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A survey on unsharp orthomodular lattices: A unifying framework

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"This paper is dedicated to Professor Anatolij Dvurečenskij on the occasion of his 75th birthday."

Abstract

In this survey we consider the variety of unsharp orthomodular lattices. We will see that there are pretty smooth conditions that neatly generalize a great deal of the theory of orthomodular lattices. In particular, we will characterize the concept of block in this framework towards an appropriate notion of commutativity. Then, we capitalize on this fact and describe a categorical equivalence between orthomodular lattices, with fixed p-filters, and the variety of unsharp orthomodular lattices.

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1 Introduction

The aim, of this survey is to provide a uniform presentation of the variety of unsharp orthomodular lattices (UOM lattices). These structures find their initial motivation in the context of an algebraic description of the effects on a Hilbert space. These objects have stirred the attention of many scholars over the last century. In particular these interests involve deep problems of non classical quantum probability, as discussed in the very seminal studies of Anatolij Dvurečenskij:

-*Signed States Properties Exhibition on a Logic* [17];
-*Selected Problems of Probability Theory on Quantum Logics* [18];
-*Joint Distributions in Quantum Logics* [19].

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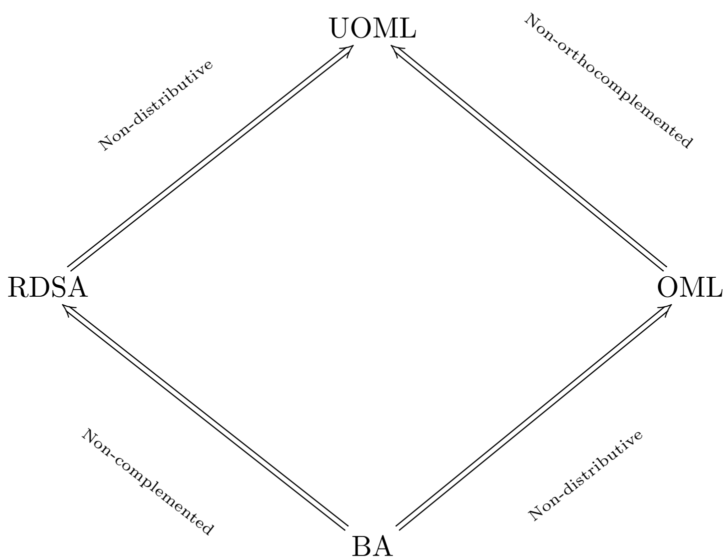
Specifically, UOM is a proper subvariety of paraorthomodular Brower-Zadeh lattices that satisfy the star (*) condition (PBZ^* lattices). In particular, UOM lattices smoothly generalize orthomodular lattices to the case where the orthocomplementation is remitted from its full strength. Indeed, UOM can be regarded as a variety that generalizes orthomodular lattices to a context in which the orthocomplement need not satisfy *tertium non-datur* and *non-contradiction* principles. UOM lattices reframe the concept of orthomodularity in an "unsharp realm", in which the orthocomplement is no longer such, but it just satisfies the Kleene condition.
Remarkably, in this context the orthomodular law is no longer equivalent to the condition:if $x \leq y'$ and $x' \wedge y = 0$ implies $x = y$, which still maintains true in PBZ^* lattices (for posets the reader may consult e.g. [11]). In other words, we will see that relaxing the orthocomplement conditions induces consequences somehow similar to what happens in residuated lattices (or substructural logics) when a structural rule is remitted: new, "multiplicative" connectives appear [22]. Indeed, in the present case the orthocomplementation is no longer such, but it splits into two different negation connectives: a fuzzy-like "prime" $'$, and an intuitionistic-like "tilde" $^\sim$.
In the concrete model of the effects on a Hilbert space, if I is the identity operator and E an effect, in PBZ^* lattices, the two unary operations $'$, and $^\sim$, stand for $I - E$ and $\ker(E)$ (and $E' = E^\sim$ if E is a projector), respectively. UOM is a variety of axiomatized concerning PBZ^* by strong de Morgan and strong Kleene axioms. These conditions allow tighter relations between UOM and the variety of orthomodular lattices (OML) on the one side, and with regular double Stone algebras (RDSA) on the other side. Intuitively,



(Types of generalization.)

Where BA is the variety of Boolean algebras. The variety of UOM forms a smooth common scenario in which orthomodular lattices and regular double Stone algebras find a unifying treatment. It may be worth also to observe that several classes of prominent importance to algebraic logic, e.g. Boolean algebras, Stone algebras, Kleene algebras, 3-valued MV algebras, regular double Heyting algebras etc. find a unique general setting [13, 29].

In this paper after providing a few basic notions in section 2, in section 3 we discuss the notions of blocks and commutation, in the context of UOM. Surprisingly enough, we will see a neat generalization of these notions as they are given for orthomodular lattices. Indeed, if in, the case of orthomodular lattices, the blocks are nothing but Boolean algebras of pairwise commuting elements, we will find out that for the case of UOM the blocks are just regular double Stone algebras, which can be alternatively regarded as 3-valued MV algebras, or Kleene algebras. Moreover, we will see that there will be an explicit characterization in terms of generalized commutativity, as in the case of orthomodular lattices.

Subsequently, in section 4, we provide a general representation of UOM. To this aim, we start with a given orthomodular lattice. Then, we generalize a long-dated construction, whose roots can be traced back to the early works of Moisil, A. Monteiro and L. Monteiro [4, 38, 39], where an idea of the method is introduced, albeit in different contexts.
Then, we use this representation to describe a full categorical equivalence between unsharp orthomodular lattices and orthomodular lattices, indexed by a p-filter, whose original proof is in [37].

2 Basic notions

Let us start this section by advising the reader that we will only mention specific facts of fundamental importance to the development of our discourse. For general notions on universal algebra, lattice theory, substructural and quantum logics we refer to [6, 8, 12, 14, 20, 21, 22, 26, 27, 32].

It is well known that in the presence of an orthocomplement, the orthomodular condition,

$$x \vee y = ((x \vee y) \wedge y') \vee y, \tag{OML}$$

is equivalent to the quasi-equation:

$$\text{if } x \leq y \text{ and } x' \wedge y = 0 \text{ then } x = y. \tag{POML}$$

In [23], the authors named (POML) *paraorthomodularity*. If the lattice is not orthocomplemented, then paraorthomodularity and orthomodularity are no longer equivalent. Again, in [23], the authors consider certain Brouwer-Zadeh lattices [10, 9], named PBZ^* lattices, where P stands for paraorthomodular (briefly, PBZ^*) that serve as abstract counterparts of lattices of effects in Hilbert spaces under the spectral ordering [15, 40]. These algebras generalize the concept of orthomodularity to the "unsharp realm" in which the orthocomplement is no longer such, but it just satisfies the (Kleene) condition:

$$x \wedge x' \leq y \vee y'. \tag{Kleene}$$

These lattices are equipped with two unary operations: a fuzzy-like complement $'$, and an intuitionistic-like a complement $^\sim$, which, if I is the identity operator and E has an effect on a Hilbert space, stand for $I - E$ and $\ker(E)$ (where $E' = E^\sim$ iff E is a projector), respectively.

Definition 2.1. [32] *A* Kleene algebra $A = (A, \wedge, \vee, ', 0, 1)$ *is an algebra, where* $(A, \wedge, \vee, 0, 1)$ *is a bounded distributive lattice and* $'$ *is an antitone and involutive operation that satisfies* (Kleene).

We say that A is *pseudo-Kleene* in case A need not satisfy distributivity. Let us now introduce in Defintion 2.2 the variety of PBZ^* lattices, which will serve as a basis for the development of this survey.

Definition 2.2. *A structure* $L = (L, \wedge, \vee, ', ^\sim, 0, 1)$ *is a* PBZ^* lattice *if the following are satisfied:*

(1) $(L, \wedge, \vee, ', 0, 1)$ *is a paraorthomodular pseudo Kleene lattice;*

(2) $x^{\sim\sim} \geq x$;

(3) $x \leq y$ *implies* $y^{\sim} \leq x^{\sim}$;

(4) $x \wedge x^{\sim} = 0$;

(5) $x^{\sim'} = x^{\sim\sim}$;

(6) $(x \wedge x')^{\sim} = x^{\sim} \vee x'^{\sim}$.

The last identity is the star condition (in symbols, *) discussed in [23].

(POML) will play an essential role in this paper because it ensures that the set of *sharp elements*, i.e. those elements satisfying $x' = x^{\sim}$, forms an orthomodular sublattice. On pain of repetition, let us recall that condition (POML) is equivalent to orthomodularity, if the equation $x' = x^{\sim}$ holds, i.e. orthomodular lattices form a (proper) subvariety of PBZ^* lattices.

As observed in [23], it is possible to define two unary operations to behave as the modal operators *necessarily* $\Box$ and *possibly* $\Diamond$ as follows:

$$\Box x = x'^{\sim} \text{ and } \Diamond x = x^{\sim\sim}. \tag{1}$$

For notational convenience we set:

$$x^{+} = x'^{\sim\sim} = x'^{\sim'}. \tag{2}$$

It may be helpful to remark that in presence of $^{\sim}$ choosing $^{+}$ instead of $'$ is nothing but a matter of choice, since they are each other term definable. In general, fixing operations $^{\dagger}, ^{\#} \in \{', ^{\sim}, ^{+}\}$, the remaining operation in this set is term definable. Actually, each property expressible using $^{\sim}$ finds a dual equivalent formulation in terms of $^{+}$.
Let us observe that for any $L \in PBZ^*$, for all $x \in L$:

$$\Box x \leq x \leq \Diamond x \text{ and } x^{\sim} \leq x' \leq x^{+}.$$

It will be helpful to note that generally $x \vee x^{\sim} \neq 1$ (dually, $x \wedge x^{+} \neq 0$).

Let us now present two further axioms that we assume in this paper in order to work in a specific variety, namely strong de Morgan (SDM) [3], and strong Kleene (SK) [34, 41]:

$$x^{\sim} \vee y^{\sim} = (x \wedge y)^{\sim}, \tag{SDM}$$

$$\text{if } \Box x = \Box y \text{ and } \Diamond x = \Diamond y, \text{ then } x = y. \tag{SK}$$

As a side remark, we notice that (SDM) expresses the fact that de Morgan condition works also for the Brower negation $^{\sim}$. In general, it is possible to show that the class of all effects of a finite-dimensional Hilbert space determines a de Morgan BZ poset [10]. We observe that (SDM) is crucial to conclude that the blocks are subalgebras [24] (see also Theorem 3.3).

Let us emphasize that, under (SDM) [25], (SK) is indeed equationally expressible, as Lemma 2.3 shows:

Lemma 2.3. [25] *Let L be a PBZ^* lattice that satisfies* (SDM)*. Then, the following are equivalent in L:*

(1) (SK);

(2) $(x \wedge \Diamond y) \wedge (\Box x \vee y) = x \wedge \Diamond y$.

Concerning (SK), it implies relatively strong consequences in our framework. For instance, the partial order is completely determined by modalities. On the one hand, in general, (SK) induces a strong dependence of the whole structure on the subalgebra formed by the sharp elements [10], which we will see that it is the largest orthomodular subalgebra.

On the other hand, (SK) enables us to maintain a tight connection with the orthomodular case and its associated properties, thereby ensuring an intriguing symmetry with more specific structures. It is straightforward to check that orthomodular lattices satisfy:

$$x' = x^{\sim},$$

and so therefore condition (SK) holds trivially true.

Now, given a PBZ^* lattice L, three subsets of the support will assume a definite relevance to the development of our discourse.

We denote by $S_K(L)$ the set of *sharp elements of* L:

$$S_K(L) = \{x \in L : x' = x^{\sim}\} = \{x \in L : x = \Diamond x\} = \{x^{\sim} : x \in L\}.$$

Let us notice that $S_K(L)$ is the largest orthomodular subalgebra of L (see e.g. [23]). It can be seen that in $S_K(L)$ the operations $'$ and $^{\sim}$ coincide, and therefore they are antitone involutions. Moreover, all sharp elements are complemented. Also, they are all stable under $\Diamond$, $\Box$, i.e. $\Diamond x = \Box x = x$. It is straightforward that in $S_K(L)$ (SK) is nothing but a trivial. Indeed, as we already mentioned $S_K(L)$ does form an orthomodular lattice.

Given $x \in L$, $\Box x$, $\Diamond x$ are its *sharp approximations*. Namely, $\Box x$ is the largest element in $S_K(L)$ under x, and dually $\Diamond x$ is the smallest element in $S_K(L)$ over x. The sharp approximations of an element of the form $x' \in L$ are (see (2)):

$$x^{\sim} \leq x' \leq x^{+}.$$

We indicate by $D^{\sim}(L)$ the set of *dense elements of* L:

$$D^{\sim}(L) = \{x \in L : x^{\sim} = 0\}.$$

We note that if $x \in S_K(L) \cap D^{\sim}(L)$, then $x = 1$. Furthermore, for all $x \in L$, $x \vee x^{\sim} \in D^{\sim}(L)$.

Dually, we define the set of *dually dense elements of* L:

$$D^{+}(L) = \{x \in L : x^{+} = 1\}.$$

We note that if $x \in S_K(L) \cap D^{+}(L)$, then $x = 0$. Moreover, for all $x \in L$, $x \wedge x^{+} \in D^{+}(L)$.

Let us recall that we will denote by UOM (*unsharp orthomodular lattices*) the variety of PBZ^* lattices that satisfies (SDM) and (SK). Precisely,

Definition 2.4. *An* unsharp orthomodular lattice *is a structure* $L = (L, \wedge, \vee, ', ^{\sim}, 0, 1)$ *which is a* PBZ^* *lattice (Definition 2.2) that satisfies* (SDM) *and* (SK).

3 Blocks and commutation

As observed in [23], PBZ^* lattices are an instead natural generalization of orthomodular lattice to a non orthocomplemented case. We will see in the present section that UOM will be a very natural generalization of orthomodular lattices to the case in which many structures of great relevance to algebraic logic (Boolean algebras, Kleene algebras, Stone algebras, MV-algebras, regular double Stone algebras etc.) find a unifying treatment.

Paralleling what happens for orthomodular lattices, in this section we introduce a relatively smooth notion of commutativity, that generalizes the corresponding notion for orthomodular structures. From this relation emerges directly a general notion of block. It is well known that in the orthomodular case every block can be regarded as a classical context, namely a Boolean algebra. We shall see that in case of UOM the situation will be perfectly reflected: every block corresponds to a "non classical" context. Namely, it can be regarded as a Kleene lattice. Indeed, we have a natural generalization of the concept of Boolean block of an orthomodular lattice to the non orthocomplemented case. Equivalently, any block can be also considered as an MV algebra, or equivalently again as a regular double Stone algebra (for a detailed discussion on the relations between regular double Stone algebras and several varieties of definite importance for algebraic logic we refer the reader to [36]).

Let us also notice that, as we will see, these non classical contexts still maintain a "classical flavour", that we will characterize in several ways.

We now introduce one of the key ideas on which the entire article relies: commutativity. Let us anticipate that, in order to introduce our concept, the commutativity relation between sharp elements plays a crucial role.

Definition 3.1. *Let L be in UOM, and $x, y \in L$. We say that x* commutes *with y, in symbols $x\bar{C}y$, if the following conditions are satisfied:*

(1) $\Box x C \Box y$ *and* $\Diamond x C \Diamond y$*;*

(2) $\Box x C \Diamond y$ *and* $\Diamond x C \Box y$*;*

Where the relation C is the commutativity relation as defined for the variety of orthomodular lattices: xCy if $x = (x \vee y) \wedge (x \vee y')$.[1]

By Definition 3.1, we can smoothly derive a correspondent notion of the block that neatly generalizes the crucial notion of a block in the orthomodular case. It is possible to consider a block M of L in UOM as a maximal subset, where $x, y \in M$ if and only if $x\bar{C}y$.

In addition, we propose another explicit definition of a block. Obviously, they are equivalent, however as we shall see the latter description of a block M will prove more useful at an operational level.

Definition 3.2. *Let L be in UOM. Given a block B in the orthomodular lattice $S_k(L)$ of sharp elements of L, a B-*block *is a set M of L such that,*

$$M = \{x \in L : \Box x \quad and \quad \Diamond x \; \in \; B\}.$$

[1] Let us remark that we are free to consider the commutativity relation as defined for the variety of orthomodular lattices since the elements of the form $\Diamond x, \Box x$ are orthomodular [5].

Whenever no confusion is possible, we simply call a B-block a *block*.

Theorem 3.3. *Let $L \in UOM$. If M is a block of L, then M is a distributive subalgebra.*

Proof. (Sketch) Since Definition 3.2, it can be seen that $^{\sim}$ is a pseudocomplement, and dually for $^{+}$. Then exploiting the results in [34, Theorem 3.2, page 240] the claim follows. □

As a consequence (see [2, 24, 34]),

Corollary 3.4. *Let L be in UOM. If M is a block of L, then M is a regular double Stone algebra.*

By the previous statements, if L is in UOM, and M a block, then M can be regarded as a Kleene lattice, whose lattice reduct, we recall, is distributive. Let us emphasize that this fact is perfectly symmetric to what happens with orthomodular lattices and their Boolean blocks.

As a side remark, let us notice that for $n \leq 4$, an n-valued Łukasiewicz algebra is indeed an n-valued MV algebra [30]. Consequently, each block can be equivalently regarded either as a Kleene lattice, or a 3-valued Łukasiewicz algebra, or a 3-valued MV algebra, or as a regular double Heyting algebra.

Lemma 3.5 shows that the commutativity relation in UOM is symmetric as in the orthomodular case.

Lemma 3.5. *Let $L \in UOM$. The following conditions are equivalent:*

(1) $x\bar{C}y$;

(2) $y\bar{C}x$;

(3) $x\bar{C}y'$.

We now discuss a result that reflects what happens between blocks and commutation, in orthomodular lattices. Indeed, we will provide equational conditions equivalent to the notion of commutation, and so to the notion of block. Let us observe that in the case of orthomodular lattices a single equation, or its dual, would suffice to capture commutation fully. In UOM, a modicum of elaboration will be needed. We have several equivalent equational conditions that characterize commutation, the one "dual" to the other up to switching $^{\sim}$ with $^{+}$.

Theorem 3.6. *Let $L \in UOM$. The following are equivalent:*

(1) $x\bar{C}y$;

(2) *L satisfies* $(x \wedge y) \vee (x \wedge y^{+}) = x$;

(3) *L satisfies* $y \vee (y^{+} \wedge x) = x \vee y$;

(4) *L satisfies* $(x \vee y) \wedge (x \vee y^{\sim}) = x$;

(5) *L satisfies* $y \wedge (y^{\sim} \vee x) = y \wedge x$.

Let us notice that Theorem 3.6 smootly generalizes well known properties of orthomodular lattices. However, we may observe that if, for the orthomodular case, the equivalences are carried over by the lattice operations and the orthocomplement, i.e. they satisfy

$$xCy \text{ iff } x = (x \wedge y) \vee (x \wedge y') \text{ iff } x = (x \vee y) \wedge (x \vee y'), \tag{3}$$

in the case of UOM both, negations $^{\sim}$, $^{+}$ are necessary to carry over the duality in (3).

Theorem 3.7. *Let $L \in UOM$. We have that, if*

(1) $\Diamond x \leq y$ *then* $x \vee (x^+ \wedge y) = y$;

(2) $y \leq \Box x$ *then* $x \wedge (x^\sim \vee y) = y$.

In other words, Theorem 3.7 says that if $x\bar{C}y$ (the fact that $\Diamond x \leq y$ and $y \leq \Box x$ would suffice for x, y to commute), then a generalized form of orthomodularity holds.

Finally, we can observe without a straightforward proof that other well-known facts that are true for orthomodular lattices can be generalized in UOM.

Indeed, from Theorem 3.7 we can derive Corollary 3.8. Corollary 3.8 expresses in several forms the notion of paraorthomodularity, which in the context of PBZ^* satisfying (SDM) is no longer equivalent to the conditions in Theorem 3.7.

Corollary 3.8. *Any $L \in UOM$ satisfies the following:*

(1) *if* $\Diamond y \leq x$ *and* $y^+ \wedge x = 0$, *then* $x = \Diamond y$;

(2) *if* $x \leq \Box y$ *and* $x \vee y^\sim = 1$, *then* $x = \Box y$.

Finally, Theorem 3.9 generalizes the state of affairs in orthomodular lattices (see, e.g. [5, Theorem 6.11]).

Theorem 3.9. *Let $L \in UOM$. Then, the following conditions are equivalent:*

(1) *L is a distributive UOM, i.e. it is a regular double Stone algebra;*

(2) *L is an MV algebra;*

(3) *for all x, y, $x\bar{C}y$;*

(4) *$\bar{C}$ is transitive;*

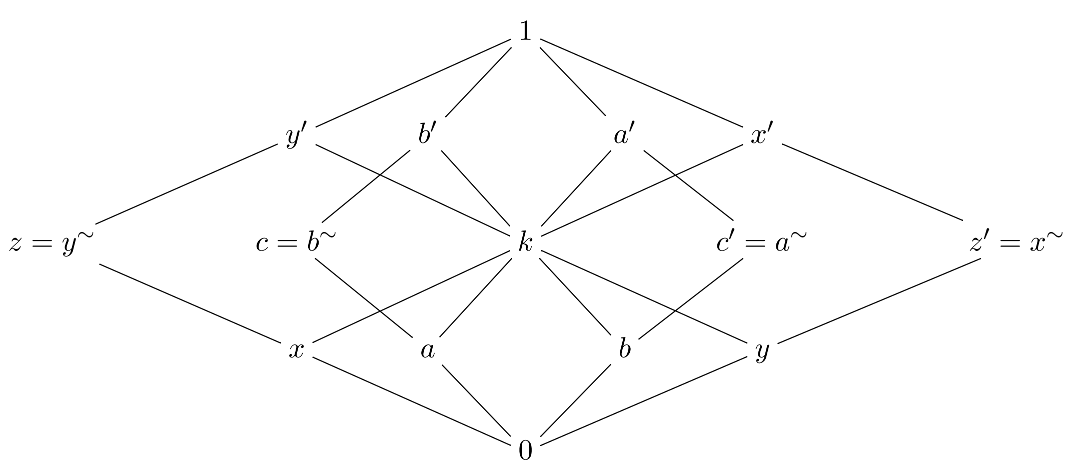
(5) *$\bar{C}$ is an equivalence relation;*

(6) *$^\sim$ is a pseudocomplement;*

(7) *$S_K(L)$ is a Boolean algebra.*

To favor a concrete intuition, we propose Example 3.10.

Example 3.10. *Let us call* 15 *the UOM lattice whose structure is described by the following Hasse diagram:*



We can see that $S_K(15)$ is the lattice whose elements are $\{0, z, z', c, c', 1\}$, while the dense elements $D^{\sim}(15)$ and the dually dense elements $D^{+}(15)$ are respectively the lattices whose elements are $\{1, y', b', a', x', k\}$ and $\{k, x, a, b, y, 0\}$, respectively. Also, the blocks of 15 are isomorphic copies of a 9-element Kleene lattice.[2] *Finally, let us note that $k = k'$, i.e. k is a fixed point.*

4 A categorical equivalence

In order to discuss the main results of the present section, we provide a general representation of UOM. To this aim we start with a given orthomodular lattice. Then, we generalize a long-dated construction, whose roots can be traced back to the early works of Moisil, A. Monteiro and L. Monteiro [4, 38, 39], where an idea of the method is introduced, albeit in different contexts.

To approach the construction, let A be an orthomodular lattice, and F a p-filter, i.e. a lattice filter F such that for all $x \in F$, and, for all $y \in A$:

$$y \vee (y' \wedge x) \in F.$$

Then, let us consider the following set:

$$\mathcal{G}(A, F) = \mathcal{G}(A) = \{(x, y) \in A^2 : (x \leq y) \,\&\, (x \vee y' \in F)\}. \tag{A}$$

With a slight abuse of language, when there is no risk of confusion, for the sake of readability, we will write $G(A)$ instead of $G(A, F)$. Moreover, we will treat the improper filter as principal: namely as the 0-generated filter.

On $\mathcal{G}(A)$ we consider $\wedge, \vee$ componentwise, and we define the following unary operations on $\mathcal{G}(A)$:

$$(x, y)' = (y', x'); \quad (x, y)^{\sim} = (y', y'); \quad (x, y)^{+} = (x', x'). \tag{4}$$

Sometimes, if specifications on the filter F are in order, we will use the notation (A, F), instead of $\mathcal{G}(A)$.

Theorem 4.1 is a bridge result that allows to extract an unsharp orthomodular lattice out of any orthomodular lattice with a designed p-filter.

Theorem 4.1. *Let A be a orthomodular lattice and F a p-filter of A. Then $G(A)$ is an unsharp orthomodular lattice.*

We will see in Lemma 4.2 that in any unsharp orthomodular lattice the orthomodular subalgebra of the sharp elements and the p-filter (up to isomorphism) of dense elements will assume a fundamental role in providing a representation theorem for the variety in question.

Lemma 4.2 is also an important tool for the categorical equivalence described in this section. In fact, for any orthomodular lattice A, it can be seen that the set of sharp elements of $\mathcal{G}(A)$ is isomorphic to A, and that the p-filter is lattice-isomorphic to the set of dense elements.

Lemma 4.2. *Let A be an orthomodular lattice and F a p-filter. Then:*

[2] For reasons of space we do not report here this 9-elements Kleene lattice. We invite the interested reader to contact the authors.

(1) $S_K(\mathcal{G}(A))$ *is an orthomodular lattice isomorphic to* A*;*

(2) $D^{\sim}(\mathcal{G}(A))$ *is lattice-isomorphic to* F*.*

Proof. (Sketch) Set $g : A \to S_K(\mathcal{G}(A))$ defined by $g(x) = (x, x)$, any element in $S_K(\mathcal{G}(A))$ is of the form (x, x). Moreover, set

$$f : D^{\sim}(\mathcal{G}(A)) \to F \text{ defined by } f((x, 1)) = x.$$

Since all elements in $D^{\sim}(\mathcal{G}(A))$ are of the form $(x, 1)$, for $x \in A$. □

In fact, any algebra L in UOM has "encoded" in its structure all the information on the orthomodular lattice and the p-filter that via the construction originates L.

By Lemma 4.2, if $L = G(A)$, for a pair (A, F), where A is an orthomodular lattice, and F a p-filter, then $(S_K(L), f(D^{\sim}(L)))$ and (A, F) are up to isomorphism the same object. We take advantage of this fact and, since no danger of confusion will be possible, we will use both $(S_K(L), f(D^{\sim}(L)))$ and (A, F) indistinctly.

We now have all the tools required to prove our representation Theorem 4.3:

Theorem 4.3. *Every unsharp orthomodular lattice is* $\mathcal{G}(A)$*, for a unique orthomodular lattice with a fixed p-filter* (A, F)*.*

Proof. (Sketch) Let us notice that in any unsharp orthomodular lattice L, $S_K(L)$ is an orthomodular lattice, and $D^{\sim}(L)$ can be regarded as a p-filter on $S_K(L)$. It can be proven that L is isomorphic to $(S_K(L), f(D^{\sim}(L)))$ via the mapping

$$\varphi : L \to (S_K(L), f(D^{\sim}(L))) \text{ defined by } \varphi(x) = (x'^{\sim}, x^{\sim\sim}).$$

□

Indeed, Theorem 4.3 shows that the representation is unique for any unsharp orthomodular lattice: the orthomodular lattice and the fixed p-filter in the construction *must be unique*.

For instance, let us observe that algebra 15 in Example 3.10 is of the form (MO_2, MO_2) (see [5]), where MO_2 is regarded as the improper filter.

Any algebra L in UOM can be thought of as a pair either of two copies of the same orthomodular lattice if L has a fixpoint (i.e. when the improper filter is considered), or as a pair of orthomodular lattices, if the p-filter is principal, i.e. if the dense elements possess a minimum. Otherwise, as a pair composed by an orthomodular lattice and a generalized dual orthomodular lattice, if no requirements on the p-filter are assumed (see the seminal work of Janowitz [31], and for general facts [5]).

Theorem 4.4. *Let* A *be an orthomodular lattice and* F *a p-filter. The following conditions hold:*

(1) $(A, \{1\})$ *is isomorphic to* A*;*

(2) *if* $F \subseteq G$*, then* (A, F) *is a subalgebra of* (A, G)*, i.e. the construction is (strictly) monotone to the lattice of p-filters;*

(3) (A, A) *is the sole construction on* A *with a fixed point for* $'$*;*

(4) *F is principal if and only if $D^{\sim}(\mathcal{G}(A))$ has a minimum.*

We remark that all the properties described for filters and dense elements can be stated in dual form for ideals and dually dense elements.

Let us now focus on a rather reasonable (in our opinion) category, whose objects are orthomodular lattices indexed by a given p-filter. On the one hand, as natural as it is, this category properly contains category of orthomodular lattices, [7, 33, 36].

On the other hand, we consider another category whose objects are unsharp orthomodular lattices, indexed by the filter of dense elements. In general, for unsharp orthomodular lattices we may dispense of these indexes. However, this notation will be expedient in easing our arguments.

Surprisingly enough, we will see that the categories of unsharp orthomodular lattices indexed by the set of their dense elements, and orthomodular lattices indexed by a p-filter are, from an abstract point of view, different perspectives on one and the same concept: they will be shown to be equivalent in categorical terms. We will capitalize on the construction discussed in section 3, which traces back to the work of Balbes and Grätzer [4]. This idea has attracted the work of many authors in different contexts, from general algebra to rough sets, see e.g. [35] and [16].

Let us now explicitly introduce the two categories that will play a quite central role in this whole paper.

Definition 4.5. *We call* orthomodular lattices with fixed p-filter*, and denote by $\mathbf{A_F}$, the category defined as follows:*

(1) *the* objects *are pairs $[A, F]$, where A is an orthomodular lattice and F is a p-filter of A;*

(2) *the* morphisms *are orthomodular homomorphisms that preserve the filters.*

Namely, if $g : [A_1, F] \to [A_2, G]$ in $\mathbf{A_F}$, then g is an orthomodular homomorphism such that $g(F) \subseteq G$.

Since no danger of confusion will be impending, to ease the discourse we will sometimes call $\mathbf{A_F}$ the category of *indexed orthomodular lattices*. Let us remark that this naming is not to be referred by any means to the concepts of indexed category or comma category [1].

We will also denote by A the category of orthomodular lattices, equipped with the usual morphisms. It will be noticed in due course that A can be regarded as a full subcategory of $\mathbf{A_F}$.

Definition 4.6. *We denote by T the category defined as follows:*

(1) *the* objects *are pairs $(L, D^{\sim}(L))$, where L is an unsharp orthomodular lattice, and $D^{\sim}(L)$ is the filter of dense elements of L;*

(2) *the* morphisms *are homomorphisms.*

Let us remark that in Definition 4.6 the homomorphisms always preserve $D^{\sim}(L)$.

Theorem 4.7 relies deeply on the representation of unsharp orthomodular lattice, and it shows that the information in the category A_F can be faithfully shifted by the functor Ψ into the category T.

Theorem 4.7. *Let $[A_1, F_1]$, $[A_2, F_2]$ be objects in $\mathbf{A_F}$ and $g : [A_1, F_1] \to [A_2, F_2]$ a morphism. Upon defining the functor $\Psi : \mathbf{A_F} \to T$ by setting:*

$$\Psi([A, F]) = \mathcal{G}(A) = (A, F), \text{ and } (\Psi(g))((x, y)) = (g(x), g(y)),$$

the following diagram commutes:

$$\begin{array}{ccc} [A_1, F_1] & \xrightarrow{\quad g \quad} & [A_2, F_2] \\ \Big\downarrow \Psi & & \Big\downarrow \Psi \\ (L_1, D^{\sim}(L_1)) & \xrightarrow{\Psi(g)} & (L_2, D^{\sim}(L_2)) \end{array} \qquad (\star)$$

Let us observe that, in general, the category A can be regarded as a full subcategory of $\mathbf{A_F}$ and T. In fact, for any system $g : A_1 \to A_2$ in A always exists a natural transformation. Indeed, consider as objects $(A_1, g^{-1}(1))$, $(A_2, \{1\})$, and (g, g) componentwise.

Theorem 4.8 describes a functor Ξ converse to Ψ, that faithfully lifts the information in T to the category $\mathbf{A_F}$.

Theorem 4.8. *Upon defining the functor* $\Xi : T \to \mathbf{A_F}$ *by setting:*

$$\Xi((L, D^{\sim}(L))) = (S_K(L), f(D^{\sim}(L))), \text{ and } \Xi(h) = h \restriction_{S_K(L)},$$

the following diagram commutes:

$$\begin{array}{ccc} (L_1, D^{\sim}(L_1)) & \xrightarrow{\quad h \quad} & (L_2, D^{\sim}(L_2)) \\ \Big\downarrow \Xi & & \Big\downarrow \Xi \\ [S_K(L_1), f(D^{\sim}(L_1))] & \xrightarrow{\Xi(h)} & [S_K(L_2), f(D^{\sim}(L_2))] \end{array} \qquad (\star\star)$$

Moreover, Ψ, Ξ *are mutually inverse functors.*

As a corollary we obtain that:

Corollary 4.9. *The categories* $\mathbf{A_F}$ *and* $\mathbf{T}$ *are equivalent. Moreover, the category* T *includes as proper full subcategories the categories of Boolean algebras, regular double Stone algebras, 3-valued MV-algebras (Łukasiewicz algebras), and regular double Heyting algebras.*

Conclusion

In this survey we have recapped several aspects of the variety of unsharp orthomodular lattices, which in our opinion are of definite relevance to algebraic logic. In fact, the variety of UOM forms a smooth common scenario in which both orthomodular lattices and regular double Stone algebras find a unifying treatment. It may be worth also to observe that several classes of prominent importance to algebraic logic, e.g. Boolean algebras, Stone algebras, Kleene algebras, 3-valued MV algebras, regular double Heyting algebras etc. find a unique general setting. In this context many important notions can be generalized, e.g. the notion of block with its relevance to the structure theory. Finally, as a consequence of this fact a categorical equivalence with orthomodular lattices indexed by a p-filter obtains.

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References

[1] J. Adámek, H. Herrlich, G.E. Strecker, *Abstract and concrete categories*, Wiley and Sons, New York, 1990.

[2] M.E. Adams, H.P. Sankappanavar, J. Vaz de Carvalho, *Regular double p-algebras*, Mathematica Slovaca, 69 (2019), 15–34. https://doi.org/10.48550/arXiv.2210.10387

[3] R. Balbes, *A survey of stone algebras*, Proceedings of the Conference on Universal Algebra, (1979), 148–170.

[4] R. Balbes, G. Grätzer, *Injective and projective stone algebras*, Duke Mathematical Journal, (1971), 339–347. https://doi.org/10.1215/S0012-7094-71-03843-9

[5] L. Beran, Orthomodular lattices: Algebraic approach, Reidel, Dordrecht, 1985. https://doi.org/10.1007/978-94-009-5215-7

[6] G. Birkhoff, *Lattice theory*, American Mathematical Society College Publications, Providence, 1967.

[7] G. Birkhoff, J. Von Neumann, *The logic of quantum mechanics*, Annals of Mathematics, 37 (1936), 823–843. https://doi.org/10.2307/1968621

[8] S. Burris, H.P. Sankappanavar, *A course in universal algebra, the millennium edition*, Springer, Verlag, 2001. https://www.math.uwaterloo.ca/~snburris/htdocs/UALG/univ-algebra.pdf

[9] G. Cattaneo, *Brouwer-Zadeh (Fuzzy-intuitionistic) posets for unsharp quantum mechanics*, International Journal of Theoretical Physics, 31 (1992), 1573–1597. https://doi.org/10.1007/BF00671774

[10] G. Cattaneo, R. Giuntini, *Some results on BZ structures from Hilbertian unsharp quantum physics*, Foundations of Physics, 25 (1995), 1147–1183. https://doi.org/10.1007/BF02055256

[11] I. Chajda, D. Fazio, H. Länger, A. Ledda, J. Paseka, *Algebraic properties of paraorthomodular posets*, Logic Journal of the IGPL, 30 (2022), 840–869. https://doi.org/10.1093/jigpal/jzab024

[12] R.L.O. Cignoli, I.M.L. d'Ottaviano, D. Mundici, *Algebraic foundations of many-valued reasoning*, Springer, Verlag, 2000. https://doi.org/10.1007/978-94-015-9480-6

[13] J.M. Cornejo, M. Kinyon, H.P. Sankappanavar, *Regular double p-algebras: A converse to a Katriňák theorem and applications*, Mathematica Slovaca, 73(6) (2023), 1373–1388. https://doi.org/10.48550/arXiv.2210.10387

[14] M.L. Dalla Chiara, R. Giuntini, R. Greechie, *Reasoning in quantum theory*, Kluwer, Dordrecht, 2004. https://doi.org/10.1007/978-94-017-0526-4

[15] H.F. De Groote, *On a canonical lattice structure on the effect algebra of a von Neumann algebra*, (2005), arXiv preprint math-ph/0410018. https://doi.org/10.48550/arXiv.math-ph/0410018

[16] I. Düntsch, *A logic for rough sets*, Theoretical Computer Science, 179 (1997), 427–436. https://www.cosc.brocku.ca/~duentsch/archive/roughlog.pdf

[17] A. Dvurečenskij, *Vyšetrovanie vlastností zovšeobecnených stavov na logike*, ÚMMT SAV, 1976, Bratislava.

[18] A. Dvurečenskij, *Vybrané problémy teórie pravdepodobnosti na kvantových logikách*, MÚ SAV, 1988, Bratislava.

[19] A. Dvurečenskij, *Joint distributions in quantum logics*, MÚ SAV, 1992, Bratislava.

[20] A. Dvurečenskij, S. Pulmannová, *New trends in quantum structures*, Springer, Verlag, 2000. https://doi.org/10.1007/978-94-017-2422-7

[21] K. Engesser, D.M. Gabbay, D. Lehmann (eds.), *Handbook of quantum logic and quantum structures*, Elsevier, Amsterdam, 2007. https://doi.org/10.1016/C2009-0-16935-0

[22] N. Galatos, P. Jipsen, T. Kowalski, H. Ono, *Residuated lattices: An algebraic glimpse at substructural logics*, Elsevier, Amsterdam, 2007.

[23] R. Giuntini, A. Ledda, F. Paoli, *A new view of effects in a Hilbert space*, Studia Logica, 104 (2016), 1145–1177. https://doi.org/10.1007/s11225-016-9670-3

[24] R. Giuntini, A. Ledda, G. Vergottini, *Generalizing orthomodularity to unsharp contexts: Properties, blocks, residuation*, Logic Journal of the IGPL, 2024. https://doi.org/10.1093/jigpal/jzae076

[25] R. Giuntini, C. Mureşan, F. Paoli, *PBZ^*-lattices: Structure theory and subvarieties*, Reports on Mathematical Logic, 55 (2020), 3–39. https://doi.org/10.48550/arXiv.1804.08136

[26] S. Givant, P. Halmos, *Introduction to Boolean algebras*, Springer, Verlag, 2009. https://doi.org/10.1007/978-0-387-68436-9

[27] G. Grätzer, *Lattice theory: First concepts and distributive lattices*, Courier Corporation, Chelmsford, 2009.

[28] R.J. Greechie, *On the structure of orthomodular lattices satisfying the chain condition*, Journal of Combinatorial Theory, 4 (1968), 210–218. https://doi.org/10.1016/S0021-9800(68)80002-X

[29] R.S. Grigolia, *Algebraic analysis of Łukasiewicz-Tarski's n-valued logical systems*, in Wójcicki R., Malinowski G. (eds.), Selected Papers on Łukasiewicz Sentential Calculi, Ossolineum, Wroclaw, (1977), 81–92.

[30] A. Iorgulescu, *Connections between MV_n-algebras and n-valued Łukasiewicz-Moisil algebras Part I*, Discrete Mathematics, 181 (1998), 155–177. https://doi.org/10.1016/S0012-365X(97)00052-6

[31] M.F. Janowitz, *A note on generalized orthomodular lattices*, Journal of Natural Sciences and Mathematics, 8 (1968), 89–94.

[32] J.A. Kalman, *Lattices with involution*, Transactions of the American Mathematical Society, 87 (1958), 485–491.

[33] G. Kalmbach, *Orthomodular lattices*, Academic Press, New York, 1983.

[34] T. Katriňák, *The structure of distributive double p-algebras. Regularity and congruences*, Algebra Universalis, 3 (1973), 238–246. https://doi.org/10.1007/BF02945123

[35] T. Katriňák, P. Mederly, *Constructions of p-algebras*, Algebra Universalis, 17 (1983), 288–316. https://doi.org/10.1007/BF01194538

[36] A. Ledda, G. Vergottini, *Categorical, structural, and model-theoretic properties of regular double Stone algebras, with applications*, Submitted, (2024).

[37] A. Ledda, G. Vergottini, *Orthomodular and unsharp orthomodular lattices: A categorical equivalence*, Submitted, 2024.

[38] G.C. Moisil, *Recherches sur les logiques non-chrysippiennes*, Annales Scientifiques de l'Université de Jassy, 26 (1940), 431–466.

[39] A. Monteiro, L.F. Monteiro, *Three-valued Łukasiewicz algebras*, Lectures given at the Universidad National del Sur, Bahía Blanca, 1963. https://inmabb.conicet.gob.ar/static/publicaciones/nlm/nlm-41.pdf

[40] M.P. Olson, *The self-adjoint operators of a von Neumann algebra form a conditionally complete lattice*, Proceedings of the American Mathematical Society, 28 (1971), 537–544. https://doi.org/10.2307/2038007

[41] J. Varlet, *A regular variety of type* $\langle 2, 2, 1, 1, 0, 0\rangle$, Algebra Universalis, 2 (1972), 218–223.