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# Strict Lyapunov Functions for the Super-Twisting Consensus Algorithm

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**Abstract**—The super-twisting algorithm is widely used in the design of controllers, observers, and differentiators. This paper investigates its application to achieve consensus among a team of peer agents characterized by first-order perturbed dynamics. Agents communicate through a directed, weight-balanced communication topology. The key contributions include employing the super-twisting algorithm to solve the finite-time robust consensus problem and developing a method for constructing a family of strict Lyapunov functions to analyze its robust stability. The analysis is based on the Lyapunov approach, and the necessary conditions for the stability of the closed-loop system are derived. Simulations supporting the theoretical analysis are also provided.

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## I. INTRODUCTION

Achieving consensus among  $n$  agents, where each follows first-order integrator dynamics

$$\dot{x}_i(t) = u_i(t), \quad i = 1, 2, \dots, n, \quad (1)$$

has been extensively studied in Systems Theory [1]–[5]. Algebraic Graph Theory and Lyapunov stability arguments have been used to analyze consensus over undirected or directed balanced graphs via the Laplacian flow (L-F) protocol:

$$u_i(t) = \sum_{j \neq i} a_{ij}(x_j(t) - x_i(t)), \quad (2)$$

where  $a_{ij}$  represents the weights of local disagreements. Clearly, (2) represents the simplest feedback interaction rule aimed at enforcing a *consensus*, i.e.,  $x_i(t) = x_j(t)$ ,  $\forall i, j$ .

With the growth of communication technologies, the decision-making consensus paradigm has become essential for coordinating geographically distributed agents, including smart grid prosumer management [6], [7], sensor network applications [8], [9], and multi-robot coordination [10], [11]. Ensuring robustness has thus become a crucial problem in consensus control [12]. Robust consensus techniques include PI-based Laplacian flows for managing biases and noise [13], randomized methods for addressing packet loss [14], and  $H_\infty$  and  $H_2$  control for handling model uncertainties [15], [16]. Nonlinear techniques, including the Variable Structure Control approach with Sliding Modes (SM), have shown

particular effectiveness [17]. Two main families of SM protocols can be identified. The signed Laplacian flow (sL-F) and the signed deviations sum flow (sDS-F), i.e.

$$\text{sL-F:} \quad u_i(t) = k \operatorname{sgn} \left( \sum_{j \neq i} a_{ij}(x_j(t) - x_i(t)) \right), \quad (3)$$

$$\text{sDS-F:} \quad u_i(t) = k \sum_{j \neq i} a_{ij} \operatorname{sgn}(x_j(t) - x_i(t)), \quad (4)$$

both belonging to the family of first-order SMs (FOSMs). While higher-order SM (HOSM) algorithms offer improved performance [18], extending these methods to higher-order consensus manifolds remains challenging. Finding Lyapunov functions for the linear L-F protocol [5] is complex, and extending this to nonlinear Lyapunov functions and integral consensus manifolds adds further difficulty. Only the twisting SM sL-F has been formally proven using Lyapunov arguments for undirected graphs [6], [19].

As for the other popular HOSM algorithm, the *super-twisting* (STW), besides [20], other works claiming to use it for consensus do not implement it in a distributed manner. Instead, they utilize it, resp., in a decentralized way [21], for designing a local observer [22], or to implement an additive second-order local integral SM term [23]. It is worth also noting that while [20] provides theorems and simulations showing the STW sL-F works if implemented in a distributed manner, it does not offer any formal proof as support.

**Main Contribution:** This paper addresses these challenges by developing a method to construct strict Lyapunov functions for the STW sL-F under directed weighted-balanced graphs, both with and without perturbations. This extends the method in [24] to a second-order consensus problem, advancing beyond [20] and [24]. Unlike [20], we consider a leaderless setting which is more difficult to study due to the lack of a reference trajectory for the agents. This makes the Lyapunov analysis more challenging because it requires recasting the consensus problem into a manifold's stability problem, as opposed to the more common stability-of-the-equilibrium theory used for the leader-tracking problem.

**Paper's Structure:** Section II introduces some basics of Graph Theory. Section III formulates the problem statement, Section IV presents the main results, and Section V provides supporting simulations. Finally, Section VI summarizes the conclusions and suggests directions for future research.

**Notations:** Let  $s \in \mathbb{C}$ , then  $\Re\{s\} \in \mathbb{R}$  and  $\Im\{s\} \in \mathbb{R}$  denote, resp., its real and imaginary parts. For  $i = 1, 2, \dots, d \in \mathbb{N}$ , a real column vector  $x \in \mathbb{R}^d$  is denoted by  $x = [x_i]$  =

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$(x_1; x_2; \dots; x_d)$ , where  $x_i \in \mathbb{R}$  is its  $i$ -th entry, while  $x^\top = (x_1, x_2, \dots, x_d)$  denotes its transpose.  $\mathbf{M} = [m_{ij}] \in \mathbb{R}^{d_1 \times d_2}$  denotes a real matrix, and  $m_{ij} \in \mathbb{R}$  is its  $(i, j)$ -th entry. To denote  $\mathbf{M}$  is positive (resp. semi-) definite we use  $\mathbf{M} \succ 0$  (resp.  $\mathbf{M} \succeq 0$ ). For  $x, y \in \mathbb{R}^d$ ,  $x^\top \cdot y$  and  $x \otimes y$  denote the scalar, and the Kronecker product. The 1-, 2- and  $\infty$ -norms of  $x$  are, resp.,  $\|x\|_1 = \sum_{i=1}^d |x_i|$ ,  $\|x\|_2 = \sqrt{x^\top \cdot x}$ ,  $\|x\|_\infty = \max_i |x_i|$ , moreover  $x^\top \cdot y \leq |x^\top \cdot y| \leq \|x\| \cdot \|y\|$ . Let  $\mathbf{1}_d = [1] \in \mathbb{R}^d$  and  $\mathbf{0}_d = [0] \in \mathbb{R}^d$ , while  $\mathbf{I}_d \in \mathbb{R}^{d \times d}$  is the identity matrix.  $\text{diag}(x) : \mathbb{R}^d \rightarrow \mathbb{R}^{d \times d}$  outputs a diagonal matrix whose entry  $(i, i)$  is  $x_i \forall i$ , 0 otherwise, while  $\text{sgn}(x) : \mathbb{R}^d \rightarrow \mathbb{R}^d$  outputs a vector whose entry  $i$  is 1 if  $x_i > 0$ , -1 if  $x_i < 0$ , 0 otherwise.

## II. BASICS OF GRAPH THEORY

Consider a set of agents  $\mathcal{V} = \{1, 2, \dots, n\}$ . Interactions can be drawn by a directed graph (di-graph)  $G(\mathcal{V}, \mathcal{E})$ , where  $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$  is the set of available interactions. Arc  $(i, j) \in \mathcal{E}$  exists if and only if  $a_{ij} > 0$ , meaning  $i$  receives information from  $j$ . Otherwise,  $a_{ij} = 0$ . The neighborhood of agent  $i$  is  $\mathcal{N}_i = \{j \in \mathcal{V} : (i, j) \in \mathcal{E}\}$ , and the adjacency matrix of  $G$  is  $\mathbf{A} = [a_{ij}]$ .

The (non-symmetric) Laplacian matrix  $\mathbf{L} = [\ell_{ij}] \in \mathbb{R}^{n \times n}$  associated with the pair  $(G, \mathbf{A})$  is defined as  $\ell_{ij} = -a_{ij}$  for  $i \neq j$  and  $\ell_{ii} = \sum_{j \in \mathcal{N}_i} a_{ij}$ . The  $i$ -th row of  $\mathbf{L}$  is denoted by  $\mathbf{L}_i$ . The ordered set of eigenvalues of  $\mathbf{L}$  is  $\Lambda(\mathbf{L}) = \{\lambda_1(\mathbf{L}), \lambda_2(\mathbf{L}), \dots, \lambda_n(\mathbf{L})\}$ , where  $|\Re\{\lambda_1\}| \leq |\Re\{\lambda_2\}| \leq \dots \leq |\Re\{\lambda_n\}|$ . By construction,  $\mathbf{L}\mathbf{1}_n = \mathbf{0}_n$  and  $\lambda_1(\mathbf{L}) = 0$ .

A di-graph  $G$  contains a rooted spanning tree if there exists an agent from which a path (formed by edges) connects it to all other agents, in which case  $\Re\{\lambda_2\} \neq 0$ . Here,  $\lambda_2(\mathbf{L})$ , known as the *algebraic connectivity*, indicates the connectivity of  $G$ . If  $G$  is strongly connected, there is a directed path from every agent to all others.

When  $\mathbf{1}_n^\top \mathbf{L} = \mathbf{0}_n^\top$ ,  $G$  is said to be weighted-balanced. In undirected graphs,  $a_{ij} = a_{ji}$ , and consequently  $\mathbf{L} = \mathbf{L}^\top \succeq 0$ .

**Lemma 1** ([5]). *Let  $G$  be a di-graph with a spanning tree and Laplacian matrix  $\mathbf{L}$ . For any  $\mathbf{Q} = \mathbf{Q}^\top \succ 0$  and  $\alpha \in \mathbb{R}_{>0}$ , there exists a matrix  $\mathbf{P}(\alpha) \in \mathbb{R}^{n \times n}$ ,  $\mathbf{P} = \mathbf{P}^\top \succ 0$  such that*

$$\mathbf{P}\mathbf{L} + \mathbf{L}^\top \mathbf{P} + \alpha(\mathbf{P}\mathbf{v}_\ell \mathbf{v}_\ell^\top + \mathbf{v}_\ell \mathbf{v}_\ell^\top \mathbf{P}) = \mathbf{Q}, \quad (5)$$

where  $\mathbf{v}_\ell$  and  $\mathbf{v}_r$  are the left and right real eigenvectors corresponding to the zero eigenvalue of  $\mathbf{L}$ ,  $\lambda_1(\mathbf{L}) = 0$ . ■

## III. PROBLEM FORMULATION

Consider a team of  $n$  perturbed agents described by:

$$\dot{x}_i(t) = u_i(t) + \pi_i(t), \quad i = 1, 2, \dots, n, \quad (6)$$

where  $x_i \in \mathbb{R}$  is the state of agent  $i$ ,  $u_i \in \mathbb{R}$  is the control input, and  $\pi_i \in \mathbb{R}$  is an unknown input, representing disturbances or uncertainties. The di-graph  $G$  captures the interaction between agents. The following assumptions will be used:

**Assumption 1.** *Di-graph  $G$  has a directed spanning tree.* □

**Assumption 2.** *Di-graph  $G$  is weighted and balanced.* □

**Assumption 3.** *Let  $\Pi$  be a known constant, perturbations  $\pi_i(t)$  are smooth and satisfy  $|\dot{\pi}_i(t)| \leq \Pi$ .* □

Consider (6), our goal is to ensure consensus among the agents' states, regardless of the unknown inputs acting in (6), within a finite time. More formally, this is expressed as:

$$\exists \quad T < \infty : x_i(t) - x_j(t) = 0, \quad \forall t \geq T, \quad \forall i, j \in \mathcal{V}. \quad (7)$$

To address task (7), we propose the following *Super-Twisting-based Laplacian-flow* interaction rule:

$$u_i = -k_1 |L_i x|^\frac{1}{2} \text{sgn}(L_i x) + w_i, \quad (8)$$

$$\dot{w}_i = -k_2 \text{sgn}(L_i x), \quad (9)$$

where  $k_1 > 0$  and  $k_2 > 0$  are positive tuning constants.

To obtain a concise representation of the overall MAS dynamics and, in accordance with Assumption 3, to explicitly account for the impact of the perturbations' time derivatives, we introduce the following change of variable:

$$(x, w) \mapsto (x, \overbrace{w + \pi}^z), \quad (10)$$

where  $x = [x_i]$ ,  $w = [w_i]$ , and  $\pi = [\pi_i]$  are three ordered vectors stacking, respectively, states  $x_i$ , the integral control  $w_i$ , and the perturbations  $\pi_i$ , while  $z = w + \pi$ .

Let  $\mathbf{L}_i$  be the  $i$ -th row of  $\mathbf{L}$ . Substituting (8) and (9) into (6), and using (10), we derive the following representation:

$$\dot{x} = -k_1 \text{diag}(|L_1 x|^\frac{1}{2}, \dots, |L_n x|^\frac{1}{2}) \text{sgn}(\mathbf{L}x) + z, \quad (11)$$

$$\dot{z} = -k_2 \text{sgn}(\mathbf{L}x) + \dot{\pi}. \quad (12)$$

**Remark 1.** *Since the right-hand side of (11)-(12) is discontinuous, their solutions must be interpreted in the sense of Filippov [25]. This interpretation involves treating the equations as differential inclusions, whose solutions are guaranteed to exist due to specific properties of the associated set-valued map. These solutions retain important properties, such as absolute continuity, as noted in [17].* ■

## IV. MAIN RESULTS

### A. Preliminary Considerations and Instrumental Results

For later convenience, and drawing inspiration from the single-input-single-output STW milestone paper [24], (11)-(12) need to be rewritten in the following coordinate space:

$$\sigma = \begin{bmatrix} \zeta \\ z \end{bmatrix} = \begin{bmatrix} \text{diag}(|L_1 x|^\frac{1}{2}, \dots, |L_n x|^\frac{1}{2}) \text{sgn}(\mathbf{L}x) \\ z \end{bmatrix} \quad (13)$$

From this, through straightforward manipulations, we obtain:

$$\dot{\sigma}(t) = \mathbf{D}(\zeta)(\mathbf{A} + \mathbf{N}(t))\sigma(t) \quad (14)$$

where  $\mathbf{D}(\zeta)$ ,  $\mathbf{A}$ , and  $\mathbf{N}(t)$  are real matrices defined as:

$$\mathbf{D}(\zeta) = \mathbf{I}_2 \otimes \text{diag}\left(\frac{1}{|\zeta_1|}, \frac{1}{|\zeta_2|}, \dots, \frac{1}{|\zeta_N|}\right), \quad (15)$$

$$\mathbf{A} = \begin{pmatrix} -\frac{k_1}{2} \mathbf{L} & \frac{1}{2} \mathbf{L} \\ -k_2 \mathbf{I}_n & \mathbf{0}_{n \times n} \end{pmatrix}, \mathbf{N}(t) = \begin{pmatrix} \mathbf{0}_{n \times n} & \mathbf{0}_{n \times n} \\ \text{diag}([\dot{\pi}_i \cdot \text{sgn}(L_i x)]) & \mathbf{0}_{n \times n} \end{pmatrix} \quad (16)$$

From (14), it is easy to observe that the STW sL-F dynamics (11)-(12) are now rewritten in an appealing form.

Moreover, this holds particularly in the unperturbed case, where the matrix  $N(t)$  is identically zero, yielding:

$$\dot{\sigma} = D(\zeta) A \sigma. \quad (17)$$

For later purposes, the following result is illustrated.

**Lemma 2.** *Let  $\lambda_j(L)$  be the eigenvalues of a generic (non-symmetric) Laplacian matrix  $L$ , with  $j = 1, 2, \dots, n$ . Consider  $A$  in (16), with  $k_1 > 0$  and  $k_2 > 0$ . To each  $\lambda_j(L)$  corresponds a pair of eigenvalues of  $A$  that are solutions of:*

$$s^2 + \left(\frac{k_1}{2} \lambda_j(L)\right)s + \frac{k_2}{2} \lambda_j(L) = 0, \quad \forall \lambda_j(L) \in \mathbb{C}. \quad (18)$$

**Proof.** Consider the matrix  $A$  in (16). The corresponding eigenvalue problem can also be written in block form as

$$\begin{pmatrix} -\frac{1}{2}k_1 L & \frac{1}{2}L \\ -k_2 I_n & \mathbf{0}_{n \times n} \end{pmatrix} \cdot \begin{bmatrix} \zeta \\ z \end{bmatrix} = s \cdot \begin{bmatrix} \zeta \\ z \end{bmatrix}, \quad (19)$$

where  $s \in \mathbb{C}$  is one of the  $2n$  eigenvalues of  $A$ . From the second line of (19), we have  $s \cdot z = -k_2 \zeta$ . Substituting  $z$  into the first line of (19), and by letting

$$\lambda = -\frac{s}{\left(\frac{k_1}{2} + \frac{k_2}{2s}\right)}, \quad \text{it yields} \quad (L - \lambda I_n) \cdot \zeta = \mathbf{0}_n. \quad (20)$$

From (20) it follows  $\lambda$  solves the eigenvalue problem of  $L$ ,  $\det(L - \lambda I_n) = 0$ . Thus, by recasting the first expression of (20), one can finally obtain (18). From that, it further results that each  $\lambda_j(L)$  gives a pair of eigenvalues for  $A$  located at

$$s_{1,2} = \frac{-\frac{k_1}{2} \lambda_j(L) \pm \sqrt{\left(\frac{k_1}{2} \lambda_j(L)\right)^2 - 8k_2 \lambda_j(L)}}{2} \in \mathbb{C}. \quad \square \quad (21)$$

Let us now provide a characterization of the eigenvalues of  $A$  as a function of the communication di-graph  $G$ , and the tuning gains  $k_1 > 0$  and  $k_2 > 0$  in the protocol (8)-(9).

**Lemma 3.** *Let  $L(G)$  be a given (non-symmetric) Laplacian matrix, and let  $A$  be as in (16), with  $k_1 > 0$  and  $k_2 > 0$ . Then, the following statements are true:*

a)  $\lambda_1(A) = \lambda_2(A) = 0$ , while  $\lambda_k(A) \in \mathbb{C}$  for all  $k \geq 3$  and for all  $L(G)$ ;

b) If  $G$  has a directed spanning tree, a necessary and sufficient condition for  $\Re\{\lambda_k(A)\} < 0$  for all  $k \geq 3$  is:

$$0 < k_2 < \left( \frac{\Re\{\lambda_j(L)\}^3 + \Re\{\lambda_j(L)\} \Im\{\lambda_j(L)\}^2}{2 \Im\{\lambda_j(L)\}^2} \right) k_1^2, \quad \forall j \geq 2; \quad (22)$$

c) If  $G$  has a directed spanning tree, a sufficient condition for  $\Re\{\lambda_k(A)\} < 0$  for all  $k \geq 3$  is:

$$0 < k_2 \leq 0.5 \cdot \Re\{\lambda_2(L)\} k_1^2; \quad (23)$$

d) If  $G$  is undirected and strongly connected, then  $\Re\{\lambda_k(A)\} \in \mathbb{R}_{<0}$  for all  $k \geq 3$ , and  $k_1 > 0$ ,  $k_2 > 0$ .

**Sketch of Proof.** The complete proof is omitted due to space limitations. The full proof will appear in the extended version of this study. However, note that: a) Follows from the properties  $L\mathbf{1}_n = \mathbf{0}_n$  and  $\lambda_1(L) = 0$ ; b) Is concluded from the application of Routh's criterion for polynomials with complex coefficients [26, Algorithm 1]; c) Represents the

upper bound of (22), as a function of  $\lambda_2(L)$  alone; d) Is a consequence of the fact that  $L = L^\top$ .  $\square$

For later usage, note that achieving (7), or equivalently  $\zeta_i = 0 \forall i$ , can be reformulated into a second-order consensus along  $\zeta$  and  $z$  as in (13)-(14), namely

$$\begin{aligned} \zeta_i(t) - \zeta_j(t) &= 0 \\ z_i(t) - z_j(t) &= 0 \end{aligned} \quad \forall t \geq T, \quad \forall i, j. \quad (24)$$

Although satisfying (24) implies only  $\zeta_i = \zeta_j = c$  where  $c$  is a constant value. Since  $\zeta_i = L_i x$ , it is easy to verify that the only  $c$  satisfying  $L_i x = L_j x$  for all  $j$  is  $c = 0$ . Let us thus rewrite (13) in terms of its *disagreement vector*, given by

$$\tilde{\delta} = \begin{bmatrix} \tilde{\delta}_1 \\ \tilde{\delta}_2 \end{bmatrix} = \overbrace{\begin{bmatrix} L_c & \mathbf{0}_{n \times n} \\ \mathbf{0}_{n \times n} & L_c \end{bmatrix}}^{H_c} \cdot \begin{bmatrix} \zeta \\ z \end{bmatrix} = \begin{bmatrix} \mathbf{0}_n \\ \mathbf{0}_n \end{bmatrix} \quad \forall t \geq T, \quad (25)$$

where the matrix

$$L_c = \left( I_n - \frac{\mathbf{1}_n \mathbf{1}_n^\top}{n} \right) \text{ satisfies } \begin{cases} \mathbf{1}_n^\top L_c = \mathbf{0}_n^\top \\ L_c \mathbf{1}_n = \mathbf{0}_n \end{cases} \implies L L_c = L \quad (26)$$

has the same properties as a Laplacian matrix normalized for  $n$  of a *complete graph*. Namely, it has an eigenvalue  $\lambda_1(L_c) = 0$  with multiplicity 1 and  $\lambda_j(L_c) = 1$  with multiplicity  $n - 1$  [13], which thus also implies

$$\|L_c\| \leq 1 \implies \|H_c\| \leq 1 \quad (27)$$

Furthermore, if  $L_c$  pre-multiplies a Laplacian matrix  $L$  of a generic *weighted-balanced* di-graph, it further holds

$$L_c L = L. \quad (28)$$

Finally, for later usage, we present the following two results.

**Corollary 1.** *Assume  $G$  is weight-balanced. Consider matrix  $A$  as in (16) and  $H_c = \text{diag}([L_c, L_c])$  as in (25)-(26). Let*

$$\tilde{A} = H_c A = \begin{pmatrix} -\frac{k_1}{2} L & \frac{1}{2} L \\ -k_2 L_c & \mathbf{0}_{n \times n} \end{pmatrix}, \quad \text{then:} \quad (29)$$

a) *The spectrum of  $\tilde{A}$  and  $A$  are identical. Thus it follows Lemma 2 holds for  $\tilde{A}$  as well;*

b) *Let  $v_r = \mathbf{1}_n$  be the right eigenvector corresponding to  $\lambda_1(L)$ . The right eigenvectors of  $\lambda_1(\tilde{A})$  and  $\lambda_2(\tilde{A})$  are*

$$v'_r = \begin{bmatrix} v_r \\ v_r \end{bmatrix} = \begin{bmatrix} \mathbf{1}_n \\ \mathbf{1}_n \end{bmatrix}, \quad v''_r = \begin{bmatrix} v_r \\ -v_r \end{bmatrix} = \begin{bmatrix} \mathbf{1}_n \\ -\mathbf{1}_n \end{bmatrix}. \quad (30)$$

c) *Let  $v_\ell = \mathbf{1}_n$  be the left eigenvector corresponding to  $\lambda_1(L)$ . The left eigenvectors of  $\lambda_1(\tilde{A})$  and  $\lambda_2(\tilde{A})$  are*

$$v'_\ell = \begin{bmatrix} \mathbf{1}_n \\ \mathbf{1}_n \end{bmatrix}, \quad v''_\ell = \begin{bmatrix} \mathbf{1}_n \\ -\mathbf{1}_n \end{bmatrix}. \quad (31)$$

**Sketch of Proof.** The complete proof is omitted due to space limitations. The full proof will appear in the extended version of this study. However, note that all the statements can be easily proved by analyzing the eigenvalue problem associated with matrix  $\tilde{A}$  while exploiting the properties of matrix  $L$  as a result of Lemma 3.  $\square$

Finally, let us now present our generalization of the recent Lemma 1 from [5], tailored for the specific case under consideration, namely, the matrix  $\tilde{\mathbf{A}}$  in (29).

**Lemma 4.** *Let  $\mathbf{L}(\mathbf{G})$  be a given (non-symmetric) Laplacian matrix, where  $\mathbf{G}$  satisfies Assumptions 1 and 2 ( $\mathbf{G}$  is weight-balanced and has a spanning tree). Consider the matrix  $\tilde{\mathbf{A}}$  in (29) with  $k_1 > 0$  and  $k_2 > 0$  satisfying either (22) or (23) (thus  $\Re\{\lambda_k(\tilde{\mathbf{A}})\} < 0 \ \forall k \geq 3$ ).*

*Then, for any  $\mathbf{Q} = \mathbf{Q}^\top \succ 0$  and  $\alpha \in \mathbb{R}_{>0}$ , there exists a matrix  $\mathbf{P} \in \mathbb{R}^{2n \times 2n}$ ,  $\mathbf{P} = \mathbf{P}^\top \succ 0$  such that*

$$\mathbf{P}\tilde{\mathbf{A}} + \tilde{\mathbf{A}}^\top \mathbf{P} - \alpha [\mathbf{P}(\mathbf{v}'_r \mathbf{v}'_\ell{}^\top + \mathbf{v}''_r \mathbf{v}''_\ell{}^\top) + (\mathbf{v}'_r \mathbf{v}'_\ell{}^\top + \mathbf{v}''_r \mathbf{v}''_\ell{}^\top)^\top \mathbf{P}] = -\mathbf{Q} \quad (32)$$

**Proof.** The proof is constructive. Let's first consider the Jordan decomposition of  $\tilde{\mathbf{A}}$ , given by:

$$\tilde{\mathbf{A}} = \mathbf{V} \mathbf{J} \mathbf{V}^{-1} \quad (33)$$

Recalling that  $\mathbf{J}$  is a Jordan matrix, and  $\mathbf{V}$  and  $\mathbf{V}^{-1}$  in (33) are composed of the right and left eigenvectors of the matrix  $\tilde{\mathbf{A}}$  respectively, we can denote those matrices as follows:

$$\mathbf{V} = (\mathbf{v}'_r, \mathbf{v}''_r, \mathbf{V}_1) \quad , \quad \mathbf{V}^{-1} = (\mathbf{v}'_\ell{}^\top; \mathbf{v}''_\ell{}^\top; \mathbf{V}_1^\dagger). \quad (34)$$

Here,  $\mathbf{v}'_r, \mathbf{v}''_r$  (resp.  $\mathbf{v}'_\ell, \mathbf{v}''_\ell$ ) denote the right (resp. left) eigenvectors corresponding to  $\lambda_1(\tilde{\mathbf{A}}) = 0$  and  $\lambda_2(\tilde{\mathbf{A}}) = 0$ . For further development, we also remark that

$$\tilde{\mathbf{A}} = \lambda_1(\tilde{\mathbf{A}}) \mathbf{v}'_r \mathbf{v}'_\ell{}^\top + \lambda_2(\tilde{\mathbf{A}}) \mathbf{v}''_r \mathbf{v}''_\ell{}^\top + \mathbf{V}_1 \mathbf{J}_1 \mathbf{V}_1^\dagger = \mathbf{V}_1 \mathbf{J}_1 \mathbf{V}_1^\dagger \quad (35)$$

since  $\lambda_1(\tilde{\mathbf{A}}) = \lambda_2(\tilde{\mathbf{A}}) = 0$ . Now, for any  $\alpha > 0$ , define

$$\mathbf{R}(\alpha) := \mathbf{V}_1 \mathbf{J}_1 \mathbf{V}_1^\dagger - \alpha [\mathbf{v}'_r \mathbf{v}'_\ell{}^\top + \mathbf{v}''_r \mathbf{v}''_\ell{}^\top], \quad (36)$$

where  $\mathbf{R}(0) = \tilde{\mathbf{A}}$  and the matrix  $\mathbf{R}(\alpha)$  satisfies

$$\mathbf{R}(\alpha) = \mathbf{V} \mathbf{J}_\alpha \mathbf{V}^{-1} \quad \text{with} \quad \mathbf{J}_\alpha = \begin{pmatrix} -\alpha \otimes \mathbf{I}_2 & \mathbf{0}_{2 \times 2n} \\ \mathbf{0}_{2n \times 2} & \mathbf{J}_1 \end{pmatrix}. \quad (37)$$

Since  $\mathbf{R}(\alpha)$  is Hurwitz  $\forall \alpha > 0$ , for any  $\alpha > 0$  and  $\mathbf{Q} = \mathbf{Q}^\top \succ 0$  there exist  $\mathbf{P} = \mathbf{P}^\top \succ 0$  such that

$$\mathbf{P}\mathbf{R}(\alpha) + \mathbf{R}(\alpha)^\top \mathbf{P} = -\mathbf{Q}, \quad (38)$$

or equivalently (32) holds.  $\square$

### B. Lyapunov Function Analysis Without Perturbations

Consider the unperturbed MAS dynamics (17), with matrices defined in (15)-(16), where  $\sigma$  is defined in (13).

**Theorem 1.** *Let  $\mathbf{G}$  be a weighted-balance di-graph containing a directed spanning-tree (Assumption 1 and 2 hold), with Laplacian matrix  $\mathbf{L}(\mathbf{G})$ . Consider  $n$  unperturbed single-integrator agents described by (1) under the STW sL-F control protocol (8)-(9). Let  $k_1 > 0$ , and let  $k_2 > 0$  satisfy either (22) or (23). Then, the agents will achieve the consensus condition (7) in a finite time.*

**Proof.** First, we demonstrate that the agents achieve (7) exponentially. Then, we establish the finite-time stability.

a) *Exponential stability:* To prove this statement we study the disagreement vector (25) as an alternative coordinate space.

It is evident that  $\tilde{\delta} = \mathbf{0}_{2n}$  implies (24), meaning  $\zeta_i - \zeta_j = \mathbf{0}_n$  and  $z_i - z_j = \mathbf{0}_n$  for all  $i, j$ . This further implies  $|\mathbf{L}_i x|^\frac{1}{2} = |\mathbf{L}_j x|^\frac{1}{2} = \mathbf{0}_n$  for all  $i, j$ . From that let  $\mathbf{Q} = \mathbf{Q}^\top \succ 0$  and  $\alpha > 0$  be arbitrarily fixed. Then, consider

$$W = \tilde{\delta}^\top \mathbf{P} \tilde{\delta} \succ 0 \quad (39)$$

as candidate Lyapunov function, where  $\mathbf{P} = \mathbf{P}^\top > 0$ . Then, by differentiating it with respect to the time, it yields

$$\dot{W}(\tilde{\delta}) = \dot{\tilde{\delta}}^\top \mathbf{P} \tilde{\delta} + \tilde{\delta}^\top \mathbf{P} \dot{\tilde{\delta}} \quad (40)$$

For later convenience, let us first substitute (25), and then (17), into (40), resulting in

$$\begin{aligned} \dot{W} &= (\mathbf{H}_c \dot{\sigma})^\top \mathbf{P} (\mathbf{H}_c \sigma) + (\mathbf{H}_c \sigma)^\top \mathbf{P} (\mathbf{H}_c \dot{\sigma}) \\ &= (\sigma^\top \mathbf{A}^\top \mathbf{D}(\zeta) \mathbf{H}_c^\top) \mathbf{P} (\mathbf{H}_c \sigma) + (\sigma^\top \mathbf{H}_c^\top) \mathbf{P} (\mathbf{H}_c \mathbf{D}(\zeta) \mathbf{A} \sigma). \end{aligned}$$

Then, recalling  $\mathbf{D}(\zeta)$  from (15), and letting  $\rho = \|\mathbf{D}(\zeta)\|$ , the following upper estimation holds

$$\dot{W} \leq \rho \cdot \left( (\sigma^\top \mathbf{A}^\top \mathbf{H}_c^\top) \mathbf{P} \mathbf{H}_c \sigma + \sigma^\top \mathbf{H}_c^\top \mathbf{P} (\mathbf{H}_c \mathbf{A} \sigma) \right). \quad (41)$$

Consider  $\mathbf{H}_c$  in (25),  $\mathbf{A}$  in (16), and recall that  $\tilde{\mathbf{A}} = \mathbf{H}_c \mathbf{A}$  as in (29). Thus, (41) can be further rewritten as

$$\dot{W} \leq \rho \cdot \left( (\sigma^\top \tilde{\mathbf{A}}^\top) \mathbf{P} \mathbf{H}_c \sigma + \sigma^\top \mathbf{H}_c^\top \mathbf{P} (\tilde{\mathbf{A}} \sigma) \right). \quad (42)$$

Moreover, since  $\mathbf{L}\mathbf{L}_c = \mathbf{L}$  implies

$$\tilde{\mathbf{A}} \mathbf{H}_c = \tilde{\mathbf{A}}, \quad (43)$$

and because of Assumption 2 implies

$$\mathbf{H}_c \tilde{\mathbf{A}} = \tilde{\mathbf{A}}, \quad (44)$$

then the quadratic form in (42) can be reformulated back in terms of the original variable  $\tilde{\delta}$ , as per (39), as follows

$$\begin{aligned} \dot{W} &\leq \rho \cdot \left( (\sigma^\top \mathbf{H}_c^\top \tilde{\mathbf{A}}^\top) \mathbf{P} \mathbf{H}_c \sigma + \sigma^\top \mathbf{H}_c^\top \mathbf{P} (\tilde{\mathbf{A}} \mathbf{H}_c \sigma) \right) \\ &= \rho \cdot \tilde{\delta}^\top (\tilde{\mathbf{A}}^\top \mathbf{P} + \mathbf{P} \tilde{\mathbf{A}}) \tilde{\delta}. \end{aligned}$$

Now, let  $\alpha > 0$ , and because of  $k_1 > 0$  and  $k_2 > 0$  satisfy either (22) or (23), by using (32), it results

$$\dot{W} \leq \rho \cdot \tilde{\delta}^\top \left( -\mathbf{Q} + \alpha [(\mathbf{v}'_r \mathbf{v}'_\ell{}^\top + \mathbf{v}''_r \mathbf{v}''_\ell{}^\top)^\top \mathbf{P} + \mathbf{P}(\mathbf{v}'_r \mathbf{v}'_\ell{}^\top + \mathbf{v}''_r \mathbf{v}''_\ell{}^\top)] \right) \tilde{\delta},$$

where, due to Assumptions 1 and Assumption 2, and in accordance with Corollary 1.c, it follows that

$$\mathbf{v}'_r \mathbf{v}'_\ell{}^\top + \mathbf{v}''_r \mathbf{v}''_\ell{}^\top = \begin{bmatrix} \mathbf{1}_n \\ \mathbf{1}_n \end{bmatrix} \begin{bmatrix} \mathbf{1}_n \\ \mathbf{1}_n \end{bmatrix}^\top + \begin{bmatrix} \mathbf{1}_n \\ -\mathbf{1}_n \end{bmatrix} \begin{bmatrix} \mathbf{1}_n \\ -\mathbf{1}_n \end{bmatrix}^\top = 2 \cdot \begin{pmatrix} \mathbf{1}_{n \times n} & \mathbf{0}_{n \times n} \\ \mathbf{0}_{n \times n} & \mathbf{1}_{n \times n} \end{pmatrix}, \quad (45)$$

and where by noticing that property  $\mathbf{L}_c \mathbf{1}_n = \mathbf{0}_n$  implies

$$\mathbf{H}_c \begin{pmatrix} \mathbf{1}_{n \times n} & \mathbf{0}_{n \times n} \\ \mathbf{0}_{n \times n} & \mathbf{1}_{n \times n} \end{pmatrix} = \begin{pmatrix} \mathbf{1}_{n \times n} & \mathbf{0}_{n \times n} \\ \mathbf{0}_{n \times n} & \mathbf{1}_{n \times n} \end{pmatrix} \mathbf{H}_c = \mathbf{0}_{2n \times 2n} \quad (46)$$

then, one concludes that (45) can be further rewritten as

$$\dot{W} \leq -\rho \cdot \tilde{\delta}^\top \mathbf{Q} \tilde{\delta} < 0, \quad \forall \tilde{\delta} \neq \mathbf{0}_{2n}. \quad (47)$$

From this, it results that (17) is exponentially stable.

b) *Finite-time stability:* Consider the Lyapunov function (39), where  $\lambda_k(\mathbf{P}) \in \mathbb{R}_{>0}$  since  $\mathbf{P} = \mathbf{P}^\top \succ 0$ . Then, it yields

$$0 < \lambda_1(\mathbf{P}) \|\tilde{\delta}\|_2^2 \leq W \leq \lambda_{2n}(\mathbf{P}) \|\tilde{\delta}\|_2^2 \quad (48)$$

$$\Rightarrow \|\tilde{\delta}\|_2^2 \geq \frac{W}{\lambda_{2n}(\mathbf{P})}, \text{ and } \|\tilde{\delta}\|_2 \leq \frac{W^{\frac{1}{2}}}{(\lambda_1(\mathbf{P}))^{\frac{1}{2}}} \quad (49)$$

Now, let us consider (47), and note that

$$\dot{W} \leq -\rho \cdot \lambda_1(\mathbf{Q}) \|\tilde{\delta}\|_2^2. \quad (50)$$

Recalling  $\rho = \|\mathbf{D}(\zeta)\|$  where matrix  $\mathbf{D}(\zeta)$  is as defined in (15), and thanks to property (27), it results there exists  $\beta > 0$  such that the following chain of inequalities holds

$$\rho = \|\mathbf{D}(\zeta)\| \geq \|\zeta\|^{-1} \geq \|\sigma\|^{-1} \geq (\beta \|\tilde{\delta}\|)^{-1} \quad (51)$$

Based on these results, it is deduced that (50) can be further lower-bounded as follows

$$\dot{W} \leq -\rho \cdot \frac{\lambda_1(\mathbf{Q})}{\lambda_{2n}(\mathbf{P})} W \leq -\frac{(\lambda_1(\mathbf{P}))^{\frac{1}{2}} \lambda_1(\mathbf{Q})}{\beta \lambda_{2n}(\mathbf{P})} W^{\frac{1}{2}} \quad (52)$$

As a result, all trajectories of (52) globally converge to the origin of the consensus manifold  $\tilde{\delta}$  in a finite-time.  $\square$

### C. Lyapunov Function Analysis With Perturbations

Conditions for deriving a strict Lyapunov function for finite-time consensus (7) under perturbations are presented.

**Theorem 2.** Let  $\mathbf{G}$  be a weighted-balance di-graph containing a directed spanning-tree (Assumption 1 and 2 hold), with Laplacian matrix  $\mathbf{L}(\mathbf{G})$ . Consider  $n$  perturbed single-integrator agents described by (6) under the STW sL-F control protocol (8)-(9). Let the gains  $k_1 > 0$  and  $k_2 > 0$  be such that either (22) or (23) holds true while the local perturbations satisfy Assumption 3, namely  $|\dot{\pi}_i| \leq \Pi$ . Let

$$\Phi = \begin{pmatrix} -\frac{k_1}{2} \mathbf{L} & \frac{1}{2} \mathbf{L} \\ -k_2 \mathbf{L}_c & \mathbf{0}_{n \times n} \end{pmatrix} - (2\alpha) \cdot \begin{pmatrix} \mathbf{1}_{n \times n} & \mathbf{0}_{n \times n} \\ \mathbf{0}_{n \times n} & \mathbf{1}_{n \times n} \end{pmatrix}, \quad (53)$$

for any  $\alpha > 0$ . Let  $\mathbf{Q} = \mathbf{Q}^\top \succ 0$  be any positive definite matrix. If there exists a matrix  $\mathbf{P} = \mathbf{P}^\top \succ 0$  satisfying

$$\Phi^\top \mathbf{P} + \mathbf{P} \Phi = -\mathbf{Q}, \quad (54)$$

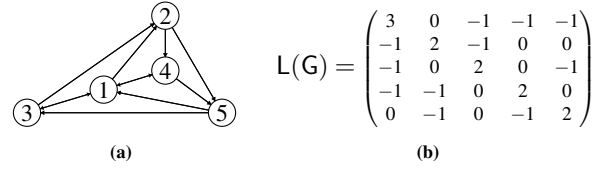
$$\mathbf{Q} - (2\Pi) \cdot \mathbf{P} \succ 0. \quad (55)$$

Then, the agents will achieve the consensus condition (7), irrespective of the presence of perturbations  $\pi_i(t)$ .

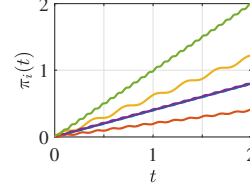
**Sketch of Proof.** The full proof is omitted due to space constraints and will be provided in the extended version of this study. Briefly, following a similar approach to the proof of Theorem 1, we differentiate (39) and apply Assumption 2, along with properties (43) and (44), where  $\tilde{\mathbf{A}}$  is given in (29). We also use the fact that  $\|\mathbf{N}(t)\| \leq \Pi$  and results from Corollary 1. This leads to:

$$\begin{aligned} \dot{W} &\leq \rho \cdot \tilde{\delta}^\top (\tilde{\mathbf{A}}^\top \mathbf{P} + \mathbf{P} \tilde{\mathbf{A}}) \tilde{\delta} \\ &\quad + \rho \cdot \sigma^\top (N(t)^\top \mathbf{H}_c^\top \mathbf{P} \mathbf{H}_c + \mathbf{H}_c^\top \mathbf{P} \mathbf{H}_c N(t)) \sigma \\ &\leq \rho \cdot \tilde{\delta}^\top (\tilde{\mathbf{A}}^\top \mathbf{P} + \mathbf{P} \tilde{\mathbf{A}}) \tilde{\delta} + (2\rho\Pi) \cdot \tilde{\delta}^\top \mathbf{P} \tilde{\delta} \\ &\leq -\rho \cdot \tilde{\delta}^\top (\mathbf{Q} - 2\Pi \cdot \mathbf{P}) \tilde{\delta}. \end{aligned} \quad (56)$$

From this, it follows that a sufficient condition for the agents to achieve (7) exponentially is that (55) holds. The proof of finite-time stability can then be directly derived using the



**Fig. 1:** Visualization of a weighted-balanced di-graph  $\mathbf{G}$  with unitary weights (a) and its corresponding Laplacian matrix  $\mathbf{L}(\mathbf{G})$  (b).



**Fig. 2:** Transient evolution of the perturbations  $\pi_i$ , where  $\pi_i$  is as in (57).

approach from the proof of Theorem 1, with  $\hat{\mathbf{Q}} = \mathbf{Q} - 2\Pi \cdot \mathbf{P}$ , where  $\mathbf{P} \succ 0$  is the solution to the LMIs (54)-(55), and  $\mathbf{Q} = \mathbf{Q}^\top \succ 0$  is any positive definite matrix.  $\square$

## V. NUMERICAL CASE STUDIES

We evaluated the proposed control protocol through numerical simulations in MATLAB®/Simulink, using the Euler fixed-step solver with a  $10^{-5}$  second sampling time.

We considered  $n = 5$  perturbed single-integrator agents as in (6), with initial conditions  $x_i(0) = -1 + i$  and  $z_i(0) = 0$  for  $i = 1, 2, \dots, 5$ . The graph  $\mathbf{G}$  was configured as shown in Figure 1a, with its Laplacian matrix  $\mathbf{L}(\mathbf{G})$  in Figure 1b. Following Assumption 3, perturbations  $\pi_i$  are unknown, with a known raw upper bound  $\Pi$  on their derivatives. In particular

$$\dot{\pi}_i(t) = a_i \sin(c_i t + d_i) + b_i \quad (57)$$

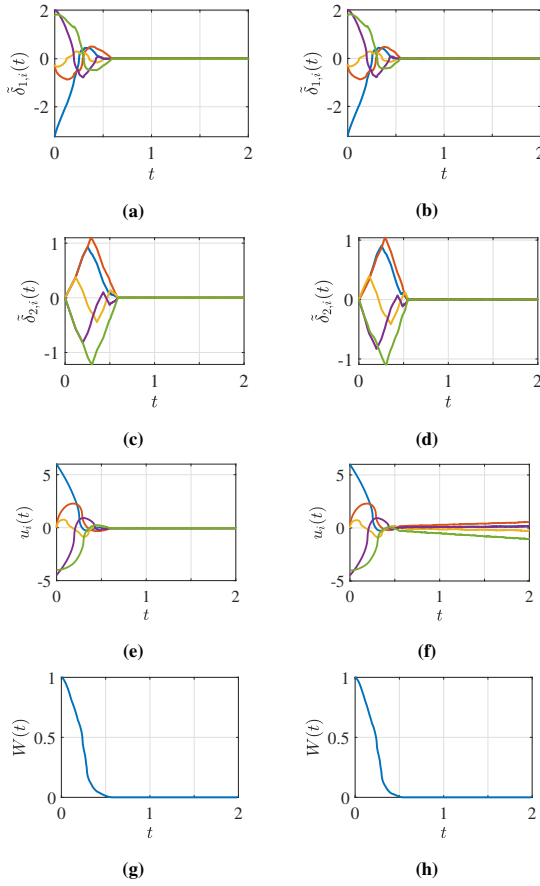
where parameters were randomly selected from the ranges  $a_i \in [-0.5, 0.5]$ ,  $b_i \in [-0.5, 0.5]$ ,  $c_i \in [0, 100]$ , and  $d_i \in [0, 2\pi]$ , ensuring  $\Pi = 1$ . Remarkably,  $\pi_i(t)$  has the potential to become unbounded, as depicted in Figure 2.

Figure 3 compares the multi-agent system's response (6) with perturbations (right column) and without perturbations (left column). In the unperturbed scenario, Theorem 1 shows that any pair of gains  $k_1 > 0$  and  $k_2 > 0$  satisfying either (22) or (23) ensures finite-time consensus as defined in (7). With  $\lambda_2(\mathbf{L}) \approx 2.34$ , we chose  $k_1 = 2$  and  $k_2 = 2$ , meeting the criteria of (23). The results indicate that both  $\tilde{\delta}_1$  and  $\tilde{\delta}_2$  in (25) converge to zero within a finite time, thus confirming the achievement of finite-time consensus.

In the perturbed scenario with  $\Pi = 1$ , Theorem 2 was validated using the feasp MATLAB® LMI solver. For  $k_1 = 2$ ,  $k_2 = 2$ ,  $\Pi = 1$ , and  $\mathbf{Q} = \mathbf{I}_{10} \succ 0$ , we solved

$$\begin{pmatrix} \Phi(\alpha)^\top \mathbf{P} + \mathbf{P} \Phi(\alpha) + (2\Pi) \mathbf{P} + \mathbf{Q} & & \\ & -\mathbf{P} & \\ & & -\alpha \end{pmatrix} \leq \gamma \cdot \mathbf{I}_{26}, \quad (58)$$

where  $\Phi(\alpha)$  is defined in (53). The solver iterated three times to solve (58), yielding  $\gamma = -0.9018$  and  $\alpha = 187.8703$ . The optimal positive definite matrix  $\mathbf{P}$  is omitted for brevity but was used to illustrate the transient behavior of the strict



**Fig. 3:** Simulation results are shown for both the absence (left column) and presence (right column) of perturbations. (a) and (b) (resp. (c) and (d)) show the disagreement vectors  $\tilde{\delta}_1$  (resp.  $\tilde{\delta}_2$ ) as defined in (25); (e) and (f) show the STW sL-F protocol  $u_i(t)$ ; (g) and (h) show the strict Lyapunov function  $W(t) = \tilde{\delta}^\top \mathbf{P} \tilde{\delta}$ , where  $\mathbf{P} = \mathbf{P}^\top \succ 0$  is obtained by solving the LMI (58).

Lyapunov function  $W(t) = \tilde{\delta}^\top \mathbf{P} \tilde{\delta}$ , as shown in the bottom subfigures of Figure 3, normalized for  $W(0)$ . The right column of Figure 3 shows that both  $\tilde{\delta}_1$  and  $\tilde{\delta}_2$  converge to zero within a finite time, confirming the achievement of finite-time consensus even in the presence of perturbations.

## VI. CONCLUSIONS

This paper presents a method for constructing strict Lyapunov functions for both unperturbed and perturbed STW sL-F consensus problems, assuming  $G$  is directed, strongly connected, and weight-balanced, with globally bounded disturbance derivatives. A simple tuning rule ensures finite-time consensus in the unperturbed case, while LMIs are derived for the perturbed case. Future work includes extending the results to leader-following consensus, removing the weight-balanced assumption, and reformulating the LMI problem to treat  $k_1$  and  $k_2$  as decision variables.

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