

BOUNDEDNESS IN A CHEMOTAXIS SYSTEM WITH CONSUMED CHEMOATTRACTANT AND PRODUCED CHEMOREPELLENT

SILVIA FRASSU AND GIUSEPPE VIGLIALORO*

Dipartimento di Matematica e Informatica
 Università di Cagliari
 Via Ospedale 72, 09124. Cagliari (Italy)

ABSTRACT. We study this zero-flux attraction-repulsion chemotaxis model, with linear and superlinear production g for the chemorepellent and sublinear rate f for the chemoattractant:

$$(\diamond) \quad \begin{cases} u_t = \Delta u - \chi \nabla \cdot (u \nabla v) + \xi \nabla \cdot (u \nabla w) & \text{in } \Omega \times (0, T_{max}), \\ v_t = \Delta v - f(u)v & \text{in } \Omega \times (0, T_{max}), \\ 0 = \Delta w - \delta w + g(u) & \text{in } \Omega \times (0, T_{max}). \end{cases}$$

In this problem, Ω is a bounded and smooth domain of \mathbb{R}^n , for $n \geq 1$, $\chi, \xi, \delta > 0$, $f(u)$ and $g(u)$ reasonably regular functions generalizing the prototypes $f(u) = Ku^\alpha$ and $g(u) = \gamma u^l$, with $K, \gamma > 0$ and proper $\alpha, l > 0$. Once it is indicated that any sufficiently smooth $u(x, 0) = u_0(x) \geq 0$ and $v(x, 0) = v_0(x) \geq 0$ produce a unique classical and nonnegative solution (u, v, w) to (\diamond) , which is defined in $\Omega \times (0, T_{max})$, we establish that for any such (u_0, v_0) , the life span $T_{max} = \infty$ and u, v and w are uniformly bounded in $\Omega \times (0, \infty)$, (i) for $l = 1$, $n \in \{1, 2\}$, $\alpha \in (0, \frac{1}{2} + \frac{1}{n}) \cap (0, 1)$ and any $\xi > 0$, (ii) for $l = 1$, $n \geq 3$, $\alpha \in (0, \frac{1}{2} + \frac{1}{n})$ and ξ larger than a quantity depending on $\chi \|v_0\|_{L^\infty(\Omega)}$, (iii) for $l > 1$, $\alpha \in (0, \frac{1}{2} + \frac{1}{n}) \cap (0, 1)$, any $\xi > 0$, and in any dimensional settings. Finally, an illustrative analysis about the effect by logistic and repulsive actions on chemotactic phenomena is proposed by comparing the results herein derived for the linear production case with those in [11].

1. PRESENTATION OF THE MODEL

This article is dedicated to the following Cauchy boundary problem

$$(1) \quad \begin{cases} u_t = \Delta u - \chi \nabla \cdot (u \nabla v) + \xi \nabla \cdot (u \nabla w) & \text{in } \Omega \times (0, T_{max}), \\ v_t = \Delta v - f(u)v & \text{in } \Omega \times (0, T_{max}), \\ 0 = \Delta w - \delta w + g(u) & \text{in } \Omega \times (0, T_{max}), \\ u_\nu = v_\nu = w_\nu = 0 & \text{on } \partial\Omega \times (0, T_{max}), \\ u(x, 0) = u_0(x), v(x, 0) = v_0(x) & x \in \bar{\Omega}, \end{cases}$$

defined in a bounded and smooth domain Ω of \mathbb{R}^n , with $n \geq 1$, $\chi, \xi, \delta > 0$ and some functions $f = f(s)$ and $g = g(s)$, sufficiently regular in their argument $s \geq 0$, and further regular initial data $u_0(x) \geq 0$ and $v_0(x) \geq 0$. Additionally, the subscript ν in $(\cdot)_\nu$ indicates the outward normal derivative on $\partial\Omega$, whereas T_{max} the maximum time up to which solutions to the system are defined.

The consideration of model (1) comes, essentially, from a natural coupling of two widely studied chemotaxis systems, largely employed in biological processes: the classical Keller–Segel model ([8, 9, 10]) idealizing aggregation phenomena in situations where certain cells (populations, organisms) are attracted by a signal they themselves absorb, and a repulsive counterpart, where the same cells are repelled in response to another substance emitted by them. More precisely, if $u = u(x, t)$ is used to denote the population density of these cells at the position x and at the time t , and $v = v(x, t)$ and $w = w(x, t)$ stand, respectively, for the concentration of the attractive and repulsive chemical signals (chemoattractant and chemorepellent), problem (1) indicates that: (a) the motion of the cells, inside an insulated domain (zero-flux on the border) and initially distributed according to the law of u_0 , results from the competition between the aggregation/repulsion impact from the cross terms $\chi u \nabla v / \xi u \nabla w$ (increasing for larger sizes of χ and ξ) and the diffusion of the cells (the Laplacian Δu); (b) the initial signal v_0 is spread, w diffuses as well but v (second equation in (1)) is consumed with a rate $f(u)$ whereas w (third equation) is proliferated with rate $g(u)$; (c) consumption and production are higher the more the cell density increases.

Purely intuitive considerations (but below we will give precise references) suggest that this interplay between the factors taking part in model (1) might lead to very different situations for the aforementioned cellular movement: from global stabilization and convergence to equilibrium of the cell distribution u , to the so-called *chemotactic collapse*, the mechanism resulting in aggregation processes for u , eventually blowing up/exploding at finite time. Mathematically, in the first case, solutions (u, v, w) are defined and bounded for all (x, t) in $\Omega \times (0, \infty)$, in the other a finite time T_{max} exists and (u, v, w) ceases to exist for larger value of T_{max} ; in particular, the component of the solution associated to the particle density becomes

2010 *Mathematics Subject Classification*. Primary: 35A01, 35K55, 35Q92. Secondary: 92C17.

Key words and phrases. Chemotaxis, Global existence, Boundedness, Nonlinear production.

* *Corresponding author*: giuseppe.viglialoro@unica.it.

unbounded approaching T_{max} , with emergencies of δ -formations. In this research we will derive criteria on the data involved in the initial-boundary value problem (1) ensuring that the life span T_{max} of its solutions is infinity and that, moreover, they are as well bounded.

2. SOME KNOWN RESULTS. CLAIM OF THE MAIN THEOREMS

2.1. A view on the state of the art. In the framework of classical Keller–Segel models, as mentioned above, (1) is a combination of the signal-production

$$(2) \quad u_t = \Delta u - \chi \nabla \cdot (u \nabla v) \quad \text{and} \quad v_t = \Delta v - v + u, \quad \text{in } \Omega \times (0, T_{max}),$$

and signal-absorption

$$(3) \quad u_t = \Delta u - \chi \nabla \cdot (u \nabla v) \quad \text{and} \quad v_t = \Delta v - uv, \quad \text{in } \Omega \times (0, T_{max}),$$

chemotaxis systems, originally derived in [8, 10] respectively. Even though the equation for u is the same, it is conceivable that the resulting evolution of each initial boundary value problem related to (2) and (3), must differ from the other, even for same fixed $\chi > 0$ and initial data u_0 and v_0 . This is essentially justified by the observation that v increases with u in problem (2), whereas it decreases in (3). Let us present some more details concerning this discussion; in particular, since we will focus on questions tied to classical solutions, in order to better establish our aims, we select only these references, among some others.

- i) For problem (2), the production of v may break the natural homogenization process of the cells, especially in terms of the size of χ related to the aggregation impact, the initial mass of the particle distribution, i.e., $m = \int_{\Omega} u_0(x) dx$, and the space dimension. Indeed, if in the one-dimensional setting blow-up phenomena are excluded (see [19]), in higher dimensions if $m\chi$ surpasses a certain critical value m_{χ} , the system might present the aforementioned chemotactic collapse, whereas for $m\chi < m_{\chi}$ no instability appears in the motion of the cells. There are many contributions dedicated to understanding this scenario. In this regard, in [4, 6, 17, 25] (and references therein cited), the interested reader can find pointers to the rich literature dealing with the existence and properties of global, uniformly bounded or blow-up (local) solutions to the Cauchy problem associated to (2). On the other hand, as far as nonlinear segregation chemotaxis models like those we are considering, when in problem (2) the production $g(u) = u$ is replaced by $g(u) \cong u^l$, with $0 < l < \frac{2}{n}$ ($n \geq 1$), uniform boundedness of all its solutions is proved in [13]. Moreover, by resorting to a simplified parabolic-elliptic version in spatially radial contexts, when the second equation is reduced to $0 = \Delta v - \mu(t) + g(u)$, with $g(u) \cong u^l$ and $\mu(t) = \frac{1}{|\Omega|} \int_{\Omega} g(u(\cdot, t))$, it is known (see [30]) that the same conclusion on the boundedness continues to be valid for any $n \geq 1$ and $0 < l < \frac{2}{n}$, whereas for $l > \frac{2}{n}$ blow-up phenomena may occur. (Let us note that the sole consideration of logistic-type restrictions, as those specified in item ii) below, does not exclude blow-up in Keller–Segel models with productions, neither in high dimensions nor in low: [26], [31].)
- ii) Conversely to what was discussed for model (2), when the chemical v responsible for gathering processes of the cells is consumed throughout the time, so far no result detecting unbounded solutions to the corresponding initial boundary-value problem to (3) is available. Such a question seems quite hard to solve, and this does not appear surprising; indeed, from comparison arguments, the second equation for the chemical immediately ensures uniform boundedness of v . Despite that, such a bound by itself is not enough to ensure that classical solutions (u, v) to (3) emanating from any sufficiently regular initial data (u_0, v_0) are uniformly bounded. Precisely, this holds true only in two-dimensional settings (as a combination of the results in [27] and [28], where a more general coupled chemotaxis-fluid model is studied); for $n \geq 3$, oppositely, the smallness assumption $\chi \|v_0\|_{L^\infty(\Omega)} \leq \frac{1}{6(n+1)}$ is required ([20]). Nevertheless, this condition does not exclude the possibility that solutions emanating from initial data, not satisfying it, may collapse in finite time. However, a way to prevent blow-up scenario to problem (3) even when $\chi \|v_0\|_{L^\infty(\Omega)}$ is larger than $\frac{1}{6(n+1)}$, consists in considering logistic sources with strong damping effect in the equation of the cells, exactly reading

$$(4) \quad u_t = \Delta u - \chi \nabla \cdot (u \nabla v) + ku - \mu u^2, \quad \text{in } \Omega \times (0, T_{max}), \quad k, \mu > 0.$$

In [11] it is indeed shown that the resulting Cauchy problem admits classical bounded solutions for arbitrarily large $\chi \|v_0\|_{L^\infty(\Omega)}$ provided μ is also larger than a certain expression depending in an increasing way on the same $\chi \|v_0\|_{L^\infty(\Omega)}$.

As far as we know, a general n -dimensional theoretical analysis tied to the attraction-repulsion chemotaxis system in the form of (1), has not been developed yet. Conversely, for $f(u) = g(u) = u$, a fully parabolic attraction-repulsion Stokes system is addressed for the two-dimensional case in [14]: here, inter alia, boundedness of classical solutions is achieved for any initial data. In addition, model (1) where the chemoattractant and chemorepellent are both produced has been proposed, as well in the fully parabolic version, in [15] for one-dimensional settings and linear proliferation, to describe the aggregation of microglia observed in Alzheimer’s disease. In particular, for the attraction-repulsion system (1) with second and third equations replaced by

$$0 = \Delta v - \beta v + f(u) \quad \text{and} \quad 0 = \Delta w - \delta w + g(u), \quad \text{in } \Omega \times (0, T_{max}), \quad \beta, \delta > 0,$$

the following is known in the literature. For linear growths of the chemoattractant and the chemorepellent, $f(u) = \alpha u$, $\alpha > 0$, and $g(u) = \gamma u$, $\gamma > 0$, we have that the value $\xi\gamma - \chi\alpha$, measuring in some sense the difference between the repulsion and attraction contributions, is critical for $n = 2$: particularly, if $\xi\gamma - \chi\alpha > 0$ (repulsion prevails over attraction), in any dimension all solutions to the model are globally bounded, whereas for $\xi\gamma - \chi\alpha < 0$ and $n = 2$ (attraction prevails over repulsion) unbounded solutions can be constructed (see [3, 12, 21, 22, 32] for some details on the issue). On the other hand, for more general production laws, respectively f and g generalizing the prototypes $f(u) = \alpha u^s$, $s > 0$, and $g(u) = \gamma u^r$, $r \geq 1$, we are only aware of the following recent result, valid for $n \geq 2$ ([23]): for every $\alpha, \beta, \gamma, \delta, \chi > 0$, and $r > s \geq 1$ (resp. $s > r \geq 1$), there exists $\xi^* > 0$ (resp. $\xi_* > 0$) such that if $\xi > \xi^*$ (resp. $\xi \geq \xi_*$), any sufficiently regular initial datum $u_0(x) \geq 0$ (resp. $u_0(x) \geq 0$ enjoying some smallness assumptions) produces a unique classical and bounded solution. In addition the same conclusion holds true for every $\alpha, \beta, \gamma, \delta, \chi, \xi > 0$, $0 < s < 1$, $r = 1$ and any sufficiently regular $u_0(x) \geq 0$.

2.2. Motivations and presentation of the Theorems. In accordance to what has been discussed above, especially in items i) and ii), with this research we mainly aim at extending the mathematical comprehension of attraction-repulsion Keller–Segel systems by giving answers to questions concerning system (1), to our knowledge, not yet studied. In this sense, since as seen a chemorepellent impact does not suffice to regularize chemotactic instabilities, we aim at essentially establishing its role on the motion of the particle density, whose kinetics is not influenced by any smoothing logistic term. Specifically, we will give sufficient conditions on the data of model (1) such that the joint actions of the consumed chemoattractant and the produced chemorepellent suffice to provide global and bounded solutions in terms of, or independently of, smallness constraints on $\chi\|v_0\|_{L^\infty(\Omega)}$. To this scope, these assumptions are fixed

$$(5) \quad f, g \in C^1(\mathbb{R}) \quad \text{with} \quad 0 \leq f(s) \leq Ks^\alpha \quad \text{and} \quad \gamma s^l \leq g(s) \leq \gamma s(s+1)^{l-1}, \quad \text{for some} \quad K, \gamma, \alpha > 0, l \geq 1 \quad \text{and all} \quad s \geq 0,$$

and the following results are shown.

Theorem 2.1. *Let Ω be a smooth and bounded domain of \mathbb{R}^n , with $n \geq 1$, and χ, δ positive. Moreover, for some $K, \gamma > 0$, let f and g fulfill (5), respectively with $\alpha \in (0, \frac{1}{2} + \frac{1}{n}) \cap (0, 1)$ and $l = 1$. Then there exists $C(n) \geq 0$, with $C(1) = C(2) = 0$ and $C(n) > 0$ for $n \geq 3$, such that for any initial data $(u_0, v_0) \in C^0(\bar{\Omega}) \times C^1(\bar{\Omega})$, with $u_0, v_0 \geq 0$ on $\bar{\Omega}$, and any $\xi > C(n)\|\chi v_0\|_{L^\infty(\Omega)}^{\frac{4}{n}}$, problem (1) admits a unique global and uniformly bounded classical solution.*

Theorem 2.2. *Let Ω be a smooth and bounded domain of \mathbb{R}^n , with $n \geq 1$, and χ, δ positive. Moreover, for some $K, \gamma > 0$, let f and g fulfill (5), respectively with $\alpha \in (0, \frac{1}{2} + \frac{1}{n}) \cap (0, 1)$ and $l > 1$. Then for any initial data $(u_0, v_0) \in C^0(\bar{\Omega}) \times C^1(\bar{\Omega})$, with $u_0, v_0 \geq 0$ on $\bar{\Omega}$, and any $\xi > 0$, problem (1) admits a unique global and uniformly bounded classical solution.*

Remark 1. *We specify the following aspects:*

- *As usual in the nomenclature, in chemotaxis models a global and uniformly bounded classical solution to problem (1) is a triplet of nonnegative functions $(u, v, w) \in (C^0(\bar{\Omega} \times [0, \infty)) \cap C^{2,1}(\bar{\Omega} \times (0, \infty)))^3$, such that for some $q > n$ and $C > 0$ this relation holds:*

$$\|u(\cdot, t)\|_{L^\infty(\Omega)} + \|v(\cdot, t)\|_{W^{1,q}(\Omega)} \leq C \quad \text{for all} \quad t \in (0, \infty).$$

- *Even though in one-dimensional settings boundedness of solutions to Keller–Segel systems is achievable by well-known reasoning, we give the proof of Theorems 2.1 and 2.2 also for $n = 1$ because it will directly result as a particular case of the n -dimensional version.*

The remaining part of the paper is structured as follows: In §3 some general and well-known preliminaries are given, whereas §4 is focused on the existence of local classical solutions (u, v, w) to problem (1). In particular, crucial properties of these solutions, and how to achieve their uniform-in-time boundedness from their L^p -boundedness, for some suitable $p > 1$, is analyzed. Successively, in §5, we associate to the local solutions, the functional $y(t) := \int_\Omega u^p + (\frac{\chi}{\gamma})^p \int_\Omega |\nabla v|^{2p}$, by means of which the desired uniform-in-time bound is proved; this will allow us to proof our results, also in the same §5. Finally, in §6 we compare [11, Theorem 1.1] and Theorem 2.1 as to discuss the boundedness issue for chemotaxis-consumption models with different smoothing reactions: a logistic source and a produced chemorepellent.

3. SOME PREPARATORY TOOLS

In this section we summarize some inequalities and further necessary results.

Lemma 3.1. *Let $A, B \geq 0$, $d_1, d_2 > 0$ and $p > 1$. Then for some $d, d_3 > 0$ we have*

$$(6) \quad A^{d_1} + B^{d_2} \geq 2^{-d}(A+B)^d - d_3,$$

and

$$(7) \quad (A+B)^p \leq 2^{p-1}(A^p + B^p).$$

Proof. The proofs can be found, respectively, in [16, Lemma 3.3] and [7, Theorem 1]. □

Lemma 3.2. *Let Ω be a bounded and smooth domain of \mathbb{R}^n , $n \geq 1$. For all $\psi \in C^2(\bar{\Omega})$, we have*

$$(8) \quad (\Delta\psi)^2 \leq n|D^2\psi|^2,$$

$$(9) \quad |D^2\psi\nabla\psi|^2 \leq |D^2\psi|^2|\nabla\psi|^2.$$

If, further, ψ satisfies $\psi_\nu = 0$ on $\partial\Omega$, then for all $p > 1$ and $\eta > 0$ one has

$$(10) \quad \|\nabla\psi\|_{L^{2p+2}(\Omega)}^2 \leq 2(4p^2 + n)\|\psi\|_{L^\infty(\bar{\Omega})}^2 \|\nabla\psi\|_{L^2(\Omega)}^{p-1} \|D^2\psi\|_{L^2(\Omega)}^2,$$

where $D^2\psi$ represents the Hessian matrix of ψ and $|D^2\psi|^2 = \sum_{i,j=1}^n \psi_{x_i x_j}^2$, whereas for some positive constant C_η

$$(11) \quad \int_{\partial\Omega} |\nabla\psi|^{2p-2} (|\nabla\psi|^2)_\nu \leq \eta \int_{\Omega} |\nabla\psi|^{2p-4} |\nabla|\nabla\psi|^2|^2 + C_\eta \left(\int_{\Omega} |\nabla\psi|^2 \right)^p.$$

Proof. Regard the proof of inequalities (8) and (9), we refer the reader to [16, Lemma 3.1]. As to (10), this is a special case of [11, Lemma 2.2], and relation (11) is derived in [11, Lemma 2.1 c)]. \square

Lemma 3.3. *Let $\Omega \subset \mathbb{R}^n$, $n \geq 1$, be a bounded and smooth domain and $\delta > 0$. Then for any nonnegative $g \in C^1(\bar{\Omega})$, the solution $0 \leq \psi \in C^{2,\kappa}(\bar{\Omega})$, $0 < \kappa < 1$, of the problem*

$$\begin{cases} 0 = \Delta\psi + g - \delta\psi & \text{in } \Omega, \\ \psi_\nu = 0 & \text{on } \partial\Omega, \end{cases}$$

has the following property: For any $\hat{c}, \sigma > 0$ and $\bar{p} \in (1, \infty)$, there exists $\tilde{c} = \tilde{c}(\sigma, \bar{p}) > 0$ such that

$$(12) \quad \hat{c} \int_{\Omega} \psi^{\bar{p}+1} \leq \sigma \int_{\Omega} g^{\bar{p}+1} + \frac{\tilde{c}}{|\Omega|^{\bar{p}}} \left(\int_{\Omega} g \right)^{\bar{p}+1}.$$

Proof. A detailed proof of (12) can be found in [23, Lemma 3.1]. (See also [29, Lemma 2.2].) \square

4. EXISTENCE OF LOCAL-IN-TIME CLASSICAL SOLUTIONS. FROM UNIFORM BOUNDEDNESS IN $L^p(\Omega)$ TO $L^\infty(\Omega)$.

Let us dedicate ourselves to the existence question of classical solutions to system (1). It is shown that such solutions are at least local and, additionally, satisfy some crucial estimates.

Lemma 4.1 (Local existence). *Let Ω be a bounded and smooth domain of \mathbb{R}^n , with $n \geq 1$, $q > n$, $\chi, \delta > 0$ and nontrivial $(u_0, v_0) \in C^0(\bar{\Omega}) \times C^1(\bar{\Omega})$, with $u_0 \geq 0$ and $v_0 \geq 0$ on $\bar{\Omega}$. Assume, moreover, that for some $\gamma, K > 0$, f and g fulfill (5), respectively with $\alpha \in (0, \frac{1}{2} + \frac{1}{n}) \cap (0, 1)$ and $l \geq 1$. Then, for any $\xi > 0$ there exist $T_{max} \in (0, \infty]$ and a unique triplet of nonnegative functions $(u, v, w) \in (C^0(\bar{\Omega} \times [0, T_{max})) \cap C^{2,1}(\bar{\Omega} \times (0, T_{max}))^3$, such that this dichotomy criterion holds true:*

$$(13) \quad \text{either } T_{max} = \infty \text{ or } \limsup_{t \rightarrow T_{max}} (\|u(\cdot, t)\|_{L^\infty(\Omega)} + \|v(\cdot, t)\|_{W^{1,q}(\Omega)}) = \infty.$$

In addition, the u -component obeys the mass conservation property, i.e.

$$(14) \quad \int_{\Omega} u(x, t) dx = \int_{\Omega} u_0(x) dx = m > 0 \quad \text{for all } t \in (0, T_{max}),$$

whilst for some $c_0 > 0$ the v -component is such that

$$(15) \quad 0 \leq v \leq \|v_0\|_{L^\infty(\Omega)} \quad \text{in } \Omega \times (0, T_{max}) \quad \text{and} \quad \int_{\Omega} |\nabla v(\cdot, t)|^2 \leq c_0 \quad \text{on } (0, T_{max}).$$

Proof. The local solvability as well as the dichotomy criterion (13) can be proved by adapting well-established approaches widely used in the frame of classical chemotaxis models (see for instance [2, Lemma 1.2], [5, Theorem 3.1] and [21, Lemma 3.1]). Moreover, comparison arguments apply to yield $u, v, w \geq 0$ in $\Omega \times (0, T_{max})$ and the first relation in (15), whereas the mass conservation property follows by integrating over Ω the first equation of (1), in conjunction with the boundary and initial conditions.

Let us, finally, derive the last claim as follows. We separate the cases $0 < \alpha \leq \frac{1}{2}$ and $\frac{1}{2} < \alpha < \min\{\frac{1}{2} + \frac{1}{n}, 1\}$. For $0 < \alpha \leq \frac{1}{2}$, from the second equation of (1), we have that an integration over Ω , the Young inequality, the bound for v given in (15) and the constraints of f in (5) lead to

$$(16) \quad \begin{aligned} \frac{d}{dt} \int_{\Omega} |\nabla v|^2 &= 2 \int_{\Omega} \nabla v \cdot \nabla(\Delta v - f(u)v) = -2 \int_{\Omega} (\Delta v)^2 + 2 \int_{\Omega} f(u)v\Delta v \\ &= -2 \int_{\Omega} (\Delta v)^2 + 2 \int_{\Omega} v(f(u) - 1)\Delta v - 2 \int_{\Omega} |\nabla v|^2 \leq - \int_{\Omega} (\Delta v)^2 - 2 \int_{\Omega} |\nabla v|^2 + \int_{\Omega} v^2(f(u) - 1)^2 \\ &\leq -2 \int_{\Omega} |\nabla v|^2 + \|v_0\|_{L^\infty(\Omega)}^2 K^2 \int_{\Omega} u^{2\alpha} + 2\|v_0\|_{L^\infty(\Omega)}^2 K \int_{\Omega} u^\alpha + \|v_0\|_{L^\infty(\Omega)}^2 |\Omega| \quad \text{on } (0, T_{max}). \end{aligned}$$

Now, since $L^1(\Omega) \subseteq L^{2\alpha}(\Omega) \subseteq L^\alpha(\Omega)$, thanks to the mass conservation property (14) we can find $c_1 > 0$ such that

$$\|v_0\|_{L^\infty(\Omega)}^2 K^2 \int_{\Omega} u^{2\alpha} + 2\|v_0\|_{L^\infty(\Omega)}^2 K \int_{\Omega} u^\alpha + \|v_0\|_{L^\infty(\Omega)}^2 |\Omega| \leq c_1 \quad \text{with } t \in (0, T_{max}),$$

so that (16) reads

$$\frac{d}{dt} \int_{\Omega} |\nabla v|^2 \leq -2 \int_{\Omega} |\nabla v|^2 + c_1 \quad \text{on } (0, T_{max}),$$

and a comparison argument entails $\int_{\Omega} |\nabla v|^2 \leq \max\{\frac{c_1}{2}, \int_{\Omega} |\nabla v_0|^2\}$ for all $t \in (0, T_{max})$.

When, indeed, $\frac{1}{2} < \alpha < \min\{\frac{1}{2} + \frac{1}{n}, 1\}$, we can pick $\frac{1}{2} < \rho < 1 - \frac{n}{2}(\alpha - \frac{1}{2})$ and set $\zeta = 1 - \rho - \frac{n}{2}(\alpha - \frac{1}{2}) > 0$. Moreover, through the Hölder inequality, taking in mind (5) and again (14), we have

$$(17) \quad \|f(u(\cdot, t))\|_{L^{\frac{1}{\alpha}}(\Omega)}^{\frac{1}{\alpha}} = \int_{\Omega} f(u)^{\frac{1}{\alpha}} \leq K^{\frac{1}{\alpha}} \int_{\Omega} u \leq K^{\frac{1}{\alpha}} m \quad \text{for all } t < T_{max}.$$

As a consequence, from the representation formula for v , we have

$$v(\cdot, t) = e^{t\Delta} v_0 - \int_0^t e^{(t-s)\Delta} f(u(\cdot, s)) v(\cdot, s) ds \quad \text{for all } t \in (0, T_{max}),$$

and aided by smoothing properties related to the Neumann heat semigroup $(e^{t\Delta})_{t \geq 0}$ (see Section 2 of [5] and Lemma 1.3 of [25]), we obtain for some $\lambda_1 > 0$, $C_S > 0$ and $c_2 > 0$, once bounds $v \leq \|v_0\|_{L^\infty(\Omega)}$ on $\bar{\Omega} \times (0, T_{max})$ and (17) are considered,

$$\begin{aligned} \|v(\cdot, t)\|_{W^{1,2}(\Omega)} &\leq \|e^{t\Delta} v_0\|_{W^{1,2}(\Omega)} + \int_0^t \|e^{(t-s)\Delta} f(u(\cdot, s)) v(\cdot, s)\|_{W^{1,2}(\Omega)} ds \\ &\leq C_S \|v_0\|_{W^{1,2}(\Omega)} + C_S \int_0^t \|(-\Delta + 1)^\rho e^{(t-s)\Delta} f(u(\cdot, s)) v(\cdot, s)\|_{L^2(\Omega)} ds \\ &\leq C_S \|v_0\|_{W^{1,2}(\Omega)} + C_S \|v_0\|_{L^\infty(\Omega)} |\Omega|^{\frac{1}{2}} \int_0^t (t-s)^{-\rho - \frac{n}{2}(\alpha - \frac{1}{2})} e^{-\lambda_1(t-s)} \|f(u(\cdot, s))\|_{L^{\frac{1}{\alpha}}(\Omega)} ds \\ &\leq c_2 \left(1 + \int_0^t (t-s)^{-\rho - \frac{n}{2}(\alpha - \frac{1}{2})} e^{-\lambda_1(t-s)} ds\right). \end{aligned}$$

By recalling the above position on ζ , we introduce the Gamma function Γ inferring $\int_0^t (t-s)^{-\rho - \frac{n}{2}(\alpha - \frac{1}{2})} e^{-\lambda_1(t-s)} ds \leq \lambda_1^{-\zeta} \Gamma(\zeta)$, so obtain the second bound in (15) with $c_0 = \max\{\frac{c_1}{2}, \int_{\Omega} |\nabla v_0|^2, c_2^2(1 + \lambda_1^{-\zeta} \Gamma(\zeta))^2\}$. \square

Remark 2. We observe that the second estimate in (15) is crucial when relation (11) and a proper version of the Gagliardo–Nirenberg inequality will be below invoked. The case $\alpha = 1$, for which this bound does not hold true, does not fit into the computations of our general machinery, and henceforth it is herein excluded.

In view of the forthcoming lemma, in order to ensure the uniform-in-time L^∞ bound of (u, v, w) , it will be sufficient in the sequel controlling the uniform-in-time L^p -norm of u , for some suitable $p > 1$.

Lemma 4.2. Under the hypotheses of Lemma 4.1 and any $\xi > 0$, let (u, v, w) be the local-in-time classical solution to problem (1). If for some $p > \max\{1, \frac{n}{2}\}$ the u -component and g belong to $L^\infty((0, T_{max}); L^p(\Omega))$, then (u, v, w) is global in time, i.e. $T_{max} = \infty$, and moreover u, v and w are uniformly bounded in $\Omega \times (0, \infty)$ (in the sense of Remark 1).

Proof. W.l.o.g., we assume $p > 1$, for $n = 1$, and $\frac{n}{2} < p < n$, for $n \geq 2$. In this way, classical regularity theory on elliptic equations in conjunction with Sobolev embedding theorems infer through the third equation of (1) that

$$w \in L^\infty((0, T_{max}); W^{2,p}(\Omega)) \text{ and } \nabla w \in L^\infty((0, T_{max}); W^{1,p}(\Omega)),$$

and so for all $2 \leq n < q < p^* := \frac{np}{n-p}$, and $q = \infty$ for $n = 1$,

$$(18) \quad w \in L^\infty((0, T_{max}); C^{[2-(n/p)]}(\bar{\Omega})) \text{ and } \nabla w \in L^\infty((0, T_{max}); L^q(\Omega)).$$

On the other hand, the hypotheses on f are such that if $u \in L^\infty((0, T_{max}); L^p(\Omega))$ also $f \in L^\infty((0, T_{max}); L^p(\Omega))$. Henceforth, we again use the variation-of-constants formula for v as to obtain, taking into account the first bound in (15), some proper $\tilde{C}_S, c_3 > 0$ producing for $\lambda_1 > 0$ as in Lemma 4.1

$$\begin{aligned} \|\nabla v(\cdot, t)\|_{L^q(\Omega)} &\leq \|\nabla e^{t\Delta} v_0\|_{L^q(\Omega)} + \int_0^t \|\nabla e^{(t-s)\Delta} f(u(\cdot, s)) v(\cdot, s)\|_{L^q(\Omega)} ds \\ &\leq \tilde{C}_S \|\nabla v_0\|_{L^q(\Omega)} + \tilde{C}_S \|v_0\|_{L^\infty(\Omega)} |\Omega|^{\frac{1}{q}} \int_0^t (1 + (t-s)^{-\frac{1}{2} - \frac{n}{2}(\frac{1}{p} - \frac{1}{q})}) e^{-\lambda_1(t-s)} \|f(u(\cdot, s))\|_{L^p(\Omega)} ds \\ &\leq c_3 \left(1 + \int_0^t (t-s)^{-\frac{1}{2} - \frac{n}{2}(\frac{1}{p} - \frac{1}{q})} e^{-\lambda_1(t-s)} ds\right). \end{aligned}$$

Further, the assumptions on q ensures that $-\frac{1}{2} - \frac{n}{2}(\frac{1}{p} - \frac{1}{q}) > -1$, so as before $\int_0^t (t-s)^{-\frac{1}{2} - \frac{n}{2}(\frac{1}{p} - \frac{1}{q})} e^{-\lambda_1(t-s)} ds$ is finite and we also get $v \in L^\infty((0, T_{max}); W^{1,q}(\Omega))$. From this inclusion and (18), since $W^{1,q}(\Omega) \hookrightarrow L^\infty(\Omega)$ for $q > n$, we immediately have that $v, w \in L^\infty((0, T_{max}); L^\infty(\Omega))$, and moreover for $\tilde{v} = \chi v - \xi w$ some positive constant C_q can be found so to get

$$(19) \quad \|\tilde{v}(\cdot, t)\|_{L^q(\Omega)} + \|\nabla \tilde{v}(\cdot, t)\|_{L^q(\Omega)} \leq C_q \quad \text{for all } t \in (0, T_{max}).$$

Subsequently, for any $(x, t) \in \Omega \times (0, T_{max})$, the first equation of (1) reads $u_t = \Delta u - \nabla \cdot (u \nabla \tilde{v})$ and for $t_0 := \max\{0, t-1\}$ we have

$$u(\cdot, t) \leq e^{(t-t_0)\Delta} u(\cdot, t_0) - \int_{t_0}^t e^{(t-s)\Delta} \nabla \cdot (u(\cdot, s) \nabla \tilde{v}(\cdot, s)) ds =: u_1(\cdot, t) + u_2(\cdot, t).$$

As to the conclusion $u \in L^\infty((0, T_{max}); L^\infty(\Omega))$, this is an adaptation of [1, Lemma 3.2], and we herewith omit it; more precisely (see also [24, Lemma 4.1]), the $L^\infty(\Omega)$ -norm of u on $(0, T_{max})$ is achieved by controlling (also with the support of $u \in L^\infty((0, T_{max}); L^p(\Omega))$, for $\frac{n}{2} < p < n$ only, (19) and (14)) a suitable norm of the cross-diffusion term $u \nabla \tilde{v}$. Finally, $u \in L^\infty((0, T_{max}); L^\infty(\Omega))$ and $v \in L^\infty((0, T_{max}); W^{1,q}(\Omega))$ imply from the dichotomy criterion (13) that necessarily we must have $T_{max} = \infty$, so that actually $u, v, w \in L^\infty((0, \infty); L^\infty(\Omega))$. \square

5. A PRIORI ESTIMATES AND PROOF OF THE THEOREMS

In this section we control the L^p -norm, $p > 1$, by establishing an absorptive differential inequality for the functional $y(t) := \int_\Omega u^p + (\frac{\chi}{\gamma})^p \int_\Omega |\nabla v|^{2p}$.

Lemma 5.1. *Let $n \geq 1, l \geq 1$ and the hypotheses of Lemma 4.1 be satisfied. Then for every $\xi > 0$ the local solution (u, v, w) to problem (1) is such that for any $p \in (\max\{l, \frac{l(nl-2)}{n}\}, \infty)$ and all $t \in (0, T_{max})$ one has:*

- For $l = 1$ and some $c_4 > 0$

$$\frac{d}{dt} \int_\Omega u^p \leq -\frac{2(p-1)}{p} \int_\Omega |\nabla u^{\frac{p}{2}}|^2 + \frac{\chi^2 p(p-1)}{2(p+1)} \left(\frac{\xi \gamma (p+1)}{2p^2 \chi^2} \right)^{-p} \int_\Omega |\nabla v|^{2(p+1)} - \frac{\xi \gamma (p-1)}{4} \int_\Omega u^{p+1} + c_4;$$

- For $l > 1$, every $\epsilon_1, \epsilon_2 > 0$ and some $c_5 > 0$

$$\frac{d}{dt} \int_\Omega u^p \leq \left[-\frac{2(p-1)}{p} + \epsilon_1 \right] \int_\Omega |\nabla u^{\frac{p}{2}}|^2 + \epsilon_2 \int_\Omega |\nabla v|^{2(p+1)} - \frac{\xi \gamma (p-1)}{4} \int_\Omega u^{p+l} + c_5.$$

Proof. Testing the first equation of problem (1) by u^{p-1} , using its boundary conditions and recalling the assumptions of g in (5), provide on $(0, T_{max})$

$$(20) \quad \begin{aligned} \frac{1}{p} \frac{d}{dt} \int_\Omega u^p &= \int_\Omega u^{p-1} u_t = -(p-1) \int_\Omega u^{p-2} |\nabla u|^2 + (p-1) \chi \int_\Omega u^{p-1} \nabla u \cdot \nabla v - \xi (p-1) \int_\Omega u^{p-1} \nabla u \cdot \nabla w \\ &\leq -(p-1) \int_\Omega u^{p-2} |\nabla u|^2 + (p-1) \chi \int_\Omega u^{p-1} \nabla u \cdot \nabla v + \frac{\xi \delta (p-1)}{p} \int_\Omega u^p w - \frac{\xi \gamma (p-1)}{p} \int_\Omega u^{p+l}, \end{aligned}$$

whereas Young's inequality infers

$$(21) \quad (p-1) \chi \int_\Omega u^{p-1} \nabla u \cdot \nabla v \leq \frac{(p-1)}{2} \int_\Omega u^{p-2} |\nabla u|^2 + \frac{\chi^2 (p-1)}{2} \int_\Omega u^p |\nabla v|^2 \quad \text{on } (0, T_{max}).$$

Now, let us analyze separately the two cases.

- Case $l = 1$. The Young inequality entails

$$(22) \quad \frac{\chi^2 (p-1)}{2} \int_\Omega u^p |\nabla v|^2 \leq \frac{\xi \gamma (p-1)}{4p} \int_\Omega u^{p+1} + \frac{\chi^2 (p-1)}{2(p+1)} \left(\frac{\xi \gamma (p+1)}{2p^2 \chi^2} \right)^{-p} \int_\Omega |\nabla v|^{2(p+1)} \quad \text{for all } t \in (0, T_{max}).$$

On the other hand, since $l = 1$, an integration over Ω of the third equation of (1), together with the mass conservation property (14), give $\int_\Omega w = \frac{m\gamma}{\delta}$ on $(0, T_{max})$. In this way, by exploiting Young's inequality, again, and relation (12) with $\psi = w, g(u) = \gamma u$ and $\bar{p} = p$, proper positive constants \hat{c}, c_6 imply

$$(23) \quad \frac{\xi \delta (p-1)}{p} \int_\Omega u^p w \leq \frac{\xi \gamma (p-1)}{4p} \int_\Omega u^{p+1} + \hat{c} \int_\Omega w^{p+1} \leq \frac{\xi \gamma (p-1)}{4p} \int_\Omega u^{p+1} + \frac{\xi \gamma (p-1)}{4p} \int_\Omega u^{p+1} + c_6 \quad \text{on } (0, T_{max}).$$

By plugging estimates (21), (22) and (23) into bound (20), and for $c_4 = pc_6$, we directly obtain the claim in view of the identity

$$\int_\Omega u^{p-2} |\nabla u|^2 = \frac{4}{p^2} \int_\Omega |\nabla u^{\frac{p}{2}}|^2 \quad \text{on } (0, T_{max}).$$

- Case $l > 1$. Let us first estimate the term $(\int_\Omega u^l)^{\frac{p+l}{l}}$: by applying the Gagliardo–Nirenberg inequality (see [18]) combined with (7), for any $\bar{c} > 0$ we can introduce a suitable constant $c_7 > 0$ and obtain

$$\bar{c} \left(\int_\Omega u^l \right)^{\frac{p+l}{l}} = \bar{c} \|u^{\frac{p}{2}}\|_{L^{\frac{2l}{p}}(\Omega)}^{\frac{2(p+l)}{p}} \leq c_7 \|\nabla u^{\frac{p}{2}}\|_{L^2(\Omega)}^{\frac{2(p+l)}{p} \theta_1} \|u^{\frac{p}{2}}\|_{L^{\frac{2}{p}}(\Omega)}^{\frac{2(p+l)}{p} (1-\theta_1)} + c_7 \|u^{\frac{p}{2}}\|_{L^{\frac{2}{p}}(\Omega)}^{\frac{2(p+l)}{p}} \quad \text{for all } t \in (0, T_{max}),$$

where for p as in our assumptions we have

$$0 < \theta_1 = \frac{1 - \frac{1}{l}}{1 + \frac{2}{np} - \frac{1}{p}} < 1 \quad \text{and} \quad 0 < \frac{(p+l)}{p} \theta_1 < 1.$$

Hence, by recalling the mass conservation property (14), the Young and above inequalities entail for $c_8, c_9 > 0$ and any $\epsilon_1 > 0$

$$(24) \quad \bar{c} \left(\int_{\Omega} u^l \right)^{\frac{p+l}{l}} \leq c_8 \left(\int_{\Omega} |\nabla u^{\frac{p}{2}}|^2 \right)^{\frac{(p+l)}{p} \theta_1} + c_8 \leq \frac{\epsilon_1}{p} \int_{\Omega} |\nabla u^{\frac{p}{2}}|^2 + c_9 \quad \text{with } t \in (0, T_{max}).$$

On the other hand, by noting that $\frac{2(p+l)}{l} < 2(p+1)$ for $l > 1$, a double application of the Young inequality leads for all $t \in (0, T_{max})$, $\epsilon_2 > 0$ and some $c_{10}, c_{11} > 0$ to

$$(25) \quad \frac{\chi^2(p-1)}{2} \int_{\Omega} u^p |\nabla v|^2 \leq \frac{\xi \gamma (p-1)}{4p} \int_{\Omega} u^{p+l} + c_{10} \int_{\Omega} |\nabla v|^2 \frac{p+l}{l} \leq \frac{\xi \gamma (p-1)}{4p} \int_{\Omega} u^{p+l} + \frac{\epsilon_2}{p} \int_{\Omega} |\nabla v|^{2(p+1)} + c_{11}.$$

Now by applying restrictions in (5) and relation (7), as well as the obvious inequality $u \leq u+1$, we have these estimates for any $\bar{p} > 1$ and some $c_{12}, c_{13}, c_{14}, c_{15} > 0$:

$$\int_{\Omega} (g(u))^{\bar{p}+1} \leq \int_{\Omega} (\gamma u (u+1)^{l-1})^{\bar{p}+1} \leq c_{12} \int_{\Omega} u^{l(\bar{p}+1)} + c_{13} \quad \text{for all } t \in (0, T_{max}),$$

and also

$$\left(\int_{\Omega} g(u) \right)^{\bar{p}+1} \leq c_{14} \left(\int_{\Omega} u^l \right)^{\bar{p}+1} + c_{15} \quad \text{for all } t \in (0, T_{max}),$$

Aided by the gained estimates, we now use a combination of Young's inequality and relation (12) with $\psi = w$ and $\bar{p} = \frac{p}{l} > 1$; we have for some $c_{16} > 0$

$$(26) \quad \begin{aligned} \frac{\xi \delta (p-1)}{p} \int_{\Omega} u^p w &\leq \frac{\xi \gamma (p-1)}{4p} \int_{\Omega} u^{p+l} + \frac{\hat{c}}{c_{12}} \int_{\Omega} w^{\bar{p}+1} \\ &\leq \frac{\xi \gamma (p-1)}{4p} \int_{\Omega} u^{p+l} + \sigma \int_{\Omega} u^{p+l} + \bar{c} \left(\int_{\Omega} u^l \right)^{\frac{p+l}{l}} + c_{16} \quad \text{on } (0, T_{max}). \end{aligned}$$

By collecting (21), (25) and (26), with $\sigma = \frac{\xi \gamma (p-1)}{4p}$, bound (20) gives the conclusion also in view of relation (24). \square

In the forthcoming lemma we adapt to our framework some derivations already developed in [11, Lemma 4.2].

Lemma 5.2. *Let $n \geq 1$, $l \geq 1$ and the hypotheses of Lemma 4.1 be satisfied. Then for every $\xi > 0$ the local solution (u, v, w) to problem (1) is such for any $p \in (1, \infty)$ and all $t \in (0, T_{max})$ one has:*

- For $l = 1$ and some $c_{17} > 0$

$$\begin{aligned} \left(\frac{\chi^2}{\gamma} \right)^p \frac{d}{dt} \int_{\Omega} |\nabla v|^{2p} + \left(\frac{\chi^2}{\gamma} \right)^p p \int_{\Omega} |\nabla v|^{2p-2} |D^2 v|^2 &\leq \frac{\xi \gamma (p-1)}{4} \int_{\Omega} u^{p+1} \\ &\quad + \frac{p}{8(4p^2 + n) \|v\|_{L^\infty(\Omega)}^2} \left(\frac{\chi^2}{\gamma} \right)^p \int_{\Omega} |\nabla v|^{2(p+1)} + c_{17}; \end{aligned}$$

- For $l > 1$, every positive ϵ_3 and some $c_{18} > 0$

$$\left(\frac{\chi^2}{\gamma} \right)^p \frac{d}{dt} \int_{\Omega} |\nabla v|^{2p} + \left(\frac{\chi^2}{\gamma} \right)^p p \int_{\Omega} |\nabla v|^{2p-2} |D^2 v|^2 \leq \frac{\xi \gamma (p-1)}{4} \int_{\Omega} u^{p+l} + \epsilon_3 \int_{\Omega} |\nabla v|^{2(p+1)} + c_{18}.$$

Proof. From the second equation of (1), we derive this pointwise identity valid for all $x \in \Omega$ and $t \in (0, T_{max})$:

$$(|\nabla v|^2)_t = 2\nabla v \cdot \nabla v_t = 2\nabla v \cdot \nabla \Delta v - 2\nabla v \cdot \nabla (f(u)v) = \Delta |\nabla v|^2 - 2|D^2 v|^2 - 2\nabla v \cdot \nabla (f(u)v).$$

Successively, multiplying this last relation by $|\nabla v|^{2p-2}$ and integrating over Ω lead to

$$(27) \quad \begin{aligned} \left(\frac{\chi^2}{\gamma} \right)^p \frac{1}{p} \frac{d}{dt} \int_{\Omega} |\nabla v|^{2p} + (p-1) \left(\frac{\chi^2}{\gamma} \right)^p \int_{\Omega} |\nabla v|^{2p-4} |\nabla |\nabla v|^2|^2 + 2 \left(\frac{\chi^2}{\gamma} \right)^p \int_{\Omega} |\nabla v|^{2p-2} |D^2 v|^2 \\ = -2 \left(\frac{\chi^2}{\gamma} \right)^p \int_{\Omega} |\nabla v|^{2p-2} \nabla v \cdot \nabla (f(u)v) + \left(\frac{\chi^2}{\gamma} \right)^p \int_{\partial \Omega} (|\nabla v|^2)^{p-1} (|\nabla v|^2)_\nu \quad \text{for all } t \in (0, T_{max}). \end{aligned}$$

Now, by virtue of the L^2 -bound in (15), we apply estimate (11) with $\psi = v$ so to obtain for $c_{19} = C_\eta c_0^p$

$$(28) \quad \left(\frac{\chi^2}{\gamma} \right)^p \int_{\partial \Omega} (|\nabla v|^2)^{p-1} (|\nabla v|^2)_\nu \leq \eta \left(\frac{\chi^2}{\gamma} \right)^p \int_{\Omega} |\nabla v|^{2p-4} |\nabla |\nabla v|^2|^2 + c_{19} \quad \text{on } (0, T_{max}).$$

Hence, an integration by parts to the right hand side term of (27) produces, also thanks to the first estimate in (15) and assumption (5),

$$\begin{aligned}
(29) \quad & -2 \left(\frac{\chi^2}{\gamma} \right)^p \int_{\Omega} |\nabla v|^{2p-2} \nabla v \cdot \nabla (f(u)v) = 2 \left(\frac{\chi^2}{\gamma} \right)^p \int_{\Omega} f(u)v |\nabla v|^{2p-2} \Delta v \\
& + 2(p-1) \left(\frac{\chi^2}{\gamma} \right)^p \int_{\Omega} f(u)v |\nabla v|^{2p-4} \nabla v \cdot \nabla |\nabla v|^2 \\
& \leq 2K \left(\frac{\chi^2}{\gamma} \right)^p \|v_0\|_{L^\infty(\Omega)} \int_{\Omega} u^\alpha |\nabla v|^{2p-2} |\Delta v| \\
& + 2K \left(\frac{\chi^2}{\gamma} \right)^p (p-1) \|v_0\|_{L^\infty(\Omega)} \int_{\Omega} u^\alpha |\nabla v|^{2p-3} |\nabla |\nabla v|^2| \quad \text{for all } t \in (0, T_{max}).
\end{aligned}$$

In addition, the Young and (8) inequalities allow us to derive for some $c_{20}, c_{21} > 0$

$$(30) \quad 2K \left(\frac{\chi^2}{\gamma} \right)^p \|v_0\|_{L^\infty(\Omega)} \int_{\Omega} u^\alpha |\nabla v|^{2p-2} |\Delta v| \leq \left(\frac{\chi^2}{\gamma} \right)^p \int_{\Omega} |\nabla v|^{2p-2} |D^2 v|^2 + c_{20} \int_{\Omega} u^{2\alpha} |\nabla v|^{2p-2} \quad \text{on } (0, T_{max})$$

and similarly for all $t \in (0, T_{max})$ and any $\eta \in (0, p-1)$

$$(31) \quad 2K \left(\frac{\chi^2}{\gamma} \right)^p (p-1) \|v_0\|_{L^\infty(\Omega)} \int_{\Omega} u^\alpha |\nabla v|^{2p-3} |\nabla |\nabla v|^2| \leq (p-1-\eta) \left(\frac{\chi^2}{\gamma} \right)^p \int_{\Omega} |\nabla v|^{2p-4} |\nabla |\nabla v|^2|^2 + c_{21} \int_{\Omega} u^{2\alpha} |\nabla v|^{2p-2}.$$

By inserting (28), (29), (30) and (31) into (27), we deduce that for some $c_{22}, c_{23} > 0$

$$(32) \quad \left(\frac{\chi^2}{\gamma} \right)^p \frac{1}{p} \frac{d}{dt} \int_{\Omega} |\nabla v|^{2p} + \left(\frac{\chi^2}{\gamma} \right)^p \int_{\Omega} |\nabla v|^{2p-2} |D^2 v|^2 \leq c_{22} \int_{\Omega} u^{2\alpha} |\nabla v|^{2p-2} + c_{23} \quad \text{on } (0, T_{max}).$$

Further, for all $l \geq 1$, the Young inequality also gives on $(0, T_{max})$

$$(33) \quad c_{22} \int_{\Omega} u^{2\alpha} |\nabla v|^{2p-2} \leq \frac{\xi \gamma (p-1)}{4p} \int_{\Omega} u^{p+l} + c_{24} \int_{\Omega} |\nabla v|^{\frac{2(p-1)(p+l)}{p+l-2\alpha}} \leq \frac{\xi \gamma (p-1)}{4p} \int_{\Omega} u^{p+l} + \frac{\epsilon_3}{p} \int_{\Omega} |\nabla v|^{2(p+1)} + c_{25},$$

where we have used that $\frac{2(p-1)(p+l)}{p+l-2\alpha} < 2(p+1)$ (recall $0 < \alpha < 1$) and ϵ_3 is an arbitrarily positive constant and $c_{24}, c_{25} > 0$. We have the two claims introducing (33) into (32), with an evident choice of ϵ_3 when $l = 1$. \square

Lemma 5.3. *Let $n \geq 1, l = 1$ and the hypotheses of Lemma 4.1 be satisfied. Then, for any $p \in (1, \infty)$ there exists $\tilde{C}(p, n) > 0$ such that for all $\xi > 0$ fulfilling*

$$(34) \quad \xi > \left(\frac{4\tilde{C}(p, n) \|\chi v_0\|_{L^\infty(\Omega)}^2}{p} \right)^{\frac{1}{p}},$$

the following holds true: For some $L > 0$ the u -component of the local solution (u, v, w) to problem (1) complies with

$$\int_{\Omega} u^p \leq L \quad \text{for all } t \in (0, T_{max}).$$

Additionally, the same conclusion is valid whenever $n \geq 1, l > 1, \xi > 0$ and all $p \in (\max\{l, \frac{l(nl-2)}{n}\}, \infty)$.

Proof. When $l = 1$, Lemma 5.1, Lemma 5.2 and relation (10), supported by the uniform bound for v in (15), imply that

$$\begin{aligned}
(35) \quad & \frac{d}{dt} \left(\int_{\Omega} u^p + \left(\frac{\chi^2}{\gamma} \right)^p \int_{\Omega} |\nabla v|^{2p} \right) + \frac{2(p-1)}{p} \int_{\Omega} |\nabla u^{\frac{p}{2}}|^2 + p \left(\frac{\chi^2}{\gamma} \right)^p \int_{\Omega} |\nabla v|^{2p-2} |D^2 v|^2 \\
& \leq \left(\frac{p}{8(4p^2+n)\|v\|_{L^\infty(\Omega)}^2} \left(\frac{\chi^2}{\gamma} \right)^p + \frac{\chi^2 p(p-1)}{2(p+1)} \left(\frac{\xi \gamma (p+1)}{2p^2 \chi^2} \right)^{-p} \right) \int_{\Omega} |\nabla v|^{2(p+1)} + c_{26} \\
& \leq \left(\frac{p}{8(4p^2+n)\|v\|_{L^\infty(\Omega)}^2} \left(\frac{\chi^2}{\gamma} \right)^p + \frac{\chi^2 p(p-1)}{2(p+1)} \left(\frac{\xi \gamma (p+1)}{2p^2 \chi^2} \right)^{-p} \right) 2(4p^2+n) \|v_0\|_{L^\infty(\Omega)}^2 \int_{\Omega} |\nabla v|^{2p-2} |D^2 v|^2 + c_{26} \\
& = \left(\frac{\chi^2}{\gamma} \right)^p \left(\frac{p}{4} + \frac{\tilde{C}(p, n)}{\xi^p} \|\chi v_0\|_{L^\infty(\Omega)}^2 \right) \int_{\Omega} |\nabla v|^{2p-2} |D^2 v|^2 + c_{26} \quad \text{on } (0, T_{max}),
\end{aligned}$$

where $c_{26} > 0$ and where for clarity we have set

$$\tilde{C}(p, n) = \frac{2^p p^{2p+1} (p-1) (4p^2+n)}{(p+1)^{p+1}}.$$

Since by our assumptions ξ satisfies restriction (34), we get that $\frac{\tilde{C}(p,n)}{\xi^p} \|\chi v_0\|_{L^\infty(\Omega)}^2 < \frac{p}{4}$; henceforth relation (35) actually reads

$$\frac{d}{dt} \left(\int_{\Omega} u^p + \left(\frac{\chi^2}{\gamma} \right)^p \int_{\Omega} |\nabla v|^{2p} \right) + \frac{2(p-1)}{p} \int_{\Omega} |\nabla u^{\frac{p}{2}}|^2 + \frac{p}{2} \left(\frac{\chi^2}{\gamma} \right)^p \int_{\Omega} |\nabla v|^{2p-2} |D^2 v|^2 \leq c_{26} \quad \text{on } (0, T_{max}).$$

On the other hand, from inequality (9) we have

$$|\nabla |\nabla v|^p|^2 = \frac{p^2}{4} |\nabla v|^{2p-4} |\nabla |\nabla v|^2|^2 = p^2 |\nabla v|^{2p-4} |D^2 v \nabla v|^2 \leq p^2 |\nabla v|^{2p-2} |D^2 v|^2,$$

so that we obtain

$$(36) \quad y'(t) + \frac{2(p-1)}{p} \int_{\Omega} |\nabla u^{\frac{p}{2}}|^2 + \frac{1}{2p} \left(\frac{\chi^2}{\gamma} \right)^p \int_{\Omega} |\nabla |\nabla v|^p|^2 \leq c_{26} \quad \text{on } (0, T_{max}).$$

Conversely, for $l > 1$, by relying again on Lemma 5.1, Lemma 5.2, any $\epsilon > 0$ and some $c_{27} > 0$ entail

$$\begin{aligned} \frac{d}{dt} \left(\int_{\Omega} u^p + \left(\frac{\chi^2}{\gamma} \right)^p \int_{\Omega} |\nabla v|^{2p} \right) + \left[\frac{2(p-1)}{p} - \epsilon_1 \right] \int_{\Omega} |\nabla u^{\frac{p}{2}}|^2 + p \left(\frac{\chi^2}{\gamma} \right)^p \int_{\Omega} |\nabla v|^{2p-2} |D^2 v|^2 \leq \\ \frac{\epsilon p \left(\frac{\chi^2}{\gamma} \right)^p}{2(4p^2 + n) \|v_0\|_{L^\infty(\Omega)}^2} \int_{\Omega} |\nabla v|^{2(p+1)} + c_{27} \quad \text{for all } t \in (0, T_{max}), \end{aligned}$$

which similarly to what has been previously done, by choosing $\epsilon_1 \in \left(0, \frac{2(p-1)}{p}\right)$ and $\epsilon \in (0, 1)$, produces an absorptive inequality similar to (36); then, both can be unified for certain positive constants a, b, c as

$$(37) \quad y'(t) + a \int_{\Omega} |\nabla u^{\frac{p}{2}}|^2 + b \int_{\Omega} |\nabla |\nabla v|^p|^2 \leq c \quad \text{on } (0, T_{max}).$$

Successively, for any $l \geq 1$, by exploiting again the Gagliardo–Nirenberg inequality, there exists a positive constant c_{28} such that

$$\int_{\Omega} u^p = \|u^{\frac{p}{2}}\|_{L^2(\Omega)}^2 \leq c_{28} \|\nabla u^{\frac{p}{2}}\|_{L^2(\Omega)}^{2\theta} \|u^{\frac{p}{2}}\|_{L^{\frac{2}{p}}(\Omega)}^{2(1-\theta)} + c_{28} \|u^{\frac{p}{2}}\|_{L^{\frac{2}{p}}(\Omega)}^2 \quad \text{for all } t \in (0, T_{max}),$$

with

$$0 < \theta = \frac{\frac{np}{2}(1 - \frac{1}{p})}{1 - \frac{n}{2} + \frac{np}{2}} < 1.$$

Taking into consideration bound (14) and introducing $c_{29} > 0$, the two above inequalities lead to

$$(38) \quad \int_{\Omega} u^p \leq c_{29} \left(\int_{\Omega} |\nabla u^{\frac{p}{2}}|^2 \right)^{\theta} + c_{29} \quad \text{on } (0, T_{max}).$$

In a similar way, with another application of the Gagliardo–Nirenberg we arrive for some $c_{30} > 0$ at

$$\int_{\Omega} |\nabla v|^{2p} = \| |\nabla v|^p \|_{L^2(\Omega)}^2 \leq c_{30} \| |\nabla |\nabla v|^p | \|_{L^2(\Omega)}^{2\theta} \| |\nabla v|^p \|_{L^{\frac{2}{p}}(\Omega)}^{2(1-\theta)} + c_{30} \| |\nabla v|^p \|_{L^{\frac{2}{p}}(\Omega)}^2 \quad \text{with } t \in (0, T_{max}).$$

Moreover, by invoking the L^2 -bound for ∇v in (15), we have for $c_{31} > 0$

$$(39) \quad \int_{\Omega} |\nabla v|^{2p} \leq c_{31} \left(\int_{\Omega} |\nabla |\nabla v|^p|^2 \right)^{\theta} + c_{31} \quad \text{with } t \in (0, T_{max}).$$

As a consequence of all of the above, by manipulating inequalities (38) and (39) and thereof inserting the results into (37), we can observe also by virtue of (6) that y satisfies this initial problem

$$\begin{cases} y'(t) \leq c_{32} - c_{33} y^{\frac{1}{\theta}}(t) & \text{for all } t \in (0, T_{max}), \\ y(0) = \int_{\Omega} u_0^p + \left(\frac{\chi^2}{\gamma} \right)^p \int_{\Omega} |\nabla v_0|^{2p}, \end{cases}$$

with c_{32}, c_{33} positive constants. Consequently, an ODE comparison principle implies that $\int_{\Omega} u^p \leq y(t) \leq \max\{y(0), \left(\frac{c_{32}}{c_{33}}\right)^{\theta}\} := L$ for all $t \in (0, T_{max})$. \square

Now we have all the necessary tools to conclude.

Proof of Theorems 2.1 and 2.2. For $l = 1$, let $\tilde{C}(p, n)$ be the constant defined in Lemma 5.3 and let us set

$$C(n) = \begin{cases} 0 & \text{if } n \in \{1, 2\}, \\ \left(\frac{\xi}{n} \tilde{C}(n/2, n)\right)^{\frac{2}{n}} & \text{if } n \geq 3. \end{cases}$$

From our hypotheses, $\xi > C(n) \|\chi v_0\|_{L^\infty(\Omega)}^{\frac{4}{n}}$, so that from continuity arguments we can always pick $p > \max\{1, \frac{n}{2}\}$ such that assumption (34) holds true. Henceforth, Lemma 5.3 ensures that the u -component of the local solution (u, v, w) to problem (1) belongs to $L^\infty((0, T_{max}); L^p(\Omega))$; since $l = 1$ also $g \in L^\infty((0, T_{max}); L^p(\Omega))$ and the claim follows by invoking Lemma 4.2. Indeed, for any $l > 1$, upon enlarging p in the same Lemma 5.3, we also can have $u, g \in L^\infty((0, T_{max}); L^p(\Omega))$ for $p > \max\{1, \frac{n}{2}\}$, and identically conclude. \square

6. LOGISTIC SOURCE VS. CHEMOREPELLENT IN CHEMOTAXIS-CONSUMPTION MODELS: WHICH ONE IS MORE EFFECTIVE TOWARD BOUNDEDNESS?

We complement this research by discussing some differences and analogies between a chemotaxis-consumption model with a logistic source and that presented here with a chemorepellent (linearly produced). To be precise, when the equation for u in problem (1) is replaced by (4), and we set $f(u) = u$ in that for v , the chemotaxis-consumption model with logistic source (indicated with P_μ below) is obtained, and in [11] boundedness of solutions is established provided μ overcomes some expression of $\chi \|v_0\|_{L^\infty(\Omega)}$. Conversely, for our investigated attraction-repulsion model P_ξ (to facilitate the comparison, we also re-write it next to P_μ) an analogous largeness restriction is moved to the parameter ξ :

$$P_\mu : \begin{cases} u_t = \Delta u - \chi \nabla \cdot (u \nabla v) + ku - \mu u^2 \\ v_t = \Delta v - uv \end{cases} \quad \text{and} \quad P_\xi : \begin{cases} u_t = \Delta u - \chi \nabla \cdot (u \nabla v) + \xi \nabla \cdot (u \nabla w) \\ v_t = \Delta v - f(u)v \\ 0 = \Delta w - \delta w + \gamma u \end{cases}.$$

In particular, some straightforward computations, show that the condition in [11, Theorem 1.1] reads

$$\begin{aligned} \mu &> \frac{4^{1/n} (n-1)n}{n+1} \left(\frac{(n-1)(4n^2+n)}{n+1} \right)^{1/n} \|\chi v_0\|_{L^\infty(\Omega)}^{\frac{2}{n}} \\ &+ \frac{2^{\frac{n-1}{2}+n+1} (2n-1)}{n+1} \left(\frac{(n-1)(2n-1)(4n^2+n)}{n+1} \right)^{\frac{n-1}{2}} \|\chi v_0\|_{L^\infty(\Omega)}^{2n} =: C_\mu(\chi \|v_0\|_{L^\infty(\Omega)}), \end{aligned}$$

whereas in Theorem 2.1 the correlated assumption appears as

$$\xi > \left(2^{2-\frac{n}{2}} \left(\frac{n}{2} - 1 \right) \left(\frac{n}{2} + 1 \right)^{-\frac{n}{2}-1} n^n (n^2 + n) \right)^{2/n} \|\chi v_0\|_{L^\infty(\Omega)}^{\frac{2}{n}} =: C_\xi(\chi \|v_0\|_{L^\infty(\Omega)}).$$

Even though a very direct comparison between models P_μ and P_ξ is not strictly possible, from Figure 1 it can be observed that quantitatively $C_\mu(\chi \|v_0\|_{L^\infty(\Omega)}) > C_\xi(\chi \|v_0\|_{L^\infty(\Omega)})$, for any value of $\chi \|v_0\|_{L^\infty(\Omega)}$.

For the sake of scientific clarity, we would like to stress that the curve trend of the function C_μ may be improved; this is essentially due to the fact that in [11] the authors prove the deduction “ $L^p \Rightarrow L^\infty$ ” for $p > n$, and not for $p > \frac{n}{2}$, as we performed in Lemma 4.2. (As known, in this context, $\frac{n}{2}$ is the smallest value toward the validity of the above implication.) In this sense, by adjusting to this choice of p the expression of C_μ , the situation is different. More precisely, for sufficiently large values of $\chi \|v_0\|_{L^\infty(\Omega)}$, for which the analysis is more interesting, $C_\mu(\chi \|v_0\|_{L^\infty(\Omega)}) \gg C_\xi(\chi \|v_0\|_{L^\infty(\Omega)})$; on the other hand, the same does not happen when $\chi \|v_0\|_{L^\infty(\Omega)}$ is small. (See Figure 2.) As a consequence, if we consider that for high values of the cell concentration in problem P_ξ the chemoattractant is consumed with a weaker law than that in P_μ ($0 < \alpha < \frac{1}{2} + \frac{1}{n}$ vs. $\alpha = 1$, respectively), this discussion seems to indicate that the introduction in the classical Keller–Segel model with consumption (3) of a linearly produced chemorepellent, has a more effective stabilizing impact on the cells’ motility than the one resulting by the introduction of a quadratic-damping logistic source.

Acknowledgments. The authors are members of the Gruppo Nazionale per l’Analisi Matematica, la Probabilità e le loro Applicazioni (GNAMPA) of the Istituto Nazionale di Alta Matematica (INdAM). GV is partially supported by the research projects *Evolutionary and stationary Partial Differential Equations with a focus on biomathematics*, funded by Fondazione di Sardegna (2019), and by MIUR (Italian Ministry of Education, University and Research) Prin 2017 *Nonlinear Differential Problems via Variational, Topological and Set-valued Methods* (Grant Number: 2017AYM8XW).

REFERENCES

- [1] N. Bellomo, A. Bellouquid, Y. Tao, and M. Winkler. Toward a mathematical theory of Keller–Segel models of pattern formation in biological tissues. *Math. Models Methods Appl. Sci.*, 25(09):1663–1763, 2015.
- [2] T. Cieślak and M. Winkler. Finite-time blow-up in a quasilinear system of chemotaxis. *Nonlinearity*, 21(5):1057–1076, 2008.
- [3] Q. Guo, Z. Jiang, and S. Zheng. Critical mass for an attraction-repulsion chemotaxis system. *Appl. Anal.*, 97(13):2349–2354, 2018.
- [4] M. A. Herrero and J. J. L. Velázquez. A blow-up mechanism for a chemotaxis model. *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4)*, 24(4):633–683, 1997.
- [5] D. Horstmann and M. Winkler. Boundedness vs. blow-up in a chemotaxis system. *J. Differential Equations*, 215(1):52–107, 2005.

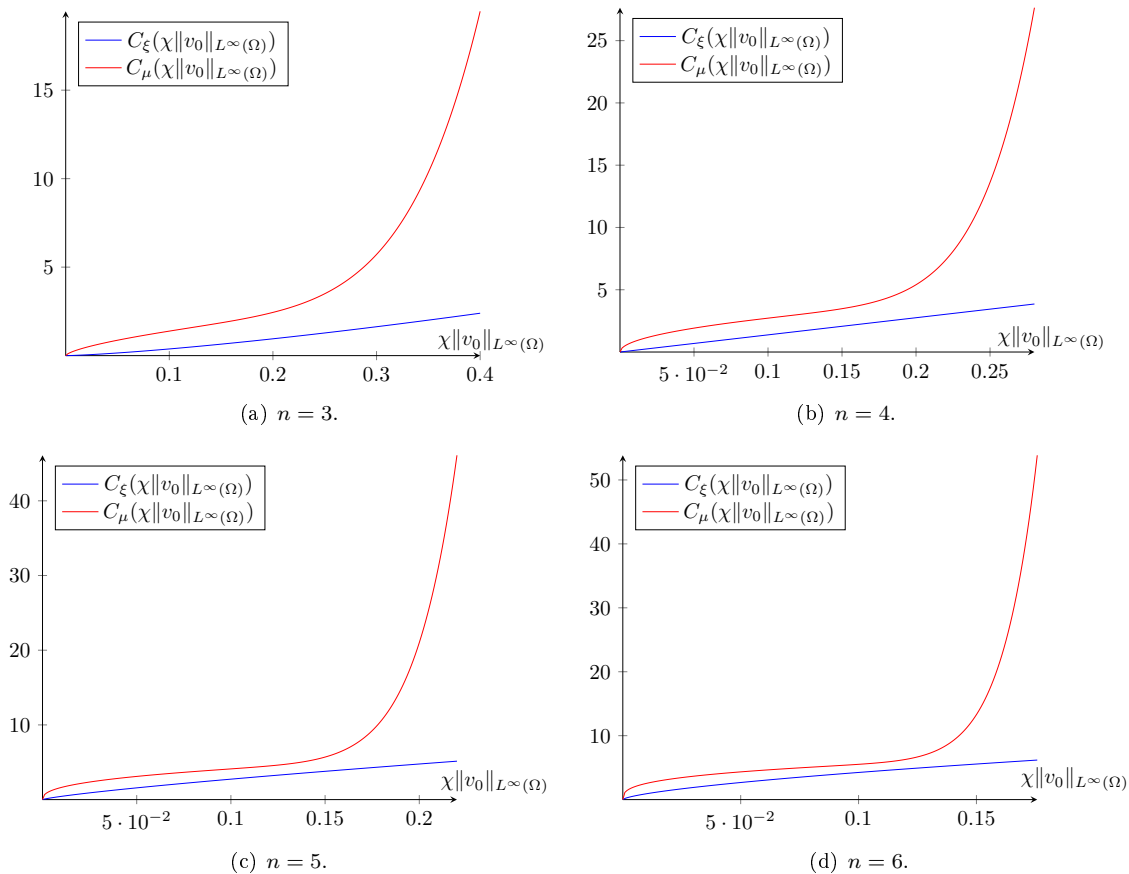


FIGURE 1. Illustration comparing for $n \in \{3, 4, 5, 6\}$ the graphs of the functions $C_\mu(\chi\|v_0\|_{L^\infty(\Omega)})$ and $C_\xi(\chi\|v_0\|_{L^\infty(\Omega)})$ computed in [11, Theorem 1.1] and in Theorem 2.1, respectively.

- [6] W. Jäger and S. Luckhaus. On explosions of solutions to a system of partial differential equations modelling chemotaxis. *Trans. Amer. Math. Soc.*, 329(2):819–824, 1992.
- [7] G. J. O. Jameson. Some inequalities for $(a + b)^p$ and $(a + b)^p + (a - b)^p$. *Math. Gaz.*, 98(541):96–103, 2014.
- [8] E. F. Keller and L. A. Segel. Initiation of slime mold aggregation viewed as an instability. *J. Theoret. Biol.*, 26(3):399–415, 1970.
- [9] E. F. Keller and L. A. Segel. Model for chemotaxis. *J. Theoret. Biol.*, 30(2):225–234, 1971.
- [10] E. F. Keller and L. A. Segel. Traveling bands of chemotactic bacteria: A theoretical analysis. *J. Theoret. Biol.*, 30(2):235, 1971.
- [11] J. Lankeit and Y. Wang. Global existence, boundedness and stabilization in a high-dimensional chemotaxis system with consumption. *Discrete Contin. Dyn. Syst.*, 37(12):6099–6121, 2017.
- [12] Y. Li and Y. Li. Blow-up of nonradial solutions to attraction-repulsion chemotaxis system in two dimensions. *Nonlinear Anal. Real World Appl.*, 30:170–183, 2016.
- [13] D.-m. Liu and Y.-s. Tao. Boundedness in a chemotaxis system with nonlinear signal production. *Appl. Math. J. Chinese Univ. Ser. B*, 31(4):379–388, 2016.
- [14] J. Liu. Stabilization in a two-dimensional attraction-repulsion Stokes system with consumption of chemoattractant. *Ann. Mat. Pura Appl.*, 1–32, 2020.
- [15] M. Luca, A. Chavez-Ross, L. Edelstein-Keshet, and A. Mogilner. Chemotactic signaling, microglia, and Alzheimer’s disease senile plaques: Is there a connection? *Bull. Math. Biol.*, 65(4):693–730, 2003.
- [16] M. Marras and G. Vigliano. Boundedness in a fully parabolic chemotaxis-consumption system with nonlinear diffusion and sensitivity, and logistic source. *Math. Nachr.*, 291(14–15):2318–2333, 2018.
- [17] T. Nagai. Blowup of nonradial solutions to parabolic-elliptic systems modeling chemotaxis in two-dimensional domains. *J. Inequal. Appl.*, 6(1):37–55, 2001.
- [18] L. Nirenberg. On elliptic partial differential equations. *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (3)*, 2(13):115–162, 1959.
- [19] K. Osaki and A. Yagi. Finite dimensional attractor for one-dimensional Keller–Segel equations. *Funkcial. Ekvacioj.*, 44(3):441–470, 2001.
- [20] Y. Tao. Boundedness in a chemotaxis model with oxygen consumption by bacteria. *J. Math. Anal. Appl.*, 381(2):521–529, 2011.
- [21] Y. Tao and Z.-A. Wang. Competing effects of attraction vs. repulsion in chemotaxis. *Math. Models Methods Appl. Sci.*, 23(1):1–36, 2013.
- [22] G. Vigliano. Explicit lower bound of blow-up time for an attraction-repulsion chemotaxis system. *J. Math. Anal. Appl.*, 479(1):1069–1077, 2019.
- [23] G. Vigliano. Influence of nonlinear production on the global solvability of an attraction-repulsion chemotaxis system. *Math. Nachr.*, doi: [10.1002/mana.201900465](https://doi.org/10.1002/mana.201900465), 2021.
- [24] G. Vigliano and T. E. Woolley. Solvability of a Keller–Segel system with signal-dependent sensitivity and essentially sublinear production. *Appl. Anal.*, 99(14):2507–2525, 2020.
- [25] M. Winkler. Aggregation vs. global diffusive behavior in the higher-dimensional Keller–Segel model. *J. Differential Equations*, 248(12):2889–2905, 2010.

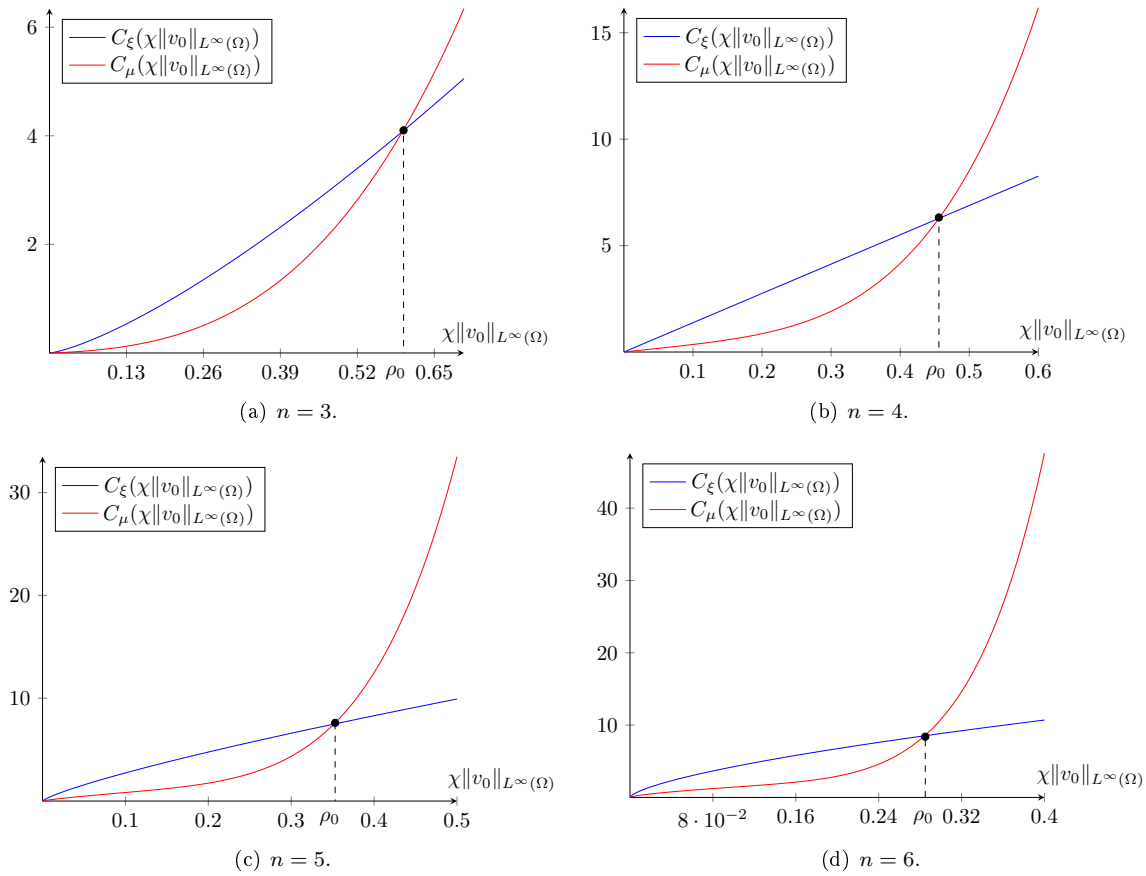


FIGURE 2. Illustration comparing for $n \in \{3, 4, 5, 6\}$ the graphs of the functions $C_\mu(\chi\|v_0\|_{L^\infty(\Omega)})$ and $C_\xi(\chi\|v_0\|_{L^\infty(\Omega)})$ computed in [11, Theorem 1.1] and in Theorem 2.1, both for $p = \frac{n}{2}$. The intersection point is such that the abscissa ρ_0 approaches 0 with n increasing. For values of $\chi\|v_0\|_{L^\infty(\Omega)}$ smaller than ρ_0 , C_ξ and C_μ are rather similar (quantitatively of the same order), whilst for larger values, C_μ is much bigger than C_ξ . This phenomenon is even more perceptible in higher dimensions.

- [26] M. Winkler. Blow-up in a higher-dimensional chemotaxis system despite logistic growth restriction. *J. Math. Anal. Appl.*, 384(2):261–272, 2011.
- [27] M. Winkler. Global large-data solutions in a chemotaxis-(Navier-)Stokes system modeling cellular swimming in fluid drops. *Comm. Partial Differential Equations*, 37(2):319–351, 2012.
- [28] M. Winkler. Stabilization in a two-dimensional chemotaxis-Navier-Stokes system. *Arch. Ration. Mech. Anal.*, 212(2):455–487, 2014.
- [29] M. Winkler. How far can chemotactic cross-diffusion enforce exceeding carrying capacities? *J. Nonlinear. Sci.*, 24(5):809–855, 2014.
- [30] M. Winkler. A critical blow-up exponent in a chemotaxis system with nonlinear signal production. *Nonlinearity*, 31(5):2031–2056, 2018.
- [31] M. Winkler. Finite-time blow-up in low-dimensional Keller-Segel systems with logistic-type superlinear degradation. *Z. Angew. Math. Phys.*, 69(40):1–25, 2018.
- [32] H. Yu, Q. Guo, and S. Zheng. Finite time blow-up of nonradial solutions in an attraction-repulsion chemotaxis system. *Nonlinear Anal. Real World Appl.*, 34:335–342, 2017.