force

$$\mathbf{F}_\nu^k(\mathbf{x}) = \sum_{i=0}^{3} \frac{x_3^i \mathbf{F}_\nu^{k(i)}(\mathbf{x}', 0)}{i!} + O((h^k)^4). \tag{31}$$

Now we consider together, both the Taylor-Young expansion of the macroscopic nominal stress tensor (30) and of the macroscopic volumic force (31) into the macroscopic equilibrium equation (23), we obtain:

$$\mathrm{div}_{\mathbf{x}'}\,\mathbf{\Pi}_0^{k(i)T} + \mathbf{\Pi}_0^{k(i+1)T}.\mathbf{e}_3 + \mathbf{F}_\nu^{k(i)} = 0 \qquad \text{in} \qquad \Omega^k, \quad \text{for}\, i = 0, 1, 2, \quad (32)$$

in summary there are $5n$ vector unknowns ($\mathbf{u}_0^{k(i)}$ for i=0, 1, 4) for 5 n vectorial equations:

a) Macroscopic equilibrium equations (32).

b) Continuity of both the macroscopic stress tensor (27) and the macroscopic displacement field (26):

$$\mathbf{u}_0^{k(i)}\left(h^k\right) = \sum_{i=0}^{4} \frac{(h^k)^i \mathbf{u}_0^{k(i)}(\mathbf{x}', 0)}{i!} = \mathbf{u}_0^{k+1}(\mathbf{x}', 0), \qquad \text{for} \quad k = 1, 2, \ldots .n-1,$$

$$\left(\sum_{i=0}^{3} \frac{(h^k)^i \mathbf{\Pi}_0^{k[i]T}(\mathbf{x}', 0)}{i!}\right).\mathbf{e}_3 = \mathbf{\Pi}_0^{k+1[0]\,T}(\mathbf{x}', 0).\mathbf{e}_3 \qquad \text{for} \quad k = 1, 2, \ldots .n-1.$$

c) Neumann type boundary conditions at both the bottom and the top layers (24) and (25).

The detail of the computation is given in the incompressible case in Du *et al.* [59] and will be simpler for this quasi incompressible case.

6 Asymptotic plate theory when $\varepsilon \ll \mu$

As explained in Caillerie [11], we have to determine first a plate model when the size of heterogenities does not vary. Then the size of the heterogeneities vanishes $\mu \to 0$ and we get a new homogenized plate model . In the first step, the size of the thickness is constant and then cannot intervene in the asymptotic process required for homogenization. Thus the size of the heterogenities goes to zero only in the mid-plane directions. If we consider a simple heterogeneity characterized by a rapidly changing periodic wavy interface between two materials which constitute the plate (Figure 2). Obviously, the shape of the heterogeneities is not conserved in the asymptotic process for homogenization since hv is constant and μ goes to zero (Figure 2). It is inconsistent. Nevertheless this method will be applicable when the geometry of the heterogeneities does not depend on the thickness direction of the plate because the asymptotic homogenization method is independent of the thickness direction of the plate and the two-scale for asymptotic analysis for homoge

nization are only in the mid-plane directions. Thus the microscopic coordinate is $\mathbf{y}'$ in the layer k. The well known example is the honeycomb composite structure. Another application of the theory of this section is possible for the leaves of Monstera Adansopnii called swiss cheese plants ([65]), because at the microscopic scale the heterogenities can be modelled by big holes in the leaves which are much bigger than the thickness of the leaves. This is confirmed in Figure 3.

Figure 2:Plate with wavy interface between the two constituents.

Figure 3: A leave of Monstera Adansopnii

6.1 Firstly plate theory

This section is similar to Section 5.2. Since the geometry of the heterogeneities only depend on the in plane direction at each point $\mathbf{x}^{'}$ indeed the plate is homogeneous in the thickness direction. So for sufficiently regular displacement field, we can write a Taylor-Young expansion in x_3

$$\mathbf{u}_{\mu}^{k}(\mathbf{x}) = \sum_{i=0}^{4} \frac{x_3^i \mathbf{u}_{\mu}^{k\,(i)}(\mathbf{x}^{'},0)}{i!} + O((h^k)^5). \tag{33}$$

According to the series expansion (33), we obtain the deformation gradient tensor $\mathbf{F}_{\mu}^{k}$ in the layer k, can also be expanded $\mathbf{u}^{k(i)}$[1]as

$$\mathbf{F}_{\mu}^{k}(\mathbf{x}) = \sum_{i=0}^{3} \frac{x_3^i \mathbf{F}_{\mu}^{k[i]}(\mathbf{x}^{'},0)}{i!} + O((h^k)^4). \tag{34}$$

in which $\mathbf{F}_{\mu}^{k[0]} = \mathbf{I} + \mathrm{grad}_{\mathbf{x}^{'}}\ \mathbf{u}_{\mu}^{k\,(0)} + \mathbf{u}_{\mu}^{k(1)} \otimes \mathbf{e}_3$ and for $i \geqslant 1$, $\mathbf{F}_{\mu}^{k[i]} = \mathrm{grad}_{\mathbf{x}^{'}}\ \mathbf{u}_{\mu}^{k\,(i)}$ $+\mathbf{u}_{\mu}^{k\,(i+1)} \otimes \mathbf{e}_3$.

We recall that the growth tensor does not depend on the thickness direction at the macroscopic level (see formula (15)). From the Taylor-Young expansion of the deformation gradient in each layer (34) and the decomposition (1), we obtain the Taylor-expansion of the elastic part of the deformation gradient

$$\mathbf{A}_{\mu}^{k}(\mathbf{x}) = \sum_{i=0}^{3} \frac{x_3^i \mathbf{A}_{\mu}^{k[i]}(\mathbf{x}^{'})}{i!} + O((h^k)^4). \tag{35}$$

in which $\mathbf{A}_{\mu}^{k[i]}(\mathbf{x}^{'}) = \mathbf{F}_{\mu}^{k[i]}(\mathbf{x}^{'},0).(\mathbf{G}_{\mu}^{k})^{-1}\left(\mathbf{x}^{'},\mathbf{y}^{'}\right)$.

From the local constitutive law (4) and the Taylor-Young expansion (35), we can deduce the Taylor-Young expansion of the microscopic nominal stress tensor in each layer.

$$\pi_\mu^k(\mathbf{x}) = \sum_{i=0}^{3} \frac{x_3^i \pi_\mu^{k[i]}(\mathbf{x}^{'})}{i!} + O((h^k)^4). \tag{36}$$

in which

$$\pi_\mu^{k[0]} = \pi_\mu^k \left(\mathbf{G}_\mu^k, \mathbf{A}_\mu^{k[0]} \right),$$

$$\pi_\mu^{k[1]} = J_{\mathbf{G}_\mu^k} (\mathbf{G}_\mu^k)^{-1}. \frac{\partial \boldsymbol{\pi}_\mu^{elk}}{\partial \mathbf{A}_\mu^k} \left(\mathbf{G}_\mu^k, \mathbf{A}_\mu^{k[0]} \right) : \mathbf{A}_\mu^{k[1]},$$

$$\pi_\mu^{k[2]} = J_{\mathbf{G}_\mu^k} (\mathbf{G}_\mu^k)^{-1}. \left(\frac{\partial \boldsymbol{\pi}_\mu^{elk}}{\partial \mathbf{A}_\mu^k} \left(\mathbf{G}_\mu^k, \mathbf{A}_\mu^{k[0]} \right) : \mathbf{A}_\mu^{k[2]} + \left(\frac{\partial^2 \boldsymbol{\pi}_\mu^{el\ k}}{\partial (\mathbf{A}_\mu^k)^2} \left(\mathbf{G}_\mu^k, \mathbf{A}_\mu^{k[0]} \right) : \mathbf{A}_\mu^{k[1]} \right) : \mathbf{A}_\mu^{k[1]} \right),$$

$$\pi_\mu^{k[3]} = J_{\mathbf{G}_\mu^k} (\mathbf{G}_\mu^k)^{-1}. \left[\frac{\partial \boldsymbol{\pi}_\mu^{el\ k}}{\partial \mathbf{A}_\mu^k} \left(\mathbf{G}_\mu^k, \mathbf{A}_\mu^{k[0]} \right) : \mathbf{A}_\mu^{k[3]} + \left[\left(\frac{\partial^3 \boldsymbol{\pi}_\mu^{el\ k}}{\partial (\mathbf{A}_\mu^k)^3} \left(\mathbf{G}_\mu^k, \mathbf{A}_\mu^{k[0]} \right) : \mathbf{A}_\mu^{k[1]} \right) : \mathbf{A}_\mu^{k[1]} \right] : \mathbf{A}_\mu^{k[1]} \right.$$

$$\left. +3 \left(\frac{\partial^2 \boldsymbol{\pi}_\mu^{el\ k}}{\partial (\mathbf{A}_\mu^k)^2} \left(\mathbf{G}_\mu^k, \mathbf{A}_\mu^{k[0]} \right) : \mathbf{A}_\mu^{k[1]} \right) : \mathbf{A}_\mu^{k[2]} \right].$$

We finally have to give the Taylor-Young expansion of the microscopic volumic force

$$\mathbf{f}_\mu^{k\nu}(\mathbf{x}, \mathbf{y}^{'}) = \sum_{i=0}^{3} \frac{x_3^i \mathbf{f}_\mu^{k\nu(i)}(\mathbf{x}^{'}, 0)}{i!} + O((h^k)^4). \tag{37}$$

Now we consider together, the Taylor-Young expansion of both the microscopic nominal stress tensor (36) and the microscopic volumic force (37) into the microscopic equilibrium equation (5), we obtain:

$$\mathrm{div}_{\mathbf{x}'}\, \pi_\mu^{k[i]T} + \pi_\mu^{k[i+1]T}.\mathbf{e}_3 + \mathbf{f}_\mu^{k\nu(i)} = 0 \quad \text{in} \quad \Omega^k, \quad \text{for } i = 0, 1, 2 \tag{38}$$

in summary there are $5n$ vector unknowns ($\mathbf{u}_\mu^{k\,(i)}$ for i=0, 1, 2, 3 and 4) for 5n vectorial equations:

- Microscopic equilibrium equations (38).
- Continuity of both the microscopic stress tensor (13) and the microscopic displacement field (12) which imply that

$$\mathbf{u}_\mu^k \left(\mathbf{x}^{'}, h^k \right) = \sum_{i=0}^{4} \frac{(h^k)^i \mathbf{u}_\mu^{k(i)}}{i!} = \mathbf{u}_\mu^{k+1}, \qquad \text{for} \quad k = 1, 2, \ldots .n-1, \tag{39}$$

$$\left(\sum_{i=0}^{3} \frac{(h^k)^i \pi_\mu^{k[i]T}}{i!} \right) .\mathbf{e}_3 = \pi_\mu^{k+1[0]T}.\mathbf{e}_3 \qquad \text{for} \quad k = 1, 2, \ldots .n-1. \tag{40}$$

- Neumann type boundary conditions at both the top and the bottom layers (8) and (9).

$$\sum_{i=0}^{3} \frac{(h^n)^i \boldsymbol{\pi}_\mu^{n[i]\,T}(\mathbf{x}^{'})}{i!}.\mathbf{e}_3 = \overline{\mathbf{t}}^{\mathrm{upp}}(\mathbf{x}^{'}) \qquad \text{on} \qquad \omega, \tag{41}$$

$$\pi_\mu^{1\,[0]\,T}(\mathbf{x}^{'}).\mathbf{e}_3 = \overline{\mathbf{t}}^{\mathrm{low}}(\mathbf{x}^{'}) \qquad \text{on} \qquad \omega. \tag{42}$$

The detail of the computation is given in the incompressible homogeneous case in Du *et al.* [57-59] and it will be simpler for this quasi incompressible case. But we only use this equation in the next section on stratified plate with homogeneous layer i.e. after homogenization of each layer of the plate.

6.2 Secondly homogenization of the plate model

Classically, we assume an asymptotic expansion of the displacement field:

$$\mathbf{u}_\mu^{k(i)}(\mathbf{x}^{'}) = \mathbf{u}_0^{k(i)}(\mathbf{x}^{'}) + \mu \mathbf{u}_1^{k(i)}\left(\mathbf{x}^{'}, \mathbf{y}^{'}\right) + \cdots,$$

in which the microscopic displacement field $\mathbf{u}_1^{k(i)}$ is $\mathbf{Y}^{'k}$ periodic in $\mathbf{y}^{'}$.

Then we can compute the first term of the asymptotic expansion of the deformation gradient:

$$\mathbf{F}_\mu^{k[0]} = \mathbf{I} + \mathrm{grad}_{\mathbf{x}'}\, \mathbf{u}_\mu^{k(0)} + \mathbf{u}_\mu^{k(1)} \otimes \mathbf{e}_3 = \mathbf{F}_0^{k[0]} + \mu(\ldots.) + \cdots,$$

with $\mathbf{F}_0^{k[0]} = \mathbf{I} + \mathrm{grad}_{\mathbf{x}'}\, \mathbf{u}_0^{k(0)} + \mathrm{grad}_{\mathbf{y}'}\, \mathbf{u}_1^{k(0)} + \mathbf{u}_0^{k(1)} \otimes \mathbf{e}_3$,
and for $i \geqslant 1$

$$\mathbf{F}_\mu^{k[i]} = \mathrm{grad}_{\mathbf{x}'}\, \mathbf{u}_\mu^{k(i)} + \mathbf{u}_\mu^{k(i+1)} \otimes \mathbf{e}_3 = \mathbf{F}^{k[i]0} + \mu(\ldots.) + \cdots,$$

with $\mathbf{F}_0^{k[i]} = \mathrm{grad}_{\mathbf{x}'}\, \mathbf{u}_0^{k(i)} + \mathrm{grad}_{\mathbf{y}'}\, \mathbf{u}_1^{k(i)} + \mathbf{u}_0^{k(i+1)} \otimes \mathbf{e}_3$.

We can note that it is not necessary to compute first and higher order terms in the asymptotic expansion of both $\mathbf{F}_\mu^{k[i]}$.

Due to the particular geometry of the heterogeneities, the asymptotic expansion of the growth tensor (15) becomes

$$\mathbf{G}_\mu^{k}(\mathbf{x}^{'}) = \mathbf{G}_0^{k}\left(\mathbf{x}^{'}, \mathbf{y}^{'}\right) + \mu \mathbf{G}_1^{k}\left(\mathbf{x}^{'}, \mathbf{y}^{'}\right) + \cdots$$

And the asymptotic expansion of the elastic part of the deformation gradient is

$$\mathbf{A}_\mu^{k[i]} = \mathbf{F}_\mu^{k[i]}.(\mathbf{G}_\mu^{k})^{-1} = \mathbf{A}_0^{k[i]} + \mu(\ldots.) + \cdots,$$

with $\mathbf{A}_0^{k[i]} = \mathbf{F}_0^{k[i]}.(\mathbf{G}_0^{k})^{-1}$.

From the previous relations, we deduce the asymptotic expansion of the nominal stress tensor

$$\boldsymbol{\pi}_\mu^{k[i]}(\mathbf{x}^{'}) = \boldsymbol{\pi}_0^{k[i]}\left(\mathbf{x}^{'}, \mathbf{y}^{'}\right) + \mu(\ldots.) + \cdots,$$

in which $\boldsymbol{\pi}_0^{k[0]}=\boldsymbol{\pi}^k(\mathbf{A}_0^{k[0]},\mathbf{G}_0^k)$,

$\boldsymbol{\pi}_0^{k[1]}=J_{\mathbf{G}_0^k}(\mathbf{G}_0^k)^{-1}.\frac{\partial\boldsymbol{\pi}^{el\,k}}{\partial\mathbf{A}^k}(\mathbf{A}_0^{k[0]}):\ \mathbf{A}_0^{k[1]}$,

$\boldsymbol{\pi}_0^{k[2]}=J_{\mathbf{G}_0^k}(\mathbf{G}_0^k)^{-1}.\left(\frac{\partial\boldsymbol{\pi}^{el\,k}}{\partial\mathbf{A}^k}(\mathbf{A}_0^{k[0]}):\mathbf{A}_0^{k[2]}+\left(\frac{\partial^2\boldsymbol{\pi}^{el\,k}}{\partial(\mathbf{A}^k)^2}(\mathbf{A}_0^{k[0]}):\mathbf{A}_0^{k[1]}\right):\mathbf{A}_0^{k[1]}\right)$,

$\boldsymbol{\pi}_0^{k[3]}=J_{\mathbf{G}_0^k}(\mathbf{G}_0^k)^{-1}.\left[\frac{\partial\boldsymbol{\pi}^{el\,k}}{\partial\mathbf{A}^k}(\mathbf{A}_0^{k[0]}):\mathbf{A}_0^{k[3]}+(\left(\frac{\partial^3\boldsymbol{\pi}^{el\,k}}{\partial(\mathbf{A}^k)^3}(\mathbf{A}_0^{k[0]}):\mathbf{A}_0^{k[1]}\right):\mathbf{A}_0^{k[1]}\right):\right.$
$\mathbf{A}_0^{k[1]}$

$\left.+3\left(\frac{\partial^2\boldsymbol{\pi}^{el\,k}}{\partial(\mathbf{A}^k)^2}(\mathbf{A}_0^{k[0]}):\mathbf{A}^{k[1]0}\right):\mathbf{A}_0^{k[2]}\right]$.

6.2.1 Microscopic problems

We have to cancel the coefficient terms of $\frac{1}{\mu}$ which appear in equilibrium equation (38) and we obtain microscopic equilibrium equations

$$\mathrm{div}_{\mathbf{y}'}\,\boldsymbol{\pi}_0^{k[i]T}=0\qquad\text{in}\qquad\mathbf{Y}^{'k}\quad\text{for}\quad i=0,1,2,3,\tag{43}$$

For the fourth microcopic problems (43),we can add average conditions

$$\left\langle\mathbf{u}_1^{k(i)}\right\rangle_{\mathbf{Y}^{'k}}=0\quad\text{for}\quad i=0,1,2,3\text{ and }4.$$

For the first problem (i=0), we easily see that $\mathbf{u}_1^{k(0)}$ can be determined in terms of $\mathbf{y}^{'}$, and terms involved by macroscopic displacement, *i.e.*, $\mathrm{grad}_{\mathbf{x}'}\,\mathbf{u}_0^{k(0)}$ and $\mathbf{u}_0^{k(1)}$. In the same manner, the three other microcopic displacement fields ($\mathbf{u}_1^{k(i)}$, for i=1, 2 and 3) can be determined in terms of macroscopic displacement fields ($\mathbf{u}_0^{k(i)}$, for i=0, 1, 2 and 3) by using the three remaining microcopic equilibrium equations (43, for i=1, 2 and 3). Indeed, the microscopic displacement are determined in terms of the macroscopic displacement fields and it remains to write the macroscopic equations satisfied by the macroscopic displacement field. Finally for the higher displacement field (i=4), we assume that there is no microscopic part for this displacement field i.e.

$$\mathbf{u}_\mu^{k(4)}(\mathbf{x}^{'k})=\mathbf{u}_0^{k(4)}(\mathbf{x}^{'k}).$$

This approximation is not a problem since this term is not significant. There is another way to proceed, it suffices to express the analytical expression of $\boldsymbol{\pi}_0^{k[4]}$ and to consider microcopic equilibrium equation (43) for $i=4$.

6.2.2 Macroscopic homogenized plate problem

This problem is similar to the one of section 5.2. Then we only explain how to establish this macroscopic problem. The three macroscopic equilibrium equations are obtained by cancelling the coefficients of the term of order μ^0 which appear in equilibrium equation (38) and by averaging them over $\mathbf{Y}^{'k}$

$$\mathrm{div}_{\mathbf{x}^{'k}}\langle\boldsymbol{\pi}_0^{k[i]}\rangle^T_{\mathbf{Y}^{'k}}+\langle\boldsymbol{\pi}_0^{k[i+1]}\rangle^T_{\mathbf{Y}^{'k}}.\mathbf{e}_3+\left\langle\mathbf{f}_0^{k\nu(i)}\right\rangle_{\mathbf{Y}^{'k}}=0\qquad\text{in}\qquad\boldsymbol{\omega},\quad\text{for } i=0,1,2.$$

The continuity of both the macroscopic stress tensors and the macroscopic displacement fields are obtained by considering the coefficients of the term of order μ^0 in equations (39) and (40) and averaging them over $\mathbf{Y}'^k$.

Finally the Neumann type boundary conditions at both the bottom and the top layers are obtained by using the coefficients of the term of order μ^0 in equations (41) and (42) and averaging them over $\mathbf{Y}'^k$.

We obtain a closed system of $5n$ vectorial unknowns ($\mathbf{u}_0^{k^{(i)}}$ for i=0, 1, 2, 3 and 4) for 5 n vectorial equations (for more details see [57-59]). In particular it was explained how to deduce the lateral boundary conditions after averaging (10) over $\mathbf{Y}'^k$ and considering (11) at zeroth order ($\mu = 0$). The Taylor-Young expansions of both the displacement field (33) and the nominal stress tensor (36) are used at zeroth order ($\mu = 0$).

7 Conclusion and discussion for applications

In this work, we consider both homogenization and reduction problem for non linear composites multilayered plates including growth theory for biologinal tissues when the plate thickness and the size of heterogeneities are not of the same order of magnitude. Homogeneization is performed by the standard asymptotic homogenization method ([55-56]). While the reduction process is obtained by using a method resulting from the Taylor-Young expansion of the displacement field through the thicknees of the plate ([7]; [62-64]). For the sake of simplicity, we have assumed that the local heterogeous microstructure is periodic. However, the local microstructure is sometimes highly random ([66], Figures 2 and 5) and then nonperiodic and a statistical volume elements should be considered in future works. The Voronoi tessellation can be used to generate an accurate representation of a non-periodic micro-structure ([67]; [68-69]).

This work is a continuation of the theoretical work of Pruchnicki [26, 27] which deal with homogenization theory of heterogeneous plate in nonlinear setting when the thickness of the plate and the size of the in plane hetereogeneities are of the same order of magnitude. The leaf vein structure gave the rigity of leaf blade which can be modeled as a plate. This microstructure is given for several kinds of leaves plant in Figure 1 and the details of the arrangement of leaf venation is given in Figures 2 and 3 of the work Portella *et al.* [70], see also Figure 1. and 2. in the work of Sack and Scoffoni [71]. Figure 1 in Li *et al.* [72] shows that the internal structure of vein is heterogeneous so a prior homogenization of leaf microstructure is necessary in order to replace it by homogeneous material.

The order of magnitude of in plane heterogeneity of the vein microstructure ([70-71]) and the order of magnitude of the leave thickness ([73], (Table 2)) are the same (approximatively about 0,15 mm). This assumption is also satisfied for alga Coleochaetales ([74], (Figure 4a)). Then the theory of homogenization presented in Pruchnicki [26, 27] can be applied. This assumption is also satisfied for a kind of Cactus given in Figure 4 and of a kind of water lilly in Figure 5.

Figure 4. Structure of a cactus.

Figure 5: Water lilly on the left side microscopic scale and unit cells and on the right side macroscopic scale.

We also give an another application for this case. The cell wall is composed of a network of cellulose microfibrils and cross-linking glycans embedded in a highly cross-linked matrix of pectin polysaccharides ([75] (Figure 3); [66] (Figure 1.d)). So it can be seen as a heteregoneous plate for which the size of the heterogeneity and the thickness of the plate are of the same order of magnitude. It seems that the getting of the macroscopic stiffness of cell wall presents a practical interest ([75] (Figure 3)).

Both the cell wall arrangement and the turgor pressure allow the plants to have a rigidity and the cell wall arrangement induces internal microstructure heterogeneities in the plants which can be numerically studied with the theory presented in Pruchnicki [26, 27] and in the present work.

Finally, the order of magnitude of the size of the heterogeneities induced by the vein structure of a leave can be much bigger than the thickness of the leave as shown in Figure 6; then the theory of Section 6 can be applied.

Figure 6: Leaves of calatea zebrena humilior

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