



Triviality results for weighted porous medium heat equations on Riemannian manifolds

Fatma Gamze Düzgün¹ 

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Abstract

We study a weighted porous medium heat equation on complete Riemannian manifolds with nonnegative Ricci curvature. Our aim is to establish Liouville-type nonexistence for nonnegative solutions. We treat both the slow diffusion regime $m > 1$ and the fast diffusion regime $0 < m < 1$. Classical solutions are considered in the slow diffusion regime, whereas the fast diffusion result is formulated for weak solutions. Under suitable integral growth assumptions on the weight function, formulated according to the natural space-time scales of the corresponding diffusion regimes, we establish Liouville-type nonexistence results. In the slow diffusion case, the assumptions are expressed in terms of the intrinsic time scale $T_R = R^{2+\alpha(m-1)}$, whereas in the fast diffusion case the standard parabolic time scale $T_R = R^2$ is used. The resulting criteria extend several known triviality results for semilinear heat equations to the porous medium setting on Riemannian manifolds.

Keywords Weighted porous medium equation · Liouville-type nonexistence · Riemannian manifolds · Slow diffusion · Fast diffusion

Mathematics Subject Classification 35B53 · 35K55 · 35K65 · 35K67 · 58J35

1 Introduction

We study the nonexistence of nontrivial nonnegative solutions to the Cauchy problem

$$\begin{cases} u_t - \Delta(u^m) = V(x, t)u^p & \text{in } M \times (0, +\infty), \\ u(\cdot, 0) = u_0 \geq 0 & \text{in } M \end{cases}, \quad (1.1)$$

where (M, g) is a smooth, connected, complete Riemannian manifold of dimension n , Δ is the Laplace–Beltrami operator on (M, g) , $m > 0$, $p > 1$, and V is a prescribed

✉ Fatma Gamze Düzgün
fatmagamze.duzgun@unica.it

¹ Dipartimento di Matematica e Informatica, Università degli Studi di Cagliari, via Ospedale 72 – Palazzo delle Scienze, 09124 Cagliari, Italy

positive function. The precise notions of classical and weak solution used in the slow and fast diffusion regimes, respectively, are given in Sect. 2. The weight function is assumed to satisfy

$$V \in C^0(M \times (0, +\infty)), \quad V > 0.$$

The aim of this paper is to derive Liouville-type nonexistence results for problem (1.1) on complete Riemannian manifolds with nonnegative Ricci curvature. We investigate separately the slow diffusion regime $m > 1$ and the fast diffusion regime $0 < m < 1$, establishing conditions on the weight function V that guarantee the triviality of nonnegative solutions. In the slow diffusion case we work with classical solutions, while in the fast diffusion case we use a weak formulation that remains meaningful on the zero set of the solution. The obtained results identify parameter ranges ensuring the triviality of nonnegative solutions and extend several known nonexistence results from the semilinear heat equation framework to weighted porous medium equations on manifolds.

Nonexistence and Liouville-type results for nonlinear parabolic equations play an important role in the qualitative analysis of evolution problems. Such results are closely related to Fujita-type phenomena and provide information on the interplay between diffusion, reaction terms and the geometry of the underlying space. In the Euclidean setting, a vast literature has been devoted to semilinear and quasilinear equations, including porous medium-type models. The porous medium equation and its variants arise in several physical and biological applications, and their mathematical theory has been extensively developed; we refer to the classical work of Aronson [1] and to the monograph of Vázquez [9] for comprehensive accounts of the subject. Questions concerning critical exponents, global existence and nonexistence of solutions for nonlinear diffusion equations have also attracted considerable attention; see, for instance, Fujita [3], Galaktionov [4], Galaktionov and Levine [5], Mitidieri and Pohozaev [6], Pohozaev and Tesei [8], and the references therein.

The influence of the geometry of the ambient space on these phenomena has been extensively investigated in the framework of complete Riemannian manifolds. In particular, several Fujita-type and Liouville-type theorems have been obtained under curvature and volume growth assumptions, highlighting the role played by the geometry in determining critical exponents and global qualitative properties of solutions. Among these contributions, Zhang [10, 11] established nonexistence results for semilinear heat equations on complete manifolds under suitable assumptions on the geometry and the weight function.

A particularly relevant contribution for the present work is due to Castorina, Mantegazza and Sciunzi [2], who considered the weighted semilinear heat equation

$$u_t = \Delta u + |u|^p V$$

on complete Riemannian manifolds with nonnegative Ricci curvature. Under a suitable weighted volume growth assumption on the positive weight V , they established a Liouville-type theorem for solutions bounded from below and having nonnegative initial data, proving the triviality of solutions in the critical range $1 < p \leq 1 + \frac{2}{\alpha}$, where α is the parameter appearing in the weighted volume condition. To the best of our

knowledge, analogous Liouville-type results for weighted porous medium equations on complete Riemannian manifolds have not been established in this generality.

Motivated by this result, we investigate the corresponding problem for the porous medium equation (1.1). The replacement of the linear diffusion operator by the non-linear term $\Delta(u^m)$ substantially changes the nature of the equation, since the diffusion becomes degenerate for $m > 1$ and singular for $0 < m < 1$. As a consequence, the slow and fast diffusion regimes exhibit fundamentally distinct behaviors and require separate analyses. In particular, unlike the semilinear framework of [2], the weighted volume growth assumptions can no longer be described by a single condition. Instead, they are formulated through two distinct conditions, reflecting the scaling properties of the corresponding equations. These conditions, denoted by (V_S) and (V_F) in the following definitions, are associated with the slow and fast diffusion regimes, respectively.

The different scaling properties of the slow and fast diffusion regimes require separate space-time growth assumptions on the weight function. In the slow diffusion case, the natural intrinsic time scale depends on both the diffusion exponent m and the geometric parameter α and is given by $T_R = R^{2+\alpha(m-1)}$. Accordingly, condition (V_S) is formulated on space-time cylinders associated with this intrinsic scale. In the fast diffusion case, we use the standard parabolic scale $T_R = R^2$, which leads to condition (V_F) . Notice that when $m = 1$, the intrinsic scale in (V_S) reduces to R^2 , and the condition formally coincides with the weighted volume assumption considered in [2]. Moreover, the threshold $m + 2/\alpha$ reduces to $1 + 2/\alpha$, recovering the corresponding semilinear result.

Definition 1.1 The weight V is said to satisfy condition (V_S) if there exist a point $x_0 \in M$ and constants $\alpha > 0$, $\tilde{C} \geq 1$, and $R_0 \geq 1$ such that, for every $R \geq R_0$, setting

$$T_R := R^{2+\alpha(m-1)},$$

the following estimates hold:

$$\int_{T_R/4}^{T_R} \int_{B_R} V(x, t)^{-\frac{1}{p-1}} d\mu dt \leq \tilde{C} R^{2+m\alpha} \quad (1.2)$$

and

$$\int_0^{T_R} \int_{B_R \setminus B_{R/2}} V(x, t)^{-\frac{m}{p-m}} d\mu dt \leq \tilde{C} R^{2+m\alpha}, \quad (1.3)$$

where $1 < m < p$ and $B_R = B_R(x_0)$ is the geodesic ball centered at x_0 with radius R .

Definition 1.2 The weight V is said to satisfy condition (V_F) if there exist a point $x_0 \in M$ and constants $\alpha > 0$, $\tilde{C} \geq 1$, and $R_0 \geq 1$ such that, for every $R \geq R_0$, the following estimates hold:

$$\int_{R^2/4}^{R^2} \int_{B_R} V(x, t)^{-\frac{1}{p-1}} d\mu dt \leq \tilde{C} R^{2+\alpha}$$

and

$$\int_0^{R^2} \int_{B_R \setminus B_{R/2}} V(x, t)^{-\frac{m}{p-m}} d\mu dt \leq \tilde{C} R^{2+\alpha},$$

where $0 < m < 1 < p$ and $B_R = B_R(x_0)$ is the geodesic ball centered at x_0 with radius R .

Remark 1.3 The weighted volume conditions (V_S) and (V_F) are compatible with the polynomial volume growth implied by the assumption $\text{Ric} \geq 0$. Indeed, by the Bishop–Gromov inequality, there exists a constant $C_0 > 0$ such that

$$\mu(B_R) \leq C_0 R^n, \quad R > 0,$$

where μ denotes the Riemannian volume measure on (M, g) and B_R is a geodesic ball of radius R in M .

In particular, let $V \equiv 1$. In the slow diffusion case, recalling that

$$T_R = R^{2+\alpha(m-1)},$$

the integrals appearing in (V_S) can be estimated by

$$T_R \mu(B_R) \leq C_0 R^{n+2+\alpha(m-1)}.$$

Therefore, condition (V_S) is satisfied whenever $\alpha \geq n$. In particular, for the natural choice $\alpha = n$, we have

$$n + 2 + n(m - 1) = 2 + mn = 2 + m\alpha,$$

and the corresponding exponent becomes

$$m + \frac{2}{\alpha} = m + \frac{2}{n},$$

in agreement with the classical exponent for the unweighted Euclidean porous medium equation with reaction.

In the fast diffusion case, the time scale is R^2 , and the integrals appearing in (V_F) can be estimated by

$$R^2 \mu(B_R) \leq C_0 R^{n+2}.$$

Thus, choosing $\alpha = n$, condition (V_F) is also consistent with the polynomial volume growth of geodesic balls.

For $V \equiv 1$ and $\alpha = n$, Theorem 1.5 yields the nonexistence range

$$1 < p \leq m + \frac{2m}{n}.$$

This is a sufficient nonexistence range. It follows from the present test-function method. Since no complementary global-existence result above this threshold is established here, we make no claim of sharpness and do not identify this endpoint as a critical Fujita exponent.

In view of the different behaviors of the slow and fast diffusion regimes, our main results are formulated separately in the following two theorems.

Theorem 1.4 *Let (M, g) be a complete n -dimensional Riemannian manifold with $\text{Ric} \geq 0$. Let $m > 1$, and let u be a nonnegative classical solution of problem (1.1) in the sense of Definition 2.1. Assume further that $V \in C^0(M \times (0, +\infty))$ is positive and satisfies condition (V_S) . If*

$$1 < m < p \leq m + \frac{2}{\alpha},$$

then

$$u \equiv 0.$$

Theorem 1.5 *Let (M, g) be a complete n -dimensional Riemannian manifold with $\text{Ric} \geq 0$. Let $0 < m < 1$, let $u_0 \in L^1_{\text{loc}}(M)$ be nonnegative, and let u be a nonnegative weak solution of problem (1.1) in the sense of Definition 2.2. Assume further that $V \in C^0(M \times (0, +\infty))$ is positive and satisfies condition (V_F) . If*

$$\frac{\alpha}{\alpha + 2} < m < 1 < p \leq m + \frac{2m}{\alpha},$$

then

$$u \equiv 0.$$

The paper is organized as follows. In Sect. 2, we specify the notions of nonnegative classical and weak solution used in the slow and fast diffusion regimes, respectively. Section 3 is devoted to the proofs of the main results, which are treated separately according to the slow and fast diffusion regimes.

2 Notion of solution

In both diffusion regimes, nonnegativity is imposed directly as part of the solution concept. In the slow diffusion case we use the following classical notion.

Definition 2.1 Let $m > 1$, $p > 1$, and let $u_0 \in C^0(M)$ be nonnegative. A function u is called a nonnegative classical solution of problem (1.1) if

$$u \in C^0(M \times [0, +\infty)), \quad \partial_t u \in C^0(M \times (0, +\infty)), \quad u^m \in C^{2,0}(M \times (0, +\infty)),$$

$u \geq 0$, $u(\cdot, 0) = u_0$, and equation (1.1) holds pointwise in $M \times (0, +\infty)$. Here $C^{2,0}$ refers to two continuous derivatives in the spatial variables, with continuous dependence on time.

For $0 < m < 1$, requiring only $u \in C^{2,1}$ would not ensure that u^m is twice differentiable at points where $u = 0$. We therefore formulate the fast diffusion result using the following weak notion.

Definition 2.2 Let $0 < m < 1$, $p > 1$, $u_0 \in L^1_{\text{loc}}(M)$ with $u_0 \geq 0$, and let $V \in C^0(M \times (0, +\infty))$ be positive. A measurable function u is called a nonnegative weak solution of problem (1.1) if

$$u \geq 0, \quad u, u^m, Vu^p \in L^1_{\text{loc}}(M \times [0, +\infty)),$$

and, for every test function $\zeta \in C^{2,1}_c(M \times [0, +\infty))$, one has

$$\int_0^\infty \int_M Vu^p \zeta \, d\mu \, dt = - \int_0^\infty \int_M u \, \partial_t \zeta \, d\mu \, dt - \int_0^\infty \int_M u^m \Delta \zeta \, d\mu \, dt - \int_M u_0(x) \zeta(x, 0) \, d\mu. \quad (2.1)$$

All the terms in (2.1) are well defined under the stated local integrability assumptions. Notice, in particular, that no pointwise differentiability of u^m is imposed at the zero set of u .

3 Liouville-type nonexistence results

Let $x_0 \in M$ be the point appearing in conditions (V_S) and (V_F) . For $R > 0$, define

$$Q_R := B_R \times (0, T_R],$$

where $B_R = B_R(x_0)$ is the geodesic ball of radius R centered at x_0 . The time scale T_R will be specified separately in the slow and fast diffusion cases.

In both cases, we use the cutoff function

$$\psi_R(x, t) = \phi_R(r) \eta_R(t),$$

where $r = d(x, x_0)$ and

$$\phi_R(r) = \phi\left(\frac{r}{R}\right), \quad \eta_R(t) = \eta\left(\frac{t}{T_R}\right).$$

The functions ϕ and η are chosen so that $0 \leq \phi, \eta \leq 1$ and

$$\begin{aligned} \phi(r) &= 1 \quad \text{for } 0 \leq r \leq \frac{1}{2}, & \phi(r) &= 0 \quad \text{for } r \geq 1, \\ -C &\leq \phi'(r) \leq 0, & |\phi''(r)| &\leq C, \end{aligned}$$

while

$$\eta(t) = 1 \quad \text{for } 0 \leq t \leq \frac{1}{4}, \quad \eta(t) = 0 \quad \text{for } t \geq 1, \quad -C \leq \eta'(t) \leq 0.$$

Clearly,

$$-\frac{C}{R} \leq \phi'_R \leq 0, \quad |\phi''_R| \leq \frac{C}{R^2}, \quad -\frac{C}{T_R} \leq \eta'_R \leq 0. \quad (3.1)$$

We define

$$J_R := \int_{Q_R} V(x, t) u^p(x, t) \psi_R^q(x, t) d\mu dt. \quad (3.2)$$

The exponent q will also be specified separately in the two diffusion regimes.

3.1 Slow diffusion case: proof of Theorem 1.4

In this case, we choose the time scale

$$T_R := R^{2+\alpha(m-1)}$$

and let

$$q := \frac{p}{p-m}$$

in (3.2).

Multiplying equation (1.1) by ψ_R^q and integrating by parts in space and time, we obtain

$$\begin{aligned} J_R &= - \int_{Q_R} u \partial_t(\psi_R^q) d\mu dt - \int_{Q_R} u^m \Delta(\psi_R^q) d\mu dt \\ &\quad - \int_{B_R} u_0(x) \phi_R^q(x) d\mu \\ &\leq - \int_{Q_R} u \partial_t(\psi_R^q) d\mu dt - \int_{Q_R} u^m \Delta(\psi_R^q) d\mu dt \\ &= - \int_{Q_R} u \phi_R^q q \eta_R^{q-1} \eta'_R d\mu dt - \int_0^{T_R} \int_{B_R} u^m \Delta \phi_R^q \eta_R^q d\mu dt. \end{aligned}$$

Here we used $\psi_R(\cdot, T_R) = 0$, $\eta_R(0) = 1$, and $u_0 \geq 0$.

Now,

$$\Delta \phi_R^q = q \phi_R^{q-1} \Delta \phi_R + q(q-1) \phi_R^{q-2} |\nabla \phi_R|^2 \geq q \phi_R^{q-1} \Delta \phi_R.$$

Thus,

$$J_R \leq - \int_{Q_R} u \phi_R^q q \eta_R^{q-1} \eta'_R d\mu dt - q \int_0^{T_R} \int_{B_R} u^m \phi_R^{q-1} \Delta \phi_R \eta_R^q d\mu dt.$$

Using the support properties of ϕ_R and η_R , we obtain

$$J_R \leq - \int_{T_R/4}^{T_R} \int_{B_R} u \phi_R^q q \eta_R^{q-1} \eta'_R d\mu dt - q \int_0^{T_R} \int_{B_R \setminus B_{R/2}} u^m \phi_R^{q-1} \Delta \phi_R \eta_R^q d\mu dt.$$

By (3.1), using $|\nabla r| = 1$ and the Laplacian comparison estimate,

$$\Delta \phi_R = |\nabla r|^2 \phi_R'' + \Delta r \phi_R' \geq \phi_R'' + \frac{n-1}{r} \phi_R' \geq -\frac{C}{R^2}$$

in $B_R \setminus B_{R/2}$.

Moreover,

$$-\eta_R'(t) \leq \frac{C}{T_R}.$$

Hence,

$$J_R \leq \frac{C}{T_R} \int_{T_R/4}^{T_R} \int_{B_R} u \phi_R^q \eta_R^{q-1} d\mu dt + \frac{C}{R^2} \int_0^{T_R} \int_{B_R \setminus B_{R/2}} u^m \phi_R^{q-1} \eta_R^q d\mu dt.$$

Since $\phi_R, \eta_R \leq 1$,

$$J_R \leq \frac{C}{T_R} \int_{T_R/4}^{T_R} \int_{B_R} u \psi_R^{q-1} d\mu dt + \frac{C}{R^2} \int_0^{T_R} \int_{B_R \setminus B_{R/2}} u^m \psi_R^{q-1} d\mu dt. \quad (3.3)$$

We now estimate the first term using Hölder's inequality with exponents p and $p/(p-1)$.

$$\int_{T_R/4}^{T_R} \int_{B_R} u \psi_R^{q-1} d\mu dt = \int_{T_R/4}^{T_R} \int_{B_R} (Vu^p \psi_R^q)^{1/p} V^{-1/p} \psi_R^{q-1-q/p} d\mu dt.$$

Since

$$q-1-\frac{q}{p} = \frac{p}{p-m} - 1 - \frac{1}{p-m} = \frac{m-1}{p-m} \geq 0,$$

and $\psi_R \leq 1$, we obtain

$$\psi_R^{q-1-q/p} \leq 1.$$

Therefore,

$$\int_{T_R/4}^{T_R} \int_{B_R} u \psi_R^{q-1} d\mu dt \leq J_R^{1/p} \left(\int_{T_R/4}^{T_R} \int_{B_R} V^{-1/(p-1)} d\mu dt \right)^{(p-1)/p}. \quad (3.4)$$

For the second term,

$$\int_0^{T_R} \int_{B_R \setminus B_{R/2}} u^m \psi_R^{q-1} d\mu dt = \int_0^{T_R} \int_{B_R \setminus B_{R/2}} (Vu^p \psi_R^q)^{m/p} V^{-m/p} \psi_R^{q-1-qm/p} d\mu dt.$$

Now,

$$q-1-\frac{qm}{p} = \frac{p}{p-m} - 1 - \frac{m}{p-m} = 0.$$

Hence,

$$\psi_R^{q-1-qm/p} = 1,$$

and Hölder's inequality gives

$$\int_0^{T_R} \int_{B_R \setminus B_{R/2}} u^m \psi_R^{q-1} d\mu dt \leq J_R^{m/p} \left(\int_0^{T_R} \int_{B_R \setminus B_{R/2}} V^{-m/(p-m)} d\mu dt \right)^{(p-m)/p}. \quad (3.5)$$

Substituting (3.4) and (3.5) into (3.3), we finally obtain

$$J_R \leq \frac{C}{T_R} J_R^{1/p} \left(\int_{T_R/4}^{T_R} \int_{B_R} V^{-\frac{1}{p-1}} d\mu dt \right)^{\frac{p-1}{p}} + \frac{C}{R^2} J_R^{m/p} \left(\int_0^{T_R} \int_{B_R \setminus B_{R/2}} V^{-\frac{m}{p-m}} d\mu dt \right)^{\frac{p-m}{p}}. \quad (3.6)$$

Applying (1.2) and (1.3) to (3.6), and recalling that

$$T_R = R^{2+\alpha(m-1)},$$

we obtain

$$J_R \leq K_1 R^{-2-\alpha(m-1)} J_R^{1/p} \left(R^{2+m\alpha} \right)^{\frac{p-1}{p}} + K_1 R^{-2} J_R^{m/p} \left(R^{2+m\alpha} \right)^{\frac{p-m}{p}},$$

where $K_1 = K_1(C, \tilde{C})$ is a positive constant.

Therefore,

$$J_R \leq K_1 R^{\frac{\alpha(p-m)-2}{p}} J_R^{1/p} + K_1 R^{\frac{m(\alpha(p-m)-2)}{p}} J_R^{m/p}. \quad (3.7)$$

First, assume the subcritical condition

$$1 < m < p < m + \frac{2}{\alpha},$$

Then

$$\alpha(p-m)-2 < 0.$$

Suppose that

$$0 < J_R \leq 1,$$

Since

$$\frac{1}{p} < \frac{m}{p}$$

we obtain

$$J_R \leq K_1 \left(R^A + R^B \right) J_R^{1/p},$$

where

$$A = \frac{\alpha(p-m)-2}{p} < 0, \quad B = \frac{m(\alpha(p-m)-2)}{p} < 0.$$

Dividing by $J_R^{1/p}$ yields

$$J_R^{\frac{p-1}{p}} \leq K_1 (R^A + R^B).$$

Letting $R \rightarrow +\infty$, we obtain

$$J_R \rightarrow 0.$$

On the other hand, if $J_R > 1$, then

$$J_R^{1/p} < J_R, \quad J_R^{m/p} < J_R,$$

since

$$\frac{1}{p} < 1, \quad \frac{m}{p} < 1.$$

Thus, dividing by $J_R > 1$ yields

$$1 \leq K_1 (R^A + R^B).$$

Since $A < 0$ and $B < 0$, the right-hand side tends to zero as $R \rightarrow +\infty$, which is impossible for sufficiently large R . Therefore, for R large enough, necessarily

$$0 < J_R \leq 1.$$

We next consider the endpoint case

$$p = m + \frac{2}{\alpha},$$

we have

$$\alpha(p - m) - 2 = 0.$$

Hence the powers of R appearing in (3.7) vanish. Therefore,

$$J_R \leq K_1 J_R^{1/p} + K_1 J_R^{m/p}.$$

By Young's inequality, for every $\varepsilon > 0$,

$$K_1 J_R^{1/p} \leq \varepsilon J_R + C_\varepsilon, \quad K_1 J_R^{m/p} \leq \varepsilon J_R + C_\varepsilon,$$

because

$$\frac{1}{p} < 1, \quad \frac{m}{p} < 1.$$

Choosing $\varepsilon > 0$ sufficiently small, we obtain

$$J_R \leq \tilde{K},$$

where $\tilde{K}$ is independent of R .

Since

$$0 \leq \chi_{Q_R} \psi_R^q \leq 1$$

and

$$\chi_{Q_R} \psi_R^q \rightarrow 1 \quad \text{pointwise in } M \times (0, +\infty),$$

Fatou's lemma gives

$$\int_0^\infty \int_M V(x, t) u^p(x, t) d\mu dt \leq \liminf_{R \rightarrow +\infty} J_R \leq \tilde{K}.$$

Hence,

$$Vu^p \in L^1(M \times (0, +\infty)).$$

We now return to the localized form of the estimate:

$$J_R \leq K_1 \left(\int_{T_R/4}^{T_R} \int_{B_R} Vu^p \psi_R^q d\mu dt \right)^{1/p} + K_1 \left(\int_0^{T_R} \int_{B_R \setminus B_{R/2}} Vu^p \psi_R^q d\mu dt \right)^{m/p}.$$

Since $Vu^p \in L^1(M \times (0, +\infty))$ and

$$B_R \times (T_R/4, T_R) \subset M \times (T_R/4, +\infty)$$

and

$$(B_R \setminus B_{R/2}) \times (0, T_R) \subset (M \setminus B_{R/2}) \times (0, +\infty),$$

we have

$$\int_{T_R/4}^{T_R} \int_{B_R} Vu^p \psi_R^q d\mu dt \rightarrow 0$$

and

$$\int_0^{T_R} \int_{B_R \setminus B_{R/2}} Vu^p \psi_R^q d\mu dt \rightarrow 0.$$

Therefore,

$$J_R \rightarrow 0 \quad \text{as } R \rightarrow +\infty.$$

Finally, the following argument applies to both the subcritical and endpoint cases. Since

$$0 \leq \chi_{Q_R} \psi_R^q \leq 1$$

and

$$\chi_{Q_R} \psi_R^q \rightarrow 1 \quad \text{pointwise in } M \times (0, +\infty),$$

Fatou's lemma yields

$$\int_0^\infty \int_M Vu^p d\mu dt \leq \liminf_{R \rightarrow +\infty} \int_{Q_R} Vu^p \psi_R^q d\mu dt.$$

Since

$$J_R = \int_{Q_R} V u^p \psi_R^q d\mu dt \longrightarrow 0,$$

we obtain

$$\int_0^\infty \int_M V u^p d\mu dt = 0.$$

Since $V > 0$ and $u \geq 0$, it follows that

$$u \equiv 0.$$

3.2 Fast diffusion case: proof of Theorem 1.5

In this case, we choose the time scale

$$T_R := R^2$$

and let

$$q := \frac{p}{p-1}$$

in (3.2).

Using $\zeta = \psi_R^q$ in (2.1), with the same smooth approximation as in the slow diffusion case, and dropping the nonpositive initial term because $u_0 \geq 0$, we obtain

$$\begin{aligned} J_R &\leq - \int_{Q_R} u \partial_t (\psi_R^q) d\mu dt - \int_{Q_R} u^m \Delta (\psi_R^q) d\mu dt \\ &= - \int_{Q_R} u(x, t) \phi_R^q(x) q \eta_R^{q-1}(t) \eta_R'(t) d\mu dt - \int_0^{R^2} \int_{B_R} u^m(x, t) \Delta \phi_R^q(x) \eta_R^q(t) d\mu dt. \end{aligned}$$

Since

$$\Delta \phi_R^q = q \phi_R^{q-1} \Delta \phi_R + q(q-1) \phi_R^{q-2} |\nabla \phi_R|^2 \geq q \phi_R^{q-1} \Delta \phi_R,$$

substitution into the preceding inequality, followed by the support properties of ϕ_R and η_R , gives

$$\begin{aligned} J_R &\leq - \int_{R^2/4}^{R^2} \int_{B_R} u \phi_R^q q \eta_R^{q-1} \eta_R' d\mu dt \\ &\quad - q \int_0^{R^2} \int_{B_R \setminus B_{R/2}} u^m \phi_R^{q-1} \Delta \phi_R \eta_R^q d\mu dt. \end{aligned}$$

The cutoff estimates and the Laplacian comparison theorem yield

$$\Delta \phi_R = |\nabla r|^2 \phi_R'' + \Delta r \phi_R' \geq \phi_R'' + \frac{n-1}{r} \phi_R' \geq -\frac{C}{R^2} \quad \text{in } B_R \setminus B_{R/2}.$$

Therefore,

$$J_R \leq \frac{C}{R^2} \left[\int_0^{R^2} \int_{B_R} u \phi_R^q \eta_R^{q-1} d\mu dt + \int_0^{R^2} \int_{B_R \setminus B_{R/2}} u^m \phi_R^{q-1} \eta_R^q d\mu dt \right].$$

Since $0 \leq \phi_R, \eta_R \leq 1$, it follows that

$$J_R \leq \frac{C}{R^2} \left[\int_0^{R^2} \int_{B_R} u \psi_R^{q-1} d\mu dt + \int_0^{R^2} \int_{B_R \setminus B_{R/2}} u^m \psi_R^{q-1} d\mu dt \right]. \quad (3.8)$$

We now apply Hölder's inequality to the two terms on the right-hand side of (3.8). For the time term, we write

$$u \psi_R^{q-1} = (V u^p \psi_R^q)^{1/p} V^{-1/p} \psi_R^{q-1-q/p}.$$

Since $q = p/(p-1)$, we have

$$q - 1 - \frac{q}{p} = 0.$$

Consequently,

$$\int_0^{R^2} \int_{B_R} u \psi_R^{q-1} d\mu dt \leq J_R^{1/p} \left(\int_0^{R^2} \int_{B_R} V(x, t)^{-\frac{1}{p-1}} d\mu dt \right)^{1/q}. \quad (3.9)$$

For the spatial term, the correct factorization is

$$u^m \psi_R^{q-1} = (V u^p \psi_R^q)^{m/p} V^{-m/p} \psi_R^{q-1-qm/p}.$$

Moreover,

$$q - 1 - \frac{qm}{p} = \frac{1-m}{p-1} > 0.$$

Hölder's inequality with conjugate exponents p/m and $p/(p-m)$ therefore gives

$$\begin{aligned} & \int_0^{R^2} \int_{B_R \setminus B_{R/2}} u^m \psi_R^{q-1} d\mu dt \\ & \leq \left(\int_0^{R^2} \int_{B_R \setminus B_{R/2}} V u^p \psi_R^q d\mu dt \right)^{m/p} \\ & \quad \times \left(\int_0^{R^2} \int_{B_R \setminus B_{R/2}} V(x, t)^{-\frac{m}{p-m}} \psi_R^{\frac{p(1-m)}{(p-1)(p-m)}} d\mu dt \right)^{\frac{p-m}{p}} \\ & \leq J_R^{m/p} \left(\int_0^{R^2} \int_{B_R \setminus B_{R/2}} V(x, t)^{-\frac{m}{p-m}} d\mu dt \right)^{\frac{p-m}{p}}, \end{aligned} \quad (3.10)$$

where in the last inequality we used $0 \leq \psi_R \leq 1$.

Substituting (3.9) and (3.10) into (3.8), we obtain

$$J_R \leq \frac{C}{R^2} J_R^{1/p} \left(\int_{R^2/4}^{R^2} \int_{B_R} V(x, t)^{-\frac{1}{p-1}} d\mu dt \right)^{1/q} + \frac{C}{R^2} J_R^{m/p} \left(\int_0^{R^2} \int_{B_R \setminus B_{R/2}} V(x, t)^{-\frac{m}{p-m}} d\mu dt \right)^{\frac{p-m}{p}}. \quad (3.11)$$

We first consider the subcritical case

$$\frac{\alpha}{\alpha+2} < m < 1 < p < m + \frac{2m}{\alpha}. \quad (3.12)$$

By condition (V_F) , inequality (3.11) implies

$$J_R \leq K_2 R^a J_R^{1/p} + K_2 R^b J_R^{m/p}, \quad (3.13)$$

where

$$a := \frac{(\alpha+2)(p-1)}{p} - 2, \quad b := \frac{(\alpha+2)(p-m)}{p} - 2.$$

Under assumption (3.12), we have $a < b < 0$.

Suppose first that $0 < J_R < 1$. Since $m < 1$, we have $J_R^{1/p} \leq J_R^{m/p}$; moreover, $a < b$ and $R \geq 1$ imply $R^a \leq R^b$. Hence (3.13) yields

$$J_R \leq 2K_2 R^b J_R^{m/p},$$

and therefore

$$J_R^{\frac{p-m}{p}} \leq 2K_2 R^b.$$

Since $b < 0$, it follows that $J_R \rightarrow 0$ as $R \rightarrow +\infty$.

If $J_R \geq 1$, then $J_R^{m/p} \leq J_R^{1/p}$, and (3.13) gives

$$J_R^{1-1/p} = J_R^{1/q} \leq 2K_2 R^b.$$

This is impossible for all sufficiently large R , because the left-hand side is at least 1, whereas $R^b \rightarrow 0$. Thus $J_R < 1$ for all sufficiently large R , and consequently $J_R \rightarrow 0$.

It remains to consider the endpoint case

$$p = m + \frac{2m}{\alpha}.$$

In this case

$$\frac{p-m}{p} = \frac{2}{\alpha+2},$$

and therefore

$$\frac{(2 + \alpha)(p - m)}{p} - 2 = 0.$$

On the other hand, since $m < 1$, we have

$$\frac{(2 + \alpha)(p - 1)}{p} - 2 = \frac{\alpha p - (\alpha + 2)}{p} = \frac{(\alpha + 2)(m - 1)}{p} < 0.$$

Hence, from (3.11) and condition (V_F) , we obtain

$$J_R \leq K_2 R^{\frac{(\alpha+2)(m-1)}{p}} J_R^{1/p} + K_2 J_R^{m/p}.$$

Since

$$\frac{1}{p} < 1, \quad \frac{m}{p} < 1,$$

Young's inequality gives, for every $\varepsilon > 0$,

$$K_2 R^{\frac{(\alpha+2)(m-1)}{p}} J_R^{1/p} \leq \varepsilon J_R + C_\varepsilon, \quad K_2 J_R^{m/p} \leq \varepsilon J_R + C_\varepsilon.$$

Choosing $\varepsilon > 0$ sufficiently small, we deduce that

$$J_R \leq \tilde{K},$$

where $\tilde{K}$ is independent of R . Since

$$0 \leq \chi_{Q_R} \psi_R^q \leq 1, \quad \chi_{Q_R} \psi_R^q \rightarrow 1 \quad \text{pointwise in } M \times (0, +\infty),$$

Fatou's lemma yields

$$\int_0^\infty \int_M V(x, t) u^p(x, t) d\mu dt < +\infty.$$

We now return to the localized estimate preceding (3.11). Using again (V_F) and the endpoint relation $p = m + 2m/\alpha$, we have

$$J_R \leq K_2 R^{\frac{(\alpha+2)(m-1)}{p}} \left(\int_{R^2/4}^{R^2} \int_{B_R} V u^p \psi_R^q d\mu dt \right)^{1/p} + K_2 \left(\int_0^{R^2} \int_{B_R \setminus B_{R/2}} V u^p \psi_R^q d\mu dt \right)^{m/p}.$$

Since $V u^p \in L^1(M \times (0, +\infty))$ and

$$B_R \times (R^2/4, R^2) \subset M \times (R^2/4, +\infty)$$

and

$$(B_R \setminus B_{R/2}) \times (0, R^2) \subset (M \setminus B_{R/2}) \times (0, +\infty),$$

both integrals on the right-hand side tend to zero as $R \rightarrow +\infty$. Moreover,

$$R^{\frac{(\alpha+2)(m-1)}{p}} \leq 1$$

for $R \geq 1$, because $m < 1$. Consequently,

$$J_R \rightarrow 0 \quad \text{as } R \rightarrow +\infty.$$

We can now conclude in both the subcritical and endpoint cases. Since

$$0 \leq \chi_{Q_R} \psi_R^q \leq 1$$

and

$$\chi_{Q_R} \psi_R^q \rightarrow 1 \quad \text{pointwise in } M \times (0, +\infty),$$

Fatou's lemma yields

$$\int_0^\infty \int_M V(x, t) u^p(x, t) d\mu dt \leq \liminf_{R \rightarrow +\infty} \int_{Q_R} V(x, t) u^p(x, t) \psi_R^q(x, t) d\mu dt = \liminf_{R \rightarrow +\infty} J_R = 0.$$

Since $V > 0$ and $u \geq 0$, we obtain

$$u \equiv 0.$$

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Declarations

Conflict of interest The author declares that there is no conflict of interest.

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