Social Sciences & Humanities Open 10 (2024) 101067

Contents lists available at ScienceDirect

Social Sciences & Humanities Open

journal homepage: www.sciencedirect.com/journal/social-sciences-and-humanities-open

Regular Article

Stabilization of chaotic dynamics emerging in an economy with international labor migration

Giovanni Bella [*], Paolo Mattana, Beatrice Venturi

Department of Economics and Business, University of Cagliari, Via S. Ignazio 17, 09123, Italy

ARTICLE INFO

Keywords:
Equilibrium indeterminacy
Chaotic dynamics
Labor migration
Chaos control

ABSTRACT

This paper examines the conditions for the emergence of chaotic dynamics in the economy described by Parello (2019) and the role of international labor migration on the global indeterminacy of the equilibrium resulting from the competition in the labor market between native workers and immigrants entering a host country. The application of the Shilnikov theorem reveals the spiraling structure of the chaotic attractor, allowing us to infer the unpredictability of long-run policies amid international labor migration. We also propose an approach to control this chaotic scenario and stabilize the economic dynamics towards a stable equilibrium by applying the algorithm proposed by Ott et al. (1990), to determine the necessary conditions and exact parametric configuration to redirect the economy along the optimal path that avoids the undesired indeterminate solution.

1. Introduction

Increasing global economic complexity has exposed the failure of linear modelling in forecasting future outcomes. Several alternatives have been proposed in the area of nonlinear dynamic systems, aiming to confirm the existence of indeterminate equilibria and eventually chaotic dynamics (see, for example, Barnett et al., 2022; Bella et al., 2017). This approach is of particular interest when applied in the field of international migration, as its complex impact on demography and economic development could influence the dynamic path towards a long-run economic growth in different ways. For example, if migration data are not stationary and become highly sensitive to initial economic endowments, or give rise to nonlinear evolutionary patterns, then the dynamics resulting from the incoming migrant labor force could produce unpredictable economic and social outcomes (James, 2003). This can lead to a chaotic behavior that cannot be explained using standard economic tools that examine the evolution of (linear) systems through a deterministic set of rules. Instead, the use of alternative instruments that can accommodate the combined presence of nonlinear systems and nondeterministic factors that produce undesired distortions amid unexpected random shocks can be more insightful.

Investigating phenomena like international migration, which can be rather episodic with seasonal ups and downs, could reveal how a complex system can exhibit irregular equilibrium patterns when influencing factors emerge. This endeavor may be able to identify the onset of a deterministic chaos, that can undermine the predictive power for short-term decision-making processes and the accuracy of future economic outcomes forecasts.

This study considers the economy described by Parello (2019), who examined the influence of international labor migration on equilibrium indeterminacy. The basic argument is that competition in the labor market between native workers and immigrants entering the host country may result in equilibrium solutions that depend on the immigration ratio, giving rise to different (i.e., multiple) equilibrium paths, which is a source of indeterminacy. As a step forward, this study demonstrates that this area of local indeterminacy is in fact also global, because the application of the Shilnikov theorem allows us to show that phenomena associated with chaotic dynamics can also emerge in a restricted set of the structural parameters of the model (see, Shilnikov, 1965; Shang & Han, 2005; Chen & Zhou, 2011). We also propose an approach to control this chaotic scenario by applying the algorithm constructed by Ott et al. (1990); henceforth, OGY.

The remainder of this paper develops as follows. In section 2, we present the variables and the system of differential equations that characterize the Parello (2019) model. In section 3, we detail the conditions for the emergence of a chaotic attractor by applying the

* Corresponding author.
E-mail addresses: bella@unica.it (G. Bella), mattana@unica.it (P. Mattana), venturi@unica.it (B. Venturi).

https://doi.org/10.1016/j.ssaho.2024.101067
Received 12 February 2024; Received in revised form 30 July 2024; Accepted 31 July 2024
Available online 3 August 2024

Shilnikov theorem and provide an associated economic interpretation. In section 4, we illustrate the OGY algorithm to break down the chaotic scenario and restore stability to the dynamic system.[1] Section 5 presents a brief conclusion summarizing the main findings of the paper. The Appendix provides all necessary proofs.

2. The model

Consider the Ramsey-like economy with endogenous labor migration outlined by Parello (2019).[2] The dynamics that characterize the competitive equilibrium are fully described in the following system of differential equations, henceforth named S:

$$\dot{k} = k\left(\frac{q-1}{h}\right) - \delta k$$

$$\dot{q} = (\rho+\delta)q - \frac{(q-1)^2}{2h} - \alpha A k^{\alpha-1}(1-\theta+\theta m^{\varrho})^{\frac{1-\alpha}{\varrho}} \quad (S)$$

$$\dot{m} = \eta(1-\alpha)\theta A k^{\alpha}(1-\theta+\theta m^{\varrho})^{\frac{1-\alpha-\varrho}{\varrho}} m^{\varrho} - \eta m w_0,$$

where k is a predetermined variable representing capital per domestic worker, and q and m are jump (non-predetermined) variables indicating the relative shadow price of installed capital and the immigration rate, respectively. As for the set of parameters, $h > 0$ measures the sensitivity of the adjustment costs to changes in total investments, $\delta \in (0,1)$ is the depreciation rate of installed capital, $\rho \in (0,1)$ is the subjective discount rate, $\alpha \in (0,1)$ is the share of capital in the production of final output, $A > 0$ is the productivity level, $\theta \in (0,1)$ is the proportion of income earned by immigrants, $\varrho \in [1,-\infty)$ is a parameter measuring the elasticity of substitution between labor inputs, $\eta > 0$ measures the sensitivity of migration to changes in wage differentials, and w_0 is the wage offered in the country of origin.

Since, at the equilibrium, $\dot{k} = \dot{q} = \dot{m} = 0$, it follows from system S that:

$$q^* = 1 + \delta h \tag{1.a}$$

$$k^* = \left[\frac{(1-\theta)\zeta^{-\frac{\varrho}{1-\alpha}}}{1-\zeta^{-\frac{\alpha\varrho}{(1-\alpha)(1-\varrho)}}\theta^{\frac{1}{1-\varrho}}\omega^{-\frac{\varrho}{1-\varrho}}}\right]^{\frac{1}{\varrho}} \tag{1.b}$$

$$m^* = \left[\frac{(1-\theta)\zeta^{-\frac{\alpha\varrho}{(1-\alpha)(1-\varrho)}}\theta^{\frac{\varrho}{1-\varrho}}\omega^{-\frac{\varrho}{1-\varrho}}}{1-\zeta^{-\frac{\alpha\varrho}{(1-\alpha)(1-\varrho)}}\theta^{\frac{1}{1-\varrho}}\omega^{-\frac{\varrho}{1-\varrho}}}\right]^{\frac{1}{\varrho}} \tag{1.c}$$

where $\zeta = [2\rho(1+h\delta)+\delta(2+h\delta)]/(2\alpha A)$ and $\omega = w_0/[(1-\alpha)A]$.

The triplet $P^* \equiv (q^*, k^*, m^*)$ represents the unique steady state of the economy described by system S, which can be approximated around the stationary equilibrium as follows:

$$\begin{pmatrix}\dot{k}\\ \dot{q}\\ \dot{m}\end{pmatrix} = \mathscr{J}\begin{pmatrix} q-q^*\\ k-k^*\\ m-m^*\end{pmatrix} \tag{2}$$

where

$$\mathscr{J} = \begin{pmatrix} 0 & \frac{k^*}{h} & 0 \\ \frac{\alpha(1-\alpha)A\zeta}{k^*} & \rho & \frac{\alpha(1-\alpha)A\omega}{k^*} \\ \frac{\eta\alpha(1-\alpha)A\omega m^*}{k^*} & 0 & -\frac{\eta(1-\alpha)A\omega[(1-\theta)(1-\varrho)+\alpha\theta m^{*\varrho}]}{1-\theta+\theta m^{*\varrho}} \end{pmatrix} \tag{3}$$

is the Jacobian matrix associated with S, at the steady state (q^*, k^*, m^*).

The eigenvalues of $\mathscr{J}$ are the solutions of the following characteristic equation:

$$\det(\lambda \mathbf{I} - \mathscr{J}) = \lambda^3 - \mathrm{Tr}(\mathscr{J})\lambda^2 + \mathrm{B}(\mathscr{J})\lambda - \mathrm{Det}(\mathscr{J}), \tag{4}$$

given the identity matrix, $\mathbf{I}$, and

$$\mathrm{Tr}(\mathscr{J}) = \rho - A(1-\alpha)\eta\omega[(1-\varrho)(1-\Theta)+\alpha\Theta] \tag{5.1}$$

$$\mathrm{B}(\mathscr{J}) = -A(1-\alpha)\left\{\frac{\alpha\zeta}{h} + \eta\rho\omega[(1-\varrho)(1-\Theta)+\alpha\Theta]\right\} \tag{5.2}$$

$$\mathrm{Det}(\mathscr{J}) = \frac{A^2(1-\alpha)^2\alpha\eta(1-\varrho)\omega\zeta(1-\Theta)}{h} \tag{5.3}$$

where $\Theta = \zeta^{-\frac{\alpha\varrho}{(1-\alpha)(1-\varrho)}}\theta^{\frac{1}{1-\varrho}}\omega^{-\frac{\varrho}{1-\varrho}}$, and $\mathrm{Tr}(\mathscr{J})$, $\mathrm{Det}(\mathscr{J})$, and $\mathrm{B}(\mathscr{J})$ represent the trace, determinant, and sum of principal minors of order 2 of $\mathscr{J}$, respectively.

Applying the Routh-Hurwitz criterion, Parello (2019) demonstrated that the number of positive roots associated with (4) is always one, which is sufficient for concluding that the equilibrium is locally indeterminate. This also implies that a continuum of equilibrium trajectories emerges around the steady state, depending on the different initial levels of the non-predetermined variable, $m(0)$. As noted by Parello, this is as a novelty in the literature on migration, where unicity and determinacy of equilibrium solution is usually expected. This also leaves space for the possibility that similar economies, starting with different initial degrees of immigration, might move towards the predicted long-term equilibrium at completely different rates.

With these results in hand, we seek to determine the conditions for the emergence of a Shilnikov chaotic attractor, and the tools necessary to control this scenario and restore economic stability. We also construct some examples using the same set of parameters employed in Parello (2019) to prove the robustness of our results, with the remainder of the study devoted to this end.

3. Emergence of Shilnikov chaos

We first consider the result obtained by Bella et al. (2017).

Definition 1. Consider the following generic dynamic system:

$$\frac{dx}{dt} = f(x,\mu), x \in \mathbb{R}^3, \mu \in \mathbb{R}^1,$$

where f is sufficiently smooth. Assume f has a hyperbolic saddle-focus equilibrium point, $\hat{x} = 0$, at the (unit vector) bifurcation parameter $\hat{\mu}$, implying that the eigenvalues of the Jacobian matrix, $J = Df$, are in the form α and $\beta \pm \gamma i$, where α, β, and γ are real constants with $\alpha\beta < 0$, and with a saddle quantity $s = |\alpha| - |\beta| \neq 0$. Let

$$\Phi \equiv \mathrm{B}(\mathscr{J}) + \mathrm{Tr}(\mathscr{J})^2 = 0 \tag{6}$$

be the bifurcating condition that separates the region of parameters where the system exhibits a saddle-focus dynamics with s > 0, from the region where the equilibrium is saddle path stable with s < 0. Then, if we can determine that when crossing the boundary Φ, a region exists where $\frac{\partial\Phi}{\partial\hat{\mu}} \neq 0$, which is associated with a saddle-focus dynamics with positive saddle quantity, then a sequence of Smale-horseshoes emerges

[1] We use Maple software to characterize the Shilnikov bifurcation curves and implement the stabilizing OGY algorithm and the MatCont package for MATLAB for the numerical computation of the orbits to produce the chaotic attractor and the derivation of the associated time series of the model variables.

[2] The full derivation of the model is detailed in Parello (2019). This study examines the conditions for the emergence of chaotic dynamics, while preserving the original mathematical notation used.

along the homoclinic orbit connecting the equilibrium to itself, giving rise to a Shilnikov chaotic attractor.

Applying condition (6) to system S allows us to characterize a bifurcating surface in terms of some critical parameters. For convenience, we keep the triplet of parameters (ρ, η, ϱ) free, setting all the others as in Parello (2019).

Denote the set of the parameters $\Delta \equiv \{(A, \alpha, \theta, w_0, h, \delta, \rho, \varrho, \eta)\}$, and assume the following:

$$\overline{\Delta} \equiv (1, 0.3, 0.3, 0.75, 0.05, 0.4, 15, 0.05, \rho, \varrho, \eta) \in \Delta \quad (7)$$

Then, equation (6) reduces to a nonlinear function of the remaining parameters, $\Phi \equiv \Phi(\rho, \varrho, \eta) = 0$, which is represented as the red curve in Fig. 1 (panel a). Assume choosing ϱ from Definition 1 as the convenient unit vector bifurcation parameter. Since can be easily determined that equation (6) monotonically increases in ϱ (with $\frac{\partial \Phi}{\partial \varrho} > 0$ and $\frac{\partial s}{\partial \varrho} > 0$), we can infer that equilibrium P^* undergoes a saddle-focus dynamics with $s > 0$ above the red surface.[3] Additionally, the gray curve in Fig. 1 (panel a) represents the bifurcating curve $\Phi(\rho, \eta)$, for the given value of ϱ assumed in Parello (2019), which subsequently represents an upward boundary for the region of parameters of interest in our analysis, at which indeterminacy of equilibrium occurs for a parametric combination satisfying $\Phi > 0$. The reason for this choice will be clarified in the following sections.

As it is clear from the three-dimensional graph in panel a of Fig. 1, a very narrow combination of the pair of parameters (ρ, η) lies between the two curves, that allows to stay simultaneously above the red curve (at which the equilibrium is a saddle-focus) and below the boundary of the gray curve (where the constraints given by $\varrho < 1$ and $\Phi > 0$ hold).

The parametric area of interest for the onset of Shilnikov chaos is also depicted in the bidimensional diagram in panel b of Fig. 1using gray-shaded lines. Therefore, the region where convergence towards the steady state appears through damping oscillating behavior occurs when $\rho \in (0.2, 1)$ and $\eta \in (0, 1)$, at any given $\varrho < 1$.

The results in panel b suggest some interesting economic interpretation concerning the restrictions obtained on the chosen parameters. Given the sensitivity of migration to changes in wage differences below unity ($\eta < 1$), if the discount rate (ρ) is high and above 0.2, economic agents value the current generation more, although the entry of a migrant labor force requires a reduction in wage differences ($\varrho < 1$) to increase the produced output level and favor economic growth. In this scenario, agents might also exhibit a desire not to smooth consumption and prefer to expend the higher wealth achieved today, leaving less for future generations perhaps with more migrant citizens in the host country; however, chaotic dynamics could then emerge. Hence, the unpredictability of policy actions to restore long-term stability becomes a problem that policymakers must address to avoid continuous and irregular episodes of oscillatory economic activity.

To ease the numerical computation required to draw the resulting chaotic attractor, we must first put system S in the following convenient normal form:

$$\begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{pmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \varepsilon_1 & \varepsilon_2 & \varepsilon_3 \end{bmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ dx^2 + gx^3 \end{pmatrix} \quad (8)$$

where (x, y, z) is a new set of coordinates that arise from the near-identity transformation, $\varepsilon_1 = \mathrm{Det}(\mathscr{J}), \varepsilon_2 = -\mathrm{B}(\mathscr{J})$, and $\varepsilon_3 = \mathrm{Tr}(\mathscr{J})$, and where d and g are combinations of the coefficients of the nonlinear terms, as demonstrated by Freire et al. (2002).[4]

Example 1. Consider the set of parameters $\overline{\Delta}$. Assume $(\rho, \eta) = (0.4, 0.3)$, to support the shaded area explained in Fig. 1, and $\varrho = 0.75$ as in Parello (2019). The equilibrium implies the triplet $(q^*, k^*, m^*) = (1.75, 0.3030035104, 0.427744831)$. Obtaining that $(\alpha, \beta) = (-0.34661948, 0.20483097)$ and $s = 0.14178851 > 0$ with $\Phi = 0.22475 > 0$, the requirements in Definition 1 for Shilnikov theorem are satisfied. Given that $\varepsilon_1 = -0.084125, \varepsilon_2 = 0.00341, \varepsilon_3 = 0.375$ and $(d, g) = (1.75, 0.1096947638)$ the numerical computation of the chaotic attractor is obtained in Fig. 2.

The dynamics of the economy along the spiraling structure of the chaotic attractor reveals periods of irregular oscillatory activity as the phase dynamics start to move away from the saddle-focus point, as confirmed by the computation of the associated Lyapunov exponent ($\mathscr{l} = 0.00141652182839505$), where the positive result demonstrates the divergence of the equilibrium trajectories from the implied steady state, indicating an unstable dynamical system with chaotic behavior.[5] The economic implication of this result leads us to conclude that a small change in the initial conditions in the degree of incoming migrants in presence of a chaotic attractor can cause a considerable change in the model dynamics over time.

This behavior is clearly outlined in Fig. 3, where we simulate the equilibrium trajectories by imposing different initial degrees of migration, $m(0)$, and prove that the chaotic attractor is robust to small variations in the bifurcation parameter. The experiment reveals that, in response to a change in the value of ϱ within the interval $(0.034, 0.048)$, solution trajectories initialized at different values of m begin to exhibit a different oscillating behavior, which remains confined in a bounded attracting region off the steady state level $m^* = 0.427744831$ found in Example 1.[6]

Therefore, we confirm that economies that start with the same set of initial conditions can follow completely different paths to achieve a common long-term equilibrium level. Moreover, given the initial value of our non-predetermined variable within the obtained chaotic attractor, a continuum of initial values of the jump variables emerges giving rise to a pattern of unpredictable admissible equilibria, which implies global indeterminacy of the equilibrium. This is clarified in Fig. 4, which presents the time profile of the migration rate, m, along the Shilnikov chaotic scenario.

The waves generated by the spiral attractor clearly exhibit periods in which the migration ratio lowers when approaching the equilibrium point through the stable arm of the saddle-focus dynamics. In this case, let us assume that the economy is stable but stagnant because of inadequate labor force. In contrast, bursts of oscillatory activity suddenly follow when the economy is in a phase of growth expectations, with an increasing need for immigrant workers to sustain the production process, which, unfortunately, push back the dynamics on the divergent spiral branch of the saddle-focus. Therefore, the policy actions adopted in presence of the economic uncertainty of achieving a stable and intended steady state, could be problematic if the economy is trapped in a chaotic attractor.

4. A way to end the chaos

This section presents the OGY algorithm, which allows us to establish the necessary parametric conditions to force a chaotic trajectory onto a desired target (a periodic orbit or a steady state of the system) using a correction mechanism, which takes the form of a small, time-dependent,

[3] Note that $\mathrm{B}(\mathscr{J})$ and $\mathrm{Tr}(\mathscr{J})$ are increasing in ϱ, which will be the same for Φ.

[4] As demonstrated by Wiggins (1991) and Freire et al. (2002), the hyper-normal form in equation (8) is topological equivalent to S. Therefore, the change of parameters in the transformed system is a local diffeomorphism, that preserves the dynamics of the original system in the new ambient space.

[5] We use a standard routine in R software to compute the Lyapunov exponent.

[6] We conduct this simulation using R software, to demonstrate the mutual movement of different initial conditions of $m(0)$ in the full range of $(0, 1)$ for different levels of the bifurcation parameter ϱ along the region of the chaotic attractor.

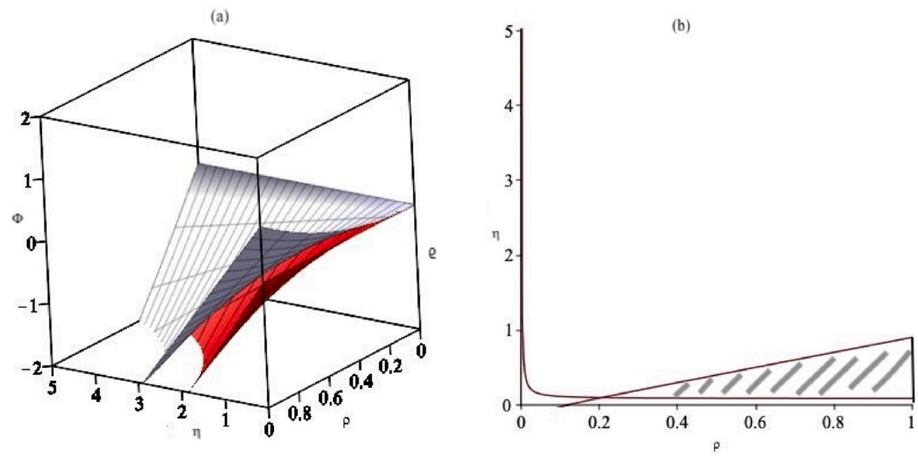

Fig. 1. The Shilnikov bifurcation curve.

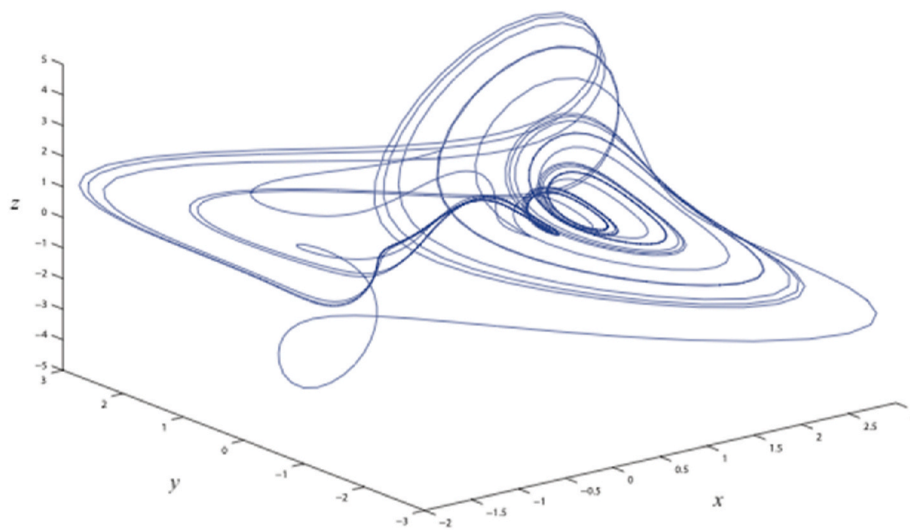

Fig. 2. The Shilnikov chaotic attractor.

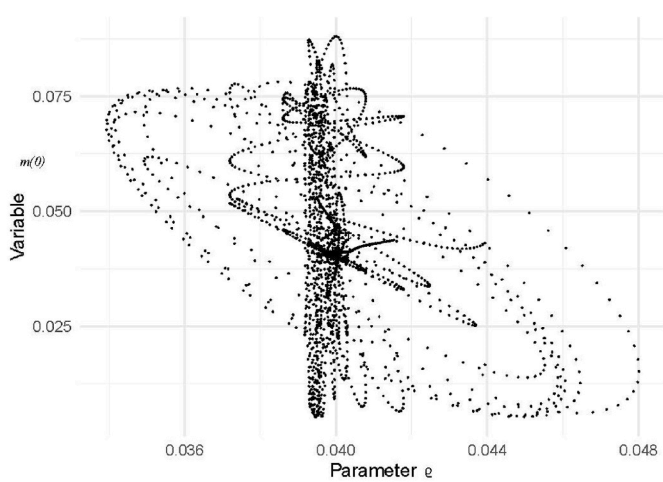

Fig. 3. Sensibility of the initial condition m(0) along the chaotic attractor.

perturbation of a certain control parameter of the model. We also reference Bella and Mattana (2020) for a detailed application of the OGY setting for controlling chaos due to bubbles in the financial markets.

Recalling the bifurcating equation (6), it is easy to understand that a convenient parameter to be tuned to change the sign of the eigenvalues is the elasticity of substitution between labor inputs, ϱ.

We find that system stabilization is obtained by searching values of ϱ at which the eigenvalues of the original system S exhibit the following structure: one negative eigenvalue and two with positive real part, which implies equilibrium stability (see Appendix 1.a). This is accomplished if we allow $\varrho > 1$, which is in contrast with Parello (2019), where it is assumed $\varrho < 1$. Our choice finds support in Behar (2023), where a new analysis and estimation procedure was constructed to determine variations in different countries' income levels based on the elasticity of substitution between skilled (residents) and unskilled (immigrants) labor. Following a recent body of research on the topic, the author's assumption is motivated by the effect of technology on the choice of labor force composition and the role of wage differences between the two cohorts of workers. The author concludes that values of $\varrho < 1$ will result in a much higher reduction in inequality, whereas values of $\varrho > 1$ would have a larger effect on the demand for skilled and unskilled workers and magnify the wage differences (see, for example, Borjas et al., 2012; Caselli & Coleman, 2006). The resulting bias-adjusted estimates provide evidence of $\varrho = 1.7$ for the whole sample of countries analysed. In brief, we summarize this trajectory as follows. When native residents save more, a higher level of capital is accumulated, which produces a need for increased labor, which may increase the need to attract immigrants. As output grows, wages also

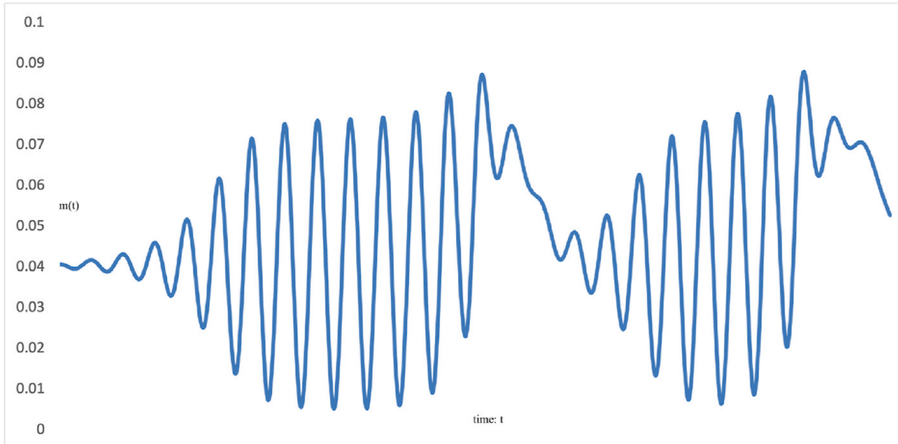

Fig. 4. Time profile of the chaotic migration rate.

grow for migrants that cost less in the labor market, and the wage differential tends to be reduced. However, when marginal productivity of capital lowers, this displaces native residents from the labor, who start to save less and consume more. This lower output pushes migrants out of the market. Marginal productivity of capital begins to increase and calls back natives, which favors the desire for savings and investments, and migrants at higher wage entry levels are once again needed. And the cycle starts anew.

We can now state the following.

Definition 2. Consider an economy with a high discount factor and low sensitivity to wage differences between natives and immigrants. Furthermore, assume that the economy evolves within a chaotic attractor. Then, if the elasticity of substitution between skilled (residents) and unskilled (immigrants) labor moves above unity, the economy can avoid irregular and cyclical behavior and steadily approaches the intended long-term equilibrium.

Appendix 1.a proves this mathematically by first looking at the region of parameters that guarantee the existence of real solutions to the cubic characteristic equation in (4). This implies a negative discriminant $D \equiv D(\varrho) < 0$, which in our case occurs at the restricted range $\varrho \in (1.1, 1.15)$, as shown in Fig. 5.

Then, within this region, the OGY algorithm permits to determine the exact value of the parameter that will allow for one negative eigenvalue and two positive real solutions, which is confirmed by the following example.

Example 2. Let us denote a set of parameters for the emergence of a chaotic attractor, at which system S has a saddle-focus equilibrium with positive saddle quantity and the Shilnikov theorem is satisfied as in Example 1. Let us select $\varrho \equiv \widehat{\varrho} = 1.1$, which is derived by the algorithm presented in Appendix 1.b to obtain the controlled system. Then $D(\widehat{\varrho}) \simeq -3.928 \bullet 10^{-9}$, and $\mathscr{J}$ has one negative eigenvalue ($\lambda_1 = -0.2456906395$) and two eigenvalues with positive real parts ($\lambda_2 = 0.02542180378, \lambda_3 = 3.490975040$).

Fig. 6 illustrates the time profile of the migration ratio in presence of a chaotic attractor (blue curve), obtained with $\varrho = 0.75 < 1$ in Example 1, and the stabilized series (red curve) obtained with $\varrho = 1.1 > 1$ in Example 2.

The economic interpretation of the result obtained in Example 2 is straightforward, and implies that an increase in the wage difference between natives and migrants produces a drop in the marginal product of installed capital and to the associated market interest rate, which also pushes higher consumption and less savings. This turns economic activity downward and restores stability by pushing the unnecessary migrant labor force out of the labor market. However, if the subsequent

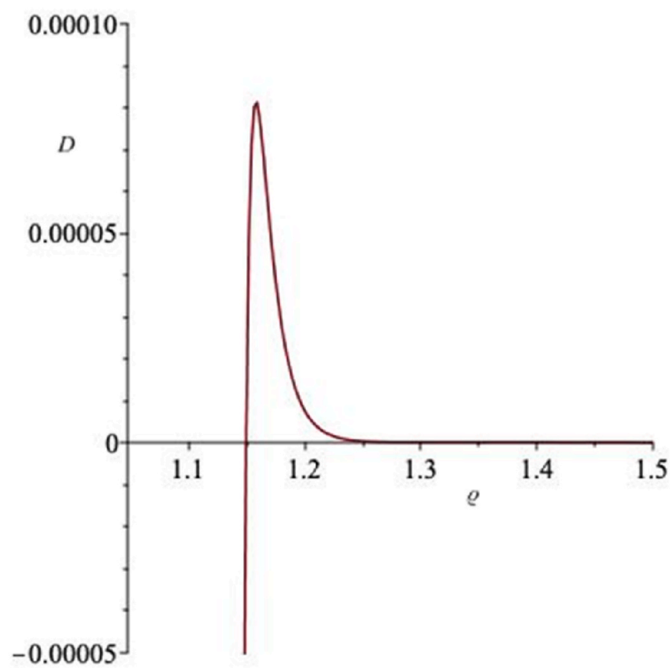

Fig. 5. Discriminant of the characteristic equation when $\varrho > 1$.

fall in wages due to economic crisis pushes back in favor of a less skilled labor force and immigration rates rise again, then the cyclical chaotic scenario might return. However, as outlined by the red curve in Fig. 6, if the elasticity of substitution between skilled (residents) and unskilled (immigrants) labor is set above unity, as suggested by the OGY algorithm, damping is finally minimized and the economy will avoid the previous irregular and cyclical behavior, subsequently converging to the steady state along a stable equilibrium trajectory.

5. Conclusions

The application of the standard Shilnikov theorem to the economy described by Parello (2019), has proved that a chaotic attractor might appear in the plausible settings of the model parameters. The existence of a chaotic attracting set has different policy implications, since the choices policy actions to achieve economic stability might be misleading and can produce undesired and uncontrollable fluctuations depending on the evolution of the migration ratio, which can generate a series of complex and even seemingly stochastic behavior in transitional

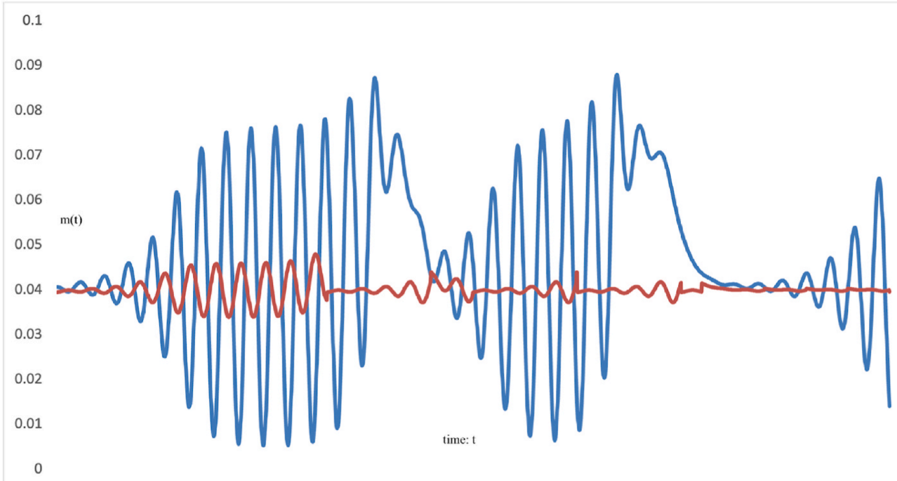

Fig. 6. Uncontrolled and controlled migration rate.

dynamics. This confirms why different countries that are similar in their fundamental structure may nevertheless save, consume and perform at extremely different growth rates.

In this regard, Fig. 1 identifies a critical surface in the parameter space, indicating that if parameters are chosen in the region outside the area formed by the implied bifurcation curves then convergence towards a unique steady state can be achieved along a stable path of the equilibrium trajectory. In contrast, if parameters fall within the narrow area between the two curves, the dynamics around the equilibrium form a saddle-focus that is associated with damped oscillating paths that produce a self-sustaining set of aperiodic cycles around the steady state. This phenomenon is of a global nature, implying that for any given initial condition of the immigration ratio, the economy is driven away from the long-term steady state along the unstable manifold of the saddle-focus and starts to fluctuate in the neighborhood of the steady state in a perpetual oscillating pattern. Therefore, any small initial difference in the non-predetermined variable of the immigration ratio will magnify over time, and economies that are initially close in terms of fundamentals will begin to follow completely different equilibrium paths. This means that policy actions intended to restore stability that are calibrated to similar economies in terms of initial endowments may produce completely different long-term outcomes, perhaps without ever attaining the intended steady state and confining the economy to the trapped region.

As commonly accepted in the literature, chaos is an especially odd outcome that may occur in nonlinear models and is associated with a motion of the dynamic trajectories towards the equilibrium that never exactly repeat themselves (e.g., Boldrin et al., 2001). High sensitivity to initial conditions makes it impossible to anticipate the future economic outcome, since any small initial unanticipated perturbation in the initial conditions could result in a permanent departure from the actual motion towards the steady state. To this end, the numerical simulation presented in Fig. 2 demonstrates that equilibrium trajectories follow different oscillating patterns at different initial degrees of immigration, which remain confined in a bounded attracting region off the steady state level for a given region of the bifurcation parameter within the critical surface implied by the Shilnikov bifurcation theorem.

Finally, applying of the OGY algorithm, we demonstrate the role of the elasticity of substitution between the two labor inputs as the appropriate policy instrument for controlling chaos and achieving the conditional saddle-path stability of the equilibrium. Furthermore, we show that if the elasticity of substitution between skilled (residents) and unskilled (immigrants) labor is set above unity, the economy will avoid the irregular and cyclical behavior associated with Shilnikov chaos, and will instead approach the intended steady state along a stable unique equilibrium trajectory.

A possible evolution of the present version of the model could attempt to reformulate the analysis introducing heterogeneous agents to overcome possible limitations due to the representative agent setting, try to endogenize the stochastic process migrants' randomly-distributed labor productivities in a nondeterministic framework, and once again investigate the conditions to ensure the uniqueness of the equilibrium to prevent unwanted self-fulfilling fluctuations. Even more interestingly, future studies could consider whether the correlation of such distributions of idiosyncratic productivities can extend the interpretation of our work in the spirit of a quantum entanglement theory by analyzing economic system entropy amid international migration. We leave these considerations to potential future research.

Funding

This research did not receive any specific grant from funding agencies in the public, commercial, or not-for-profit sectors.

CRediT authorship contribution statement

Giovanni Bella: Writing – original draft, Formal analysis, Conceptualization. **Paolo Mattana:** Formal analysis, Conceptualization. **Beatrice Venturi:** Formal analysis, Conceptualization.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Acknowledgments

We wish to thank the anonymous referees for their valuable and constructive suggestions that helped us to improve the quality of the paper. Our gratitude is also extended to Giulia Contu and Marco Ortu for their generous and helpful insights.

Appendix A. Supplementary data

Supplementary data to this article can be found online at https://doi.org/10.1016/j.ssaho.2024.101067.

References

Barnett, W., Bella, G., Ghosh, T., Mattana, P., & Venturi, B. (2022). Shilnikov chaos, low interest rates, and New Keynesian macroeconomics. *Journal of Economic Dynamics and Control, 134*, Article 104291.

Behar, A. (2023). The elasticity of substitution between skilled and unskilled labor in developing countries: A directed technical change perspective. *IMF working papers 2023/165*. International Monetary Fund.

Bella, G., & Mattana, P. (2020). Chaos control in presence of financial bubbles. *Economics Letters, 193*, Article 109314.

Bella, G., Mattana, P., & Venturi, B. (2017). Shilnikov chaos in the Lucas model of endogenous growth. *Journal of Economic Theory, 172*, 451–477.

Boldrin, M., Nishimura, K., Shigoka, T., & Yano, M. (2001). Chaotic equilibrium dynamics in endogenous growth models. *Journal of Economic Theory, 96*, 97–132.

Borjas, G., Grogger, J., & Hanson, G. (2012). Comment: On estimating elasticities of substitution. *Journal of the European Economic Association, 10*(1).

Caselli, F., & Coleman, W. (2006). The world technology frontier. *The American Economic Review, 96*(3).

Chen, B., & Zhou, T. (2011). Shilnikov homoclinic orbits in two classes of 3D autonomous nonlinear systems. *International Journal of Modern Physics B, 25*(20), 2697–2712.

Freire, E., Gamero, E., Rodriguez-Luis, A. J., & Algaba, A. (2002). A note on the triple-zero linear degeneracy: Normal forms, dynamical and bifurcation behaviors of an unfolding. *International Journal of Bifurcation and Chaos, 12*, 2799–2820.

James, F. (2003). Comment on Exact solutions to chaotic and stochastic systems. *Chaos: International Journal of Nonlinear Science, 13*, 123.

Ott, E., Grebogi, C., & Yorke, J. A. (1990). Controlling chaos. *Physical Review Letters, 64*, 1196–1199.

Parello, C. P. (2019). Equilibrium indeterminacy in one-sector small open economies: The role of international labor migration. *Macroeconomic Dynamics, 23*, 1528–1562.

Shang, D., & Han, M. (2005). The existence of homoclinic orbits to saddle-focus. *Applied Mathematics and Computation, 163*, 621–631.

Shilnikov, L. P. (1965). A case of the existence of a denumerable set of periodic motions. *Soviet mathematics - Doklady, 6*, 163–166.

Wiggins, S. (1991). *Introduction to applied nonlinear dynamical systems and chaos*. New York: Springer-Verlag.