Advances in Mathematics 448 (2024) 109677

Contents lists available at ScienceDirect

Advances in Mathematics

journal homepage: www.elsevier.com/locate/aim

Moment problem for algebras generated by a nuclear space

Maria Infusino[a,*], Salma Kuhlmann[b], Tobias Kuna[c], Patrick Michalski[b]

[a] *Dipartimento di Matematica e Informatica, Università degli Studi di Cagliari, Via Ospedale 72, 09124 Cagliari, Italy*
[b] *Fachbereich Mathematik und Statistik, Universität Konstanz, Universitätstrasse 10, 78457 Konstanz, Germany*
[c] *Dipartimento di Ingegneria, Scienze dell' Informazione e Matematica, Università degli Studi dell'Aquila, Via Vetoio, 67100, L'Aquila, Italy*

ARTICLE INFO

Article history:
Received 19 March 2023
Received in revised form 6 February 2024
Accepted 7 April 2024
Available online 20 May 2024
Communicated by Dan Voiculescu

MSC:
primary 44A60, 46M40, 28C20

Keywords:
Moment problem
Infinite dimensional moment problem
Nuclear space
Projective limit
Prokhorov's condition

ABSTRACT

We establish a criterion for the existence of a representing Radon measure for linear functionals defined on a unital commutative real algebra A, which we assume to be generated by a vector space V endowed with a Hilbertian seminorm q. Such a general criterion provides representing measures with support contained in the space of characters of A whose restrictions to V are $q-$continuous. This allows us in turn to prove existence results for the case when V is endowed with a nuclear topology. In particular, we apply our findings to the symmetric tensor algebra of a nuclear space.

* Corresponding author.
E-mail addresses: maria.infusino@unica.it (M. Infusino), salma.kuhlmann@uni-konstanz.de (S. Kuhlmann), tobias.kuna@univaq.it (T. Kuna), patrick.michalski@googlemail.com (P. Michalski).

https://doi.org/10.1016/j.aim.2024.109677

0. Introduction

Given a unital commutative (not necessarily finitely generated) $\mathbb{R}$–algebra A and a linear subspace V of A, we say that A *is generated by* V if there exists a set of generators G of A such that V is the linear span of G, i.e., $V = span(G)$. Equivalently, A is generated by V if V contains a set of generators of A. This article deals with the moment problem for A generated by a vector space V which is endowed with a topology τ_V compatible with the addition and the scalar multiplication, namely (V, τ_V) is a topological vector space. Moreover, we always assume that the *character space* of A, i.e., the set $X(A)$ of all $\mathbb{R}$–algebra homomorphisms from A to $\mathbb{R}$, is non-empty and we always endow $X(A)$ with the weakest (Hausdorff) topology $\tau_{X(A)}$ on $X(A)$ such that for each $a \in A$ the function $\hat{a}\colon X(A) \to \mathbb{R}$, $\alpha \mapsto \alpha(a)$ is continuous and consider on $X(A)$ the Borel $\sigma-$algebra $\mathcal{B}(\tau_{X(A)})$ w.r.t. $\tau_{X(A)}$. Our main question is the following.

Main Question. *Let (V, τ_V) be a topological vector space and A an algebra generated by V such that $\{\alpha \in X(A) : \alpha \restriction_V \text{ is } \tau_V - \text{continuous}\} \neq \emptyset$. Given a linear functional L on A with $L(1) = 1$, does there exist a Radon measure ν on $X(A)$ with support contained in $\{\alpha \in X(A) : \alpha \restriction_V \text{ is } \tau_V - \text{continuous}\}$ such that*

$$L(a) = \int_{X(A)} \hat{a}(\alpha)\mathrm{d}\nu(\alpha) \qquad \text{for all } a \in A? \tag{0.1}$$

If a Radon measure ν as in (0.1) does exist, then we call ν a *representing Radon measure for* L. We recall that a *Radon measure* ν on $X(A)$ is a non-negative measure on $\mathcal{B}(\tau_{X(A)})$ that is locally finite and inner regular w.r.t. compact subsets of $X(A)$. The *support of* ν, denoted by $\mathrm{supp}(\nu)$, is the smallest closed subset C of $X(A)$ for which $\nu(X(A) \setminus C) = 0$. Given $K \subseteq X(A)$ closed, we say that ν is a $K-$*representing measure for* L if both (0.1) holds and $\mathrm{supp}(\nu) \subseteq K$.

The main difficulty is to understand how different choices of τ_V as well as different topological properties of L impact the solvability of the Main Question and the support of the corresponding representing measures. In this article we first focus on the case when τ_V is the topology generated by a Hilbertian seminorm (i.e., a seminorm induced by a symmetric positive semidefinite bilinear form) and then consider the case when (V, τ_V) is a nuclear space, as nuclear topologies are generated by a system of Hilbertian seminorms. Let us stress that in this article we also cover instances when K is non-compact and even non-bounded (for the compact case, see [19] and references therein).

An early study of the Main Question for (V, τ_V) nuclear can be found e.g. in [13], [2, Chapter 5, Section 2], [3], [8], [12], [24, Section 12.5 and 15.1], [1], where A is the symmetric (tensor) algebra $S(V)$ of V. This is a very natural choice as, on the one hand, any algebra generated by V is isomorphic to a quotient of $S(V)$ by an ideal and, on the other hand, $S(V)$ is isomorphic to the ring of polynomials having as variables the coordinate vectors with respect to a basis of V and the character space $X(S(V))$

of $S(V)$ can be identified with the algebraic dual V^* of V. In fact, in those works the nuclearity assumption on V allows to get the existence of representing measures with support contained in V', where V' is the topological dual of V. More recently, the role of the nuclearity assumption on V was discussed in [26, Sections 5 and 6], [10] and [15, Section 3] while in [16] and [17] a better localization of the support was obtained for a specific choice of the nuclear space, namely $V = \mathcal{C}_c^\infty(\mathbb{R}^n)$, i.e., the space of infinitely differentiable functions on $\mathbb{R}^n$ with compact support. Note that for $V = \mathbb{R}^n$, the Main Question for $A = S(V)$ coincides with the classical (finite dimensional) moment problem, see e.g., [21], [25], [15].

Let us describe the two main results in this article.

First, when τ_V is the topology generated by a Hilbertian seminorm q, we derive in Theorem 2.5 a criterion for the existence of a representing measure with support contained in the characters of A whose restrictions to V are $q-$continuous (see also Remark 2.9 and Theorem 2.8). The criterion is based on the projective limit approach to the moment problem introduced in [18], that is, we build the representing measure for L on A from representing measures for L restricted to finitely generated subalgebras of A. In fact, we prove that a representing measure for L exists if and only if for any finitely generated subalgebra S of A the restriction $L\restriction_S$ is represented by a Radon measure ν_S such that the family of all ν_S's is concentrated w.r.t. another $q-$continuous Hilbertian seminorm p, which has finite trace with respect to q. The concentration of a family of measures is a classical concept in measure theory and is crucial for us, because it ensures the applicability of our projective limit approach in [18] by implying a Prokhorov type condition.

Second, in Theorem 2.10 we show that, when A itself is endowed with a Hilbertian seminorm q and there exists $C > 0$ such that $L(a^2) \leq Cq(a)^2$ for all $a \in A$, it is enough to check the conditions in our criterion only on a dense subalgebra of A to get the existence of a representing measure for L with support contained in the $q-$continuous characters of A (see also Theorem 2.11).

These two main results are based on two Hilbertian seminorms q and p. We investigate different choices of them in terms of the functional L. For example, a natural choice for p is the Hilbertian seminorm induced by L, i.e., $s_L(a) := \sqrt{L(a^2)}$ for all $a \in A$. For this choice, the concentration of the ν_S's holds automatically and so we get more concrete sufficient conditions for the existence of a representing measure for L in Corollary 2.13 and Corollary 2.14. We then exploit in Corollary 2.16 the choice of s_L to demonstrate how one can give sufficient conditions only in terms of L to guarantee existence of representing measures for $L\restriction_S$ for all finite subalgebras S. Those corollaries all reveal the fundamental role played by the Hilbertian seminorm q. Thus, in Section 2.3, we explore the case when no Hilbertian seminorm q on V is pre-given. In particular, in Corollary 2.20 we give conditions under which one can construct a suitable q and derive a solution for the Main Question in this case.

Another setting in which it is always possible to obtain a suitable q is when (V, τ_V) is a nuclear space. Therefore, in Section 2.4, we prove analogous results for the Main Question when A is generated by a nuclear space (V, τ_V), see Corollary 2.21, Corollary 2.22 and Corollary 2.23. From those corollaries, some of the results in literature mentioned above can be retrieved.

The structure of the paper is as follows.

In Section 1, we present our general context, thereby providing definitions and notations. In particular, we review the notions of Hilbertian seminorm and nuclear space in Subsection 1.1. In Subsection 1.2, we state and prove Lemma 1.6 (about the support localization of a Radon probability measure on a finite dimensional space with a Hilbertian seminorm), which we need for the proof of our first main result Theorem 2.5. Section 2 contains our main results, as described above. Subsection 2.1 is dedicated to the concept of p-concentration of a family of Radon measures for a given seminorm p, which is exploited in the subsequent Subsections 2.2, 2.3 and 2.4 when studying the Main Question for V endowed with the topology τ_V induced by a Hilbertian seminorm (respectively, a nuclear topology). In Section 3 we apply our main results to the case when A is the symmetric algebra $S(V)$ of a nuclear space (V, τ_V), see Corollary 3.1 and Corollary 3.2. In Theorem 3.3, we consider the case when some of the sufficient conditions for the existence of the representing measure for L on $S(V)$ are only given on a total subset E of the nuclear space (V, τ_V). Then the nuclearity allows us to obtain a Hilbertian norm q on V but, in order to apply our criterion Theorem 2.10 to the dense sub-algebra $S(\operatorname{span}(E))$, we need a Hilbertian seminorm $\tilde{q}$ on $S(V)$, which we construct in Lemma 3.5. We note that Theorem 3.3 is a generalization of the classical solution to the Main Question when $A = S(V)$ with (V, τ_V) nuclear due to Berezansky and Kondratiev. Finally, in Subsection 4.1 of Appendix 4, we explain the relation between the notion of trace of a Hilbertian seminorm w.r.t. another and the classical definition of trace of a positive continuous operator on a Hilbert space. We then compare in Subsection 4.2 the definition of nuclear space used in this article (due to Yamasaki [30]) with that due to Grothendieck [11] and Mityagin [22], as well as with the definitions of this concept given by Berezansky and Kondratiev in [2, p. 14] and by Schmüdgen in [26, p. 445] (this comparison is needed in Section 3). We also provide in Subsection 4.3 a complete proof of the measure theoretical identity (2.7), which we exploited in the proof of Theorem 2.10.

1. Preliminaries

In this section we collect some fundamental concepts, notations, and results which we will repeatedly use in the following.

Throughout this article A denotes a unital commutative $\mathbb{R}$–algebra with non-empty character space $X(A)$.

A subset $Q \subseteq A$ is a *quadratic module (in A)* if $1 \in Q, Q + Q \subseteq Q$, and $A^2 Q \subseteq Q$. The set $\sum A^2$ of all finite sums of squares of elements in A is the smallest quadratic module in A. The *non-negativity* set of a quadratic module Q is defined as

$$K_Q := \{\alpha \in X(A) : \hat{a}(\alpha) \geq 0 \text{ for all } a \in Q\} \subseteq X(A),$$

which is closed. Given $C \subseteq X(A)$ closed, the set

$$\mathrm{Pos}(C) := \{a \in A : \hat{a}(\alpha) \geq 0 \text{ for all } \alpha \in C\}$$

is a quadratic module with $K_{\mathrm{Pos}(C)} = C$ (see, e.g. [18, Proposition 2.1-(i)]).

Throughout this article each linear functional $L\colon A \to \mathbb{R}$ is assumed to be *normalized*, that is, $L(1) = 1$.

Given a quadratic module Q in A, we say that a linear functional $L\colon A \to \mathbb{R}$ is *Q–positive* if $L(Q) \subseteq [0, \infty)$. In particular, each $\sum A^2$–positive linear functional $L\colon A \to \mathbb{R}$ satisfies the Cauchy–Bunyakovsky–Schwarz inequality, i.e.,

$$L(ab)^2 \leq L(a^2)L(b^2) \qquad \text{for all } a, b \in A. \tag{1.1}$$

Throughout this article we consider A generated by an $\mathbb{R}$–vector space V endowed with a *locally convex* topology, namely a topology induced by a family of seminorms. Therefore, let us recall that a function $p\colon V \to [0, \infty)$ is a *seminorm* if $p(\lambda v) = |\lambda|\, p(v)$ and $p(v+w) \leq p(v)+p(w)$ for all $\lambda \in \mathbb{R}$ and all $v, w \in V$. We denote by $B_r(p)$ the closed semi-ball of radius $r > 0$ centered at the origin in (V, p), i.e., $B_r(p) := \{v \in V : p(v) \leq r\}$.

A linear functional $l\colon V \to \mathbb{R}$ is *continuous* w.r.t. a seminorm p on V if there exists $C > 0$ such that $|l(v)| \leq Cp(v)$ for all $v \in V$. We denote by V'_p the topological dual of (V, p), i.e., the collection of all $p-$continuous linear functionals on V, while V^* denotes the algebraic dual of V. The operator seminorm p' on V'_p is defined as $p'(\ell) := \sup_{v \in B_1(p)} |\ell(v)| < \infty$. The weak topology on the algebraic dual (resp., topological dual) of (V, p) is the weakest topology on V^* (resp., on V'_p) such that for each $v \in V$ the evaluation function $\mathrm{ev}_v\colon V^* \to \mathbb{R}$ (resp., $V'_p \to \mathbb{R}$) is continuous.

We will often use the restriction map $\phi_V : X(A) \to V^*$ defined by $\phi_V(\alpha) = \alpha \restriction_V$, $\forall \alpha \in X(A)$. Note that ϕ_V is continuous as $X(A)$ is endowed with $\tau_{X(A)}$ and V^* with the weak topology.

We recall that the *spectrum of a seminorm* p is defined as

$$\mathfrak{sp}(p) := \{\alpha \in X(A) : \alpha \text{ is } p\text{–continuous}\}.$$

More generally, for each $C > 0$ we define

$$\mathfrak{sp}_C(p) := \{\alpha \in X(A) : |\alpha(a)| \leq Cp(a),\ \forall a \in A\},$$

which is compact in $X(A)$, as it is closed and continuously embeds into the product $\prod_{a \in A}[-Cp(a), Cp(a)]$. Note that the spectrum $\mathfrak{sp}(p) = \bigcup_{n \in \mathbb{N}} \mathfrak{sp}_n(p)$, which provides that $\mathfrak{sp}(p)$ is $\sigma-$compact in $X(A)$ and so Borel measurable.

1.1. Hilbertian seminorms and nuclear spaces

Throughout this section V will denote a real vector space.

Definition 1.1. A seminorm p on V is called *Hilbertian* if it is induced by a symmetric positive semidefinite bilinear form $\langle\cdot,\cdot\rangle$ on V, i.e., $p(v) = \sqrt{\langle v, v\rangle}$ for all $v \in V$.

Note that a seminorm p on V is Hilbertian if and only if p fulfills the parallelogram law, i.e., $p(v+w)^2 + p(v-w)^2 = 2p(v)^2 + 2p(w)^2$ for all $v, w \in V$, in which case the bilinear form $\langle\cdot,\cdot\rangle_p$ is uniquely determined by p via the polarization identity:

$$\langle v, w\rangle = \tfrac{1}{2}\left(p(v+w)^2 - p(v)^2 - p(w)^2\right) \quad \text{for all } v, w \in V. \tag{1.2}$$

For this reason, in the following we denote the positive semidefinite bilinear form inducing p by $\langle\cdot,\cdot\rangle_p$.

The term "Hilbertian seminorm", used e.g. in [29] and [30], is also sometimes replaced by the term "prehilbertian seminorm" according to the Bourbaki's tradition [7, V.4, Definition 3]. Both terms hint to the fact that this type of seminorms can be always used to construct a Hilbert space (see Remark 4.5).

Let us also observe that there always exists an Hilbertian seminorm on every non-trivial vector space V. Indeed, if $(e_i)_{i\in I}$ is an algebraic basis of V then for any $x = \sum_{i\in I} x_i e_i \in V$ and $y = \sum_{i\in I} y_i e_i \in V$ we can define $\langle x, y\rangle := \sum_{i\in I} x_i y_i$. As only finitely many summands are unequal to zero, the sum is finite and $p(x) := \sqrt{\langle x, x\rangle}$ defines a Hilbertian seminorm on V.

Let us now introduce the notion of trace of a Hilbertian seminorm w.r.t. another one (see [7, V.58, No. 9]) which will be fundamental in the definition of a nuclear space used in this article. To this purpose, let us recall that given a Hilbertian seminorm p on V, a subset E of V is called:

- *p–orthogonal* if $\langle e_1, e_2\rangle_p = 0$ for all distinct elements $e_1, e_2 \in E$.
- *p–orthonormal* if E is p–orthogonal and $p(e) = 1$ for all $e \in E$.

In particular, a p–orthonormal set E is said to be a *complete p–orthonormal system* of V if E is total in V, i.e., $\overline{span(E)}^p = V$. Such a system is also known as *orthonormal basis* of V.

Definition 1.2. Let p and q be two Hilbertian seminorms on V. The *trace of p w.r.t. q* is denoted by $\mathrm{tr}(p/q)$ and defined as

$$\mathrm{tr}(p/q) := \begin{cases} \sup\limits_{E\in \mathrm{FON}(q)} \sum\limits_{e\in E} p(e)^2, & \text{if } \ker(q) \subseteq \ker(p) \\ \infty, & \text{otherwise} \end{cases},$$

where $\mathrm{FON}(q)$ denotes the collection of all finite q–orthonormal subsets of V.

When there exists $C > 0$ such that $p \leq Cq$ the following characterization of the trace of p w.r.t. q holds (by combining Proposition 4.6 and (4.2) in Appendix 4.1):

$$\forall E \text{ complete } q\text{−orthonormal system in } V, \ \operatorname{tr}(p/q) = \sum_{e \in E} p(e)^2. \tag{1.3}$$

The following properties are immediate from the Definition 1.2.

Lemma 1.3. *Let p and q be two Hilbertian seminorms on V with $\operatorname{tr}(p/q) < \infty$. Then:*

(i) $p^2 \leq \operatorname{tr}(p/q) q^2$.
(ii) $\forall \varepsilon, \delta > 0, \ \operatorname{tr}(\varepsilon p/\delta q) = \left(\frac{\varepsilon}{\delta}\right)^2 \operatorname{tr}(p/q)$.
(iii) $\forall$ *W subspace of V,* $\operatorname{tr}(p\restriction_W / q\restriction_W) \leq \operatorname{tr}(p/q)$.

We are equipped now with all notions needed to introduce the definition of a nuclear space due to Yamasaki (see [30, Definition 20.1]), which we are going to adopt in this article.

Definition 1.4. A locally convex space (V, τ) is called *nuclear* if τ is induced by a directed family $\mathcal{P}$ of Hilbertian seminorms on V such that for each $p \in \mathcal{P}$ there exists $q \in \mathcal{P}$ with $\operatorname{tr}(p/q) < \infty$.

Definition 1.4 is equivalent to the more traditional ones in [11] and [22], which we report in Appendix 4.2 for the convenience of the reader (see Definition 4.10 and Definition 4.11).

Note that a nuclear topology can be always constructed on every vector space V. However, this nuclear topology has typically no relation with a pre-given topology τ_V on V. However, when (V, τ_V) is a separable locally convex space with a Schauder basis, there exists a dense subspace U of V on which a nuclear topology stronger than $\tau_V \restriction_U$ can be constructed.

1.2. Probabilities on finite dimensional Hilbertian seminormed spaces

In the following we introduce a fundamental result about the support localization of a Radon measure defined on the dual of a finite dimensional real vector space, namely Lemma 1.6, which is inspired by [29, Fundamental lemma (p. 24)] and will play a crucial role in the proof of our main result Theorem 2.5. For this, let us recall two properties of the Gaussian measure on a finite dimensional real vector space endowed with a Hilbertian seminorm (see [29, p. 26-28] for a proof).

Proposition 1.5. *Let q be a Hilbertian seminorm on an n−dimensional $\mathbb{R}$−vector space V with* $\ker(q) = \{0\}$ *and E a complete q−orthonormal system of V. Let γ be the Gaussian measure on V, i.e.,*

$$\mathrm{d}\gamma(v) := (2\pi)^{-\frac{n}{2}} \exp\left(-\tfrac{1}{2}q(v)^2\right) \mathrm{d}\lambda(v),$$

where λ is the measure on V corresponding to the Lebesgue measure on $\mathbb{R}^n$ under the identification $V \to \mathbb{R}^n, v \mapsto (\langle v, e\rangle_q)_{e\in E}$. Then the following properties hold

(i) $\int \langle v, w\rangle_q^2 \mathrm{d}\gamma(v) = 1$ *for all $w \in V$ such that $q(w) = 1$.*
(ii) $\gamma(\{v \in V : |\ell(v)| \geq 1\}) \geq 7^{-1}$ *for all $\ell \in V'$ with $q'(\ell) \geq 1$, where q' denotes the operator seminorm on V'.*

Lemma 1.6. *Let p and q Hilbertian seminorms on a finite dimensional $\mathbb{R}$–vector space such that $\mathrm{tr}(p/q) < \infty$ and let μ be a probability measure on V'.*
If for every $\varepsilon > 0$ there exists a $\delta_\varepsilon > 0$ such that $\mu(\{l \in V' : |l(v)| \geq 1\}) \leq \varepsilon$ for all $v \in B_{\delta_\varepsilon}(p)$, then for these $\varepsilon, \delta_\varepsilon$ we have that

$$\mu(B_1(q')) \geq 1 - 7(\varepsilon + \mathrm{tr}(p/\delta_\varepsilon q)),$$

where q' denotes the operator seminorm on V'.

Proof. Let μ be a probability measure on V'. W.l.o.g. we can assume $\ker(q) = \{0\}$, because otherwise for each complement W of $\ker(q)$ in V we have μ–almost surely that $B_1(q') = \{l \in V' : |l(w)| \leq q(w)\ \forall\ w \in W\}$ holds. Indeed, on the one hand, it is immediate that $B_1(q') = \{l \in V' : q'(l) \leq 1\} = \{l \in V' : |l(v)| \leq q(v) \text{ for all } v \in V\} \subseteq \{l \in V' : |l(w)| \leq q(w) \text{ for all } w \in W\}$. On the other hand, since $p^2 \leq \mathrm{tr}(p/q)q^2$, we have that $\ker(q) \subseteq \ker(p) \subset B_\delta(p)$ for all $\delta > 0$ and so the assumption on μ in Lemma 1.6 ensures that $\mu(\{l \in V' : |l(v)| \geq 1\}) = 0$ for all $v \in \ker(q)$, i.e., $\mu(\{l \in V' : |l(v)| = 0,\ \forall v \in \ker(q)\}) = 1$, which immediately provides that $\mu\left\{B_1(q') \setminus \{l \in V' : |l(w)| \leq q(w) \text{ for all } w \in W\}\right\} = 0$.

Consider

$$D := (\mu \times \gamma)(\{(l, v) \in V' \times V : |l(v)| \geq 1\}),$$

where $\mu \times \gamma$ denotes the product measure between the given measure μ on V' and the Gaussian measure γ on V. Now, let $l \in V' \setminus B_1(q')$. Then Fubini's theorem on the one hand, combined with Proposition 1.5-(ii), provides that

$$D = \int_{V'} \gamma(\{v \in V : |l(v)| \geq 1\})\mathrm{d}\mu(l) \geq 7^{-1}\mu(V' \setminus B_1(q')) = 7^{-1}\left(1 - \mu(B_1(q'))\right),$$

and, on the other hand, combined with the assumption yields that

$$D = \int_{V} \mu(\{l \in V' : |l(v)| \geq 1\})\mathrm{d}\gamma(v) \leq \varepsilon\gamma(B_{\delta_\varepsilon}(p)) + \gamma(V \setminus B_{\delta_\varepsilon}(p)). \tag{1.4}$$

Moreover, by [7, V, §4.8, Theorem 2], there exists a complete $q-$orthonormal system E of V that is $p-$orthogonal. In particular, $p(v)^2 = \sum_{e\in E} \langle v, e\rangle_q^2 p(e)^2$ holds for all $v \in V$, which combined with Proposition 1.5-(i) and (1.3) gives

$$\gamma(V \setminus B_{\delta_\varepsilon}(p)) \leq {\delta_\varepsilon}^{-2} \int_V p(v)^2 \mathrm{d}\gamma(v) = \delta_\varepsilon^{-2} \sum_{e\in E} p(e)^2 \int_V \langle v, e\rangle_q^2 \mathrm{d}\gamma(v) = \delta_\varepsilon^{-2} \mathrm{tr}(p/q).$$

The latter together with (1.4) and Lemma 1.3-(ii) provides

$$D \leq \varepsilon + \delta_\varepsilon^{-2} \mathrm{tr}(p/q) = \varepsilon + \mathrm{tr}(p/\delta_\varepsilon q). \tag{1.5}$$

Combining (1.4) and (1.5) yields the assertion. □

2. Main results

In this section we are going to present our main results concerning the Main Question for a unital commutative real algebra A generated by a vector space V first endowed with a Hilbertian seminorm q and then with a nuclear topology. More precisely, in Subsection 2.2 we first establish a criterion for the existence of a representing measure with support contained in the set of characters of A whose restrictions to V are $q-$continuous (see Theorem 2.5 and Remark 2.9, as well as Theorem 2.8). When the seminorm q is defined on the full algebra A, i.e., $A = V$, this result provides in particular a criterion for the existence of a representing measure on the Gelfand spectrum of q. We actually show that when $L(a^2) \leq Cq(a)^2$ for all $a \in A$ for some $C > 0$ then it is enough to check the latter criterion just on a dense subalgebra of A (see Theorem 2.10). Moreover, in Lemma 2.18 we provide an explicit bound on L which guarantees the existence of a Hilbertian seminorm q on A satisfying our criteria.

Exploiting our general criteria, in Corollary 2.16, we identify more concrete sufficient conditions on L and q for the existence of such a representing measure for L. Those allow us to clarify in Subsection 2.4 the relation between the solvability of the Main Question and the presence of a nuclear topology on V. Our general criteria are based on the projective limit approach introduced in [18] which allows to reduce the Main Question to a family of finite-dimensional moment problems whose solutions satisfy a concentration condition to which we dedicate Subsection 2.1.

2.1. The concentration condition

Let A be a unital commutative $\mathbb{R}$-algebra generated by a linear subspace $V \subseteq A$ such that $X(A) \neq \emptyset$, and L a normalized linear functional on A.

As already mentioned, in proving our main results for the Main Question we will exploit the projective limit approach we developed in [18]. This is based on the construction of $(X(A), \tau_{X(A)})$ together with the maps $\{\pi_S : S \in J\}$ as the projective limit of the projective system of Hausdorff spaces $\{(X(S), \tau_{X(S)}), \pi_{S,T}, J\}$, where

$$J := \{\langle W\rangle : W \text{ finite dimensional subspace of } V\}$$

is ordered by inclusion, $\langle W\rangle$ denotes the subalgebra of A generated by W, $\tau_{X(S)}$ is the weak topology on $X(S)$, for any S, T subalgebras of A with $S \subseteq T$ the map $\pi_{S,T} : X(T) \to X(S)$ is the natural restriction and $\pi_S := \pi_{S,A}$. The corresponding projective system of measurable spaces is given by $\{(X(S), \mathcal{B}(\tau_{X(S)})), \pi_{S,T}, J\}$, where $\mathcal{B}(\tau_{X(S)})$ is the Borel $\sigma-$algebra w.r.t. $\tau_{X(S)}$. Recall that this means that $\pi_{S,T}$ is measurable for all $S \subseteq T$ in J and that $\pi_{S,T} \circ \pi_{T,R} = \pi_{S,R}$ for all $S \subseteq T \subseteq R$ in J. Note that replacing in the above construction J by a cofinal subset J' of J, i.e., for every $i \in J$ there exists some $j \in J'$ such that $j \geq i$, does not change the projective limit (see e.g. [6, III., §7.2, Proposition 3], [18, Proposition 1.3].)

Roughly speaking, in [18], we establish that there exists a representing Radon measure for L on A supported in $(X(A), \mathcal{B}(\tau_{X(A)}))$ if and only if for each $S \in J$ there exists a representing Radon measure ν_S supported in $(X(S), \mathcal{B}(\tau_{X(S)}))$ such that $\{\nu_S : S \in J\}$ fulfills the so-called Prokhorov condition. In the next subsection we will exploit this result when studying the Main Question for V endowed with the topology τ_V induced by a Hilbertian seminorm and we will exploit the given topological structure on V to prove that the Prokhorov condition (see [18, Section 1.2] and references therein) is satisfied whenever $\{\nu_S : S \in J\}$ fulfills the following concentration property.

Definition 2.1. Given a seminorm p on V and for each $S \in J$ a Radon measure ν_S on $(X(S), \mathcal{B}(\tau_{X(S)}))$, we say that $\{\nu_S : S \in J\}$ is $p-$*concentrated* (or *concentrated w.r.t.* p) if

$$\forall \varepsilon > 0\, \exists \delta > 0\colon \forall S \in J, \forall a \in B_\delta(p) \cap S, \nu_S(\{\alpha \in X(S) : |\alpha(a)| \geq 1\}) \leq \varepsilon. \tag{2.1}$$

This definition is an adaptation to our setting of the notion of *continuity for cylindrical measures* introduced in [9, 16, Chapter IV, Section 1.4]. It also easily relates to the notion of *concentrations of cylindrical measures* in [27, Definition 1, p.192]. In fact, (2.1) is weaker than assuming that the cylindrical quasi-measure associated to $\{\nu_S : S \in J\}$ is cylindrically concentrated on $\{\mathfrak{sp}_C(p) : C > 0\}$, namely $\forall \varepsilon > 0\, \exists \delta > 0\colon \forall S \in J$, $\nu_S(\pi_S(\mathfrak{sp}_\delta(p))) \geq 1 - \varepsilon$.

Let us now provide a useful characterization of the $p-$concentration of a collection of Radon measures.

Proposition 2.2. *Given a seminorm p on V and for each $S \in J$ a Radon measure ν_S on $(X(S), \mathcal{B}(\tau_{X(S)}))$, we have that $\{\nu_S : S \in J\}$ is $p-$concentrated if and only if the following holds*

$$\forall \varepsilon > 0\, \exists \gamma > 0 \colon \forall S \in J, \forall a \in S \cap V, \nu_S(\{\alpha \in X(S) : |\alpha(a)| \leq \gamma p(a)\}) \geq 1 - \varepsilon. \tag{2.2}$$

Proof. Suppose (2.1) holds and fix $\varepsilon > 0$. Taking $0 < \delta' < \delta$ with δ as in (2.1), we have that (2.2) holds for $\gamma = \frac{1}{\delta'}$. In fact, for any $S \in J$, let $b \in S \cap V$ and distinguish the following two cases.

- If $p(b) \neq 0$, then $\frac{\delta' b}{p(b)} \in B_\delta(p) \cap S$ and so (2.1) provides that $\nu_S(\{\alpha \in X(S) : |\alpha(b)| \geq \frac{p(b)}{\delta}\}) \leq \varepsilon$, which implies $\nu_S(\{\alpha \in X(S) : |\alpha(b)| \leq \frac{p(b)}{\delta'}\}) \geq 1 - \varepsilon$.
- If $p(b) = 0$, then clearly $span(b) \subseteq B_\delta(p) \cap V \cap S$ and so (2.1) gives that $\forall \lambda > 0$, $\forall a \in span(b)$, $\nu_S\left(\{\alpha \in X(S) : |\alpha(a)| \geq 1\}\right) \leq \lambda$, i.e., $\forall a \in span(b)$, $\nu_S\left(\{\alpha \in X(S) : |\alpha(a)| < 1\}\right) = 1$. Then

$$\forall r > 0,\ \nu_S\left(\left\{\alpha \in X(S) : |\alpha(b)| < \frac{1}{r}\right\}\right) = 1,$$

  and so we get $\nu_S(\{\alpha \in X(S) : |\alpha(b)| = 0\}) = 1$, which in particular gives that $\nu_S(\{\alpha \in X(S) : |\alpha(b)| \leq \frac{p(b)}{\delta'}\}) = 1 \geq 1 - \varepsilon$.

Conversely, suppose (2.2) holds and fix $\varepsilon > 0$. Taking γ as in (2.2), we have that (2.1) holds for $\delta \leq \frac{1}{\gamma}$. In fact, for any $S \in J$, let $b \in B_\delta(p) \cap S$ and distinguish the following two cases.

- If $p(b) \neq 0$, then (2.2) provides that $\nu_S(\{\alpha \in X(S) : |\alpha(b)| \leq \gamma p(b)\}) \geq 1 - \varepsilon$ which implies $\nu_S(\{\alpha \in X(S) : |\alpha(b)| < 1\}) \geq 1 - \varepsilon$.
- If $p(b) = 0$, then (2.2) provides that $\nu_S(\{\alpha \in X(S) : |\alpha(b)| = 0\}) \geq 1 - \varepsilon$, i.e., $\nu_S(\{\alpha \in X(S) : |\alpha(b)| > 0\}) \leq \varepsilon$, which implies $\nu_S(\{\alpha \in X(S) : |\alpha(b)| \geq 1\}) \leq \varepsilon$. □

Remark 2.3. If ν is a Radon measure on $X(A)$ s.t. $\{\pi_{S\#}\nu : S \in J\}$ is $p-$concentrated, then

$$\nu(\{\alpha \in X(A) : |\alpha(b)| = 0\ \forall b \in \ker(p)\}) = 1, \tag{2.3}$$

where $\pi_{S\#}\nu$ denotes the pushforward measure of ν w.r.t. π_S. Indeed, using the same argument as in the proof of Proposition 2.2, we can show that $\forall b \in \ker(p), \nu(\{\alpha \in X(A) : |\alpha(b)| = 0\}) = \pi_{\langle b\rangle\#}\nu(\{\alpha \in X(\langle b\rangle) : |\alpha(b)| = 0\}) = 1$. This together with the fact that $\{\alpha \in X(A) : |\alpha(b)| = 0\}$ is a closed subset of $X(A)$ and ν a Radon measure yields (2.3) by [27, Part I, Chapter I, 6.(a)].

Let us establish now a sufficient condition for the $p-$concentration of a collection of representing measures, which we will often exploit in the rest of the article.

Lemma 2.4. *Let A be an algebra generated by a linear subspace V, p a seminorm on V and L a normalized linear functional on A. If for each $S \in J$ there exists a representing measure ν_S for $L\restriction_S$ and*

$$\exists\, C > 0 \,:\, L(a^2) \leq Cp(a)^2 \qquad \textit{for all } a \in V \tag{2.4}$$

holds, then $\{\nu_S : S \in J\}$ is $p-$concentrated.

Proof. Let $\varepsilon > 0$ and take $\delta := \sqrt{\frac{\varepsilon}{C}}$. Then for all $a \in B_\delta(p) \cap S$

$$\nu_S(\{\alpha \in X(S) : |\alpha(a)| \geq 1\}) \leq \int_{X(S)} \hat{a}^2 d\nu_S = L(a^2) \leq Cp(a)^2 \leq C\delta^2 \leq \varepsilon$$

i.e., $\{\nu_S : S \in J\}$ fulfills (2.1). □

With a similar proof Lemma 2.4 holds with (2.4) replaced by the following condition:

$$\forall \varepsilon > 0\ \exists\, C > 0 : L(a^2) \leq Cp(a)^2 + \varepsilon \qquad \text{for all } a \in V. \tag{2.5}$$

2.2. The case when τ_V generated by a Hilbertian seminorm

Theorem 2.5. *Let A be an algebra generated by a linear subspace $V \subseteq A$, q a Hilbertian seminorm on V such that $\{\alpha \in X(A) : \alpha\restriction_V$ is q–continuous$\} \neq \emptyset$ and $J := \{\langle W\rangle :$ W finite dimensional subspace of $V\}$. Let L be a normalized linear functional on A.*

There exists a representing Radon measure ν for L with support contained in $\{\alpha \in X(A) : \alpha\restriction_V$ is q–continuous$\}$ if and only if there exists a Hilbertian seminorm p on V with $\operatorname{tr}(p/q) < \infty$ and for each $S \in J$ there exists a representing Radon measure ν_S for $L\restriction_S$ with support contained in $X(S)$ and such that $\{\nu_S : S \in J\}$ is $p-$concentrated.

Proof. For each $S \in J$, let ν_S be a representing Radon measure for $L\restriction_S$ with support contained in $X(S)$ and p be a Hilbertian seminorm p on V with $\operatorname{tr}(p/q) < \infty$ such that $\{\nu_S : S \in J\}$ is $p-$concentrated. Let us first show that the family $\{\nu_S : S \in J\}$ fulfills the so-called Prokhorov condition by means of the characterization in [18, Proposition 1.18], that is, we aim to show that for all $\varepsilon > 0$ and for all $S \in J$, there exists $K^{(S)} \subseteq X(S)$ compact such that $\nu_S(K^{(S)}) \geq 1 - \varepsilon$ and $\pi_{S,T}(K^{(T)}) \subseteq K^{(S)}$ for all $T \in J$ with $S \subseteq T$.

For any $\varepsilon > 0$, since $\{\nu_S : S \in J\}$ is $p-$concentrated and $\operatorname{tr}(p/q) < \infty$, we can take $\delta_\varepsilon > 0$ as in (2.1) and set $r_\varepsilon := (\delta_\varepsilon\sqrt{\varepsilon})^{-1}\sqrt{\operatorname{tr}(p/q)}\, q$. For each $S \in J$, define $K^{(S)} := \{\alpha \in X(S) : |\alpha(v)| \leq r_\varepsilon(v)$ for all $v \in S \cap V\}$. Then $K^{(S)}$ is compact in $X(S)$ as it is closed and embeds into the compact product $\prod_{v\in S\cap V}[-r_\varepsilon(v), r_\varepsilon(v)]$ via the continuous map $\alpha \mapsto (\alpha(v))_{v\in S\cap V}$. Now for any $S \subseteq T$ in J the inclusion $\pi_{S,T}(K^{(T)}) \subseteq K^{(S)}$ holds by definition and for each $S \in J$ the estimate $\nu_S(K^{(S)}) \geq 1 - 14\varepsilon$ holds by Lemma 2.6 below. Hence, the family $\{\nu_S : S \in J\}$ fulfills Prokhorov's condition and so we can apply [18, Theorem 3.9-(ii)], which guaranteed the existence of a representing Radon measure ν for L with support contained in $X(A)$. It remains to show that the support of ν is contained in $\{\alpha \in X(A) : \alpha\restriction_V \text{ is } q\text{–continuous}\}$. For this, set $\phi_V : X(A) \to V^*, \alpha \mapsto \alpha\restriction_V$ and

$$K_\varepsilon := \{\alpha \in X(A) : |\alpha(v)| \leq r_\varepsilon(v) \text{ for all } v \in V\} = \bigcap_{S\in J} \pi_S^{-1}(K^{(S)}).$$

Then [5, Propositions 7.2.2-(i) and 7.2.5-(iii)] and Lemma 2.6 imply that

$$\nu(K_\varepsilon) = \lim_{S\in J} \nu_S(K^{(S)}) \geq 1 - 14\varepsilon.$$

Since $K_\varepsilon \subseteq {\phi_V}^{-1}(V'_{r_\varepsilon}) = {\phi_V}^{-1}(V'_q)$ for all $\varepsilon > 0$ this yields that $\nu({\phi_V}^{-1}(V'_q)) = 1$, i.e., ν has support contained in ${\phi_V}^{-1}(V'_q) = \{\alpha \in X(A) : \alpha\restriction_V \text{ is } q\text{–continuous}\}$.

Conversely, let ν be a representing Radon measure for L with support contained in $\{\alpha \in X(A) : \alpha\restriction_V \text{ is } q\text{–continuous}\}$. Then, for each $S \in J$, the push-forward $\nu_S := \pi_{S\#}\nu$ is a representing Radon measure for $L\restriction_S$ with support contained in $X(S)$. For each $n \in \mathbb{N}$, set $K_n := {\phi_V}^{-1}(B_n(q'))$ and define

$$p(v)^2 := \sum_{n=1}^{\infty} \frac{1}{n^4} \int_{K_n} \hat{v}^2 \mathrm{d}\nu \qquad \text{for all } v \in V. \tag{2.6}$$

It is easy to verify that p defines a Hilbertian seminorm on V. Then for each $E \in \mathrm{FON}(q)$ we have that

$$\sum_{e\in E} p(e)^2 \overset{(2.6)}{=} \sum_{n=1}^{\infty} \frac{1}{n^4} \int_{K_n} \sum_{e\in E} \hat{e}^2 \mathrm{d}\nu \overset{\text{Lemma 2.7}}{\leq} \sum_{n=1}^{\infty} \frac{1}{n^4} \int_{K_n} n^2 \mathrm{d}\nu \leq \sum_{n=1}^{\infty} \frac{1}{n^2} < 2,$$

that is, $\mathrm{tr}(p/q) \leq 2 < \infty$. To show that $\{\nu_S : S \in J\}$ is p–concentrated, let $\varepsilon > 0$ and take $n \in \mathbb{N}$ such that $\nu(X(A) \setminus K_n) \leq 2^{-1}\varepsilon$. Then there exists $\delta > 0$ such that $n^4\delta^2 \leq 2^{-1}\varepsilon$ and so, for each $S \in J$ and each $v \in B_\delta(p) \cap S$, we obtain that

$$\begin{aligned}\nu_S(\{\alpha \in X(S) : |\alpha(v)| \geq 1\}) &\leq \nu(\{\alpha \in X(A) : |\alpha(v)| \geq 1\} \cap K_n) + \nu(X(A) \setminus K_n) \\ &\leq \int_{K_n} \hat{v}^2 \mathrm{d}\nu_S + 2^{-1}\varepsilon \leq n^4 p(v)^2 + 2^{-1}\varepsilon \leq \varepsilon,\end{aligned}$$

i.e., (2.1) holds. □

Lemma 2.6. *For each $S \in J$, the estimate $\nu_S(K^{(S)}) \geq 1 - 14\varepsilon$ holds.*

Proof. Let $S \in J$ and $I := \{W \subseteq S \cap V : W \text{ finite dimensional subspace of } S \cap V\}$. Let $W \in I$ and consider the continuous restriction map $\phi_W : X(S) \to W'$. Then the push-forward $\mu'_W := \phi_{W\#}\nu_S$ is a probability measure on W' satisfying the following inequality for the ε and δ_ε as in the first part of the proof of Theorem 2.5:

$$\mu'_W(\{l \in W' : |l(w)| \geq 1\}) = \nu_S(\{\alpha \in X(S) : |\alpha(w)| \geq 1\}) \leq \varepsilon$$

for all $w \in B_{\delta_\varepsilon}(p\restriction_W)$. Hence, the assumption of Lemma 1.6 holds for these ε and δ_ε and thus,

$$\nu_S({\phi_W}^{-1}(B_1(r_\varepsilon\restriction'_W))) = \mu'_W(B_1(r_\varepsilon\restriction'_W))) \geq 1 - 7(\varepsilon + \mathrm{tr}(p\restriction_W / \delta_\varepsilon r_\varepsilon\restriction_W)) = 1 - 14\varepsilon$$

as $\mathrm{tr}(p\restriction_W / \delta_\varepsilon r_\varepsilon\restriction_W) \leq \mathrm{tr}(p/\delta_\varepsilon r_\varepsilon)$ and, by Lemma 1.3-((ii)), $\mathrm{tr}(p/\delta_\varepsilon r_\varepsilon) \leq \varepsilon$.

Since $K^{(S)} = \bigcap_{W\in I} {\phi_W}^{-1}(B_1(r_\varepsilon\restriction_{W'}))$ by definition, [5, Propositions 7.2.2-(i) and 7.2.5-(iii)] imply that $\nu_S(K^{(S)}) = \lim_{W\in I} \nu_S({\phi_W}^{-1}(B_1(r_\varepsilon\restriction_{W'}))) \geq 1 - 14\varepsilon$. □

Lemma 2.7. *Let $n \in \mathbb{N}$ and $E \in \mathrm{FON}(q)$. Then $\sum\limits_{e\in E} \hat{e}(\alpha) \leq n^2$ for all $\alpha \in K_n$.*

Proof. Let $\alpha \in K_n$ and set $H := \mathrm{span}(E)$ (for convenience, set $\alpha = \alpha\restriction_H$ and $q = q\restriction_H$). Since $E \in \mathrm{FON}(q)$ is finite, the space (H, q) is Hilbertian and E is a complete $q-$orthonormal system. In particular, $\alpha\restriction_H \in B_n(q\restriction'_H)$ and by the Riesz representation theorem there exists $a \in H$ such that $\alpha(x) = \langle x, a\rangle_q$ for all $x \in H$ and $q(a) = q'(\alpha) \leq n$. Therefore,

$$\sum_{e\in E} \hat{e}(\alpha)^2 = \sum_{e\in E} \alpha(e)^2 = \sum_{e\in E} \langle e, a\rangle_q^2 = q(a)^2 \leq n^2$$

yields the assertion. □

Using exactly the same proof scheme but exploiting [18, Corollary 3.11-(ii)] instead of [18, Theorem 3.9-(ii)], it is easy to obtain the following more general version of Theorem 2.5 including the localization of the support of the representing measure.

Theorem 2.8. *Let A be an algebra generated by a linear subspace $V \subseteq A$, $K \subseteq X(A)$ closed, q a Hilbertian seminorm on V such that $\{\alpha \in K : \alpha\restriction_V$ is $q-$continuous$\} \neq \emptyset$ and $J := \{\langle W\rangle : W$ finite dimensional subspace of $V\}$. Let L be a normalized linear functional on A and Q a quadratic module such that $K = K_Q$.*

There exists a representing Radon measure ν for L with support contained in $\{\alpha \in K : \alpha\restriction_V$ is $q-$continuous$\}$ if and only if there exists a Hilbertian seminorm p on V with $\mathrm{tr}(p/q) < \infty$ and for each $S \in J$ there exists a representing Radon measure ν_S for $L\restriction_S$ with support contained in $K_{Q\cap S}$ and such that $\{\nu_S : S \in J\}$ is $p-$concentrated.

Remark 2.9.

(1) If in Theorem 2.5 (resp. Theorem 2.8) we assume for each $S \in J$ the *uniqueness* of the representing measure for $L\restriction_S$ with support contained in $X(S)$ (resp. in $K_{Q\cap S}$), then

by [18, Remark 3.12-(ii)] we get the uniqueness of the corresponding representing measure for L.

(2) If in Theorem 2.5 (resp. Theorem 2.8) we take $V = A$, we obtain a criterion for the existence of a representing measure for L with support contained in $\mathfrak{sp}(q)$ (resp. on $K_Q \cap \mathfrak{sp}(q)$).

(3) Combining Theorem 2.5 (resp. Theorem 2.8) and Remark 2.3, it is easy to see that if there exists a representing measure for L with support contained in $\{\alpha \in X(A) : \alpha\restriction_V \text{ is } q\text{-continuous}\}$ then L vanishes on $\ker(p)$ and so on $\ker(q)$.

When A is endowed with a Hilbertian seminorm q and there exists $C > 0$ such that $L(a^2) \leq Cq(a)^2$ for all $a \in A$, we can characterize the representing measures for L with support contained in $\mathfrak{sp}(q)$ only through conditions on a dense subalgebra of A.

Theorem 2.10. *Let A be an algebra, q a Hilbertian seminorm on A with $\mathfrak{sp}(q) \neq \emptyset$, B a subalgebra of A which is dense in (A, q) and $I := \{\langle W\rangle : W$ finite dimensional subspace of $B\}$. Let L be a normalized linear functional on A for which there exists $C > 0$ such that $L(a^2) \leq Cq(a)^2$ for all $a \in A$.*

There exists a representing Radon measure ν for L with support contained in $\mathfrak{sp}(q)$ if and only if there exists a Hilbertian seminorm p on B with $\mathrm{tr}(p/q\restriction_B) < \infty$ and for each $S \in I$ there exists a representing Radon measure ν_S for $L\restriction_S$ with support contained in $X(S)$ such that $\{\nu_S : S \in I\}$ is p-concentrated.

Proof. By the density of B in (A, q), there is a one-to-one correspondence between the set of all q-continuous characters of A and the set of all q-continuous characters of B, which will therefore both denote simply by $\mathfrak{sp}(q)$. Moreover, let $\tau_{\mathfrak{sp}(q)^A}$ (resp. $\tau_{\mathfrak{sp}(q)^B}$) be the weakest topology on $\mathfrak{sp}(q)$ which makes $\hat{a} : \mathfrak{sp}(q) \to \mathbb{R}, \alpha \mapsto \alpha(a)$ continuous for all $a \in A$ (resp. for all $a \in B$) and by $\mathcal{B}(\tau_{\mathfrak{sp}(q)^A})$ (resp. $\mathcal{B}(\tau_{\mathfrak{sp}(q)^B})$) the associated Borel σ-algebra. We refer to Appendix 4.3 for the proof that

$$\mathcal{B}(\tau_{\mathfrak{sp}(q)^A}) = \mathcal{B}(\tau_{\mathfrak{sp}(q)^B}), \tag{2.7}$$

and so we will not distinguish between the measurable spaces $(\mathfrak{sp}(q), \mathcal{B}(\tau_{\mathfrak{sp}(q)^A}))$ and $(\mathfrak{sp}(q), \mathcal{B}(\tau_{\mathfrak{sp}(q)^B}))$, which will be both simply denoted by $(\mathfrak{sp}(q), \mathcal{B}(\tau_{\mathfrak{sp}(q)}))$.

Suppose that there exists a representing Radon measure ν for L with support contained in $\mathfrak{sp}(q)$. Then, applying Theorem 2.5 for $V = A = B$, we get that there exists a Hilbertian seminorm p on B with $\mathrm{tr}(p/q) < \infty$ and for each $S \in I$ there exists a representing Radon measure ν_S for $L\restriction_S$ with support contained in $X(S)$ such that $\{\nu_S : S \in I\}$ is p-concentrated.

Conversely, suppose there exists a Hilbertian seminorm p on B with $\mathrm{tr}(p/q) < \infty$ and for each $S \in I$ there exists a representing Radon measure ν_S for $L\restriction_S$ with support contained in $X(S)$ such that $\{\nu_S : S \in I\}$ is p-concentrated. Then, applying Theorem 2.5 for $V = A = B$ (see also Remark 2.9-(2)), we obtain that there exists a representing measure ν for $L\restriction_B$ with support contained in $\mathfrak{sp}(q)$. We aim to prove that ν is actually a representing measure for L, so it remains to show that $L(a)=\int \hat{a}(\alpha)d\nu(\alpha),\ \forall a\in A\setminus B$.

Let $a\in A\setminus B$. By the density of B in (A,q), there exists a sequence $(b_n)_{n\in\mathbb{N}}\subseteq B$ such that $q(b_n-a)\to 0$ as $n\to\infty$. Hence, for any $\alpha\in\mathfrak{sp}(q)$ we get $\alpha(b_n)\to\alpha(a)$ as $n\to\infty$, i.e., $\lim_{n\to\infty}\hat{b_n}(\alpha)=\hat{a}(\alpha)$. Then, using Fatou's lemma, we obtain that

$$\int\limits_{\mathfrak{sp}(q)}\hat{a}(\alpha)^2d\nu=\int\limits_{\mathfrak{sp}(q)}\lim_{n\to\infty}\hat{b_n}(\alpha)^2d\nu\leq\liminf_{n\to\infty}\int\limits_{\mathfrak{sp}(q)}\hat{b_n}(\alpha)^2d\nu=\liminf_{n\to\infty}L(b_n^2)=L(a^2),$$

where in the last equality we used the Cauchy–Bunyakovsky–Schwarz inequality and the existence of $C>0$ such that $L(a^2)\leq Cq(a)^2$ for all $a\in A$. Hence, $\hat{a}\in\mathcal{L}^2(\mathfrak{sp}(q),\mathcal{B}(\tau_{\mathfrak{sp}(q)}))$ and so $\hat{a}\in\mathcal{L}^1(\mathfrak{sp}(q),\mathcal{B}(\tau_{\mathfrak{sp}(q)}))$.

Then for any $M>0$ we have that

$$\begin{aligned}\int\limits_{\mathfrak{sp}(q)}|\hat{a}(\alpha)-\hat{b_n}(\alpha)|d\nu(\alpha)\leq&\int\limits_{\mathfrak{sp}(q)}\left|\hat{a}(\alpha)\mathbb{1}_{\{\beta:|\hat{a}(\beta)|\leq M\}}(\alpha)-\hat{b}_n(\alpha)\mathbb{1}_{\{\beta:|\hat{b}_n(\beta)|\leq M\}}(\alpha)\right|d\nu(\alpha)\\&+\int\limits_{\mathfrak{sp}(q)}\left|\hat{a}(\alpha)\mathbb{1}_{\{\beta:|\hat{a}(\beta)|\leq M\}}(\alpha)-\hat{a}(\alpha)\right|d\nu(\alpha)\\&+\int\limits_{\mathfrak{sp}(q)}\left|\hat{b_n}(\alpha)\mathbb{1}_{\{\beta:|\hat{b}_n(\beta)|\leq M\}}(\alpha)-\hat{b}_n(\alpha)\right|d\nu(\alpha)\\=&\int\limits_{\mathfrak{sp}(q)}\left|\hat{a}(\alpha)\mathbb{1}_{\{\beta:|\hat{a}(\beta)|\leq M\}}(\alpha)-\hat{b}_n(\alpha)\mathbb{1}_{\{\beta:|\hat{b}_n(\beta)|\leq M\}}(\alpha)\right|d\nu(\alpha)\\&+\int\limits_{\mathfrak{sp}(q)}|\hat{a}(\alpha)|\,\mathbb{1}_{\{\beta:|\hat{a}(\beta)|>M\}}(\alpha)d\nu(\alpha)\\&+\int\limits_{\mathfrak{sp}(q)}\left|\hat{b}_n(\alpha)\right|\mathbb{1}_{\{\beta:|\hat{b}_n(\beta)|>M\}}(\alpha)d\nu(\alpha)\end{aligned}\tag{2.8}$$

Using that $\mathbb{1}_{\{\beta:|\hat{b}_n(\beta)|>M\}}(\alpha)\leq\frac{1}{M}|\hat{b}_n(\alpha)|$ and that ν is a representing measure for $L\restriction_B$, we easily see that:

$$\int\limits_{\mathfrak{sp}(q)}\left|\hat{b}_n(\alpha)\right|\mathbb{1}_{\{\beta:|\hat{b}_n(\beta)|>M\}}(\alpha)\leq\frac{1}{M}\int\limits_{\mathfrak{sp}(q)}\hat{b}_n(\alpha)^2d\nu(\alpha)=\frac{L(b_n^2)}{M}\to\frac{L(a^2)}{M},\ \text{as } n\to\infty.$$

Therefore, passing to the limit for $n\to\infty$ in (2.8), we get that for any $M>0$:

$$\lim_{n\to\infty}\int\limits_{\mathfrak{sp}(q)}|\hat{a}(\alpha)-\hat{b_n}(\alpha)|d\nu(\alpha)\leq\int\limits_{\mathfrak{sp}(q)}|\hat{a}(\alpha)|\,\mathbb{1}_{\{\beta:|\hat{a}(\beta)|>M\}}(\alpha)d\nu(\alpha)+\frac{L(a^2)}{M}\tag{2.9}$$

Since $\hat{a} \in \mathcal{L}^1(\mathfrak{sp}(q), \mathcal{B}(\tau_{\mathfrak{sp}(q)}))$ and $|\hat{a}(\alpha)| \mathbb{1}_{\{\beta:|\hat{a}(\beta)|>M\}}(\alpha) \leq |\hat{a}(\alpha)|$ for all $M > 0$, we can apply the dominated convergence theorem, which ensures that:

$$\lim_{M\to\infty} \int\limits_{\mathfrak{sp}(q)} |\hat{a}(\alpha)| \mathbb{1}_{\{\beta:|\hat{a}(\beta)|>M\}}(\alpha) d\nu(\alpha) = \int\limits_{\mathfrak{sp}(q)} \lim_{M\to\infty} |\hat{a}(\alpha)| \mathbb{1}_{\{\beta:|\hat{a}(\beta)|>M\}}(\alpha) d\nu(\alpha) = 0$$

Hence, passing to the limit for $M \to \infty$ in (2.9), we obtain that:

$$\lim_{n\to\infty} \int\limits_{\mathfrak{sp}(q)} |\hat{a}(\alpha) - \hat{b_n}(\alpha)| d\nu(\alpha) = 0$$

and so

$$L(a) = \lim_{n\to\infty} L(b_n) = \lim_{n\to\infty} \int\limits_{\mathfrak{sp}(q)} \hat{b_n}(\alpha) d\nu(\alpha) = \int\limits_{\mathfrak{sp}(q)} \hat{a}(\alpha) d\nu(\alpha). \quad \square$$

With a similar proof, it is possible to show the following more general version of Theorem 2.10.

Theorem 2.11. *Let A be an algebra, $K \subseteq X(A)$ closed, q a Hilbertian seminorm on A with $\mathfrak{sp}(q) \cap K \neq \emptyset$, B a subalgebra of A which is dense in (A, q) and $I := \{\langle W \rangle :$ W finite dimensional subspace of $B\}$. Let L be a normalized linear functional on A for which there exists $C > 0$ such that $L(a^2) \leq Cq(a)^2$ for all $a \in A$ and Q a quadratic module such that $K = K_Q$.*

There exists a representing Radon measure ν for L with support contained in $\mathfrak{sp}(q) \cap K$ if and only if there exists a Hilbertian seminorm p on B with $\operatorname{tr}(p/q) < \infty$ and for each $S \in I$ there exists a representing Radon measure ν_S for $L\restriction_S$ with support contained in $K_{Q\cap S}$ such that $\{\nu_S : S \in I\}$ is $p-$concentrated.

Remark 2.12. All the results in this subsection also hold if the index set J (respectively I) is replaced by a cofinal subset of it as the correspondent projective limit does not change (see e.g. [6, III., §7.2, Proposition 3], [18, Proposition 1.3].)

2.3. A natural choice of a Hilbertian seminorm on A

Given a normalized $\sum A^2$–positive linear functional L, the map $(a, b) \mapsto L(ab)$ defines a symmetric positive semidefinite bilinear form and so the following is a natural Hilbertian seminorm on A

$$s_L(a) := \sqrt{L(a^2)} \text{ for all } a \in A. \tag{2.10}$$

Then, combining Lemma 2.4 with our main results Theorem 2.8 and Theorem 2.11, we easily obtain the following two results.

Corollary 2.13. *Let A be an algebra generated by a linear subspace $V \subseteq A$, $K \subseteq X(A)$ closed and $J := \{\langle W \rangle : W$ finite dimensional subspace of $V\}$. Let L be a normalized $\sum A^2$–positive linear functional L on A and Q a quadratic module such that $K = K_Q$.*

If there exists a Hilbertian seminorm q on V such that $\operatorname{tr}(s_L \restriction_V /q) < \infty$ and $\{\alpha \in K : \alpha \restriction_V$ is q–continuous$\} \neq \emptyset$ and for each $S \in J$ there exists a representing Radon measure ν_S for $L \restriction_S$ with support contained in $K_{Q \cap S}$, then there exists a representing measure for L with support contained in $\{\alpha \in K : \alpha \restriction_V$ is q–continuous$\}$.

Proof. Since $L(a^2) = s_L(a)^2$ for all $a \in V$, (2.4) holds for $p = s_L$ and $C = 1$. Hence, Lemma 2.4 ensures that $\{\nu_S : S \in I\}$ is s_L–concentrated. This together with the assumption $\operatorname{tr}(s_L \restriction_V /q) < \infty$ allows us to apply Theorem 2.8 for $p = s_L$, ensuring that there exists a representing Radon measure ν for L with support contained in $\{\alpha \in K : \alpha \restriction_V$ is q–continuous$\}$. □

Corollary 2.14. *Let (A, τ) be a locally convex topological algebra, B a sub-algebra of A which is dense in (A, τ) and $I := \{\langle W \rangle : W$ finite dimensional subspace of $B\}$. Let L be a normalized $\sum A^2$–positive linear functional on A and Q a quadratic module such that $K = K_Q$.*

If there exists a $\tau-$continuous Hilbertian seminorm q on A with $\operatorname{tr}(s_L/q) < \infty$ and $\mathfrak{sp}(q) \cap K \neq \emptyset$ and if for each $S \in I$ there exists a representing Radon measure ν_S for $L \restriction_S$ with support contained in $K_{Q \cap S}$, then there exists a representing measure for L with support contained in $\mathfrak{sp}(q) \cap K$.

Proof. The τ–continuity of q and the density of B in (A, τ) provide that B is dense in (A, q). Moreover, since $L(a^2) = s_L(a)^2$ for all $a \in A$, (2.4) holds for $p = s_L$, $C = 1$ and $V = A$, so Lemma 2.4 ensures that $\{\nu_S : S \in I\}$ is s_L-concentrated. Then, by Theorem 2.11, there exists a representing measure for L with support contained in $\mathfrak{sp}(q) \cap K$. □

Remark 2.15. In Corollary 2.13 and Corollary 2.14 we could actually replace A with $A/\ker(s_L)$, because it is readily seen from the Cauchy-Schwartz inequality that L vanishes on $\ker(s_L)$. Moreover, we have that $V/\ker(s_L \restriction_V) = V/(\ker(s_L) \cap V) = V/\ker(s_L)$ and, whenever $\operatorname{tr}(s_L \restriction_V /q) < \infty$ for some Hilbertian norm q, the space $V/\ker(s_L)$ endowed with the quotient seminorm induced by s_L (and also denoted by s_L with a slight abuse of notation) is separable. Hence, whenever these techniques work, $(V, s_L \restriction_V)$ is essentially a separable space.

Let us now exploit Corollary 2.13 to obtain more concrete sufficient conditions for the existence of a representing measure for L in presence of a fixed Hilbertian seminorm q on A.

Corollary 2.16. *Let A be an algebra generated by a linear subspace $V \subseteq A$, L a normalized linear functional on A, Q a quadratic module in A and q a Hilbertian seminorm on A with $\{\alpha \in K_Q : \alpha\restriction_V \text{ is } q\text{–continuous}\} \neq \emptyset$. If*

(a) $L(Q) \subseteq [0, \infty)$,
(b) for each $v \in V$, $\sum_{n=1}^{\infty} \frac{1}{\sqrt[2n]{L(v^{2n})}} = \infty$,
(c) $\operatorname{tr}(s_L\restriction_V /q) < \infty$,

then there exists a unique representing Radon measure ν for L with support contained in $\{\alpha \in K_Q : \alpha\restriction_V \text{ is } q\text{–continuous}\}$.

Proof. Let $J := \{\langle W\rangle : W \text{ finite dimensional subspace of } V\}$. By [18, Theorem 3.17-(i)], the assumptions (a) and (b) guarantee that for each $S \in J$ there exists a unique representing Radon measure ν_S for $L\restriction_S$ with support contained in $K_{Q\cap S}$. This together with the assumption (c) allows us to apply Corollary 2.13, ensuring that there exists unique representing Radon measure ν for L with support contained in $\{\alpha \in K_Q : \alpha\restriction_V$ is q–continuous$\}$. □

Remark 2.17. Corollary 2.16 still holds if we replace the assumptions (a) and (b) with the assumption (a') $L(Pos(K_Q)) \subseteq [0, +\infty)$ and in the proof we use [18, Theorem 3.14] instead of [18, Theorem 3.17]. However, under this replacement, the uniqueness is not anymore ensured.

The following lemma provides an explicit construction of a Hilbertian seminorm q as required in Corollary 2.13 and so in Corollary 2.16.

Lemma 2.18. *Let A be an algebra generated by a linear subspace $V \subseteq A$ and p a Hilbertian seminorm on V such that there exists a complete $p-$orthonormal system $\{e_n : n \in \mathbb{N}\}$ in V. Choose $(\lambda_n)_{n\in\mathbb{N}} \subseteq \mathbb{R}_{>0}$ such that $\sum_{n=1}^{\infty} \lambda_n^2 < \infty$. Then*

$$q(v) := \sqrt{\sum_{n=1}^{\infty} \lambda_n^{-2} \langle v, e_n\rangle_p^2}, \quad \forall v \in V$$

defines a Hilbertian seminorm on $U := \left\{v \in V : \sum_{n=1}^{\infty} \lambda_n^{-2}\langle v, e_n\rangle_p^2 < \infty\right\}$ such that $\operatorname{tr}(p\restriction_U /q) < \infty$ and U is dense in (V, p).

Proof. For each $a, b \in U$, let us define $\langle a, b\rangle_q := \sum_{n=1}^{\infty} \lambda_n^{-2}\langle a, e_n\rangle_p\langle b, e_n\rangle_p$ (note that the Cauchy-Schwartz inequality provides that $\langle a, b\rangle_q < \infty$, since $\langle v, v\rangle_q < \infty$ for all $v \in U$ by the definition of U). Then $\langle\cdot, \cdot\rangle_q$ is a symmetric positive semidefinite bilinear form on $U \times U$ and thus, $q(v) = \sqrt{\langle v, v\rangle_q}$ for all $v \in U$ defines a Hilbertian seminorm on U.

As $\{e_n : n \in \mathbb{N}\}$ is a complete $p-$orthonormal system in V, we have that Parseval's equality holds and so

$$\forall\ v \in U, \quad p(v)^2 = \sum_{n=1}^{\infty} \langle v, e_n \rangle_p^2 = \sum_{n=1}^{\infty} \lambda_n^2 \lambda_n^{-2} \langle v, e_n \rangle_p^2 \leq \left(\sup_{n \in \mathbb{N}} \lambda_n^2 \right) q(v)^2,$$

i.e., $\forall v \in U$, $p(v) \leq Cq(v)$ where $C := \sup_{n \in \mathbb{N}} \lambda_n^2$ is finite because of the assumption $\sum_{n=1}^{\infty} \lambda_n^2 < \infty$.

Moreover, since for all $i, j \in \mathbb{N}$ we have

$$\langle \lambda_i e_i, \lambda_j e_j \rangle_q = \sum_{n=1}^{\infty} \lambda_n^{-2} \langle \lambda_i e_i, e_n \rangle_p \langle \lambda_j e_j, e_n \rangle_p = \sum_{n=1}^{\infty} \lambda_n^{-2} (\lambda_i \delta_{i,n})(\lambda_j \delta_{j,n}) = \delta_{i,j},$$

the set $\{\lambda_n e_n : n \in \mathbb{N}\}$ is $q-$orthonormal. Moreover, for all $v \in U$ and all $n \in \mathbb{N}$ we have that $\langle v, \lambda_n e_n \rangle_q = \lambda_n^{-1} \langle v, e_n \rangle_p$ and so

$$\forall\ v \in U, \quad \langle v, v \rangle_q^2 = \sum_{n=1}^{\infty} \lambda_n^{-2} \langle v, e_n \rangle_p^2 = \sum_{n=1}^{\infty} \langle v, \lambda_n e_n \rangle_q^2,$$

i.e., Parseval's equality is satisfied, which is equivalent to say that $\{\lambda_n e_n : n \in \mathbb{N}\}$ is a complete $q-$orthonormal system in U by [7, Chapter 5, §2.3, Proposition 5]. Hence, using (1.3), we get that

$$\operatorname{tr}(p \restriction_U / q) = \sum_{n=1}^{\infty} p(\lambda_n e_n)^2 = \sum_{n=1}^{\infty} \lambda_n^2 p(e_n)^2 = \sum_{n=1}^{\infty} \lambda_n^2 < \infty \tag{2.11}$$

As U contains $\{e_n : n \in \mathbb{N}\}$, we get that U is dense in V. □

Remark 2.19.

(i) If (V, p) is Hausdorff and contains a countable total subset, then the existence of a complete $p-$orthonormal system in V is guaranteed by [7, Chapter 5, §2.4, Corollary p. V.24]. In particular, such a system exists when (V, p) is Hausdorff and separable.

(ii) If $\{f_n : n \in \mathbb{N}\}$ is another complete $p-$orthonormal system in V, then we get that $\sum_{n=1}^{\infty} \lambda_n^{-2} \langle v, f_n \rangle_p^2 < \infty$ for all $v \in TU$ where T is the linear operator $T\colon V \to V$ given by $e_n \mapsto f_n$ for all $n \in \mathbb{N}$. Indeed, since T maps a complete $p-$orthonormal system to another complete $p-$orthonormal system, T is orthogonal and so for all $v \in U$ we get that $\sum_{n=1}^{\infty} \lambda_n^{-2} \langle Tv, f_n \rangle_p^2 = \sum_{n=1}^{\infty} \lambda_n^{-2} \langle Tv, Te_n \rangle_p^2 = \sum_{n=1}^{\infty} \lambda_n^{-2} \langle v, e_n \rangle_p^2 < \infty$.

(iii) Iterating the construction in Lemma 2.18, we can show that there exists a dense subset U of (V, p) such that U can be equipped with a nuclear topology stronger than the one inherited from p.

Combining Lemma 2.18 with Corollary 2.13, we obtain the following corollary.

Corollary 2.20. *Let A be an algebra generated by a linear subspace $V \subseteq A$ and L a normalized $\sum A^2$–positive linear functional on A such that there exists a complete s_L–orthonormal system $\{e_n : n \in \mathbb{N}\}$ in V. Choose U and q as in Lemma 2.18, and set $J(U) := \{\langle W \rangle : W$ finite dimensional subspace of $U\}$.*

If $\{\alpha \in X(A) : \alpha \restriction_U$ is s_L–continuous$\} \neq \emptyset$ and for each $S \in J(U)$ there exists a representing Radon measure ν_S for $L \restriction_S$, then there exists a representing measure for $L \restriction_{\langle U \rangle}$ with support contained in $\{\alpha \in X(\langle U \rangle) : \alpha \restriction_U$ is q–continuous$\}$.

Proof. The assumptions ensure that we can apply Lemma 2.18 to $(V, s_L \restriction_V)$, which provides a Hilbertian seminorm q on a dense subset U of $(V, s_L \restriction_V)$ such that $\mathrm{tr}(s_L \restriction_U / q) < \infty$. Then $\{\alpha \in X(A) : \alpha \restriction_U$ is s_L–continuous$\} \subseteq \{\alpha \in X(\langle U \rangle) : \alpha \restriction_U$ is q–continuous$\}$ and so $\{\alpha \in X(\langle U \rangle) : \alpha \restriction_U$ is q–continuous$\} \neq \emptyset$. Hence, we can apply Corollary 2.13 to $L \restriction_{\langle U \rangle}$ and get the conclusion. □

In the above corollary the Hilbertian seminorm q on V is not pre-given as in Theorem 2.5 (resp. Theorem 2.8), but explicitly constructed through Lemma 2.18. The price to pay for this is that we obtain an integral representation for the starting linear functional L not on the whole of A but just on the subalgebra $\langle U \rangle$ of A. Note that the latter subalgebra is actually dense in (A, s_L) (or more in general in (A, p) when p is defined on the whole of A, see Lemma 4.17 in the Appendix) and so Corollary 2.20 provides a representing measure for L restricted to a dense subalgebra of (A, s_L). However, the representing measure is supported on characters whose restrictions to U lie in the topological dual of (U, q) and the density of $\langle U \rangle$ in (A, s_L) does not allow us to show that is supported on characters whose restrictions to V are in the topological dual of $(V, s_L \restriction_V)$ and so to get an integral representation for L on the full A. The latter effect is not an artefact of the techniques used here. Indeed, if $V = \ell_2$ is endowed with the usual norm $\|\cdot\|_{\ell_2}$ which makes it a Hilbert space, then the associated Gaussian measure (which is the product of infinitely many one-dimensional standard Gaussian measures) gives rise to a functional L on $S(V)$ and the Gaussian measure is the only measure representing L. As $s_L \restriction_V = \|\cdot\|_{\ell_2}$, the Gaussian measure cannot be supported on $\{\alpha \in X(S(V)) : \alpha \restriction_V$ is $s_L \restriction_V$ –continuous$\}$ because it is well-known that this set has measure zero (see e.g. [2, Theorem 1.3].

In the case when $U = V$, Corollary 2.20 provides a representing measure for the starting L on the whole A. This is for example the case when (V, τ_V) is separable nuclear and $s_L \restriction_V$ is τ_V–continuous, as analyzed in more details in the next subsection.

2.4. Results on the Main Question for τ_V nuclear

Corollary 2.13 and Corollary 2.16 nicely apply to the case when the generating subspace of the algebra is endowed with a nuclear topology.

Corollary 2.21. *Let (V, τ_V) be a nuclear space, A an algebra generated by V, $J := \{\langle W \rangle : W$ finite dimensional subspace of $V\}$ and $K \subseteq X(A)$ closed. Let L be a normalized $\sum A^2$–positive linear functional L on A and Q a quadratic module such that $K = K_Q$.*

If for each $S \in J$ there exists a representing Radon measure ν_S for $L\restriction_S$ with support contained in $K_{Q\cap S}$ and $s_L \restriction_V$ is τ_V–continuous, then for each Hilbertian seminorm q on V s.t. $\{\alpha \in K : \alpha\restriction_V$ is q–continuous$\} \neq \emptyset$ and $\mathrm{tr}(s_L \restriction_V /q) < \infty$, there exists a representing measure for L with support contained in $\{\alpha \in K : \alpha\restriction_V$ is q–continuous$\}$.

Proof. As (V, τ_V) is nuclear and $s_L \restriction_V$ is τ_V–continuous, using Lemma 1.3 and the directedness of the generating family for τ, we can easily derive that there exists a τ_V–continuous Hilbertian seminorm q on V such that $\mathrm{tr}(s_L \restriction_V /q) < \infty$. Thus, we can apply Corollary 2.13 and get the desired conclusion. □

Using exactly the same argument but replacing Corollary 2.13 with Corollary 2.16, we obtain the following.

Corollary 2.22. *Let (V, τ_V) be a nuclear space, A generated by V, L a normalized linear functional on A such that $L(\sum A^2) \subseteq [0, \infty)$ and Q a quadratic module in A. If*

(a) $L(Q) \subseteq [0, \infty)$,
(b) for each $v \in V$, $\sum_{n=1}^{\infty} \frac{1}{\sqrt[2n]{L(v^{2n})}} = \infty$,
(c) $s_L\restriction_V$ *is* τ_V*–continuous,*

then, for each Hilbertian seminorm q on V s.t. $\{\alpha \in K_Q : \alpha\restriction_V$ is q–continuous$\} \neq \emptyset$ and $\mathrm{tr}(s_L \restriction_V /q) < \infty$, there exists a unique representing Radon measure ν for L such that $\nu(\{\alpha \in K_Q : \alpha\restriction_V$ is q–continuous$\}) = 1$.

We can retrieve [26, Theorem 13] from Corollary 2.22 applied to $Q = \sum A^2$. Indeed, in [26, Theorem 13] the assumption (ii) exactly corresponds to (a) and (b) of Corollary 2.22 for $Q = \sum A^2$ (the alternative assumption (i) corresponds to (a') in Remark 2.17), and the assumption of the existence of a τ−continuous seminorm q on V such that $L(v^2) \leq q(v)^2$ for all $v \in V$ guarantees that $s_L\restriction_V (a) \leq q(v)$ for all $v \in V$, i.e., also (c) in Corollary 2.22 is satisfied.

Note that if there exists a τ−continuous Hilbertian seminorm q on A such that $L(a^2) \leq q(a)^2$ for all $a \in A$, then not only s_L is τ−continuous but also L itself is τ–continuous, since by the Cauchy-Schwarz inequality we have that

$$|L(a)|^2 = |L(1 \cdot a)|^2 \leq L(1)L(a^2) \leq q(a)^2 \qquad \text{for all } a \in A.$$

Viceversa, the continuity of L on certain classes of nuclear topological algebra provides the continuity of s_L, allowing us to establish the following result.

Corollary 2.23. *Let (A, τ) be a locally convex nuclear topological algebra which is also barrelled (respectively, has also jointly continuous multiplication), L a $\tau-$continuous linear functional on A and Q a quadratic module in A. If*

(a) $L(Q) \subseteq [0, \infty)$,
(b) for each $v \in V$, $\sum_{n=1}^{\infty} \frac{1}{\sqrt[2n]{L(v^{2n})}} = \infty$,

then, for each Hilbertian seminorm q on V s.t. $K_Q \cap \mathfrak{sp}(q) \neq \emptyset$ and $\operatorname{tr}(s_L/q) < \infty$, there exists a unique $(K_Q \cap \mathfrak{sp}(q))$–representing Radon measure ν for L.

Proof. Let us first observe that the τ–continuity of s_L is ensured both when (A, τ) is barrelled and has jointly continuous multiplication. Indeed, in the first case the τ–continuity of L ensures the existence of a Hilbertian seminorm q on A such that $L(a^2) \leq q(a)^2$ for all $a \in A$ (see, e.g. [26, Lemma 14]) and so, as observed above, s_L is τ–continuous. In the second case, the joint continuity of the multiplication provides the existence of a τ–continuous seminorm p on A such that $L(ab) \leq p(a)p(b)$ for all $a, b \in A$ and so $s_L(a)^2 = L(a^2) \leq p(a)^2$ for all $a \in A$, which shows that s_L is τ–continuous.

Hence, in both cases we can apply Corollary 2.22 for $V = A$ and get the desired conclusion. □

We can easily retrieve [26, Theorem 15] from Corollary 2.23 applied to $Q = \sum A^2$.

3. The case of the symmetric algebra of a nuclear space

Let us apply the results of Section 2 to the case when A is the symmetric algebra $S(V)$ with (V, τ_V) nuclear. Corollary 2.22 immediately gives the following result.

Corollary 3.1. *Let (V, τ_V) be a nuclear space, L a normalized linear functional on $S(V)$ and Q a quadratic module in $S(V)$. If*

(a) $L(Q) \subseteq [0, \infty)$,
(b) for each $v \in V$, $\sum_{n=1}^{\infty} \frac{1}{\sqrt[2n]{L(v^{2n})}} = \infty$,
(c) $s_L\restriction_V$ *is τ–continuous*

then, for each Hilbertian seminorm q on V such that $\operatorname{tr}(s_L\restriction_V /q) < \infty$ and $\{\alpha \in K_Q : \alpha\restriction_V \text{ is } q\text{–continuous}\} \neq \emptyset$, there exists a unique representing Radon measure ν for L such that $\nu(\{\alpha \in K_Q : \alpha\restriction_V \text{ is } q\text{–continuous}\}) = 1$.

We can retrieve [26, Theorem 16] from Corollary 3.1 applied to $Q = \sum A^2$. Indeed, the definition of nuclear space in [26, p. 445] is covered by Definition 1.4 (for more details see Remark 4.16), [26, Theorem 16] the assumption (ii) exactly correspond to (a) and

(b) of Corollary 3.1 for $Q = \sum A^2$ (the alternative assumption (i) corresponds to (a') in Remark 2.17), and the assumption of the existence of a τ−continuous seminorm q on V such that $L(v^2) \leq q(v)^2$ for all $v \in V$ guarantees that $s_L \restriction_V (a) \leq q(v)$ for all $v \in V$, i.e., also (c) in Corollary 3.1 is satisfied.

If to each $v \in V$ we associate the operator $A_v(w) = vw$ for any $w \in S(V)$, then we can also retrieve [8, Theorem 4.3, (i) ↔ (iii)] for such operators from the version of Corollary 3.1 with (a) and (b) replaced by (a') in Remark 2.17 by taking $L = T$ and $K = \overline{\mathcal{Z}}$ (see also [15, Theorem 3.11]).

Corollary 3.1 also allows to easily prove the following result.

Corollary 3.2. *Let (V, τ_V) be a nuclear space with τ_V induced by a directed family of seminorms $\mathcal{P}$ on V, L a normalized linear functional on $S(V)$ and Q a quadratic module in $S(V)$. If*

(a) $L(Q) \subseteq [0, \infty)$,
(b) for each $v \in V$, $\sum_{n=1}^{\infty} \frac{1}{\sqrt[2n]{L(v^{2n})}} = \infty$,
(c) for each $d \in \mathbb{N}$, there exists $p \in \mathcal{P}$ such that the restriction $L\colon S(V)_d \to \mathbb{R}$ is $\overline{p}_d$–continuous, where $\overline{p}_d$ is the quotient seminorm on the d−th homogeneous component $S(V)_d$ of $S(V)$ induced by the projective tensor seminorm $p^{\otimes d}$,

then, for each Hilbertian seminorm q on V such that $\operatorname{tr}(s_L \restriction_V / q) < \infty$ and $\{\alpha \in K_Q : \alpha \restriction_V$ is q–continuous$\} \neq \emptyset$, there exists a unique representing Radon measure ν for L such that $\nu(\{\alpha \in K_Q : \alpha \restriction_V$ is q–continuous$\}) = 1$.

Proof. Since $L \restriction_{S(V)_2}$ is $\overline{p}_2$–continuous for some $p \in \mathcal{P}$, there exists $C > 0$ such that $L(v^2) \leq C\overline{p}_2(v)$ for all $v \in V$. Moreover, as $\overline{p}_d$ comes from the projective tensor seminorm $p^{\otimes d}$, we easily get that $\overline{p}_d(v^d) \leq p(v)^d$ holds for all $v \in V$ and all $d \in \mathbb{N}$, see e.g. [10, Lemma 3.1]. Using the latter for $d = 2$, we obtain that $L(v^2) \leq C\overline{p}_2(v) \leq Cp(v)^2$ for all $v \in V$, namely that the Hilbertian seminorm $s_L \restriction_V$ is p–continuous and so τ_V–continuous. Hence, the conclusion follows at once from Corollary 3.1. □

Using Theorem 2.10 instead of Corollary 2.22, we can prove a slight generalization of the classical solution to the Main Question for (V, τ_V) nuclear in [2, Chapter 5, Theorem 2.1] (cf. [2, Chapter 5, Section 2.3] and [3]).

Theorem 3.3. *Let (V, τ_V) be a Hausdorff separable nuclear space with τ_V induced by a directed family $\mathcal{P}$ of Hilbertian seminorms on V, L a normalized linear functional on $S(V)$ and K a closed subset of V^*. For any $n \in \mathbb{N}$ and $s \in \mathcal{P}$, let $\tilde{s}^{(n)}$ the Hilbertian* *seminorm on* $S(V)_n$ *given by* $\widetilde{s}^{(n)}(b) := \sqrt{\sum_{i=1}^{N}\sum_{j=1}^{N}\langle b_{i1}, b_{j1}\rangle_s \cdots \langle b_{in}, b_{jn}\rangle_s}$ *for any* $b := \sum_{i=1}^{N} b_{i1}\cdots b_{in} \in S(V)_n$ *with* $N \in \mathbb{N}$ *and* $b_{ik} \in V$ *for* $k = 1, \dots, n$. *If*

(1) $L(Q) \subseteq [0, \infty)$, *where* Q *is a quadratic module of* $S(V)$ *such that* $K = K_Q$
(2) there exists a countable subset E *of* V *whose linear span is dense in* (V, τ_V) *such that* $\sum_{k=1}^{\infty} \frac{1}{\sqrt[2k]{L(v^{2k})}} = \infty$ *for all* $v \in E$
(3) For any $d \in \mathbb{N}$, *there exists* $p_{2d} \in \mathcal{P}$ *such that the restriction* $L\colon S(V)_{2d} \to \mathbb{R}$ *is* $\widetilde{p_{2d}}^{(2d)}$*−continuous*
(4) $K \cap V'_{q_2} \neq \emptyset$, *where* $q_2 \in \mathcal{P}$ *is such that* $\operatorname{tr}(p_2/q_2) < \infty$,

then there exists a representing Radon measure μ *for* L *with support contained in* $K \cap V'_{q_2}$.

Remark 3.4. If for each d the map $V \to \mathbb{R}, v \mapsto L(v^d)$ is τ_V−continuous, then the assumption (3) in Theorem 3.3 holds. Indeed, the τ_V−continuity of the map $V \to \mathbb{R}, v \mapsto L(v^d)$ implies that for any d there exists a $r_d \in \mathcal{P}$ such that $|L(v^d)| \leq 1$ for all $v \in V$ with $r_d(v) \leq 1$. Then $\left|L\left(\left(\frac{v}{r_d(v)}\right)^d\right)\right| \leq 1$ for all $v \in V$ and so

$$\left|L(v^d)\right| \leq r_d(v)^d, \forall v \in V.$$

By using the multivariate polarization identity, this in turn provides that

$$|L(v_1 \cdots v_d)| \leq \frac{d^d}{d!} r_d(v_1) \cdots r_d(v_d), \forall v_1, \dots, v_d \in V.$$

Then, since (V, τ) is nuclear, Lemma 3.5-(1) below ensures that for any $p_d \in \mathcal{P}$ with $\operatorname{tr}(r_d/p_d) < \infty$ we have

$$|L(a)| \leq \frac{(\operatorname{tr}(r_d/p_d)d)^d}{d!} \widetilde{p}_d^{(d)}(a), \forall a \in S(V)_d$$

and hence, in particular, $L \restriction_{S(V)_{2d}}$ is $\widetilde{p}_{2d}^{(2d)}$−continuous.

Lemma 3.5. *Let* (V, τ_V) *be a separable nuclear space with* τ_V *induced by a directed family of Hilbertian seminorms* $\mathcal{P}$ *on* V *and* L *a normalized linear functional.*

(1) If for some $d \in \mathbb{N}$, *there exists* $r \in \mathcal{P}$ *and* $\widetilde{C}_{L,d}$, *such that*

$$|L(v_1 \dots v_d)| \leq \widetilde{C}_{L,d}\, r(v_1) \dots r(v_d) \qquad \forall v_1, \dots, v_d \in V,$$

then for any $s \in \mathcal{P}$ *with* $\operatorname{tr}(r/s) < \infty$ *we have that*

$$|L(a)| \leq \widetilde{C}_{L,d} \left(\operatorname{tr}(r/s)\right)^d \widetilde{s}^{(d)}(a) \qquad \forall a \in S(V)_d.$$

(2) *Let $\ell \in V^*$ for some $r \in \mathcal{P}$ and α_ℓ the character on $S(V)$ associated to ℓ, which is uniquely determined by defining $\alpha_\ell(v_1 \dots v_d) := \ell(v_1)\dots\ell(v_d)$ for all $d \in \mathbb{N}$ and $v_1, \dots, v_d \in V$. If $\ell \in V_r'$ for some $r \in \mathcal{P}$, then the associate character α_ℓ on $S(V)$ is such that for any $s \in \mathcal{P}$ with $\operatorname{tr}(r/s) < \infty$ and any $d \in \mathbb{N}$ the following holds*

$$|\alpha_\ell(a)| \leq \left(r'(\ell)\operatorname{tr}(r/s)\right)^d \widetilde{s}^{(d)}(a) \qquad \forall a \in S(V)_d.$$

(3) *If the assumption (3) in Theorem 3.3 holds with continuity constant $C_{L,2d}$ and $(\lambda_d)_{d\in\mathbb{N}_0}$ is a sequence of real numbers such that*

$$\sum_{d=0}^{\infty} \lambda_d^{-2} < \infty, \tag{3.1}$$

then the seminorm defined by

$$\tilde{p}(a)^2 := \lambda_0^2|a^{(0)}|^2 + \sum_{d=1}^{\infty} \lambda_d^2 C_{L,2d} \left(\widetilde{p_{2d}}^{(d)}(a^{(d)})\right)^2, \quad \forall\ a := \sum_{d=0}^{\infty} a^{(d)} \in S(V) \tag{3.2}$$

is Hilbertian and

$$|L(a)|^2 \leq L(a^2) \leq \left(\sum_{d=0}^{\infty} \lambda_d^{-2}\right) \tilde{p}(a)^2 \qquad \text{for all } a \in S(V).$$

(4) *Let $C_{L,d}$, $(\lambda_d)_{d\in\mathbb{N}_0}$ and $\tilde{p}$ as in (3), and for each $d \in \mathbb{N}$ take a seminorm $q_{2d} \in \mathcal{P}$ such that $\operatorname{tr}(p_{2d}/q_{2d}) < \infty$ (such a seminorm always exists by nuclearity). If $(\eta_d)_{d\in\mathbb{N}_0}$ is a sequence of real numbers such that*

$$\sum_{d=1}^{\infty} \frac{\lambda_d^2}{\eta_d^2} C_{L,2d} \operatorname{tr}(p_{2d}/q_{2d})^d < \infty, \tag{3.3}$$

then the seminorm defined by

$$\tilde{q}(a)^2 := \eta_0^2|a^{(0)}|^2 + \sum_{d=1}^{\infty} \eta_d^2 \left(\widetilde{q_{2d}}^{(d)}(a^{(d)})\right)^2, \quad \forall\ a := \sum_{d=0}^{\infty} a^{(d)} \in S(V) \tag{3.4}$$

is Hilbertian and such that $\operatorname{tr}(\tilde{p}/\tilde{q}) < \infty$.

(5) *Let $C_{L,d}$, $(\lambda_d)_{d\in\mathbb{N}_0}$ and $\tilde{p}$ as in (3), and for each $d \in \mathbb{N}$ take a seminorm $q_{2d} \in \mathcal{P}$ such that $\operatorname{tr}(p_{2d}/q_{2d}) < \infty$ for all $d \in \mathbb{N}$ and also $\operatorname{tr}(q_2/q_{2d}) < \infty$ for all $d \in \mathbb{N}$ with $d \geq 2$ (such a seminorm always exists by nuclearity). If $(\eta_d)_{d\in\mathbb{N}_0}$ is a sequence of real numbers fulfilling (3.3) and*

$$\sum_{d=1}^{\infty} \frac{c^{2d}}{\eta_d^2} < \infty, \forall c > 0, \tag{3.5}$$

then $\ell \in V^*$ is q_2-continuous if and only if α_ℓ is $\tilde{q}$ continuous, i.e., V'_{q_2} and $\mathfrak{sp}(\tilde{q})$ are isomorphic, where $\tilde{q}$ is as in (3.4).

Proof.

(1) This is a direct consequence of the multilinear Schwartz kernel theorem for nuclear spaces, see e.g. [4, Lemma 6.1 and Theorem 6.1].

(2) By the $r-$continuity of ℓ, we obtain that $|\alpha_\ell(v_1 \dots v_d)| \leq r'(\ell)^d r(v_1)\dots r(v_d)$ for all $v_1, \dots, v_d \in V$. Hence, the result directly follows from (1) applied for L replaced with α_ℓ.

(3) Let $d \in \mathbb{N}$ and $b := \sum_{i=1}^{N} b_{i1} \cdots b_{id} \in S(V)_d$ with $N \in \mathbb{N}$ and $b_{ik} \in V$ for $k = 1, \dots, d$. Since the assumption (3) of Theorem 3.3 holds and $b^2 \in S(V)_{2d}$, we have that $L(b^2) \leq C_{2d}\widetilde{p_{2d}}^{(2d)}(b^2)$. Moreover, since $b^2 = \sum_{i=1}^{N}\sum_{h=1}^{N} b_{i1}\cdots b_{id}b_{h1}\cdots b_{hd}$, we obtain that

$$\begin{aligned}
\widetilde{p_{2d}}^{(2d)}(b^2) &= \sqrt{\sum_{i=1}^{N}\sum_{h=1}^{N}\langle b_{i1}, b_{j1}\rangle_{p_{2d}}\cdots\langle b_{id}, b_{jd}\rangle_{p_{2d}}\sum_{j=1}^{N}\sum_{k=1}^{N}\langle b_{h1}, b_{k1}\rangle_{p_{2d}}\cdots\langle b_{hd}, b_{kd}\rangle_{p_{2d}}}\\
&= \sqrt{\left(\sum_{i=1}^{N}\sum_{h=1}^{N}\langle b_{i1}, b_{j1}\rangle_{p_{2d}}\cdots\langle b_{id}, b_{jd}\rangle_{p_{2d}}\right)^2}\\
&= \sum_{i=1}^{N}\sum_{h=1}^{N}\langle b_{i1}, b_{j1}\rangle_{p_{2d}}\cdots\langle b_{id}, b_{jd}\rangle_{p_{2d}} = \widetilde{p_{2d}}^{(d)}(b)^2
\end{aligned}$$

Hence, we get that

$$L(b^2) \leq C_{2d}\widetilde{p_{2d}}^{(2d)}(b^2) \leq C_{2d}\widetilde{p_{2d}}^{(d)}(b)^2, \quad \forall b \in S(V)_d. \tag{3.6}$$

Let $(\lambda_d)_{d\in\mathbb{N}_0}$ as in (3.1) and $a := \sum_{d=0}^{\infty} a^{(d)} \in S(V)$. Then there exists $D_a \in \mathbb{N}$ such that $a^{(d)} = 0$ for all $d > D_a$ and so we get that

$$\begin{aligned}
|L(a)|^2 \leq |L(a^2)| \leq\ & \sum_{d=0}^{D_a}\sum_{j=0}^{D_a}|L(a^{(d)})L(a^{(j)})| \leq \sum_{d=0}^{D_a}\sum_{j=0}^{D_a}\sqrt{L\left(\left(a^{(d)}\right)^2\right)}\sqrt{L\left(\left(a^{(j)}\right)^2\right)}\\
=\ & \left(\sum_{d=0}^{D_a}\sqrt{L\left(\left(a^{(d)}\right)^2\right)}\right)^2\\
\leq\ & \left(\sum_{d=0}^{\infty}\lambda_d^{-2}\right)\left(\sum_{d=0}^{D_a}\lambda_d^2 L\left(\left(a^{(d)}\right)^2\right)\right)
\end{aligned}$$

$$\overset{(3.6)}{\leq} \left(\sum_{d=0}^{\infty} \lambda_d^{-2}\right) \left(\lambda_0^2 |a^{(0)}| + \sum_{d=1}^{D_a} \lambda_d^2 C_{2d} \widetilde{p_{2d}}^{(d)} \left(a^{(d)}\right)^2\right)$$
$$= \left(\sum_{d=0}^{\infty} \lambda_d^{-2}\right) \tilde{p}(a)^2.$$

(4) Since (V, τ_V) is Hausdorff and separable, we have that for each $d \in \mathbb{N}$ there exists a complete $q_{2d}-$orthonormal system E_d in V (see Remark 2.19).

Then $\mathcal{B} := \left\{\frac{1}{\eta_n} e_{i_1} \cdots e_{i_n} : n \in \mathbb{N}_0, e_{i_1}, \ldots, e_{i_n} \in E_n\right\}$ is a complete $\tilde{q}-$orthonormal system in $S(V)$ and thus, for any $(\lambda_d)_{d\in\mathbb{N}_0}$ as in (3.1) and $(\eta_d)_{d\in\mathbb{N}_0}$ as in (3.3), we obtain that

$$\begin{aligned}
\operatorname{tr}(\tilde{p}/\tilde{q}) &= \sum_{e\in\mathcal{B}} \tilde{p}(e)^2 = \frac{\lambda_0^2}{\eta_0^2} + \sum_{d=1}^{\infty} \sum_{e_{i_1},\ldots,e_{i_d}\in E_d} \frac{\lambda_d^2 C_{2d} \widetilde{p_{2d}}^{(2d)} (e_{i_1}\cdots e_{i_d})^2}{\eta_d^2} \\
&= \frac{\lambda_0^2}{\eta_0^2} + \sum_{d=1}^{\infty} \sum_{e_{i_1},\ldots,e_{i_d}\in E_d} \frac{\lambda_d^2 C_{2d} p_{2d}(e_{i_1})^2 \cdots p_{2d}(e_{i_d})^2}{\eta_d^2} \\
&= \frac{\lambda_0^2}{\eta_0^2} + \sum_{d=1}^{\infty} \frac{\lambda_d^2}{\eta_d^2} C_{2d} \sum_{e_{i_1}\in E_d} p_{2d}(e_{i_1})^2 \cdots \sum_{e_{i_d}\in E_d} p_{2d}(e_{i_d})^2 \\
&= \frac{\lambda_0^2}{\eta_0^2} + \sum_{d=1}^{\infty} \frac{\lambda_d^2}{\eta_d^2} C_{2d} \operatorname{tr}(p_{2d}/q_{2d})^d < \infty.
\end{aligned}$$

Hence, $\operatorname{tr}(\tilde{p}/\tilde{q}) < \infty$.

(5) Let us first show why the existence of a seminorm q_{2d} with the properties as in the statement is guaranteed by the nuclearity of V. As $\mathcal{P}$ is directed, for each $d \geq 2$ there exists a seminorm $r_{2d} \in \mathcal{P}$ such that $p_{2d} \leq r_{2d}$ and $q_2 \leq r_{2d}$. Then, by the nuclearity of V, we can choose a $q_{2d} \in \mathcal{P}$ such that $\operatorname{tr}(r_{2d}/q_{2d}) < \infty$ and hence, by definition of trace, $\operatorname{tr}(p_{2d}/q_{2d}) < \infty$ and $\operatorname{tr}(q_2/q_{2d}) < \infty$ for all $d \geq 2$.

Let ℓ be q_2-continuous. Then, for any $d \geq 2$, we get that

$$|\alpha_\ell((a^{(d)})^2)| \overset{(2)}{\leq} (q_2'(\ell)\operatorname{tr}(q_2/q_{2d}))^{2d}\, \widetilde{q_{2d}}^{(2d)}((a^{(d)})^2), \quad \forall a^{(d)} \in S(V)_d,$$

while, for $d = 1$, we have that

$$|\alpha_\ell((a^{(1)})^2)| = \ell(a^{(1)})^2 \leq q_2'(\ell) q_2(a^{(1)})^2, \quad \forall\, a^{(1)} \in S(V)_1 = V.$$

Moreover, arguing as in (3), it is easy to see that for all $d \in \mathbb{N}$

$$\widetilde{q_{2d}}^{(2d)}((a^{(d)})^2) = \widetilde{q_{2d}}^{(2d)}(a^{(d)})^2, \quad \forall a^{(d)} \in S(V)_d.$$

Now, for any $a := \sum_{d=0}^{\infty} a^{(d)} \in S(V)$, there exists $D_a \in \mathbb{N}$ such that $a^{(d)} = 0$ for all $d > D_a$. Thus, setting $\tilde{\eta}_d := \eta_d q_2'(\ell)^{-d} \left(1 + \mathrm{tr}(q_2/q_{2d})\right)^{-d}$ for all $d \in \mathbb{N}_0$ and exploiting the previous three inequalities, we get that

$$\begin{aligned}
|\alpha_\ell(a)|^2 \leq |\alpha_\ell(a^2)| &\leq \left(\sum_{d=0}^{\infty} \tilde{\eta}_d^{-2}\right) \left(\sum_{d=0}^{D_a} \tilde{\eta}_d^2 \; \alpha_\ell\left(\left(a^{(d)}\right)^2\right)\right) \\
&\leq \left(\sum_{d=0}^{\infty} \tilde{\eta}_d^{-2}\right) \left(\tilde{\eta}_0^2 |a^{(0)}| + \tilde{\eta}_1^2 q_2'(\ell)^2 q_2(a^{(1)})^2 \right. \\
&\qquad \left. + \sum_{d=2}^{D_a} \tilde{\eta}_d^2 \left(q_2'(\ell)\mathrm{tr}(q_2/q_{2d})\right)^{2d} \; \widetilde{q_{2d}}^{(d)}\left(a^{(d)}\right)^2\right) \\
&\leq \left(\sum_{d=0}^{\infty} \tilde{\eta}_d^{-2}\right) \tilde{q}(a)^2,
\end{aligned}$$

which provides the $\tilde{q}-$continuity of α_ℓ since $\left(\sum_{d=0}^{\infty} \tilde{\eta}_d^{-2}\right) < \infty$ by (3.5).

Conversely, if α_ℓ is $\tilde{q}-$continuous, then there exists $C \geq 0$ such that

$$|\ell(v)| = |\alpha_\ell(v)| \leq C\tilde{q}(v) = C\eta_1 q_2(v), \quad \forall v \in V. \quad \square$$

Proof of Theorem 3.3. Let $I' := \{\langle F \rangle : F \text{ finite subset of } E\}$.

Recall that $(X(S(V)), \tau_{X(S(V))})$ is isomorphic to V^* equipped with the weak topology. Then, by the generalization of the classical Nussbaum theorem to any finitely generated algebra (see e.g. [18, Theorem 3.16]), the assumptions (1) and (2) ensure that for each $S \in I$ there exists a unique $K_{Q\cap S}$–representing measure ν_S for $L\restriction_S$. Moreover, the separability and the nuclearity of (V, τ_V) as well as the assumptions (1) and (3) ensure that we can apply Lemma 3.5 and get two Hilbertian seminorms $\tilde{p}$ and $\tilde{q}$ on $S(V)$ such that $\mathrm{tr}(\tilde{p}/\tilde{q}) < \infty$ and $L(a^2) \leq \left(\sum_{d=0}^{\infty} \lambda_d^{-2}\right) \tilde{p}(a)^2$ for all $a \in S(V)$. Thus, by Lemma 2.4, $\{\nu_S : S \in I\}$ is $\tilde{p}$-concentrated.

Also the density of *span*(E) in (V, τ_V) given by assumption (2) implies the density of $S(\mathit{span}(E))$ in $(S(V), \tilde{q})$. Then, by Lemma 3.5-(5) we have that $\mathfrak{sp}(\tilde{q})$ is isomorphic to V'_{q_2}. Thus, exploiting also the assumption (4), the conclusion follows by applying Theorem 2.11 to $A := S(V)$, $q = \tilde{q}$, $B := S(\mathit{span}(E))$, $p = \tilde{p}$ but with I replaced by its cofinal subset I' (cf. Remark 2.12). $\square$

We can retrieve [2, Chapter 5, Theorem 2.1] from Theorem 3.3, because their definition of nuclear space (V, τ) is covered by Definition 1.4 (for more details see Remark 4.14), their regularity assumption on the starting sequence [2, Chapter 5, Section 2.1, p.52] corresponds to Theorem 3.3-(3), their positivity assumption [2, Chapter 5, (2.1)] is equivalent to Theorem 3.3-(1), and those together with their growth condition in [2, Chapter 5, (2.5)] imply that Theorem 3.3-(2) holds (see Proposition 3.6 below). For the convenience of the reader, we restate their growth condition in our setting:

$$\exists\, E \subseteq V \text{countable s.t. } span(E) \text{ is dense in } (V,\tau) \text{ and } C\{z_k\} \text{ is quasi-analytic} \tag{3.7}$$

$$\text{where } z_k := \left(\sup_{v\in E} p_{2k}(v)\right)^k \sqrt{\sup_{v_1,\dots,v_{2k}\in E}\left(\frac{|L(v_1\cdots v_{2k})|}{\widetilde{p_{2k}}^{(2k)}(v_1\cdots v_{2k})}\right)} \quad \forall k\in\mathbb{N},$$

p_{2k} and $\widetilde{p_{2k}}^{(2k)}$ are as in Theorem 3.3-(3).

Note that this stronger condition is introduced in [2, Chapter 5, Theorem 2.1] to face the problem that the sum of two infinitely differentiable functions each belonging to a maybe different quasi-analytic class is not necessarily belonging to one of the two quasi-analytic classes or to any quasi-analytic class at all, see [20, Theorem XII].

We now prove in detail the above mentioned implication.

Proposition 3.6. *Let (V,τ_V) be a separable nuclear space with τ_V induced by a directed family of seminorms $\mathcal{P}$ on V, $L\colon S(V)\to\mathbb{R}$ linear such that (1) and (3) of Theorem 3.3 hold. If (3.7) is fulfilled, then Theorem 3.3-(2) holds.*

Proof. Let us preliminarily observe that for each $k\in\mathbb{N}$ and each $v\in V$ we have

$$\sqrt{L(v^{2k})} \le z_k. \tag{3.8}$$

Since the class $\mathcal{C}\{z_k\}$ is quasi-analytic and (3.8) holds, also $\mathcal{C}\{\sqrt{L(v^{2k})}\}$ is quasi-analytic. This together with the log-convexity of $(\sqrt{L(v^{2k})})_{k\in\mathbb{N}}$ ensures, by the Denjoy-Carleman theorem, that $\sum_{k=1}^{\infty}\frac{1}{\sqrt[2k]{L(v^{2k})}}=\infty$ holds, which provides the conclusion. (For a review about log-convexity and quasi-analyticity see e.g. [14]) $\square$

4. Appendix

In the following we first explain the relation between the notion of trace of a Hilbertian seminorm w.r.t. to another and the classical definition of trace of a positive continuous operator on a Hilbert space. Then we compare the definition of nuclear space used in this article due to Yamasaki [30] with the more traditional ones due to Grothendieck [11] and Mityagin [22], and with the definitions of this concept given by Berezansky and Kondratiev in [2, p. 14] and by Schmüdgen in [26, p. 445], whose results we compared to ours in Section 3. Finally, we provide a complete proof of the measure theoretical identity (2.7), which we exploited in the proof of Theorem 2.10.

4.1. Trace of positive continuous operators on Hilbert spaces

Let us start by recalling the definition of trace of a positive continuous operator on a Hilbert space, which we also denote with the symbol tr.

Definition 4.1 *(cf. [7, V.50, (24')])*. Given a Hilbert space $(H, \langle\cdot,\cdot\rangle)$, the *trace* of a continuous and positive operator $f\colon H \to H$ is defined as

$$\operatorname{tr}(f) := \sup_{e_1,\ldots,e_n} \sum_{i=1}^{n} \langle e_i, f(e_i)\rangle, \tag{4.1}$$

where n ranges over $\mathbb{N}$ and $e_1, \ldots, e_n \in H$ ranges over the set of all finite sequences that are orthonormal w.r.t. $\langle\cdot,\cdot\rangle$.

In fact, by [7, V.48, Lemma 2], we have that for every complete orthonormal system $\{e_i : i \in \Omega\}$ in H the following holds

$$\operatorname{tr}(f) = \sum_{i\in\Omega} \langle e_i, f(e_i)\rangle. \tag{4.2}$$

If $D \subseteq H$ is dense, then there exists a complete orthonormal system in H that is contained in D. Therefore, in (4.1) it suffices to let $e_1, \ldots, e_n \in H$ range over the set of all finite sequences in D that are orthonormal w.r.t. $\langle\cdot,\cdot\rangle$.

For the convenience of the reader, we also recall here some fundamental classes of operators that will be needed in showing the relation between traces mentioned above.

Definition 4.2. Given a Hilbert space $(H, \langle\cdot,\cdot\rangle)$, we say that a bounded linear operator $f : H \to H$ is *trace-class* if $tr(\sqrt{f^*f}) < \infty$, where f^* denotes the adjoint of f. The positive bounded operator $\sqrt{f^*f}$ is called *absolute value* of f.

Definition 4.3. Given two Hilbert spaces (H_1, p_1) and (H_2, p_2), we say that a continuous operator $f : H_1 \to H_2$ is

(1) *Hilbert-Schmidt (or quasi-nuclear)* if $\operatorname{tr}(f^*f) < \infty$.
(2) *nuclear* if there exist $(v_n)_{n\in\mathbb{N}} \subseteq H_1$ and $(w_n)_{n\in\mathbb{N}} \subseteq H_2$ such that

$$\sum_{n=1}^{\infty} p_1(v_n)p_2(w_n) < \infty \quad \text{and} \quad f(\cdot) = \sum_{n=1}^{\infty} \langle\cdot, v_n\rangle_{p_1} w_n.$$

Note that $(v_n)_{n\in\mathbb{N}} \subseteq (H_1, p_1)$ and $(w_n)_{n\in\mathbb{N}} \subseteq (H_2, p_2)$ can be chosen to be orthogonal (see, e.g., [28, Corollary, p. 494]).

Proposition 4.4. *Let $f\colon (H_1, p_1) \to (H_2, p_2)$ be a nuclear operator. If $H \subseteq H_1$ closed, then $f\restriction_H\colon H \to \overline{f(H)}$ is also nuclear.*

Proof. Since f is nuclear, there exists $(v_n)_{n\in\mathbb{N}} \subseteq (H_1, p_1)$ and $(w_n)_{n\in\mathbb{N}} \subseteq (H_2, p_2)$ orthogonal such that $f(\cdot) = \sum_{n=1}^{\infty} \langle\cdot, v_n\rangle_{p_1} w_n$ and $\sum_{n=1}^{\infty} p_1(v_n)p_2(w_n) < \infty$. Then

$f(v_n) = \langle v_n, v_n \rangle_{p_1} w_n$ for all $n \in \mathbb{N}$, since $(v_n)_{n\in\mathbb{N}} \subseteq (H_1, p_1)$ is orthogonal, and so $(w_n)_{n\in\mathbb{N}} \subseteq \overline{f(H)}$. Furthermore, for each $n \in \mathbb{N}$ there exist $x_n \in H, y_n \in H^\perp$ such that $v_n = x_n + y_n$. Thus,

$$f(x) = \sum_{n=1}^{\infty} \langle x, x_n + y_n \rangle_{p_1} w_n = \sum_{n=1}^{\infty} \langle x, x_n \rangle_{p_1} w_n \quad \text{for all } x \in H.$$

Moreover, $\langle x_n, y_n \rangle_{p_1} = 0$ implies that $p_1(x_n) \leq p_1(x_n + y_n) = p_1(v_n)$ for all $n \in \mathbb{N}$ and hence, $\sum_{n=1}^{\infty} p_1(x_n)p_2(w_n) \leq \sum_{n=1}^{\infty} p_1(v_n)p_2(w_n) < \infty$. □

We are ready now to relate Definition 1.2 and Definition 4.1.

Remark 4.5. A Hilbertian seminorm p on a real vector space V can be always used to construct a Hilbert space out of V. Indeed, p induces a seminorm on $V_p := V/\ker(p)$ given by $v + \ker(p) \mapsto p(v)$ and denoted, with a slight abuse of notation, also by p. Thus, (V_p, p) is a pre-Hilbert space, as p clearly induces an inner product on V_p. Now, V_p is dense in the completion $\overline{V}_p$ of (V_p, p) and so p extends to a norm $\overline{p}$ on $\overline{V}_p$ which makes $(\overline{V}_p, \overline{p})$ a Hilbert space.

Proposition 4.6. *Let p and q be two Hilbertian seminorms on a real vector space V.*

(1) If $\ker(q) \subseteq \ker(p)$ then $u\colon V_q \to V_p, v + \ker(q) \mapsto v + \ker(p)$ is well-defined. Note that u is injective iff $\ker(q) = \ker(p)$.
(2) If there exists $C > 0$ such that $p \leq Cq$, then u is continuous and uniquely continuously extends to $\overline{u}\colon (\overline{V}_q, \overline{q}) \to (\overline{V}_p, \overline{p})$. Moreover, $\overline{u}$ is injective iff for any Cauchy sequence (v_n) in V_q s.t. $u(v_n)$ converges to 0 in p we have that v_n converges to 0 in q.
(3) $\operatorname{tr}(p/q) < \infty$ if and only if $\overline{u}$ is Hilbert-Schmidt, i.e., $\operatorname{tr}(\overline{u}^*\overline{u}) < \infty$, where $\overline{u}^*$ denotes the adjoint of $\overline{u}$.

Proof. (1) For any $v \in V$, let us set for convenience $[v]_q := v + \ker(q)$ and $[v]_p := v + \ker(p)$. Recalling the notation and the properties introduced in Remark 4.5, it is easy to see that $\ker(q) \subseteq \ker(p)$ implies (1), because under this assumption $[x]_q = [y]_q$ implies $x - y \in \ker(p)$ and so $[x]_p = [y]_p$, i.e., $u([x]_q) = u([y]_q)$.

Moreover, suppose that u is injective and that there exists $v \in \ker(p) \setminus \ker(q)$. Then $[v]_q \neq [0]_q$ and $[v]_p = [0]_p$. Hence, on the one hand the injectivity of u ensures that $u([v]_q) \neq [0]_p$, but on the other hand $u([v]_q) = [v]_p = [0]_p$ which leads to a contradiction. Conversely, if $\ker(p) = \ker(q)$, then u is the identity which is clearly injective.

(2) Suppose there exists $C > 0$ such that $p \leq Cq$. Then $\ker(q) \subseteq \ker(p)$ and so u is well-defined by (1). Also, for any $[v]_q \in V_q$ we have $p(u([v]_q)) = p([v]_p) = p(v) \leq Cq(v) = Cq([v]_q)$, i.e., u is continuous and so can be uniquely extended to the completions giving the desired $\overline{u}$.

For proving the second part of (2), suppose that $\overline{u}$ is injective and let (v_n) be a Cauchy sequence in V_q s.t. $p(u(v_n)) \to 0$. Then, by completeness, there exists $w \in \overline{V}_q$ such that $\overline{q}(v_n - w) \to 0$ and so, by continuity of $\overline{u}$, $\overline{p}(\overline{u}(v_n) - \overline{u}(w)) \to 0$. Therefore, $\overline{p}(\overline{u}(w)) \leq \overline{p}(\overline{u}(v_n) - \overline{u}(w)) + \overline{p}(\overline{u}(v_n)) = \overline{p}(u(v_n) - \overline{u}(w)) + p(u(v_n)) \to 0$, i.e., $\overline{p}(\overline{u}(w)) = 0$ that is $\overline{u}(w) = 0$. Hence, the injectivity of $\overline{u}$ implies that $w = 0$ and so that $q(v_n) = \overline{q}(v_n) = \overline{q}(v_n - w) \to 0$.

Conversely, suppose that for any Cauchy sequence (v_n) in V_q s.t. $p(u(v_n)) \to 0$ we have $q(v_n) \to 0$. If $w \in \ker(\overline{u}) \subseteq \overline{V}_q$, then there exists a Cauchy sequence (w_n) in V_q converging to w, i.e., $\overline{q}(w_n - w) \to 0$. By continuity of $\overline{u}$, we have that $\overline{p}(\overline{u}(w_n) - \overline{u}(w)) \to 0$ but $\overline{u}(w) = 0$ and so $p(u(w_n)) = \overline{p}(\overline{u}(w_n)) \to 0$. Hence, our assumption implies $q(w_n) \to 0$ and so $\overline{q}(w) \leq \overline{q}(w_n - w) + q(w_n) \to 0$, which is equivalent to $w = 0$ and so provides the injectivity of $\overline{u}$.

(3) directly follows from the following observation

$$\begin{aligned}
\operatorname{tr}(\overline{u}^*\overline{u}) &\overset{(4.1)}{=} \sup_{e_1,\ldots,e_n} \sum_{i=1}^{n} \langle [e_i]_q, \overline{u}^*\overline{u}([e_i]_q)\rangle_{\overline{q}} \\
&= \sup_{e_1,\ldots,e_n} \sum_{i=1}^{n} \langle \overline{u}([e_i]_q), \overline{u}([e_i]_q)\rangle_{\overline{p}} \\
&= \sup_{e_1,\ldots,e_n} \sum_{i=1}^{n} \langle [e_i]_p, [e_i]_p\rangle_{\overline{p}} = \sup_{e_1,\ldots,e_n} \sum_{i=1}^{n} \langle e_i, e_i\rangle_p \overset{\text{Def.1.2}}{=} \operatorname{tr}(p/q),
\end{aligned}$$

where n ranges over $\mathbb{N}$ and $e_1, \ldots, e_n \in V$ ranges over the set of all finite sequences that are orthonormal w.r.t. $\langle \cdot, \cdot \rangle_q$. $\square$

Proposition 4.7. *Let p and q be two Hilbertian seminorms on V. If $\ker(p) = \ker(q)$ and $\operatorname{tr}(p/q) < \infty$, then V_q is separable.*

Proof. Suppose that V_q is not separable. Then there exists $(e_j)_j \in J$ orthonormal basis of V_q with J uncountable. Since $q(e_j) = 1$ for all $j \in J$ and $\ker(p) = \ker(q)$, we have that $p(e_j) > 0$ for all $j \in J$. However, $\operatorname{tr}(p/q) < \infty$ implies that $\sup_{n\in\mathbb{N}} \sup_{j_1,\ldots,j_n \in J} \sum_{k=1}^{n} p(e_{j_k})^2 < \infty$ and so for all but countably many $n-$tuples in $(e_j)_j \in J$ we have $\sum_{k=1}^{n} p(e_{j_k})^2 = 0$, which contradicts the fact that $p(e_j) > 0$ for all $j \in J$. $\square$

Corollary 4.8. *Let p and q be two Hilbertian seminorms on V s.t. there exists $C > 0$ such that $p \leq Cq$ then $\overline{u}$ injective and Hilbert-Schmidt implies that V_q is separable.*

Corollary 4.9. *Let A be an algebra generated by a linear subspace $V \subseteq A$, and L a normalized linear functional on A such that $L(\sum A^2) \subseteq [0, \infty)$. If q is a Hilbertian seminorm q on V such that $\operatorname{tr}(s_L \restriction_V /q) < \infty$, then the space $V/\ker(s_L)$ endowed with the quotient seminorm induced by s_L (and also denoted by s_L with a slight abuse of notation) is separable.*

Proof. Let us endow V_q with the quotient seminorm induced by q, which we also denote by q with a slight abuse of notation. As $\mathrm{tr}(s_L \restriction_V /q) < \infty$, Lemma 1.3-(i) ensures the $q-$continuity of $s_L \restriction_V$ and so that $\ker(q) \subset \ker(s_L)$. Then the quotient seminorm induced on V_q by $s_L \restriction_V$ actually reduces to itself, i.e., $\forall v \in V$, $\inf\{s_L(v+w) : w \in \ker(q)\} = s_L(v)$, and can be continuously extended to a norm $\overline{p}_L$ on the completion $\overline{V_q}$ of V_q. Hence, both $(\overline{V_q}, \overline{p}_L)$ and $(\overline{V_q}, \overline{q})$ are Hilbert spaces.

Consider $\overline{\ker(s_L)}$ in $(\overline{V_q}, \overline{q})$ and denote by $\tilde{q}$ the quotient norm induced on $\overline{V_q}/\overline{\ker(s_L)}$ by $\overline{q}$ (respectively by $\tilde{s_L}$ the quotient norm induced on $\overline{V_q}/\overline{\ker(s_L)}$ by $\overline{s_L}$). Then $\tilde{q}$ is a Hilbertian norm as $(\overline{V_q}, \overline{q})$ is a Hilbert space and we have the following orthogonal decomposition $\overline{V_q} = \overline{\ker(s_L)} \oplus \overline{\ker(s_L)}^{\perp}$. Then $(\overline{V_q}/\overline{\ker(s_L)}, \tilde{q})$ is a Hilbert space. Hence, denoting by π the orthogonal projection $\overline{V_q}$ onto $\overline{\ker(s_L)}^{\perp}$, we get $\overline{V_q}/\ker(\pi) \cong \pi(\overline{V_q})$, i.e., $\overline{V_q}/\overline{\ker(s_L)} \cong \overline{\ker(s_L)}^{\perp}$. Exploiting this isomorphism and the fact that $\ker(s_L)$ is closed in $(\overline{V_q}, \overline{q})$, it is easy to see that any finite $\tilde{q}$-orthonormal subset $\{\tilde{e_j}\}_{j=1,\dots,J}$ with $J \in \mathbb{N}$ in $\overline{V_q}/\overline{\ker(s_L)}$ provides a finite $\overline{q}-$orthonormal subset $\{h_j\}_{j=1,\dots,J} := \pi^{-1}\left(\{\tilde{e_j}\}_{j=1,\dots,J}\right)$ in $\overline{V_q}$. By density, for each $n \in \mathbb{N}$ and each $j \in \{1, \dots, J\}$, we can choose $h_j^{(n)} \in V$ such that $\overline{q}(h_j - h_j^{(n)}) \leq 1/n$. Orthogonalizing $\{h_j^{(n)}\}_{j\in\{1,\dots,J\}}$ via the Gram-Schmidt process, we obtain a $q-$orthogonal subset $\{e_j^{(n)}\}_{j\in\{1,\dots,J\}}$ in V defined inductively by $e_1^{(n)} := h_1^{(n)}$ and $e_k^{(n)} := h_k^{(n)} - \sum_{j=1}^{k-1} \frac{\langle h_k^{(n)}, e_j^{(n)}\rangle_q}{\langle e_j^{(n)}, e_j^{(n)}\rangle_q} e_j^{(n)}$ for all $k \geq 2$. Defining $\tilde{e}_k^{(n)} := \frac{e_k^{(n)}}{q(e_k^{(n)})}$ for all $k \in \mathbb{N}$, we get a $q-$orthonormal subset in V. So for each $k \in \{1, \dots, J\}$ as $n \to \infty$ we get inductively that $\overline{q}(e_k^{(n)} - h_k) \to 0$ and hence $\overline{s_L}(e_k^{(n)} - h_k) \to 0$. Thus, for each $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that $\overline{s_L}(e_j^{(N)} - h_j) \leq \varepsilon/J$ for all j and so

$$\sum_{j=1,\dots,J} \tilde{s_L}(\tilde{e_j}) = \sum_{j=1,\dots,J} \overline{s_L}(h_j) \leq \sum_{j=1,\dots,J} s_L(e_j^{(N)}) + \varepsilon \leq \mathrm{tr}(s_L \restriction_V /q) + \varepsilon.$$

As this holds for an arbitrary finite $\tilde{q}$-orthonormal subset $\{\tilde{e_j}\}_{j=1,\dots,J}$, we have by the definition of the trace that $\mathrm{tr}(\tilde{s_L}/\tilde{q}) \leq \mathrm{tr}(s_L \restriction_V /q)+\varepsilon$, which together with $\mathrm{tr}(s_L \restriction_V /q) < \infty$ implies $\mathrm{tr}(\tilde{s_L}/\tilde{q}) < \infty$. Then, since the kernel of $\tilde{s_L}$ in $\overline{V_q}/\ker(s_L)$ is clearly trivial, so it is the kernel of $\tilde{q}$ in $\overline{V_q}/\ker(s_L)$. Hence, Proposition 4.7 provides that $\left(\overline{V_q}/\ker(s_L)\right)/\ker(\tilde{q})$ is separable, i.e., $\overline{V_q}/\ker(s_L)$ is separable. As the latter space is also metric, we have that its subspace $V_q/\ker(s_L)$ is also separable. Moreover, since $\ker(q) \subseteq \ker(s_L) \subseteq V$, we get that $V_q/\ker(s_L) \cong V/\ker(s_L)$ and so $V/\ker(s_L)$ is also separable. □

4.2. Other definitions of nuclear space

The definition of nuclear space used in this article, namely Definition 1.4, is due to Yamasaki but it is equivalent to the more traditional definitions of nuclear space due to Grothendieck [11] and Mityagin [22], which we report here for the convenience of the reader (see e.g. [30, Theorems A.1, A.2] for a proof of these equivalences).

Definition 4.10. A TVS (V, τ) is called *nuclear* if τ is induced by a family $\mathcal{P}$ of seminorms on V such that for each $p \in \mathcal{P}$ there exists $q \in \mathcal{P}$ and there exist sequences $(v_n)_{n\in\mathbb{N}} \subseteq V, (l_n)_{n\in\mathbb{N}} \subseteq V^*$ with the following property

$$\sum_{n=1}^{\infty} p(v_n)q'(l_n) < \infty \text{ for all } v \in V \text{ with } v = \sum_{n=1}^{\infty} l_n(v)v_n \text{ w.r.t. } p,$$

here q' denotes the dual norm of q.

Definition 4.11. A TVS (A, τ) is called *nuclear* if τ is induced by a family $\mathcal{P}$ of seminorms on V such that for each $p \in \mathcal{P}$ there exists $q \in \mathcal{P}$ with $d_n(U_q, U_p) \in \mathcal{O}(n^{-\lambda})$ for some $\lambda > 0$, where U_p denotes the (closed) semiball of p and $d_n(U_q, U_p)$ denotes *n–dimensional width of U_q w.r.t. U_p*, that is $d_n(U_q, U_p)$ is defined as

$$\inf\{c > 0 : U_q \subseteq V_{n-1} + cU_p \text{ for some } V_{n-1} \subseteq V \text{ with } \dim(V_{n-1}) = n - 1\}.$$

Using Proposition 4.6, it is easy to establish that Definition 1.4 coincides with the following one.

Definition 4.12. A TVS (V, τ) is called *nuclear* if τ is induced by a directed family $\mathcal{P}$ of Hilbertian seminorms on V such that for each $p \in \mathcal{P}$ there exists $q \in \mathcal{P}$ with $\ker(q) \subseteq \ker(p)$ and the continuous extension $\overline{u}\colon (\overline{V}_q, \overline{q}) \to (\overline{V}_p, \overline{p})$ of the canonical map $u : (V_q, q) \to (V_p, p)$ is Hilbert-Schmidt, i.e., $\operatorname{tr}(\overline{u}^*\overline{u}) < \infty$.

This equivalent reformulation of Definition 1.4 allows more easily to see its relation with the definitions of this concept given by Berezansky and Kondratiev in [2, p. 14] and by Schmüdgen in [26, p. 445] whose results we compare to ours in Section 3.

Definition 4.13 *(cf. [2, p. 14]).* Let I be a directed index set and $(H_i, p_i)_{i\in I}$ a family of Hilbert spaces such that $V := \bigcap_{i\in I} H_i$ is dense in each (H_i, p_i) and for all $i, j \in I$ there exists $k \in I$ with $i, j \leq k$ and $(H_k, p_k) \subseteq (H_i, p_i)$ as well as $(H_k, p_k) \subseteq (H_j, p_j)$. The space V endowed with the topology τ induced by $\mathcal{P} := \{p_i : i \in I\}$ is called nuclear if for each $i \in I$ there exists $j \geq i$ in I such that the embedding $(H_j, p_j) \subseteq (H_i, p_i)$ is Hilbert-Schmidt.

Remark 4.14. Berezansky and Kondratiev's Definition 4.13 of nuclear space is covered by Definition 4.12 (and thus, by Definition 1.4). Indeed, let (V, τ) be a nuclear space with defining family $(H_i, p_i)_{i\in I}$ in the sense of Definition 4.13. Since for each $i \in I$ we have that $\ker(p_i) = \{o\}$, we get $V_{p_i} = V$. This together with the fact that V is dense in each (H_i, p_i) ensures that the completion $(\overline{V}_{p_i}, \overline{p}_i)$ is isomorphic to (H_i, p_i) for all $i \in I$. Thus, for $i \leq j$ in I the embedding $(H_j, p_j) \subseteq (H_i, p_i)$, which is Hilbert-Schmidt by assumption, coincides with $\overline{u}\colon (\overline{V}_{p_j}, \overline{p}_j) \to (\overline{V}_{p_i}, \overline{p}_i)$. Hence, (V, τ) is nuclear in the sense of Definition 4.12.

Definition 4.15 *(cf. [26, p. 445]).* Let $(H_n, p_n)_{i\in\mathbb{N}}$ be a sequence of Hilbert spaces such that $(H_n, p_n) \subseteq (H_m, p_m)$ for all $m \leq n$ in $\mathbb{N}$. The space $V := \bigcap_{n=1}^{\infty} H_n$ endowed with the topology τ induced by $\mathcal{P} := \{p_n : n \in \mathbb{N}\}$ is called nuclear if for each $m \in \mathbb{N}$ there exists $n \in \mathbb{N}$ such that the embedding $(H_n, p_n) \subseteq (H_m, p_m)$ is nuclear (see Definition 4.3-(2)).

Remark 4.16. Schmüdgen's Definition 4.15 of nuclear space is covered by Definition 4.12 (and thus, by Definition 1.4). Indeed, let (V, τ) be a nuclear space with defining family $(H_n, p_n)_{n\in\mathbb{N}}$ in the sense of Definition 4.15. For each $n \in \mathbb{N}$, since $\ker(p_n) = \{o\}$, we get that $V_{p_n} = V \subseteq H_n$ and so that the completion $(\overline{V}_{p_n}, \overline{p}_n)$ is isomorphic to a closed subspace of (H_n, p_n). As the embedding $(H_m, p_m) \subseteq (H_n, p_n)$ is nuclear, also its restriction r to $\overline{V}_{p_m}$ is nuclear by Proposition 4.4. The continuity of r guarantees that $r(\overline{V}_{p_m}) \subseteq \overline{r(V_{p_m})} = \overline{r(V)} = \overline{V} = \overline{V}_{p_n}$ and so the map r coincides with $\overline{u}\colon (\overline{V}_{p_m}, \overline{p}_m) \to (\overline{V}_{p_n}, \overline{p}_n)$. Hence, $\overline{u}$ is nuclear. Then [28, Theorem 48.2] ensures that $\operatorname{tr}(\sqrt{\overline{u}^*\overline{u}}) < \infty$, i.e., $\overline{u}$ is a trace-class operator (see Definition 4.2). Since the family of trace-class operators on a Hilbert space forms an ideal in the space of bounded operators on the same space (see e.g. [23, Theorem VI.19]), we have that $tr(\overline{u}^*\overline{u}) < \infty$, i.e., $\overline{u}$ is Hilbert-Schmidt. Thus, (V, τ) is also nuclear in the sense of Definition 4.12.

4.3. Two auxiliary results

We provide here a proof of (2.7), which we used in the proof of Theorem 2.10 as well as a result Lemma 4.17 about dense subalgebras of topological algebras which we exploited in the analysis of Corollary 2.20.

Proof of (2.7). For notational convenience, let $\tau^A := \tau_{\mathfrak{sp}(q)^A}$ and $\tau^B := \tau_{\mathfrak{sp}(q)^B}$. We preliminarily observe that

$$q'(\alpha) := \sup_{a\in A\,:\,q(a)\leq 1} |\alpha(a)| = \sup_{a\in B\,:\,q(a)\leq 1} |\alpha(a)|, \quad \forall \alpha \in \mathfrak{sp}(q)$$

and so q' is lower semi-continuous w.r.t. both τ^A and τ^B. Hence, all sublevel sets of q' are closed in both $(\mathfrak{sp}(q), \tau^A)$ and $(\mathfrak{sp}(q), \tau^B)$, i.e., $B_n(q')^c \in \tau^A \cap \tau^B$, for all $n \in \mathbb{N}$, which gives in turn $B_n(q') \in \mathcal{B}(\tau^A) \cap \mathcal{B}(\tau^B)$. This together with the following two properties

(i) $\tau^A \cap B_n(q') = \tau^B \cap B_n(q'), \quad \forall\, n \in \mathbb{N}$.
(ii) $\mathcal{B}(\tau^C) \cap B_n(q') = \mathcal{B}(\tau^C \cap B_n(q')), \quad \forall\, n \in \mathbb{N}, C \in \{A, B\}$,

provide that

$$\mathcal{B}(\tau^A) \cap B_n(q') \overset{(ii)}{=} \mathcal{B}(\tau^A \cap B_n(q')) \overset{(i)}{=} \mathcal{B}(\tau^B \cap B_n(q')) \overset{(ii)}{=} \mathcal{B}(\tau^B) \cap B_n(q') \subseteq \mathcal{B}(\tau^B).$$

The latter ensures that if $Y \in \mathcal{B}(\tau^A)$ then $Y \cap B_n(q') \in \mathcal{B}(\tau^B)$ for all $n \in \mathbb{N}$ and so that $Y = \bigcup_{n\in\mathbb{N}} Y \cap B_n(q') \in \mathcal{B}(\tau^B)$, i.e., $\mathcal{B}(\tau^A) \subseteq \mathcal{B}(\tau^B)$. The opposite inclusion easily follows from $\tau^B \subset \tau^A$. Hence, $\mathcal{B}(\tau^A) = \mathcal{B}(\tau^B)$. □

It remains to show (i) and (ii).

Proof of (i). Let $n \in \mathbb{N}$. Since $\tau^B \subset \tau^A$, we have that $\tau^A \cap B_n(q') \supseteq \tau^B \cap B_n(q')$. For the opposite inclusion, let $\alpha \in \mathfrak{sp}(q) \cap B_n(q')$ and recall that for any $C \in \{A, B\}$ a basis of neighborhoods of α in the topology $\tau^C \cap B_n(q')$ is given by

$$\{U_{c_1,\dots,c_k;\lambda} : k \in \mathbb{N}, c_1, \dots, c_k \in C, \lambda > 0\},$$

where

$$U_{c_1,\dots,c_k;\lambda}(\alpha) := \{\gamma \in \mathfrak{sp}(q) \,:\, |\hat{c_j}(\gamma) - \hat{c_j}(\alpha)| < \lambda \text{ for } j = 1, \dots, k \text{ and } q'(\gamma) < n\}$$

We need to show that for any $k \in \mathbb{N}$, $a_1, \dots, a_k \in A$ and $\varepsilon > 0$ there exist $b_1, \dots, b_k \in B$ and $\delta > 0$ such that $U_{b_1,\dots,b_k;\delta}(\alpha) \subseteq U_{a_1,\dots,a_k;\varepsilon}(\alpha)$.

Fixed $k \in \mathbb{N}$, $a_1, \dots, a_k \in A$ and $\varepsilon > 0$, by the density of B in (A, q) we can always choose $b_1, \dots, b_k \in B$ such that $q(a_j - b_j) < \frac{\varepsilon}{3n}$ for $j = 1, \dots, k$. Then taking $\delta < \frac{\varepsilon}{3}$ we have that for any $\beta \in U_{b_1,\dots,b_k;\delta}(\alpha)$ and any $j \in \{1, \dots, k\}$ the following holds:

$$\begin{aligned}|\beta(a_j) - \alpha(a_j)| &\leq |\beta(a_j) - \beta(b_j)| + |\beta(b_j) - \alpha(b_j)| + |\alpha(b_j) - \alpha(a_j)| \\ &\leq nq(a_j - b_j) + \delta + nq(b_j - a_j) < \varepsilon\end{aligned}$$

i.e., $\beta \in U_{a_1,\dots,a_k;\varepsilon}(\alpha)$ and hence $U_{b_1,\dots,b_k;\delta}(\alpha) \subseteq U_{a_1,\dots,a_k;\varepsilon}(\alpha)$. □

Proof of (ii). Let $n \in \mathbb{N}$ and $C \in \{A, B\}$.

We have already showed that $B_n(q') \in \mathcal{B}(\tau^C)$ and so $\tau^C \cap B_n(q') \subseteq \mathcal{B}(\tau^C)$, which in turn implies that $\mathcal{B}(\tau^C) \cap B_n(q') \supseteq \mathcal{B}(\tau^C \cap B_n(q'))$.

Now let $i : \mathfrak{sp}(q) \cap B_n(q') \cap \to \mathfrak{sp}(q)$ be the identity map. On the hand, the continuity of $i : (\mathfrak{sp}(q) \cap B_n(q'), \tau^C \cap B_n(q')) \cap \to (\mathfrak{sp}(q), \tau^C)$ provides that $i : (\mathfrak{sp}(q) \cap B_n(q'), \mathcal{B}(\tau^C \cap B_n(q'))) \cap \to (\mathfrak{sp}(q), \mathcal{B}(\tau^C))$ is measurable. On the other hand, $\mathcal{B}(\tau^C) \cap B_n(q')$ is the smallest $\sigma-$algebra on $\mathfrak{sp}(q) \cap B_n(q')$ making i measurable. Hence, we have that $\mathcal{B}(\tau^C) \cap B_n(q') \subset \mathcal{B}(\tau^C \cap B_n(q'))$. □

Lemma 4.17. *Let A be an algebra generated by a linear subspace $V \subseteq A$ and τ a topology on A such that (A, τ) is a topological algebra. If U is a subspace of V which is dense in $(V, \tau\restriction_V)$, then $\langle U \rangle$ is dense in (A, τ).*

Proof. Let $w \in U$ and $v \in V$. Then there exists a net $(v_\alpha)_{\alpha\in I}$ with $v_\alpha \in U$ such that $v_\alpha \to v$. Since the multiplication is separately continuous, we have that $wv_\alpha \to wv$ and hence $wv \in \overline{\langle U \rangle}$.

Now let us take also $u \in V$.

We will show by induction on n that

$$v_1, \ldots, v_n \in V \Rightarrow v_1 \cdots v_n \in \overline{\langle U \rangle}, \forall n \in \mathbb{N}, \tag{4.3}$$

which implies that $\langle V \rangle = \overline{\langle U \rangle}$ and so the conclusion $A = \overline{\langle U \rangle}$.

Let us first show the base case $n = 2$. If $v_1, v_2 \in V$ then there exist nets $(u_\alpha)_{\alpha \in I}$ and $(w_\beta)_{\beta \in J}$ with $u_\alpha, w_\beta \in U$ such that $u_\alpha \to v_1$ and $w_\beta \to v_2$. Since the multiplication is separately continuous, for each $u \in U$, we have that $uw_\beta \to uv_2$ and hence $uv_2 \in \overline{\langle U \rangle}$. In particular each $u_\alpha v_2 \in \overline{\langle U \rangle}$ and, using again the separate continuity of the multiplication, $u_\alpha v_2 \to v_1 v_2$. Hence, $v_1 v_2 \in \overline{\langle U \rangle}$.

Suppose now that (4.3) holds for a fixed n and let $v_1, \ldots, v_{n+1} \in V$. Then there exists $(g_\alpha)_{\alpha \in I}$ with $g_\alpha \in U$ such that $g_\alpha \to v_{n+1}$. Moreover, by inductive assumption, $v_1 \cdots v_n \in \overline{\langle U \rangle}$ and so there exists $(h_\beta)_{\beta \in J}$ with $h_\beta \in \langle U \rangle$ such that $h_\beta \to v_1 \cdots v_n$. Then for any $u \in U$, by the separate continuity of the multiplication, we have that $h_\beta u \to v_1 \cdots v_n \cdot u$ and hence $v_1 \cdots v_n \cdot u \in \overline{\langle U \rangle}$. In particular for each $\alpha \in I$ we get that $v_1 \cdots v_n \cdot g_\alpha \in \overline{\langle U \rangle}$. Thus, using again the separate continuity of the multiplication, we obtain that $v_1 \cdots v_n \cdot g_\alpha \to v_1 \cdots v_n \cdot v_{n+1}$ and so $v_1 \cdots v_n \cdot v_{n+1} \in \overline{\langle U \rangle}$. $\square$

Acknowledgments

The authors would like to thank the anonymous referee for the valuable suggestions and Victor Vinnikov for the helpful discussions. They are indebted to the Baden–Württemberg Stiftung for the financial support to this work by the Eliteprogramme for Postdocs. This work was also partially supported by the Ausschuss für Forschungsfragen (AFF) and by the Fondazione di Sardegna within the project MAPS (grant number F73C23001550007). Maria Infusino and Tobias Kuna are both members of the GNAMPA group of INdAM.

References

[1] S. Albeverio, F. Herzberg, The moment problem on the Wiener space, Bull. Sci. Math. 132 (1) (2008) 7–18.

[2] Yu.M. Berezansky, Yu.G. Kondratiev, Spectral Methods in Infinite-Dimensional Analysis. Vol. II (Transl. from the 1988 Russian original), Mathematical Physics and Applied Mathematics, vol. 12/2, Kluwer Academic Publishers, Dordrecht, 1995.

[3] Yu.M. Berezansky, S.N. Šifrin, A generalized symmetric power moment problem (Russian), Ukr. Mat. Ž. 23 (1971) 291–306.

[4] Yu.M. Berezansky, Z.G. Sheftel, G.F. Us, Functional Analysis. Vol. II, Birkháuser, Basel, 1996.

[5] V.I. Bogachev, Measure Theory. Vol. II, Springer, Berlin, 2007.

[6] N. Bourbaki, Elements of Mathematics. Theory of Sets, Hermann/Addison-Wesley Publishing Co., Paris/Reading, MA etc., 1968.

[7] N. Bourbaki, Elements of Mathematics. Topological Vector Spaces, Chapters 1–5, Springer, Berlin, 2003.

[8] H.J. Borchers, J. Yngvason, Integral representations for Schwinger functionals and the moment problem over nuclear spaces, Commun. Math. Phys. 43 (3) (1975) 255–271.

[9] I.M. Gel'fand, N.Ya. Vilenkin, Generalized Functions. Vol. 4: Applications of Harmonic Analysis, Academic Press, New York and London, 1964.

[10] M. Ghasemi, M. Infusino, S. Kuhlmann, M. Marshall, Moment problem for symmetric algebras of locally convex spaces, Integral Equ. Oper. Theory 90 (3) (2018) 29, 19 pp.

[11] A. Grothendieck, Produits tensoriels topologiques et espaces nucléaires, Mem. Am. Math. Soc., vol. 16, Am. Math. Soc., Providence, RI, 1955.

[12] G.C. Hegerfeldt, Extremal decomposition of Wightman functions and of states on nuclear *-algebras by Choquet theory, Commun. Math. Phys. 45 (2) (1975) 133–135.

[13] A.G. Kostyuchenko, B.S. Mityagin, Positive definite functionals on nuclear spaces, Tr. Mosk. Mat. Obŝ. 9 (1960) 283–316.

[14] M. Infusino, Quasi-analyticity and determinacy of the full moment problem from finite to infinite dimensions, Chap. 9, in: Stochastic and Infinite Dimensional Analysis, in: Trends in Mathematics, Birkhäuser, 2016, pp. 161–194.

[15] M. Infusino, S. Kuhlmann, Infinite dimensional moment problem: open questions and applications, in: Ordered Algebraic Structures and Related Topics, in: Contemporary Mathematics, vol. 697, Amer. Math. Soc., Providence, RI, 2017, pp. 187–201.

[16] M. Infusino, T. Kuna, A. Rota, The full infinite dimensional moment problem on semi-algebraic sets of generalized functions, J. Funct. Anal. 267 (5) (2014) 1382–1418.

[17] M. Infusino, T. Kuna, The full moment problem on subsets of probabilities and point configurations, J. Math. Anal. Appl. 483 (1) (2020) 123551, 29 pp.

[18] M. Infusino, S. Kuhlmann, T. Kuna, P. Michalski, Projective limit techniques for the infinite dimensional moment problem, Integral Equ. Oper. Theory 94 (2) (2022) 12, 44 pp.

[19] M. Infusino, S. Kuhlmann, T. Kuna, P. Michalski, An intrinsic characterization of moment functionals in the compact case, Int. Math. Res. Not. (3) (2023) 2281–2303.

[20] S. Mandelbrojt, Analytic functions and classes of infinitely differentiable functions, Rice Inst. Pamphlet 29 (1) (1942).

[21] M. Marshall, Positive polynomials and sums of squares, Math. Surveys Monogr., vol. 146, American Mathematical Society, Providence, RI, 2008, xii+187 pp.

[22] B.S. Mityagin, Approximate dimension and bases in nuclear spaces, Usp. Mat. Nauk 16 (4) (1961) 59–127.

[23] M. Reed, B. Simon, Methods of Modern Mathematical Physics, vol. 1, Functional Analysis, Second edition, Academic Press, New York, London, 1980.

[24] K. Schmüdgen, Unbounded Operator Algebras and Representation Theory, Operator Theory: Advances and Applications, vol. 37, Birkhäuser Verlag, Basel, 1990.

[25] K. Schmüdgen, The Moment Problem, Grad. Texts in Math., vol. 277, Springer, Cham, 2017.

[26] K. Schmüdgen, On the infinite dimensional moment problem, Ark. Mat. 56 (2) (2018) 441–459.

[27] L. Schwartz, Radon Measures on Arbitrary Topological Spaces and Cylindrical Measures, Tata Inst. Fund. Res. Stud. Math., vol. 6, Oxford University Press, London, 1973.

[28] F. Trèves, Topological Vector Spaces, Distributions and Kernels, Academic Press, New York-London, 1967.

[29] Y. Umemura, Measures on infinite dimensional vector spaces, Publ. Res. Inst. Math. Sci., Kyoto Univ., Ser. A 1 (1965) 1–47.

[30] Y. Yamasaki, Measures on Infinite-Dimensional Spaces, Series in Pure Mathematics, vol. 5, World Scientific Publishing Co., Singapore, 1985.