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ON WEAK AND STRICT RELATIVES KÄHLER MANIFOLDS

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ABSTRACT. We study Kähler manifolds that are (weak) relatives, that is, Kähler manifolds which share a (locally isometric) submanifold. In particular, we prove that if two Kähler manifolds are weak relatives and one of them is projective, then they are relatives. Moreover, we introduce the notion of strict relatives Kähler manifolds and provide several nontrivial examples.

1. INTRODUCTION AND STATEMENTS OF THE MAIN RESULTS

The study of holomorphic isometries of Kähler manifolds provides one of the most striking examples of rigidity in Kähler geometry. For instance, in his seminal work [2] Calabi proved that it does not exist a holomorphic isometry between finite dimensional complex space forms of different type. After Calabi, the topic has received great interest and many weakening of the holomorphic isometry condition have been studied. One of these generalizations is the following: which Kähler manifolds share a Kähler submanifold? This question was first considered and answered in the negative in the case of complex space forms by Umehara in [16]. Kähler manifolds which share a Kähler submanifold were christened relatives by Di Scala and Loi [7] who proposed their systematic study. In particular, they gave the following definition.

Definition 1 ([7]). *Two Kähler manifolds M_1 and M_2 are* ***relatives*** *(or are related) if there exists a Kähler manifold X and two holomorphic isometries $\varphi_i : X \longrightarrow M_i$ for $i = 1, 2$.*

Observe further that, since we are not assuming the compactness of X, we can consider φ_i to be a holomorphic isometric embedding by restricting to a open neighbourhood of a point in X. Umehara [16] proved that complex space forms with holomorphic sectional curvature of different sign are not relatives extending the aforementioned Calabi's result. Moreover, in [7] Di Scala and Loi proved the same result for a bounded domain and a projective Kähler manifold. The reader can refer to [18] and references therein for a recent survey on this topic. Even more recently, several generalizations of these results have been proven by Loi and Mossa [12, 13, 14], and Cheng and Hao [4]. This rigidity phenomenon is peculiar of Kähler manifolds as very often the aforementioned results rely heavily on the compatibility of the Riemannian metric and the complex structure. It is then natural to weaken this notion and study which pairs of Kähler manifolds share a Riemannian submanifold which is not necessarily a holomorphic submanifold.

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Definition 2 ([7]). *Two Kähler manifolds M_1 and M_2 are **weak relatives** (or are weakly related) if there exists two locally isometric Kähler manifolds X_1 and X_2 of complex dimension≥ 2 and holomorphic isometries $\varphi_i : X_i \longrightarrow M_i$ for $i = 1, 2$.*

Observe that the condition on the dimension of the submanifolds is necessary. In fact, any isometry between two one dimensional Kähler manifolds is necessarily holomorphic or antiholomorphic, see Lemma 6 below. Notice also that weak relatives manifolds are genuinely relatives when the manifolds X_1 and X_2 in the definition are in fact locally biholomorphically isometric. This is not always the case. For instance, it suffices to consider a 2-dimensional Kähler manifold (M, g) with discrete isometry group G and a continuous family of parallel complex structures. If J_1 and J_2 are two complex structure which lie in different G-orbits, then the Kähler manifolds (M, g, J_1) and (M, g, J_2) are weak relatives but not relatives. For example, this is the case for hyperKähler manifolds. For dimension larger than 2 it is not clear whether there exist weak relatives manifolds which are not relatives.

Our main theorem asserts that being weak relatives is not in fact a weaker condition than being relatives when one of the Kähler manifolds considered is projective, that is, admits a holomorphic isometry into $\mathbb{C}\mathrm{P}^N$.

Theorem 3. *If a projective manifold M_1 and a Kähler manifold M_2 are weak relatives, then they are relatives.*

This result should be compared to [7, Lemma 2.4 and Remark 2.5] where the same conclusion is obtained when M_2 is a homogeneous bounded domain of nonpositive holomorphic bisectional curvature. Here we have no assumptions on the metric of M_2 nor on its curvature. Theorem 3 allows us to prove that certain pairs of Kähler manifolds are not weak relatives whenever we can show that they are not relatives and one of them is projective. In particular, it was proven in [14, Theorem 1.2] that a complex projective space and the product of a flat Kähler manifold with a homogeneous bounded domain are not relatives. From this we can draw the following corollary.

Corollary 4. *A projective Kähler manifold X and a product $\mathbb{C}^N/\Gamma \times \Omega$ of a flat Kähler manifold with a homogeneous bounded domain Ω are not weak relatives.*

For instance, a complex torus with a projective Kähler metric and a flat complex torus are not weak relatives. Notice that this cannot be deduced without Theorem 3, although it was already known to Umehara [16] that they are not relatives.

Notice that, if there exists a holomorphic isometric immersion $M_1 \longrightarrow M_2$ of Kähler manifolds, then trivially M_1 and M_2 are relatives. However, the relation of being relatives does appear to be of local nature. In fact, as observed after Definition 1, one can always restrict to an open set where the immersions are embeddings. Even more so, in many cases the obstruction to being relatives can be detected in any open set of a complex curve $U \subset \mathbb{C} \longrightarrow M_i$ around a point, see for instance [14]. In light of this, it is surprising that the only examples of related Kähler manifolds M_1, M_2 in the literature are due to the existence of a holomorphic isometry $M_1 \longrightarrow M_2$, or viceversa. That is, the known examples rely on a solution of the original problem of Calabi and not strictly on its generalization. We find that studying related Kähler manifolds which cannot be immersed into one another could give a better understanding of this relation and how it differs from Calabi's original problem. For this reason we propose the following

Definition 5. *Two Kähler manifolds M_1 and M_2 are **strict relatives** (or are strictly related) if they are relatives and there exists no local holomorphic isometry of one into the other.*

Observe that, if a Kähler manifold is related to a product of Kähler manifolds, then it is related to its factors. Therefore, it is rather easy to produce examples in which one of the manifolds considered is a product, cf. Example 1. Considering this, we provide several examples of strictly related irreducible Kähler manifolds. Namely, Example 2 and Example 3 exhibits two pairs of irreducible noncompact Kähler manifolds with different curvature assumptions. In Example 4 we produce an instance in which both Kähler manifolds are compact. Finally, in Example 5 we consider a pair where both Kähler manifolds are irreducible but only one is compact. These are collected in Section 3 below.

2. Weak relatives are relatives

The aim of this section is to prove Theorem 3. In order to do so we recall a result on isometries of Kähler manifolds. Although this is a well known result, to the best of the author's knowledge, there is no complete proof in the literature. Therefore we include here a simple proof for the reader's convenience.

Lemma 6. *Let $\varphi : M_1 \longrightarrow M_2$ be an isometry between irreducible Kähler manifolds M_1 and M_2. If $M_1$is not Ricci-flat, then φ is either holomorphic or anti-holomorphic.*

Proof. Denote by J_1 (respectively J_2) the complex structure on M_1 (resp. M_2). Let $A := \varphi^*(J_2)$ be the endomorphism of TM_1 given by the pullback of the complex structure of M_2. Since φ is an isometry and M_2 is Kähler, we have $\nabla A = 0$. Thus, for all $p \in M_1$, $A_p : T_pM_1 \longrightarrow T_pM_1$ commutes with the Lie algebra $\mathfrak{hol}^o_p(M_1)$ of the restricted holonomy group $\mathrm{Hol}^o_p(M_1)$ of the metric g_1 at $p \in M_1$. Clearly, the same holds true for J_1. Now a result due to Lichnerowicz (see for instance [5, Proposition 4.1]) ensures that the Lie algebra $\mathfrak{hol}^o_p(M_1)$ of $\mathrm{Hol}^o_p(M_1)$ coincides with its normalizer in the Lie algebra $\mathfrak{so}(T_pM_1)$ since M_1 is irreducible and not Ricci-flat. As a consequence, A_p and J_1 belong to the Lie algebra $\mathfrak{hol}^o_p(M_1)$. Therefore, we can identify A_p with an automorphism of $\mathbb{C}^n \cong T^{1,0}_pM_1$ such that J_1 is identified with multiplication by i. By irreducibility of M_1, the Lie algebra $\mathfrak{hol}^o_p(M_1)$ acts irreducibly on T_pM_1. Therefore Schur's lemma implies that $A_p = \lambda\,\mathrm{Id}$ for some $\lambda \in \mathbb{C}$. Thus we have $\lambda^2\,\mathrm{Id} = A_p^2 = -\,\mathrm{Id}$. We conclude $\lambda = \pm i$ so that $A_p = \pm J_1$ for all $p \in M_1$. In other words, $A = \varphi^*(J_2) = \pm J_1$ as wanted. □

Proof of Theorem 3. Let M_1 be a projective manifold, that is, M_1 admits a holomorphic isometry $\iota : M_1 \longrightarrow \mathbb{C}\mathrm{P}^N$ into a complex projective space. Suppose now the Kähler M_2 is a weak relative of M_1. Namely, there exists locally isometric Kähler manifolds X_1 and X_2 and holomorphic isometries $\varphi_i : X_i \longrightarrow M_i$ for $i = 1, 2$. Denote by φ a local isometry between X_1 and X_2. Since we are only interested in the local behaviour, without loss of generality, we can assume that φ is a global isometry between X_1 and X_2. We want to show that we can modify φ so to obtain a holomorphic isometry $\widetilde{\varphi} : X_1 \longrightarrow X_2$ which implies that M_1 and M_2 are relatives.

Let $X_1 = F \times N_1 \times \cdots \times N_k$ be the de Rham decomposition of X_1 where we collected in F all the Ricci-flat factors. By a result of Hulin [9] (see [1] for a recent generalization) a Ricci-flat Kähler manifold does not admit a holomorphic isometry into a complex projective space. Therefore the factor F cannot appear. Now φ restricts to each factor N_j as an isometry of Kähler manifolds because $\varphi(N_j)$ must be an irreducible factor in the de Rham decomposition of X_2. Denote this restriction by $\varphi_j := \varphi_{|N_j}$. By Lemma 6, φ_j is either holomorphic or anti-holomorphic. Now, let

$$\widetilde{\varphi}_j = \begin{cases} \varphi_j & \text{if } \varphi_j \text{ is holomorphic;} \\ \overline{\varphi}_j & \text{if } \varphi_j \text{ is anti-holomorphic.} \end{cases}$$

where $\overline{\varphi}_j$ denotes the conjugate of φ_j. We conclude that $\widetilde{\varphi} := \widetilde{\varphi}_1 \times \cdots \times \widetilde{\varphi}_k$ is a holomorphic isometry between X_1 and X_2 and this concludes the proof. □

3. STRICLTLY RELATED KÄHLER MANIFOLDS

As mentioned in Section 1, there has been a lack of examples of strict relatives Kähler manifolds in the literature. In this section we provide several new examples of Kähler manifolds (M_1, g_1) and (M_2, g_2) which are strict relatives, that is, which are related but such that one cannot be locally holomorphically isometrically immersed into the other. We begin by the simplest case and later provide gradually more restrictive examples.

Example 1. Consider $M_1 = \mathbb{C}^n$ endowed with the flat metric and the Kähler product $M_2 = \mathbb{C} \times \mathbb{C}\mathrm{P}^m$ so that g_2 is the sum of the standard flat metric with the Fubini-Study metric. Clearly, the complex line $\mathbb{C}$ (endowed with the flat metric) can be embedded in both M_1 and M_2.

Notice that M_2 does not admit a local holomorphic isometry in M_1 because of a classical result of Calabi [2] on holomorphic isometries of complex space forms. On the other hand, if we assume $n > m+1 \geq 2$, then M_1 cannot be holomorphically isometrically immersed in M_2 for dimensional reasons. Moreover, M_1 cannot be holomorphically isometrically immersed in M_2 even for $n = m+1$ because such an immersion would be a local isometry which is obstructed by the holomorphic sectional curvature. Hence, for $n \geq m+1 \geq 2$, M_1 and M_2 are strict relatives.

In the previous example, the common Kähler submanifold $\mathbb{C}$ is a Kähler factor of both M_1 and M_2 so it trivially embeds in them. It is natural to ask whether there exist strictly related irreducible manifolds. In the following example we provide one such instance where the manifolds are non-compact .

Example 2. Let $M_1 = \mathbb{C}^k \# \overline{\mathbb{C}\mathrm{P}}^k$ be the blow-up of $\mathbb{C}^k$ at the origin endowed with a multiple ηg_S of the Burns-Simanca metric g_S such that $0 < \eta \in \mathbb{R}$. Recall that this metric is scalar flat, see for instance [3, 10]. Moreover, consider a homogeneous Kähler manifold $M_2 = \mathbb{C}^n \rtimes \mathbb{C}\mathrm{H}^m$ being the complex (but not Kähler) product of $\mathbb{C}^n$ with the disc $\mathbb{C}\mathrm{H}^m$ endowed with a large enough multiple λg of a homogeneous Kähler metric g, see for example [17]. Now the complex line $\mathbb{C}$ (endowed with the flat metric) can be embedded in both M_1 (see [13, Theorem 1.1]) and M_2 (in a fiber of the holomorphic fibration, i.e. $z \mapsto (z, z_2, \ldots, z_n, x)$).

It was proven in [11] that M_2, being a simply connected homogeneous Kähler manifold, can be holomorphically isometrically immersed into $\mathbb{C}\mathrm{P}^\infty$ when endowed with a large enough real multiple λg of any homogeneous Kähler metric g. We can then assume that λ is such. Furthermore, an open set U in M_1 admits a holomorphic isometric immersion into $\mathbb{C}\mathrm{P}^\infty$ only if η is a positive integer. This follows from the fact that any such immersion of U into $\mathbb{C}\mathrm{P}^\infty$ can be extended to a global holomorphic isometric immersion of M_1 by Calabi's extension theorem [2], being M_1 simply connected. On the other hand, any projectively induced metric must be integral. This only happens if η is an integer because $0 \neq [\omega_S] \in H^2(M_1; \mathbb{Z})$ where ω_S is the Kähler form associated to g_S.

We can now assume that $n + m > k$ to prevent the existence of an immersion $M_2 \longrightarrow M_1$. In order to rule out the existence of an immersion of M_1 into M_2 it is enough to choose a large enough $\lambda \in \mathbb{R} \setminus \mathbb{Z}$. Notice that this argument also excludes the case $n + m = k$ because in that case an immersion is a local isometry.

We give now an example in which the manifolds M_1 and M_2 are still irreducible and noncompact as in Example 2 but not scalar flat as in the case of the Burns-Simanca metric.

Example 3. Let $M_1 = \mathbb{C}\mathrm{H}^n$ with the hyperbolic metric and let M_2 be a bounded symmetric domain of dimension $m \leq m$ and rank≥ 2 equipped with the Bergman metric. Both M_1 and M_2 admit $\mathbb{C}\mathrm{H}^1$ as a totally geodesic submanifold, see for instance [8, Chapter VI].

Clearly, if $n > m$, M_1 does not admit a local immersion in M_2 for dimensional reasons. As in the previous example, this holds also when $n = m$ because a local holomorphic isometry $M_2 \longrightarrow M_1$ would force M_2 to have constant holomorphic sectional curvature. On the other hand, if M_2 admitted a local holomorphic isometric immersion into M_1, then it would admit one into the infinite dimensional simply connected space form $\ell^2(\mathbb{C})$ by composing with the immersion $\mathbb{C}\mathrm{H}^n \longrightarrow \ell^2(\mathbb{C})$. However, by [6, Theorem 3.3] this is only possible when M_2 is (a Kähler product of copies of) $\mathbb{C}\mathrm{H}^k$. Therefore, for $n \geq m$, M_1 and M_2 are strictly related.

One would expect that compactness does not play a role in a problem of local nature as finding non strict relatives. Indeed, our next examples exhibits two compact Kähler manifold which are strict relatives.

Example 4. Let $M_1 = \mathbb{C}\mathrm{P}^n$ with the Fubini-Study metric and let $M_2 = Q^m$ where Q^m is the m-dimensional quadric in $\mathbb{C}\mathrm{P}^{m+1}$ with the projective homogeneous metric. Both M_1 and M_2 admit $\mathbb{C}\mathrm{P}^1$ as a totally geodesic submanifold, see for instance [8, Chapter VII].

If we assume $n < m$, M_2 does not admit a local immersion in M_1 for dimensional reasons. As in the previous example, also the case $n = m$ is ruled out because the existence of a local holomorphic isometry $M_2 \longrightarrow M_1$ would imply that M_2 has constant holomorphic sectional curvature. Viceversa, by [15, Theorem 1] M_1 can be (locally) holomorphically isometrcally immersed into M_2 only if $m \geq 2n$. Thus, for $n \leq m < 2n$, M_1 and M_2 are strict relatives.

We conclude this section by showing an example of two strictly related irreducible Kähler manifolds such that only one is compact.

Example 5. Let $M_1 = \mathbb{C}\mathrm{P}^n$ with the Fubini-Study metric. The definition of the manifold M_2 is slightly more involved. Namely, M_2 as a complex manifold is the projectivization of the tangent bundle of $\mathbb{C}\mathrm{H}^2$. This can be seen as the noncompact homogeneous Kähler manifold $SU(2,1)/U(1) \times U(1)$ with the natural $SU(2,1)$-action on the tangent bundle of $\mathbb{C}\mathrm{H}^2$. Let z_1, z_2 be coordinates on $\mathbb{C}\mathrm{H}^2$ and w_0, w_1 be homogeneous coordinates on the fiber $\mathbb{C}\mathrm{P}^1$. Then consider the homogeneous metric g_2 on M_2 given in the dense open set $\{w_0 \neq 0\}$ by

$$g_2 = -2i\partial\overline{\partial}\log(1 - |z_1|^2 - |z_2|^2) + i\partial\overline{\partial}\log(1 + |w|^2 - |z_1 + z_2 w|^2)$$

where $w = w_1/w_0$. Notice that M_2 is the complex product of $\mathbb{C}\mathrm{P}^1$ and $\mathbb{C}\mathrm{H}^2$ being a bundle over a contractible space but the Kähler metric is not the product metric.

One easily sees that $\mathbb{C}\mathrm{P}^1$ embeds into M_2 as the fiber over $z_1 = z_2 = 0$ because the metric restricted to that fiber reads $g_{2|_{\mathbb{C}\mathrm{P}^1}} = \frac{i}{2}\partial\overline{\partial}\log(1 + |w|^2)^2 = \omega_{FS}$. Thus both M_1 and M_2 admit $\mathbb{C}\mathrm{P}^1$ as a Kähler submanifold, the embedding $\mathbb{C}\mathrm{P}^1 \longrightarrow M_1$ being the totally geodesic one. If we assume $n \geq 3$, M_1 does not admit a local holomorphic isometric immersion in M_2 for dimensional reasons. In particular, the case $n = 3$ is ruled out because the existence of a local holomorphic isometry $M_1 \longrightarrow M_2$ would imply that M_2 has constant holomorphic sectional curvature in a neighbourhood of a point.

On the other hand, one can show [19] that the metric g_2 is induced by the full immersion $F : M_2 \longrightarrow \mathbb{C}\mathrm{P}^\infty$ given in the coordinates above by

$$F(z_1, z_2, w) = \left(1, w, \ldots, \sqrt{\binom{k+1}{j+1}(k-j+1)} z_1^j z_2^{k-j} \left(z_1 - \frac{j+1}{k-j+1} z_2 w\right), \ldots\right)$$

for $k \geq 0$ and $0 \leq j \leq k$. This prevents the existence of a local holomorphic isometry $U \subset M_2 \longrightarrow M_1$. In fact, if such an immersion existed, we would get two local holomorphic isometries of M_2 into $\mathbb{C}\mathrm{P}^\infty$. By Calabi's rigidity these are related by a rigid transformation of $\mathbb{C}\mathrm{P}^\infty$ which is impossible because one isometry is full and the other is contained in a proper totally geodesic submanifold. Therefore, for $n \geq 3$, M_1 and M_2 are strict relatives.

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