Boundedness criteria for a class of indirect (and direct) chemotaxis-consumption models in high dimensions

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Abstract

In a bounded and smooth domain Ω of $\mathbb{R}^n$, $n \geq 5$, we, mainly, consider for some ξ, χ, δ positive and $T_{max} \in (0, \infty]$ the zero-flux chemotaxis model with indirect signal absorption

$$u_t = \xi\Delta u - \chi\nabla \cdot (u\nabla v), \quad v_t = \Delta v - wv, \quad w_t = -\delta w + u, \quad \text{in } \Omega \times (0, T_{max}),$$

equipped with sufficiently regular initial data $u(x, 0) = u_0(x) \geq 0$, $v(x, 0) = v_0(x) \geq 0$ and $w(x, 0) = w_0(x) \geq 0$. We establish the existence of $\xi^* = \xi^*(n) > 1$ such that whenever $\chi\|v_0\|_{L^\infty(\Omega)}$ obeys certain constraints, functions of n and ξ $(0 < \xi < \xi^*)$, the initial-boundary value problem has a unique classical solution in $\Omega \times (0, \infty)$, which is bounded. In the frame of both direct and indirect chemotaxis models, our work (partially) improves and generalizes known results.

Keywords: Chemotaxis, Indirect Consumption, Global existence, Boundedness.
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1. Introduction and presentation of the main result

We study the following problems

$$\begin{cases} u_t = \xi\Delta u - \chi\nabla \cdot (u\nabla v) & \text{in } \Omega \times (0, T_{max}), \\ v_t = \Delta v - wv & \text{in } \Omega \times (0, T_{max}), \\ w_t = -\delta w + u & \text{in } \Omega \times (0, T_{max}), \\ u_\nu = v_\nu = 0 & \text{on } \partial\Omega \times (0, T_{max}), \\ u(x, 0) = u_0(x),\ v(x, 0) = v_0(x),\ w(x, 0) = w_0(x) & x \in \bar{\Omega}, \end{cases} \tag{1}$$

and

$$\begin{cases} u_t = \xi\Delta u - \chi\nabla \cdot (u\nabla v) & \text{in } \Omega \times (0, T_{max}), \\ v_t = \Delta v - uv & \text{in } \Omega \times (0, T_{max}), \\ u_\nu = v_\nu = 0 & \text{on } \partial\Omega \times (0, T_{max}), \\ u(x, 0) = u_0(x),\ v(x, 0) = v_0(x) & x \in \bar{\Omega}, \end{cases} \tag{2}$$

defined in a bounded and smooth domain Ω of $\mathbb{R}^n$, with $n \geq 3$, $\xi, \chi > 0$ and regular initial data $u_0(x) \geq 0$, $v_0(x) \geq 0$ and $w_0(x) \geq 0$. Additionally, the subscript ν in $(\cdot)_\nu$ indicates the outward normal derivative on $\partial\Omega$, whereas T_{max} is the maximal time up to which solutions to the systems are defined.

Problem (1) may be interpreted as the idealization of a chemotaxis-consumption mechanism, employed in biological processes, involving certain cells and signals, the last ones having an important influence on the motion of the cells themselves. More precisely, if $u = u(x, t)$ is used to denote the population density of the cells at the position x and at the time t, and $v = v(x, t)$ and $w = w(x, t)$ stand for the concentrations of as much chemical signals, by the identity $\xi\Delta u - \chi\nabla \cdot (u\nabla v) = \nabla \cdot (\xi\nabla u - \chi u\nabla v)$, problem (1) indicates that: (a) the migration process of the cells, inside an insulated domain (zero-flux on the border) and initially distributed according to the law of u_0, results from the competition in the flux $\xi\nabla u - \chi u\nabla v$ between the diffusion of the cells (throughout to the term $\xi\nabla u$), larger for higher ξ, with the aggregation impact from the cross terms $\chi u\nabla v$, increasing for larger sizes of χ; (b) the initial attractive signal v_0 is spread in time and v is linearly consumed by w; (c) the *indirect* signal w linearly increases with the cell distribution u, and at its initial configuration is given by w_0. (See [1] for real applications in quite close situations.)

On the other hand, system (1) idealizes a more complex mechanism than the classical chemotaxis Keller-Segel model with *direct* consumption, i.e. model (2). (See the seminal papers by Keller and Segel [2, 3, 4].)

With respect to the signal-production version of (2), where $-uv$ is replaced by $-v + u$, when $\xi = 1$ unbounded solutions can be constructed; see, for instance, [5, 6, 7]). Conversely, when v is consumed in the time, so far no result detecting unbounded solutions is available. More specifically, any sufficiently regular initial data (u_0, v_0) provide uniformly bounded solutions only in two-dimensional settings (from [8] and [9], where a coupled chemotaxis-fluid model is studied); for $n \geq 3$, oppositely, smallness assumptions of the form $\chi\|v_0\|_{L^\infty(\Omega)} \leq c(n)$, for some $c(n) > 0$, are required (see [10]). Additionally, in the case $\xi > 0$, the constant $c(n)$ is generalized to $\gamma(\xi, n)$, and in [11] it is established that $\gamma(1, n) > c(n)$. Nevertheless, this does not exclude that solutions to models (1) and (2) emanating from other couples $(\chi, \|v_0\|_{L^\infty(\Omega)})$ may not collapse.

As far as the aim of this research is concerned, it focuses on partial extensions and/or improvements of results already available in the literature, and dealing with both direct and indirect chemotaxis models of the type in (1) and (2).

More precisely, let us formally present our conclusion:

Theorem 1.1. *For any $\delta > 0$, $n \geq 5$, some $r > n$ and $\beta \in (0,1)$, let Ω be a smooth and bounded domain of $\mathbb{R}^n$. Then, there exist $\underline{\xi} = \underline{\xi}(n), \xi^* = \xi^*(n)$ with $0 < \underline{\xi} < \xi^*$, and $\underline{\alpha} = \underline{\alpha}(n,\xi), \overline{\alpha} = \overline{\alpha}(n,\xi)$ and $\alpha^* = \alpha^*(n,\xi)$ such that whenever $\chi > 0$ and $(u_0, v_0, w_0) \in (C^0(\bar{\Omega}), W^{1,r}(\Omega), C^\beta(\bar{\Omega}))$ are nontrivial initial data with $u_0, v_0, w_0 \geq 0$ on $\bar{\Omega}$ complying with one of the following assumptions i) $\underline{\alpha} < \chi\|v_0\|_{L^\infty(\Omega)} < \overline{\alpha}$ or $0 < \chi\|v_0\|_{L^\infty(\Omega)} < \underline{\alpha}$ for $0 < \xi < \underline{\xi}$, ii) $0 < \chi\|v_0\|_{L^\infty(\Omega)} < \overline{\alpha}$ for $\underline{\xi} \leq \xi \leq 1$, iii) $0 < \chi\|v_0\|_{L^\infty(\Omega)} < \alpha^*$ for $1 \leq \xi < \xi^*$, problem (1) admits a unique global classical solution (u, v, w), nonnegative and uniformly bounded in time.*

Remark 1.2. *Once assumptions i), ii) and iii) are explicitly written (according to what is indicated in the proof of Theorem 1.1 below), we can present the forthcoming comments, valid for $n \geq 5$ and which we consider worthwhile:*

- *Theorem 1.1 generalizes and improves [12, Theorem 1.1], where the boundedness issue for problem (1) is addressed only for $\xi = \chi = 1$; indeed, from ii) and iii) we have $\overline{\alpha}(n,1) = \alpha^*(n,1) = \frac{4}{n-2}\sqrt{\frac{n-4}{n}} > \frac{1}{3n}$. (For completeness, we mention that in [12] also a corresponding asymptotic analysis is discussed.)*

- *As specified at page 4, Theorem 1.1 remains valid also for the classical model (2) with direct consumption and, subsequently, we can compare it with [11, Theorem 2.6]. In particular, on the basis of $\underline{\alpha}(n,\xi) = \frac{2\left(n-n\xi-2\sqrt{(n-4)n\xi}\right)}{(n-2)n}$ and $\overline{\alpha}(n,\xi) = \frac{2\left(n-n\xi+2\sqrt{(n-4)n\xi}\right)}{(n-2)n} \nearrow \frac{2}{n-2}$ as $\xi \nearrow 0$, for small values of ξ, conditions in i) also improve (cyan and green shadow zones in Figure 1) that in [11], reading*

$$\chi\|v_0\|_{L^\infty(\Omega)} < \gamma(n,\xi) = \sqrt{\frac{\xi}{2(n+1)}}\left(\pi + 2\arctan\left(\frac{1-\xi}{2}\sqrt{\frac{2(n+1)}{\xi}}\right)\right),$$

where $\gamma(n,\xi) \nearrow 0$ with $\xi \nearrow 0$ (magenta shadow zone).

2. Local solvability and boundedness criterion. Proof of Theorem 1.1

Let us now focus on the well-known local solvability of systems (1) and (2), and on boundedness criteria on their local solutions. (We give some hints for the proof only for the indirect model.)

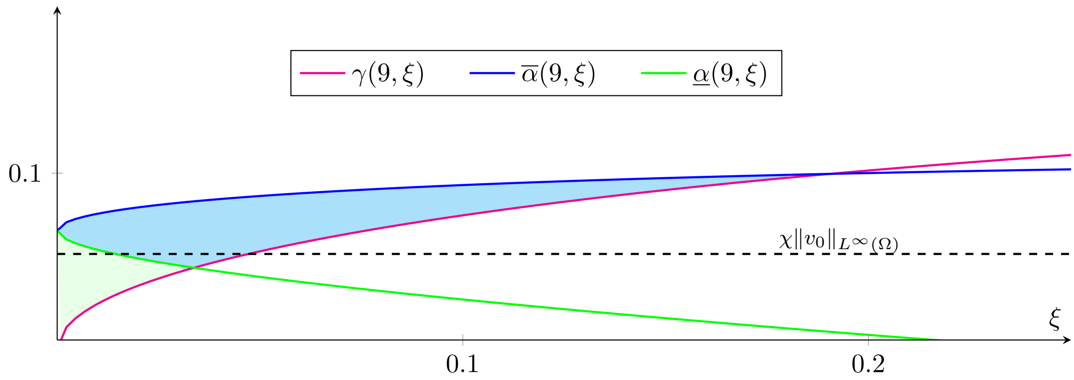

Figure 1: Overview about globality and boundedness of solutions to model (2), for $n = 9$.

Lemma 2.1 (Local existence and boundedness criterion)**.** *Let Ω be a bounded and smooth domain of $\mathbb{R}^n$, with $n \geq 3$, $r > n$, $\xi, \chi, \delta > 0$, $\beta \in (0,1)$ and $(u_0, v_0, w_0) \in (C^0(\bar{\Omega}), W^{1,r}(\Omega), C^\beta(\bar{\Omega}))$ any nontrivial initial data with $u_0, v_0, w_0 \geq 0$ on $\bar{\Omega}$. Then there exists a unique triplet of nonnegative functions*

$$\begin{cases} u \in C^0(\bar{\Omega} \times [0, T_{max})) \cap C^{2,1}(\bar{\Omega} \times (0, T_{max})) \\ v \in C^0([0, T_{max}); W^{1,r}(\Omega)) \cap C^{2,1}(\bar{\Omega} \times (0, T_{max})) \\ w \in C^0(\bar{\Omega} \times [0, T_{max})) \cap C^{0,1}(\bar{\Omega} \times (0, T_{max}))) \end{cases}$$

solving problem (1) *with $T_{max} \in (0, \infty]$. Moreover, u and v obey*

$$\int_\Omega u(x,t)dx = \int_\Omega u_0(x)dx = m > 0 \ \text{ for all } t \in (0, T_{max}) \ \text{ and } \ 0 \leq v \leq \|v_0\|_{L^\infty(\Omega)} \text{ in } \Omega \times (0, T_{max}), \quad (3)$$

and whenever $u \in L^\infty((0, T_{max}); L^k(\Omega))$ for some $k > \frac{n}{2}$, it also holds that $u, v, w \in L^\infty((0,\infty); L^\infty(\Omega))$.

Proof. The issues of local existence, and properties of (u, v, w) are essentially proved in [12, Lemma 2.1]. As to the extensibility and boundedness conclusion $u \in L^\infty((0,\infty); L^\infty(\Omega))$, as well as to the direct chemotaxis model (2), it is a consequence of $v \in L^\infty((0, T_{max}); W^{1,q}(\Omega))$ for $n < q < r < \frac{nk}{n-k}$, which is connected to (3) and the use of the theory of semigroups. (See details, for instance, in [13, Lemma 4.2].) Henceforth,with uniform boundedness of u in our hands, let us show that for the indirect case we also have $w \in L^\infty((0,\infty); L^\infty(\Omega))$. By setting $C_1 := \|u(\cdot, t)\|_{L^\infty(\Omega)}$, for all $t \in (0, \infty)$, the third equation of problem (1) yields

$$w_t \leq -\delta \left(w - \frac{C_1}{\delta} \right),$$

which by an ODI comparison principle implies

$$\|w(\cdot, t)\|_{L^\infty(\Omega)} \leq \max \left\{ \|w_0\|_{L^\infty(\Omega)}, \frac{C_1}{\delta} \right\}.$$

Obviously, the second relation in (3) implies $v \in L^\infty((0,\infty); L^\infty(\Omega))$ as well. $\square$

From now on we will make reference to these quantities, defined whenever $k > 2$ and $\xi > 0$:

$$\underline{\xi} = \underline{\xi}(k) = \frac{(3k-4) - \sqrt{8(k^2 - 3k + 2)}}{k}, \quad \xi^* = \xi^*(k) = \sqrt{\frac{8(k-1)}{k}} - 1, \quad (4)$$

and

$$\begin{cases} \alpha_0 = \alpha_0(k,\xi) = \frac{2}{k}, \quad \underline{\alpha} = \underline{\alpha}(k,\xi) = \frac{k(1-\xi) - \sqrt{4k\xi(k-2)}}{k(k-1)}, \quad \overline{\alpha} = \overline{\alpha}(k,\xi) = \frac{k(1-\xi) + \sqrt{4k\xi(k-2)}}{k(k-1)}, \\ \\ \alpha_1 = \alpha_1(k,\xi) = \frac{k(\xi-1) + 2\sqrt{k(-k\xi + 2k - 2)}}{k(k-1)}, \qquad \alpha^* = \alpha^*(k,\xi) = \frac{1}{k-1}\sqrt{\frac{8(k-1) - k(\xi+1)^2}{k}}. \end{cases} \quad (5)$$

Lemma 2.2. *Let $k > 2$ and $\xi > 0$. Then, accordingly to definitions* (4) *and* (5)*, these relations hold:*

$$\min\{\alpha_0, \overline{\alpha}, \alpha_1\} = \overline{\alpha} \quad \textit{for all } 0 < \xi < 1 \textit{ and } \min\{\alpha_0, \alpha^*\} = \alpha^* \quad \textit{for all } 1 \leq \xi < \xi^*.$$

Moreover, $0 < \underline{\xi} < 1$ and $\underline{\alpha}(\xi)$ is positive for any $0 < \xi < \underline{\xi}$.

Proof. The concluding part of this Lemma is trivial. Hence, let us analyze separately the cases $0 < \xi < 1$ and $1 \leq \xi < \xi^*$. For $0 < \xi < 1$, it is seen that α_0, α_1 and $\overline{\alpha}$ are well defined and positive. Also $\min\{\alpha_0, \alpha_1\} = \alpha_1$ is evident. Conversely, let us show that $\min\{\overline{\alpha}, \alpha_1\} = \overline{\alpha}$; to this aim, let us establish that $\Lambda(\xi) := \alpha_1(\cdot, \xi) - \overline{\alpha}(\cdot, \xi) > 0$ for $0 < \xi < 1$. We have these facts: $\Lambda(0) > 0$ and $\Lambda(\xi) = 0$ in $0 < \xi \leq 1$ if and only if $\xi = \frac{k-2}{k}$ and $\xi = 1$. Moreover, after some computations, one has that $\Lambda'(\frac{k-2}{k}) = 0$ and $\Lambda''(\frac{k-2}{k}) = \frac{1}{(k-1)(k-2)} > 0$. Thereafter, $\frac{k-2}{k}$ is a minimum for Λ and we conclude. The case $1 \leq \xi < \xi^*$ follows invoking very similar arguments. □

We are in a position to justify our main conclusion.

Proof of Theorem 1.1. For any $n \geq 5$, let us define $\underline{\xi} = \underline{\xi}(\frac{n}{2})$, $\xi^* = \xi^*(\frac{n}{2})$, $\alpha_0 = \alpha_0(\frac{n}{2}, \xi)$, $\underline{\alpha} = \underline{\alpha}(\frac{n}{2}, \xi)$, $\overline{\alpha} = \overline{\alpha}(\frac{n}{2}, \xi)$, $\alpha_1 = \alpha_1(\frac{n}{2}, \xi)$ and $\alpha^* = \alpha^*(\frac{n}{2}, \xi)$. From our hypotheses, by using continuity arguments, for some $k > \frac{n}{2} > 2$, we can fix $\sigma = \sigma(\xi) > 0$ (through Lemma 2.2) in the following manner:

$$\text{for } 0 < \xi < \underline{\xi},\ \max\left\{\|v_0\|_{L^\infty(\Omega)}, \frac{\underline{\alpha}}{\chi}\right\} < \sigma < \frac{\overline{\alpha}}{\chi}; \quad \text{for } \underline{\xi} \leq \xi < 1,\ \|v_0\|_{L^\infty(\Omega)} < \sigma < \frac{\overline{\alpha}}{\chi}; \tag{6}$$

$$\text{for } 1 \leq \xi < \xi^*,\ \|v_0\|_{L^\infty(\Omega)} < \sigma < \frac{\alpha^*}{\chi}. \tag{7}$$

From bounds (3), (6), (7) and again Lemma 2.2, we have that $\sigma - v$ is positive and finite for all $(x, t) \in \bar{\Omega} \times [0, T_{max})$, and moreover $\sigma < \frac{\alpha_0}{\chi} = \frac{2}{k\chi}$. To conclude it is sufficient to show that for the local solution (u, v, w) to problem (1), there is $L > 0$ such that

$$\int_\Omega \frac{u^k}{\sigma - v} \leq L \quad \text{on } (0, T_{max}).$$

Indeed, by virtue of $\int_\Omega \frac{u^k}{\sigma} \leq \int_\Omega \frac{u^k}{\sigma - v}$ on $(0, T_{max})$, the above estimate would provide the claim thanks to Lemma 2.1.

With this information at our disposal, by adapting ideas of [9], let us now exploit the first equation of problem (1): some integrations by parts in conjunction with the zero-flux conditions on the boundary yield

$$\begin{aligned}
\frac{d}{dt}\int_\Omega \frac{u^k}{\sigma - v} &= k\int_\Omega \frac{u^{k-1}u_t}{\sigma - v} + \int_\Omega \frac{u^k v_t}{(\sigma - v)^2} = k\int_\Omega \frac{u^{k-1}}{\sigma - v}(\xi\Delta u - \chi\nabla\cdot(u\nabla v)) + \int_\Omega \frac{u^k}{(\sigma - v)^2}(\Delta v - vw) \\
&= -k\xi\int_\Omega \nabla\left(\frac{u^{k-1}}{\sigma - v}\right)\cdot\nabla u + k\chi\int_\Omega \nabla\left(\frac{u^{k-1}}{\sigma - v}\right)u\cdot\nabla v - \int_\Omega \nabla\left(\frac{u^k}{(\sigma - v)^2}\right)\cdot\nabla v - \int_\Omega \frac{u^k vw}{(\sigma - v)^2} \\
&= -k(k-1)\xi\int_\Omega \frac{u^{k-2}}{\sigma - v}|\nabla u|^2 - k(\xi + 1)\int_\Omega \frac{u^{k-1}}{(\sigma - v)^2}\nabla u\cdot\nabla v + k(k-1)\chi\int_\Omega \frac{u^{k-1}}{\sigma - v}\nabla u\cdot\nabla v \\
&\quad + k\chi\int_\Omega \frac{u^k}{(\sigma - v)^2}|\nabla v|^2 - 2\int_\Omega \frac{u^k}{(\sigma - v)^3}|\nabla v|^2 - \int_\Omega \frac{u^k vw}{(\sigma - v)^2} \quad \text{for all } t \in (0, T_{max}).
\end{aligned}$$

The previous expression (also if one deals with problem (2)) can also be reorganized as

$$\begin{aligned}
\frac{d}{dt}\int_\Omega \frac{u^k}{\sigma - v} \leq{}& -k(k-1)\xi\int_\Omega \frac{u^{k-2}}{\sigma - v}|\nabla u|^2 - \int_\Omega u^k\left(\frac{2}{(\sigma - v)^3} - \frac{k\chi}{(\sigma - v)^2}\right)|\nabla v|^2 \\
&+ \int_\Omega u^{k-1}\left(\frac{k(k-1)\chi}{\sigma - v} - \frac{k(\xi + 1)}{(\sigma - v)^2}\right)\nabla u\cdot\nabla v \quad \text{for } t < T_{max}.
\end{aligned} \tag{8}$$

Recalling that $\sigma < \frac{2}{k\chi}$ for all $0 < \xi < \xi^*$, the integrand function of the second term on the right-hand side in estimate (8) is positive: as a matter of fact $\frac{k\chi(\sigma-v)}{2} \leq \frac{k\chi\sigma}{2} < 1$ for all $(x,t) \in \bar{\Omega} \times [0, T_{max})$. Now, the Young inequality ensures that the third integral on the right-hand side of (8) is rephrased on $(0, T_{max})$ as

$$\int_\Omega u^{k-1}\left(\frac{k(k-1)\chi}{\sigma - v} - \frac{k(\xi+1)}{(\sigma-v)^2}\right)\nabla u \cdot \nabla v \leq \int_\Omega u^k \left(\frac{2}{(\sigma-v)^3} - \frac{k\chi}{(\sigma-v)^2}\right)|\nabla v|^2 + \int_\Omega u^{k-2} h(v)|\nabla u|^2, \quad (9)$$

with

$$h(v) = \frac{(k(k-1)\chi(\sigma-v) - k(\xi+1))^2}{(\sigma-v)(8-4k\chi(\sigma-v))} \quad \text{for all } (x,t) \in \bar{\Omega} \times [0, T_{max}).$$

In this way, by plugging estimate (9) into (8) we arrive at

$$\frac{d}{dt}\int_\Omega \frac{u^k}{\sigma - v} \leq \int_\Omega u^{k-2}\left(h(v) - \frac{k(k-1)\xi}{\sigma - v}\right)|\nabla u|^2 \quad \text{on } (0, T_{max}). \quad (10)$$

In order to deal with values of $\xi > 0$ arbitrarily small, herein we define the quotient

$$\frac{h(v)}{\frac{k(k-1)}{\sigma-v}} = \frac{k(k-1)\chi^2(\sigma-v)^2 + \frac{k(\xi+1)^2}{k-1} - 2k\chi(\xi+1)(\sigma-v)}{8-4k\chi(\sigma-v)} =: \frac{h_1(v)}{h_2(v)}$$

(and not the apparently more natural $h(v)/\frac{\xi k(k-1)}{\sigma-v}$), so obtaining

$$h_1(v) - h_2(v) = k(k-1)\chi^2(\sigma-v)^2 + \frac{k(\xi+1)^2}{k-1} - 8 + 2k\chi(\sigma-v)(1-\xi). \quad (11)$$

Since the sign of $h_1(v) - h_2(v)$ also depends on $(1-\xi)$, at this point we have to distinguish the cases $0 < \xi < 1$ and $1 \leq \xi < \xi^*$, starting from the last one being more direct.

- **Case** $1 \leq \xi < \xi^*$. Estimate (11) becomes

  $$h_1(v) - h_2(v) \leq k(k-1)\chi^2\sigma^2 + \frac{k(\xi+1)^2}{k-1} - 8 =: -C_2, \quad (12)$$

  where $C_2 = 8 - k(k-1)\chi^2\sigma^2 - \frac{k(\xi+1)^2}{k-1} > 0$ from (7), whilst $h_2(v) \geq 8 - 4k\chi\sigma =: C_3 > 0$ due to $\sigma < \frac{2}{k\chi}$; henceforth, we can find a positive constant C_4 such that

  $$\frac{h_1(v)}{h_2(v)} \leq 1 - \frac{C_2}{C_3} =: 1 - C_4 \quad \text{or equivalently} \quad h(v) \leq (1-C_4)\frac{k(k-1)}{\sigma - v}. \quad (13)$$

  Now by using (13) in estimate (10), we get

  $$\frac{d}{dt}\int_\Omega \frac{u^k}{\sigma-v} \leq k(k-1)(1-\xi-C_4)\int_\Omega \frac{u^{k-2}}{\sigma - v}|\nabla u|^2 \quad \text{for all } t \in (0, T_{max}). \quad (14)$$

- **Case** $0 < \xi < 1$. By refraining from neglecting $(1-\xi)$, the difference of $h_1(v)$ and $h_2(v)$ in estimate (11) is controlled similarly to (12) with

  $$\tilde{C}_2 := 8 - k(k-1)\chi^2\sigma^2 - \frac{k(\xi+1)^2}{k-1} - 2k\chi\sigma(1-\xi),$$

  positive for $\sigma < \frac{\alpha_1}{\chi}$, in turn satisfied in view of Lemma 2.2 and (6). Therefore, likewise the previous case, there exists $\tilde{C}_4 = \frac{\tilde{C}_2}{C_3} > 0$ such that

  $$h(v) \leq (1-\tilde{C}_4)\frac{k(k-1)}{\sigma - v},$$

  so yielding bound (14), where C_4 is now $\tilde{C}_4$.

With (14) in our hands, we suddenly have that $(1 - \xi - C_4) \leq 0$ for $\xi \geq 1$. When $0 < \xi < \underline{\xi}$ some computations lead to

$$1 - \xi - \tilde{C}_4 = \frac{k(k-1)\chi^2\sigma^2 - 2k\chi\sigma(1-\xi) - 8\xi + \frac{k(\xi+1)^2}{k-1}}{8 - 4k\chi\sigma},$$

which is negative from $\frac{\underline{\underline{\alpha}}}{\chi} < \sigma < \frac{\overline{\alpha}}{\chi}$. Finally, for $\underline{\xi} < \xi < 1$ this remains consistent also for $\underline{\alpha}$ nonpositive.

In all these cases, we can conclude because (14) can be seen as

$$\frac{d}{dt}\int_\Omega \frac{u^k}{\sigma - v} \leq 0 \quad \text{for all } t \in (0, T_{max}) \text{ or also } \int_\Omega \frac{u^k}{\sigma - v} \leq \int_\Omega \frac{u_0^k}{\sigma - v_0} =: L \quad \text{on } (0, T_{max}).$$

□

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