An algebraic analysis of implication in non-distributive logics

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Abstract

In this paper, we introduce the concept of a (lattice) skew Hilbert algebra as a natural generalization of Hilbert algebras. This notion allows a unified treatment of several structures of prominent importance for mathematical logic, e.g. (generalized) orthomodular lattices, and MV-algebras, which admit a natural notion of implication. In fact, it turns out that skew Hilbert algebras play a similar role for (strongly) sectionally pseudocomplemented posets as Hilbert algebras do for relatively pseudocomplemented ones. We will discuss basic properties of closed, dense and weakly dense elements of skew Hilbert algebras and their applications, and we will provide some basic results on their structure theory.

Keywords: Hilbert algebras, skew Hilbert algebras, pseudocomplemented lattices, sectionally pseudocomplemented lattices, orthomodular lattices, implication algebras

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1 Introduction

Hilbert [22] was the first to single out the importance of the implicative fragment of classical logic, namely the calculus obtained from classical propositional logic by assuming as axioms a given set of classical tautologies containing just the connective implication. This logical system, later called *propositional calculus of positive implication* [23], was revealed to be amenable to smooth algebraic investigations by means of Henkin's implicative models [21]. Their duals, known as Hilbert algebras after Diego's works on the topic [15, 16], have been the subject of intensive and incredibly deep inquiries over the past years; see, e.g. [4–6].

We introduce skew Hilbert algebras and lattice skew Hilbert algebras with the aim of generalizing the $\{\wedge, \vee, \rightarrow, 1\}$ resp. $\{\rightarrow, 1\}$-reduct of algebras which, unlike Heyting algebras, are not distributive and so non-pseudocomplemented although they can be equipped with a term-definable implication-like connective. Indeed, our theory frames for the first time in a common class of first-order structures (generalized) orthomodular lattices (and their implicative reducts) [1], Hilbert and Heyting algebras, basic algebras and their subvariety of MV-algebras [10] (see Corollary 4.23). As a consequence, our contribution will single out minimal properties of algebras we abstract from that actually determine their universal algebraic features.

In this article, we aim to generalize the concept of a Hilbert algebra to a context in which the 'distributivity of implication over itself' ((H5) in Definition 2.1) need not hold in general. Indeed, Hilbert algebras provide the equivalent algebraic semantics in the sense of Blok and Pigozzi [3] of the implication fragment of intuitionistic propositional logic. Therefore, if we stick to the original notion, a large number of prominent structures for algebraic logic, e.g. generalized orthomodular lattices, orthomodular lattices and MV-algebras, which indeed admit a natural notion of implication [18], are left out. It seems therefore quite natural to try to extend the very notion of Hilbert algebra to a wider framework. We will see that skew Hilbert algebras play a similar role for (strongly) sectionally pseudocomplemented posets [13], i.e. posets with a top element and in which every principal order filter is a pseudocomplemented poset, as Hilbert algebras do for relatively pseudocomplemented ones. In particular, we highlight the connections between skew Hilbert algebras and orthomodular implication algebras [11]. Precisely, we will show that orthomodular implication algebras are indeed (term equivalent to) a subvariety of skew Hilbert algebras, axiomatized by two further identities. Then, making use of this result, we axiomatize the class of generalized orthomodular lattices within the class of skew Hilbert algebras, introduced by Janowitz in [26] (see also [1]), and we show that they form in fact a variety. As a consequence, we obtain a characterization of orthomodular lattices in the framework of skew Hilbert algebras. Moreover, we show that (strong) skew Hilbert algebras can be regarded as proper generalizations of lattices with sectional antitone involutions (see [10]) which are term-equivalent to basic algebras.

Let us now summarize the discourse of the paper. In Section 2, we dispatch all the necessary preliminaries on Hilbert algebras, bounded posets with operations, and (sectionally) pseudocomplemented posets. In Section 3, we discuss the concept of a (lattice) skew Hilbert algebra and we provide some examples thereof. In Section 4, we discuss how some structures having an underlying poset with sectional antitone operations, like orthomodular implication algebras, and lattices with sectional antitone involutions, can be framed within the theory of skew Hilbert algebras. In Section 5, we describe basic properties of closed, dense and weakly dense elements of skew Hilbert algebras. In Section 6, we will investigate the structure theory for the variety of lattice skew Hilbert algebras. Furthermore, we introduce the concept of a deductive system on a skew Hilbert algebra. A full characterization thereof will follow. Finally, we introduce a notion of 'order-compatible' congruence for skew Hilbert algebras that need not be lattice-ordered. Then, we show that, also in this case, many of the results from the lattice-ordered case still hold.

2 Basic concepts

The concept of a Hilbert algebra was introduced by Diego (see [15, 16]) and studied intensively by Rudeanu (see [28, 29]). Let us recall its definition.

DEFINITION 2.1
A *Hilbert algebra* is an algebra $(A, *, 1)$ of type $(2, 0)$ satisfying the following identities and quasi-identities for all $x, y, z \in A$:

(H1) $x * x \approx 1$;
(H2) if $x * y = y * x = 1$, then $x = y$;
(H3) if $x * y = y * z = 1$, then $x * z = 1$;
(H4) $x * (y * x) \approx 1$;
(H5) $(x * (y * z)) * ((x * y) * (x * z)) \approx 1$.

Because of (H1) – (H3) the binary relation $\leq$ on A defined by

$$x \leq y \text{ if and only if } x * y = 1 \ (x, y \in A)$$

is a partial order relation on A. From (H1) and (H4) we conclude $x * 1 \approx x * (x * x) \approx 1$; thus, 1 is the top element of $(A, \leq)$. Hence, a Hilbert algebra can be alternatively defined as a poset $(A, \leq, *, 1)$ with top element 1 and with a binary operation $*$ satisfying the following conditions:

- $x \leq y$ if and only if $x * y = 1$,
- $x \leq y * x$,
- $x * (y * z) \leq (x * y) * (x * z)$;

see, e.g. [15, 16, 28, 29].

Let us now recall several useful concepts from the theory of posets. Other concepts used in this paper are taken from monographs [2, 20].

Let $\mathbf{P} = (P, \leq)$ be a poset, $a, b, c \in P$ and $A, B \subseteq P$.

By $A \leq B$ we mean $x \leq y$, for all $x \in A$ and all $y \in B$. Instead of $A \leq \{b\}$, $\{a\} \leq B$ and $\{a\} \leq \{b\}$ we simply write $A \leq b$, $a \leq B$ and $a \leq b$, respectively.

Now, we define the *lower* and *upper cone* of A as follows:

$$L(A) := \{x \in P \mid x \leq A\},$$
$$U(A) := \{x \in P \mid x \geq A\}.$$

Instead of $L(A \cup B)$, $L(A \cup \{b\})$, $L(\{a, b\})$ and $L(U(A))$, we simply write $L(A, B)$, $L(A, b)$, $L(a, b)$ and $LU(a)$, respectively. Similarly, we proceed in analogous cases.

The element c is called the *relative pseudocomplement* of a with respect to b, in symbols $c = a \circ b$, if c is the greatest element x of P satisfying $L(a, x) \subseteq L(b)$. If $x \circ y$ exists for all $x, y \in P$, then $\mathbf{P}$ is called *relatively pseudocomplemented* and $\circ$ is called *relative pseudocomplementation*.

Relatively pseudocomplemented posets were investigated by three of the present authors in [12]. Relatively pseudocomplemented posets which are meet-semilattices are often called *implicative semilattices* or *Brouwerian semilattices* [27].

It was shown by Rudeanu [28] that the class of relatively pseudocomplemented posets is a proper subclass of the class of Hilbert algebras. In fact, a Hilbert algebra $(A, *, 1)$ is a relatively pseudocomplemented poset if and only if for all $x, y \in A$, $x * y$ is the relative pseudocomplement of x with respect to y.

If a relatively pseudocomplemented poset $(A, \leq)$ is a lattice, then it is called a *relatively pseudocomplemented lattice*; see [2]. In such a case, for all $x, y \in A$, $x \circ y$ is the greatest element z of A satisfying $x \wedge z \leq y$.

It is well known that every relatively pseudocomplemented lattice is distributive; see, e.g. [20]. In order to extend relative pseudocomplementation to non-distributive lattices, the first author introduced in [7] the so-called *sectionally pseudocomplemented lattices*. Recall that a lattice $(L, \vee, \wedge)$ is called *sectionally pseudocomplemented* if each of its intervals $[y)$ is pseudocomplemented, or formally, if for every $a, b \in L$, there exists a greatest element c of L satisfying $(a \vee b) \wedge c = b$. This element c is called the *sectional pseudocomplement* of a with respect to b and will be denoted by $a * b$.

Of course, every relatively pseudocomplemented lattice is sectionally pseudocomplemented but, for instance, the five-element non-modular lattice $\mathbf{N}_5$ is sectionally pseudocomplemented but not relatively pseudocomplemented; see [7, 13] for examples and details.

The concept of a sectionally pseudocomplemented lattice was generalized to posets in [32] as follows: let $(A, \leq)$ be a poset and $a, b \in A$. An element c of A is called the *sectional pseudocomplement* of a with respect to b if it is the greatest element x of A satisfying $L(U(a, b), x) = L(b)$. This element c will be denoted by $a * b$. A poset $(A, \leq)$ is called *sectionally pseudocomplemented* if for all $x, y \in A$ $x * y$ exists. Of course, in the case of lattices, this concept coincides with the above one introduced for lattices. A unary operation $'$ on A is called

- *antitone* if $x \leq y$ implies $y' \leq x'$,
- an *involution* if $x'' \approx x$,

for all $x, y \in A$. A unary operation $'$ on a bounded poset $(A, \leq, 0, 1)$ is called

- a *complementation* if $L(x, x') = \{0\}$ and $U(x, x') = \{1\}$,

for all $x \in A$. An *orthoposet* is a bounded poset $(A, \leq, ', 0, 1)$ with an antitone involution $'$ which is a complementation. An *ortholattice* is a lattice which is an orthoposet.

Let us recall the following result from [13].

PROPOSITION 2.2
The class of sectionally pseudocomplemented lattices forms a variety which is determined by the lattice axioms and the following identities:

- $z \vee y \leq x * ((x \vee y) \wedge (z \vee y))$,
- $(x \vee y) \wedge (x * y) \approx y$.

The following two results were also proved in [13].

PROPOSITION 2.3
Let $(P, \leq, *, 1)$ be a sectionally pseudocomplemented poset with top element 1. Then the following hold for all $x, y, z \in P$:

(i) $x \leq y$ if and only if $x * y = 1$;
(ii) $1 * x \approx x$;
(iii) $x * (y * x) \approx 1$;

(iv) if $y * x = 1$, then $x * ((x * y) * y) = 1$;
(v) if $x * y = 1$, then $(y * z) * (x * z) = 1$.

REMARK 2.4
By (i) and (iii), we derive $x \leq y*x$ and also $x \leq (y*x)*x$. Therefore, if $x \leq y$, then $x \leq y \leq (x*y)*y$ whence $x \leq (x*y)*y$. By (iv), we have that $x \leq (x*y)*y$ provided x and y are comparable with each other. In order to avoid this rather restrictive condition, we define a sectionally pseudocomplemented poset $(P, \leq, *, 1)$ with top element 1 is called a *strongly sectionally pseudocomplemented poset* if it satisfies the identity for all $x, y \in P$:

(vi) $x \leq (x * y) * y$.

PROPOSITION 2.5
An algebra $(P, *, 1)$ of type $(2, 0)$ can be organized into a sectionally pseudocomplemented poset if and only if it satisfies

- $x * x \approx x * 1 \approx 1$,
- $x * y = y * x = 1$ implies $x = y$,
- $x * y = y * z = 1$ implies $x * z = 1$,
- $L(U(x, y), x * y) \approx L(y)$,
- $L(U(x, y), z) = L(y)$ implies $z * (x * y) = 1$.

The last two conditions are formulated with respect to the partial order relation $\leq$ defined by $x \leq y$ if and only if $x * y = 1$ $(x, y \in P)$.

3 Skew Hilbert algebras

As mentioned above, the class of relatively pseudocomplemented posets is a proper subclass of the class of Hilbert algebras. The aim of this section is to discuss the concept of a *skew Hilbert algebra*. We will see in Section 4 that skew Hilbert algebras play the same role that Hilbert algebras play in the implicative fragment of intuitionistic propositional logic, w.r.t. basic algebras of which orthomodular lattices and MV-algebras are proper subvarieties (see [10]). In other words, we will show that lattices with sectional antitone involutions (which are term-equivalent to basic algebras) give rise quite naturally to skew Hilbert algebras, once sectional involutions are interpreted as parameters for implications. Also, we will see that generalized orthomodular lattices can be regarded as a proper subvariety of lattice skew Hilbert algebras. As a consequence another proof of the fact that these algebras form a variety is obtained.

DEFINITION 3.1
A *skew Hilbert algebra* is a poset $\mathbf{S} = (S, \leq, *, 1)$ with a binary operation $*$ and a constant 1 satisfying the following conditions:

(S1) $x \leq y$ if and only if $x * y = 1$;
(S2) if $y * x = 1$, then $x * ((x * y) * y) = 1$;
(S3) if $x * y = 1$, then $(y * z) * (x * z) = 1$;
(S4) $L(U(x, y), x * y) = L(y)$.

If $\mathbf{S}$ satisfies the identity

(S2') $x * ((x * y) * y) \approx 1$

instead of (S2), then it is called a *strong skew Hilbert algebra*. If $(S, \leq)$ is a lattice and **S** satisfies conditions (S1), (S2'), (S3) and (S4), then **S** is called a *lattice skew Hilbert algebra*.

If one wants to remove from Definition 3.1 the explicit assumption of an underlying partial order, one can just remove $\leq$ from the signature, add three further axioms (indeed quasi-equations) to the definition of skew Hilbert algebras which ensure the reflexivity, anti-symmetry and transitivity of the relation '$x \leq y$ if $x * y = 1$', namely

- $x * x = 1$,
- $x * y = 1$ and $y * x = 1$ imply $x = y$,
- $x * y = 1$ and $y * z = 1$ imply $x * z = 1$

and replacing (S4) by a suitable first-order condition. It is worth noticing that every lattice skew Hilbert algebra is strong.

From (S4), we obtain $x \in L(x) = L(U(y,x), y * x)$ whence $x \leq y * x$, i.e.

(1) $x * (y * x) \approx 1$.

From (S1) and (1), and the reflexivity of $\leq$, we conclude $x * 1 \approx x * (x * x) \approx 1$, i.e.

(2) $x * 1 \approx 1$,

and hence, 1 is the top element of the poset $(S, \leq)$. From the fact that 1 is the top element and (S4) we finally obtain $L(1 * x) = L(U(1,x), 1 * x) = L(x)$, i.e.

(3) $1 * x \approx x$.

EXAMPLE 3.2

Every poset with top element can be converted into a strong skew Hilbert algebra. Namely, if $(S, \leq, 1)$ is a poset with top element 1 and $*$ denotes the binary operation on S defined by

$$x * y := \begin{cases} 1, & \text{if } x \leq y; \\ y, & \text{otherwise,} \end{cases}$$

then $(S, \leq, *, 1)$ is a strong skew Hilbert algebra.

REMARK 3.3

If $(S, \leq, *, 1)$ is a skew Hilbert algebra, then according to (S4)

$$L(U(x,y), x * y) = L(y)$$

which shows that the infimum $U(x,y) \wedge (x * y)$ exists and hence the previous is equivalent to the equality

$$U(x,y) \wedge (x * y) = y.$$

(Here, $U(x,y) \wedge (x * y)$ means the infimum of the subset $U(x,y) \cup \{x * y\}$ of the poset $(S, \leq)$.) Thus, in case $x \geq y$, we obtain $x \wedge (x * y) = y$.

Concerning the relationship between skew Hilbert algebras, Hilbert algebras and sectionally pseudocomplemented posets, the following can be said.

Remark 3.4

- A skew Hilbert algebra is a Hilbert algebra if and only if it satisfies (H5).
- A (strong) skew Hilbert algebra is a (strongly) sectionally pseudocomplemented poset if and only if it satisfies the condition

$$L(U(x,y),z) = L(y) \text{ entails } z * (x * y) = 1.$$

  In other words, a skew Hilbert algebra is a sectionally pseudocomplemented poset if and only if $x * y$ is the sectional pseudocomplement of x with respect to y.

Comparing Definition 3.1 with Propositions 2.3 and 2.5, we conclude immediately that every (strongly) sectionally pseudocomplemented poset can be considered as a (strong) skew Hilbert algebra. The following example shows in part (b) that the converse assertion need not be true.

A natural question to ask is the following: is every skew Hilbert algebra whose underlying poset is a lattice strong, i.e. does it satisfy identity (S2') automatically? Part (a) in the following example shows that this is not the case.

Example 3.5

(a) If $\mathbf{L} = (L, \vee, \wedge)$ denotes the lattice shown in Figure 1

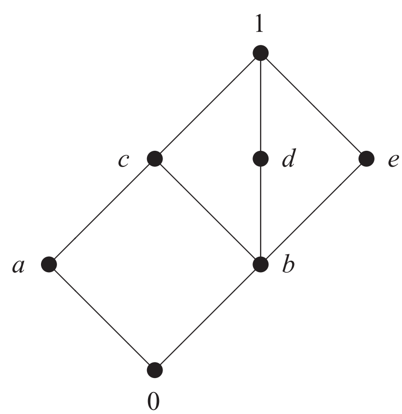

Figure 1

and $*$ the binary operation on L defined by

$*$	0	a	b	c	d	e	1
0	1	1	1	1	1	1	1
a	d	1	d	1	d	e	1
b	a	a	1	1	1	1	1
c	0	a	b	1	d	e	1
d	a	a	e	c	1	e	1
e	0	a	d	c	d	1	1
1	0	a	b	c	d	e	1

then $(L, \leq, *, 1)$ is a skew Hilbert algebra which is not strong since

$$a \not\leq e = d * b = (a * b) * b.$$

$(L, \leq, *, 1)$ is not a Hilbert algebra since

$$a * (0 * e) = a * 1 = 1 \not\leq e = d * e = (a * 0) * (a * e).$$

$(L, \vee, \wedge)$ is not a sectionally pseudocomplemented lattice since the sectional pseudocomplement of a with respect to b does not exist.

(b) Let $\mathbf{P} = (P, \leq, 1)$ denote the poset with top element 1 shown in Figure 2

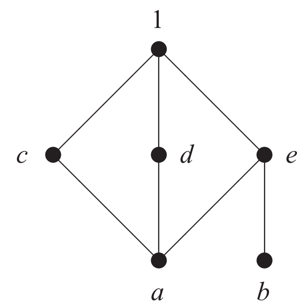

FIGURE 2

and $*$ the binary operation on P defined by

$*$	a	b	c	d	e	1
a	1	b	1	1	1	1
b	c	1	c	d	1	1
c	a	b	1	d	e	1
d	a	b	c	1	e	1
e	a	b	c	d	1	1
1	a	b	c	d	e	1

Then $(P, \leq, *, 1)$ is a skew Hilbert algebra. Clearly, it is not a lattice, and it is not a sectionally pseudocomplemented poset since there is no sectional pseudocomplement of c with respect to a. Moreover, $(P, \leq, *, 1)$ is not a strong skew Hilbert algebra since

$$b \not\leq a = c * a = (b * a) * a.$$

We close this section by showing that lattice skew Hilbert algebras form a variety.

THEOREM 3.6
Let $\mathbf{L} = (L, \vee, \wedge, *, 1)$ be a lattice with a binary operation $*$ and a constant 1. Then $\mathbf{L}$ is a lattice skew Hilbert algebra if and only if it satisfies the following identities:

(L1) $x * (x \vee y) \approx 1$,
(L2) $x * ((x * y) * y) \approx 1$,
(L3) $((x \vee y) * z) * (x * z) \approx 1$,
(L4) $(x \vee y) \wedge (x * y) \approx y$.

PROOF. Concerning the right-to-left direction, note that (L4) implies directly (S4). By (L1), one has $x * x = x * (x \vee x) = 1$. By (L4), we have $x \wedge 1 = (x \vee x) \wedge (x * x) = x$. Therefore, 1 is the top element. If $x \leq y$, then, by (L1), $x * y = x * (x \vee y) = 1$. Conversely, if $x * y = 1$, then (L4) yields $x \leq x \vee y = x \vee y \wedge (x * y) = y$. This shows that (S1) and (S3) hold. Identity (L2) coincides with (S2'), while (L3) and (S1) imply (S3). The left-to-right direction is straightforward and it is left to the reader. □

Analogous to the case of skew Hilbert algebras, from (L4), we obtain

$$x = (y \vee x) \wedge (y * x) \leq y * x,$$

i.e. (1), and from (L1) and (1), we conclude

$$x * 1 \approx x * (x * (x \vee x)) \approx x * (x * x) \approx 1,$$

i.e. (2), and hence 1 is the top element of the lattice $(L, \vee, \wedge)$. Finally, from (2) and (L4), we obtain

$$1 * x \approx (1 \vee x) \wedge (1 * x) \approx x,$$

i.e. (3).

COROLLARY 3.7
The class $\mathcal{V}$ of lattice skew Hilbert algebras forms a variety determined by the identities for lattices and identities (L1) – (L4).

It is evident that the variety of lattice Hilbert algebras and the variety of sectionally pseudocomplemented lattices (see Proposition 2.2) are subvarieties of $\mathcal{V}$. Precisely, a lattice skew Hilbert algebra is a sectionally pseudocomplemented lattice if and only if it satisfies the identity $z \vee y \leq x * ((x \vee y) \wedge (z \vee y))$. It is worth noticing that this identity is not satisfied by the lattice skew Hilbert algebra from Example 3.5 since

$$e \vee 0 = e \nleq b = c * b = c * (c \wedge e) = c * ((c \vee 0) \wedge (e \vee 0)).$$

REMARK 3.8
In contrast to both (a) and (b) in Example 3.5, sectionally pseudocomplemented lattices satisfy the identity $x \leq (x * y) * y$. This can be seen as follows: if $(P, \vee, \wedge, *)$ is a sectionally pseudocomplemented lattice and $a, b \in P$, then

$$((a * b) \vee b) \wedge (a \vee b) = (a * b) \wedge (a \vee b) = (a \vee b) \wedge (a * b) = b,$$

and hence,

$$a \leq a \vee b \leq (a * b) * b.$$

4 Applications

The aim of this section is to provide several motivating examples of skew Hilbert algebras. In particular, we highlight the connections between skew Hilbert algebras, orthomodular implication algebras [11], generalized orthomodular lattices [26] (see also [1]) and lattices with sectional antitone involutions. Precisely, we will show that (lattice) skew Hilbert algebras represent a common generalization of the aforementioned structures, once a suitable notion of implication is defined and their implicative subreduct is taken into account. In fact, orthomodular implication algebras are indeed (term equivalent to) a subvariety of skew Hilbert algebras. Then, making use of this result, we axiomatize the class of generalized orthomodular lattices within the class of skew Hilbert algebras. As a consequence, we obtain a characterization of orthomodular lattices in the framework thereof. Finally, we show that lattices with sectional antitone involutions (basic algebras) can be framed within our general theory. Since MV-algebras are indeed associative basic algebras, our results can be clearly shifted to MV-algebras. Among other results, we have an alternative proof of the fact that the aforementioned algebras are completely specified by means of their implicational reduct together with, eventually, lattice operations.

In [11], two of the present authors together with R. Halaš introduced the concept of an orthomodular implication algebra. The main motivation for discussing this notion was generalizing to the case of orthomodular lattices the fact that in Boolean algebras the properties of the implication operation can be modelled by a so-called implication algebra. This structure itself can be considered as a join-semilattice with 1, whose principal filters are Boolean algebras. In what follows, we will assume the notion of an orthomodular lattice understood. The reader may consult [1] for an account. As discussed in [11], orthomodular implication algebras can be axiomatized as follows.

DEFINITION 4.1
An *orthomodular implication algebra* is an algebra $\mathbf{A} = (A, \cdot, 1)$, of type $(2, 0)$ such that the following conditions hold:

(O1) $xx = 1$,
(O2) $x(yx) = 1$,
(O3) $(xy)x = x$,
(O4) $(xy)y = (yx)x$,
(O5) $(((xy)y)z)(xz) = 1$,
(O6) $(((((((((xy)y)z)z)z)x)x)z)x)x = (((xy)y)z)z$.

(It should be remarked that axiom (O6) comes from the fact that this axiom is satisfied by the implication operation in orthomodular lattices.) For the reader's convenience, let us recall that, setting $x \vee y = (xy)y$, for any orthomodular implication algebra $\mathbf{A}$, $(A, \vee, 1)$ is a join-semilattice with top element 1, whose induced order is specified by $x \leq y$ if and only if $xy = 1$.

DEFINITION 4.2
[11, Definition 5] An *orthomodular join-semilattice* is an algebra of the form $\mathbf{A} = (A, \vee, 1, ({}^p : p \in A))$ where $(A, \vee, 1)$ is a join-semilattice with greatest element 1 and for each $p \in A$, ^{p} is a unary operation on $[p, 1]$ such that $([p, 1], \vee, \wedge_p, {}^p, p, 1)$ is an orthomodular lattice where $\wedge_p$ denotes the meet operation corresponding to the partial order induced by $\vee$.

LEMMA 4.3
[11, Theorems 2 and 3] Let $\mathbf{A} = (A, \cdot, 1)$ and $\mathbf{B} = (B, \vee, 1, ({}^p : p \in B))$ be an orthomodular implication algebra and an orthomodular join-semilattice, respectively. Setting, for any $x, y \in A$ and $z \in [x, 1]$

$$x \vee y := (xy)y \text{ and } z^x := zx,$$

and, for any $x, y \in B$,

$$x \cdot y := (x \vee y)^y,$$

the following hold:

(i) $\mathcal{S}(\mathbf{A}) = (A, \vee, 1, (^p : p \in A))$ is an orthomodular join-semilattice;
(ii) $\mathcal{A}(\mathbf{B}) = (B, \cdot, 1)$ is an orthomodular implication algebra.

Interestingly enough, any orthomodular implication algebra naturally gives rise to a skew Hilbert algebra, as the following theorem shows.

THEOREM 4.4
Let $\mathbf{A} = (A, \cdot, 1)$ be an orthomodular implication algebra. Then, upon setting $x * y = (x \vee y)y$, and $x \leq y$ if and only if $xy = 1$, $\mathbf{A} = (A, \leq, *, 1)$ is a skew Hilbert algebra, which satisfies

(i) $(x * y) * y = (y * x) * x$,
(ii) $((((((((((x * y) * y) * z) * z) * z) * x) * x) * z) * x) * x = (((x * y) * y) * z) * z$.

Conversely, if $\mathbf{S} = (S, \leq, *, 1)$ is a skew Hilbert algebra satisfying (i) and (ii), then setting $x \cdot y = (x \vee y) * y$, where $x \vee y := (x * y) * y$, one has that $(S, \cdot, 1)$ is an orthomodular implication algebra.

PROOF. (S1) If $a \leq b$, then $a * b = (a \vee b)b = bb = 1$. Conversely, if $(a \vee b)b = a * b = 1$, then $a \vee b \leq b$, i.e. $a \leq b$. (S2) Suppose that $b * a = 1$. Hence, $b \leq a$. Consider $a * ((a * b) * b)$. Then,

$$\begin{aligned} a * ((a * b) * b) &= (a \vee ((((a \vee b)b) \vee b)b))((((a \vee b)b) \vee b)b) \\ &= (a \vee (((ab) \vee b)b))(((ab) \vee b)b) \\ &= (a \vee ((ab)b))((ab)b) \\ &= (a \vee (a \vee b))(a \vee b) \\ &= (a \vee a)a \\ &= aa = 1. \end{aligned}$$

(S3) Suppose that $a \leq b$. Now, $(b*c)*(a*c) = ((b \vee c)c)*((a \vee c)c)$. Since $a \leq b$, $a \vee c \leq b \vee c$. Then by [11,Theorem (ix)], $(b \vee c)c \leq (a \vee c)c$. Consequently, $(b*c)*(a*c) = (((b \vee c)c) \vee ((a \vee c)c))((a \vee c)c) = ((a \vee c)c)((a \vee c)c) = 1$. (S4) $L(U(a, b), a*b) = L(U(a, b), (a \vee b)b) = L(U(a, b), (a \vee b)^b)$, in the interval $[b, 1]$, which is an orthomodular lattice (see the proof of [11,Lemma 4 (ix)]). Therefore, $L(U(a, b), (a \vee b)^b) = L(a \vee b, (a \vee b)^b) = L(b)$, since $(a \vee b) \wedge (a \vee b)^b = b$. Concerning condition (i), note that $(x * y) * y = (((x \vee y)y) \vee y)y = ((x \vee y)y)y = ((x \vee y)x)x = (((x \vee y)x) \vee x)x = (y * x) * x$ by (O4). (ii) follows similarly by (O6). Concerning the converse direction, we prove that, setting, for any $a, b \in S$, $a \vee b := (a * b) * b$ and $a^p := a * p$, for any $p \in S$ and $a \in [p, 1]$, $(S, \vee, 1, (^p : p \in S))$ is an orthomodular join-semilattice. Therefore, by applying Lemma 4.3(ii), one has that, setting $x \cdot y = (x \vee y) * y$, $(S, \cdot, 1)$ is an orthomodular implication algebra. Let $a, b \in S$. Clearly, $a, b \leq (a * b) * b = (b * a) * a$, by (i). Now, suppose that $a, b \leq c$. Then, by applying (S3) twice, one has $(a * b) * b \leq (c * b) * b = (b * c) * c = 1 * c = c$. We conclude that $(S, \vee, 1)$ is a join-semilattice with 1 as its top element. Let $p \in S$. Clearly, the operation ^{p} on the interval $[p, 1]$, is an antitone involution by (S3) and $(a * p) * p = a \vee p = a$, for any $a \in [p, 1]$. Moreover, setting, for any $x, y \in [p, 1]$, $x \wedge_p y := (x^p \vee y^p)^p$, it is easily seen that $\wedge_p$ is the meet operation whose dual is $\vee$ on $[p, 1]$. Since, for any $x \in [p, 1]$, one has that $x \wedge x^p = x \wedge (x * p) = p$ (by (S4)), one has that $x \wedge_p (x * p) = p$ and so $x \vee x^p = (x^p \wedge_p x)^p = p^p = 1$. We conclude that $([p, 1], \vee, \wedge_p, ^p, p, 1)$ is an ortholattice. Finally, assume that $p \leq x \leq y$. By (ii), one has

$$\begin{aligned} y &= (x \vee y) \vee p \\ &= (((x * y) * y) * p) * p \\ &= (((((((((x * y) * y) * p) * p) * p) * x) * x) * p) * x) * x \\ &= (((((x \vee y) \vee p) * p) \vee x) * p) \vee x \\ &= (y^p \vee x)^p \vee x \\ &= (y \wedge_p x^p) \vee x. \end{aligned}$$

Therefore, $([p,1], \vee, \wedge_p, ^p, p, 1)$ is an orthomodular lattice. □

We note that any orthomodular implication algebra induces a strong skew Hilbert algebra. However, this algebra, in general, may not be lattice-ordered, since the underlying poset could be a join-semilattice only.

DEFINITION 4.5
A *sectional orthomodular lattice* is a structure $\mathbf{A} = (A, \vee, \wedge, 0, (^p : p \in A))$ such that $(A, \vee, \wedge, 0)$ is a lattice with a bottom element 0 and, for any $p \in A$, $^p : [0,p] \to [0,p]$ is an antitone involution on $([0,p], \leq)$ such that $([0,p], \vee, \wedge, ^p, 0, p)$ is an orthomodular lattice.

For the reader's convenience, let us recall the notion of generalized orthomodular lattice, which will play a relevant role in the development of the present section.

DEFINITION 4.6
[26] A *generalized orthomodular lattice* is a sectional orthomodular lattice $\mathbf{A} = (A, \vee, \wedge, 0, (^p : p \in A))$ satisfying, for any $x, y, p \in A$, the following additional condition:

$$x \leq y \leq p \text{ entails } x^y = x^p \wedge y.$$

From now on, we will denote by $\mathcal{GOML}$, the class of generalized orthomodular lattices.

LEMMA 4.7
Let $\mathbf{A} = (A, \vee, \wedge, 0, (^p : p \in A))$ be a sectional orthomodular lattice. Then $\mathbf{A}$ is generalized orthomodular if and only if it satisfies

$$(x \wedge a)^a \approx (x \wedge a)^{a \vee b} \wedge a.$$

PROOF. Note that, since $x \wedge a \leq a \leq a \vee b$, from the above condition, one has $(x \wedge a)^a = (x \wedge a)^{a \vee b} \wedge a$. The converse direction is trivial. □

Given a lattice $\mathbf{A}$, let us denote by $\mathbf{A}^\partial = (A, \vee^\partial, \wedge^\partial)$ the dual of $\mathbf{A}$, i.e. the lattice obtained from $\mathbf{A}$ by setting, for any $x, y \in A$, $x \leq^{\mathbf{A}^\partial} y$ if and only if $y \leq^{\mathbf{A}} x$. Clearly, if $\mathbf{A}$ is an orthomodular lattice, then its lattice dual $\mathbf{A}^\partial$ equipped with an antitone involution defined in the obvious way, is again an orthomodular lattice.

REMARK 4.8
Let $\mathbf{A} = (A, \vee, \wedge, 0, ({}^p : p \in A))$ be a sectional orthomodular lattice. It is easily seen that, once endowed with unary operations inherited by $\mathbf{A}$, $\mathbf{A}^\partial$ is an orthomodular join-semilattice (with 0 as its greatest element) which is also a lattice.

Lemma 4.9 shows that it is possible to frame by means of two natural identities the theory of generalized orthomodular lattices within the class of orthomodular join-semilattices.

LEMMA 4.9
Let $\mathbf{A} = (A, \vee, \wedge, 0, ({}^p : p \in A))$ be a generalized orthomodular lattice. Then $\mathbf{A}^\partial = (A, \vee^\partial, \wedge^\partial, 0, ({}^p : p \in A))$ is a (lattice-ordered) orthomodular join-semilattice satisfying, for any $x, y, z \in A$,

$$(x \vee y)^y = (x \vee y)^{y \wedge z} \vee y. \tag{B}$$

Conversely, for any lattice-ordered orthomodular join-semilattice $\mathbf{A} = (A, \vee, \wedge, 1, ({}^p : p \in A))$ satisfying (B), $\mathbf{A}^\partial = (A, \vee^\partial, \wedge^\partial, 1, ({}^p : p \in A))$ is a generalized orthomodular lattice.

PROOF. Clearly, $(A, \vee^\partial, 0)$ is a join-semilattice. Moreover, from Definition 4.6 and Lemma 4.7, we have that $\mathbf{A}^\partial = (A, \vee^\partial, \wedge^\partial, 0, ({}^p : p \in A))$ is an orthomodular join-semilattice that satisfies equation (B). The converse is immediate. □

Making use of Lemma 4.9, we obtain the following theorem.

THEOREM 4.10
Let $\mathbf{A} = (A, \vee, \wedge, 0, ({}^p : p \in A))$ be a generalized orthomodular lattice. Then setting, for any $x, y \in A$, $xy = (x \vee^\partial y)^y$, $\mathcal{A}(\mathbf{A}^\partial) = (A, \cdot, 0)$ is a lattice-ordered orthomodular implication algebra satisfying

$$xy \approx ((x \vee y)(y \wedge z)) \vee y. \tag{B*}$$

Conversely, any lattice-ordered orthomodular implication algebra satisfying (B*) induces a generalized orthomodular lattice.

PROOF. By Lemma 4.9, $\mathbf{A}^\partial = (A, \vee^\partial, \wedge^\partial, 0, ({}^p : p \in A))$ is a lattice-ordered orthomodular join-semilattice satisfying equation (B). Hence, by Lemma 4.3, $\mathcal{A}(\mathbf{A}^\partial) = (A, \cdot, 0)$ is an orthomodular implication algebra satisfying (B*). Conversely, if $\mathbf{A} = (A, \cdot, 1)$ is a lattice-ordered orthomodular implication algebra satisfying (B*), then, by Lemma 4.3, $\mathcal{S}(\mathbf{A}) = (A, \vee, \wedge, 1, ({}^p : p \in A))$ is a lattice-ordered orthomodular join-semilattice satisfying (B), and so $(\mathcal{S}(\mathbf{A}))^\partial$ is a generalized orthomodular lattice, by Lemma 4.9. □

The following corollary is a direct consequence of Theorem 4.4.

COROLLARY 4.11
$\mathcal{GOML}$ is term equivalent to the variety of lattice skew Hilbert algebras satisfying the following identities:

(i) $(x * y) * y \approx (y * x) * x$,
(ii) $((((((((x * y) * y) * z) * z) * z) * x) * x) * z) * x) * x \approx (((x * y) * y) * z) * z$,
(iii) $x * y \approx ((x \vee y)(y \wedge z)) \vee y$.

It is well known that orthomodular lattices are generalized orthomodular lattices with top element 1. Therefore, the above results characterize orthomodular lattices in the variety of lattice skew Hilbert algebras with bottom element 0.

COROLLARY 4.12
$\mathcal{GOML}$ forms a variety. The variety $\mathcal{OML}$ of orthomodular lattices is term equivalent to the variety of lattice skew Hilbert algebras with bottom element 0 satisfying conditions (i)–(iii) of Corollary 4.11.

In what follows, given an orthomodular lattice $\mathbf{A}$, we will denote by $\mathcal{H}(\mathbf{A})$ its associated skew Hilbert algebra. Note that, in general, if $\mathbf{A}$ is an orthomodular lattice, then $\mathcal{H}(\mathbf{A})$ need not be a relatively sectionally pseudocomplemented lattice as the next example shows.

EXAMPLE 4.13
Consider the orthomodular lattice $\mathbf{MO}_2$ depicted in Figure 3 with $*$ defined as follows:

$*$	0	a	a'	b	b'	1
0	1	1	1	1	1	1
a	a'	1	a'	b	b'	1
a'	a	a	1	b	b'	1
b	b'	a	a'	1	b'	1
b'	b	a	a'	b	1	1
1	0	a	a'	b	b'	1

Then $(\mathrm{MO}_2, \vee, \wedge, *, 0, 1)$ is a lattice skew Hilbert algebra. However, it is not sectionally pseudocomplemented: the first condition of Proposition 2.2 fails. Indeed, for $x = a$, $y = 0$, and $z = b$, one has

$$b \vee 0 \not\leq a' = a * 0 = a * (a \wedge b) = a * ((a \vee 0) \wedge (b \vee 0)),$$

i.e. the first condition of Proposition 2.2 fails.

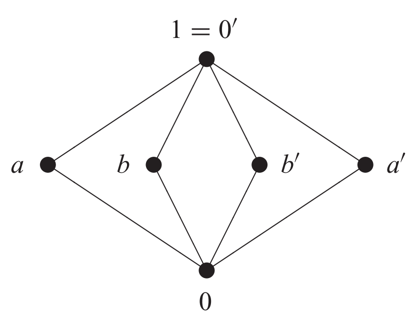

FIGURE 3

The next theorem shows that, indeed, orthomodular lattices inducing sectionally pseudocomplemented lattices are Boolean.

THEOREM 4.14
Let $\mathbf{A} = (A, \vee, \wedge, ', 0, 1)$ be an orthomodular lattice. Then $\mathcal{H}(\mathbf{A})$ is sectionally pseudocomplemented if and only if $\mathbf{A}$ is a Boolean algebra.

PROOF. Clearly, any Boolean algebra is sectionally pseudocomplemented by setting $x * y = x' \vee y$. Concerning the converse direction, by Remark 3.4, for any $a, b \in A$, $a * b$ is the sectional pseudocomplement of a with respect to b. Therefore, $x * 0 = (x \vee 0)' \vee 0 = x'$ is the largest element $c \in A$ such that $x \wedge c = 0$. Hence, we have that the following condition is fulfilled:

$$x \wedge y = 0 \quad \text{if and only if} \quad y \leq x'.$$

In other words, $\mathbf{A}$ is uniquely complemented, i.e. $\mathbf{A}$ is in fact a Boolean algebra. □

As has been pointed out above, orthomodular lattices induce prominent examples of (lattice) strong skew Hilbert algebras by setting $x * y := (x \vee y)' \vee y$. However, it is conceivable to wonder whether any orthomodular lattice can be endowed with a $*$ operation satisfying certain different, preferable conditions. Indeed, given a bounded poset $\mathbf{P} = (P, \leq, ', 0, 1)$ with a unary operation $'$ satisfying $0' \approx 1$, it seems reasonable to define

$$x * y := \begin{cases} 1 & \text{if } x \leq y, \\ x' & \text{if } y = 0, \\ y & \text{otherwise,} \end{cases} \tag{1}$$

for all $x, y \in P$, and then check whether $\mathbb{S}(\mathbf{P}) := (P, \leq, *, 1)$ is a skew Hilbert algebra. The following result provides a smooth characterization of bounded posets with a unary operation which lend themselves to accommodate the construction in condition (1).

THEOREM 4.15
Let $\mathbf{P} = (P, \leq, ', 0, 1)$ be a bounded poset with a unary operation $'$ satisfying $0' \approx 1$ and $1' \approx 0$. Then the following are equivalent:

- (i) $\mathbb{S}(\mathbf{P})$ is a skew Hilbert algebra;
- (ii) $\mathbb{S}(\mathbf{P})$ is a strong skew Hilbert algebra;
- (iii) for all $x \in P$ the following hold:
  - (a) $x' = 1$ if and only if $x = 0$,
  - (b) $'$ is antitone,
  - (c) $x \leq x''$,
  - (d) $L(x, x') = \{0\}$.

PROOF. Let $\mathbb{S}(\mathbf{P}) = (P, \leq, *, 1)$ and $a, b, c \in P$. Then,

$$1 * b = \begin{cases} 1 = b & \text{if } 1 \leq b, \\ 1' = 0 = b & \text{if } b = 0, \\ b & \text{if } 1 \not\leq b \neq 0. \end{cases}$$

This shows $1 * x \approx x$ which will be used in the sequel. (i) $\Rightarrow$ (iii):

- (a) Because of (S1), the following are equivalent: $a' = 1$; $a * 0 = 1$; $a \leq 0$; $a = 0$.
- (b) Because of (S1) and (S3), anyone of the following statements implies the next one: $a \leq b$; $a * b = 1$; $(b * 0) * (a * 0) = 1$; $b' * a' = 1$; $b' \leq a'$.
- (c) Because of (S1) and (S2), anyone of the following statements implies the next one: $0 \leq a$; $0 * a = 1$; $a * a'' = a * ((a * 0) * 0) = 1$; $a \leq a''$.

(d) Because of (S4), we have $L(a, a') = L(U(a), a') = L(U(a, 0), a * 0) = L(0) = \{0\}$.

(iii) ⇒ (ii):

(S1) Because of (a), we have

$$a * b = \begin{cases} 1 & \text{if } a \leq b, \\ a' \neq 1 & \text{if } a \nleq b = 0, \\ b \neq 1 & \text{if } a \nleq b \neq 0. \end{cases}$$

(S2') Because of (c), we have

$$a * ((a * b) * b) = \begin{cases} a * (1 * b) = a * b = 1 & \text{if } a \leq b, \\ a * (a' * 0) = a * a'' = 1 & \text{if } b = 0, \\ a * (b * b) = a * 1 = 1 & \text{if } a \nleq b \neq 0. \end{cases}$$

(S3) If $a * b = 1$, then $a \leq b$ because of (S1) and hence

$$(b * c) * (a * c) = \begin{cases} (b * c) * 1 = 1 & \text{if } a \leq c, \\ b' * a' = 1 & \text{if } c = 0, \\ c * c = 1 & \text{if } a \nleq c \neq 0 \end{cases}$$

because of (b).

(S4) Because of (d), we have $L(U(a, b), a * b) =$

$$= \begin{cases} L(U(b), 1) = LU(b) = L(b) & \text{if } a \leq b, \\ L(U(a, 0), a') = L(U(a), a') = L(a, a') = \{0\} = L(0) = L(b) & \text{if } b = 0, \\ L(U(a, b), b) = L(U(a, b), U(b)) = LU(b) = L(b) & \text{if } a \nleq b \neq 0. \end{cases}$$

(ii) ⇒ (i): This is trivial. □

As it has been pointed out above, orthomodular join-semilattices can be framed within the theory of (strong) skew Hilbert algebras. It is natural to ask whether any skew Hilbert algebra can be regarded as a poset having sectional antitone operations. In the sequel, we provide a positive answer by proving that any skew Hilbert algebra can be regarded as a poset whose sections can be endowed with a Brouwerian pseudocomplement. As a consequence, we conclude that skew Hilbert algebras can be regarded as a proper generalization of lattices with sectional antitone involutions [10].

Definition 4.16

A poset with *sectional Brouwerian pseudocomplements* is a structure $\mathbf{A} = (A, \leq, 1, (^p : p \in A))$ such that $(A, \leq, 1)$ is a poset with top element 1, and, for any $p \in A$, $([p, 1], \leq, ^p, 1)$ is a poset with a unary operation ^{p} such that, for any $x, y \in [p, 1]$,

(BP1) $x \leq y$ implies $y^p \leq x^p$,

(BP2) $x \leq x^{pp}$,

(BP3) $L(x, x^p) = L(p)$.

In the sequel, we will denote by $\mathcal{PSB}$, the class of posets with sectional Brouwerian pseudocomplements. The next remark shows that the above definition makes sense.

REMARK 4.17
Note that any poset $\mathbf{A} = (A, \leq, 1, (^p : p \in A))$ with sectional Brouwerian pseudocomplements satisfies, for any $x \in A$,

$$1^x \approx x \quad \text{and} \quad x^x \approx 1.$$

Indeed, let $p \in A$. One has $L(1^p) = L(1, 1^p) = L(p)$. Moreover, $1^p = p$ entails $1 \leq 1^{pp} = p^p$. We conclude $p^p = 1$. Therefore, if $a \in [p, 1]$, then $a \leq 1$ entails $p = 1^p \leq a^p$, i.e. $[p, 1]$ is closed under ^{p}.

PROPOSITION 4.18
Let $\mathbf{S} = (S, \leq, *, 1)$ be a skew Hilbert algebra. Then setting, for any $p \in S$ and $x \in [p, 1]$, $x^p := x * p$, $(S, \leq, 1, (^p : p \in S))$ is a poset with sectional Brouwerian pseudocomplements.

PROOF. Let $\mathbf{S} = (S, \leq, *, 1)$ be a skew Hilbert algebra. (BP1) holds by (S3). (BP2) is a consequence of (S2), while (BP3) follows from (S4), since $x \geq p$. □

Proposition 4.18 shows that every skew Hilbert algebra naturally gives rise to a poset with sectional Brouwerian pseudocomplements. However, if conditions (BP1) and (BP2) are supposed to hold for all elements of a poset $\mathbf{A}$ with sectional Brouwerian pseudocomplements, then $\mathbf{A}$ can be turned into a skew Hilbert algebra. Moreover, this correspondence is one-to-one.

DEFINITION 4.19
A poset $\mathbf{A}$ with sectional Brouwerian pseudocomplements is said to be *strong* if conditions (BP1) and (BP2) hold for any $x, y, p \in A$.

We denote by $s\mathcal{PSB}$ the class of strong posets with sectional Brouwerian pseudocomplements.

THEOREM 4.20
Let $\mathbf{S} = (S, \leq, *, 1)$ and $\mathbf{A} = (A, \leq, 1, (^p : p \in A))$ be a strong skew Hilbert algebra and a strong poset with sectional Brouwerian pseudocomplements, respectively.

(i) Setting, for any $x, y \in S$, $x^y := x * y$, one has that

$$\mathbb{B}(\mathbf{S}) = (S, \leq, 1, (^p : p \in A))$$

is a strong poset with sectional Brouwerian pseudocomplements.

(ii) Setting, for any $x, y \in A$, $x * y := x^y$, one has that

$$\mathbb{H}(\mathbf{A}) = (A, \leq, *, 1)$$

is a strong skew Hilbert algebra.

(iii) $\mathbb{H}(\mathbb{B}(\mathbf{S})) = \mathbf{S}$ and $\mathbb{B}(\mathbb{H}(\mathbf{A})) = \mathbf{A}$.

PROOF. (i) By Proposition 4.18, $\mathbb{B}(\mathbf{S})$ is a poset with sectional Brouwerian pseudocomplements. Moreover, it is easily seen that, by (S3) and (S2'), (BP1) and (BP2) hold for any $x, y, p \in A$. (ii) We prove that $\mathbb{H}(\mathbf{A})$ satisfies (S1), (S3), (S4) and (S2'). Concerning (S1), assume that $a, b \in A$ are such that $a \leq b$. Then one has that $1 = b^b \leq a^b$, by Remark 4.17, and so $a * b = 1$. Conversely, if $a^b = 1$, then $a \leq a^{bb} = 1^b = b$, again by Remark 4.17. (S2') follows by (S1) upon noticing that in any strong poset with sectional Brouwerian pseudocomplements $x \leq (x^y)^y$, for any $x, y \in A$. (S3) follows directly by (S1) and the fact that, for any $x, y, z \in A$, we have that $x \leq y$ implies $y^z \leq x^z$.

Concerning (S4), note that $a^b \leq 1$ entails $b = 1^b \leq a^{bb}$. Therefore, $a, b \leq a^{bb}$ implies $a^{bb} \in U(a, b)$, $L(U(a, b)) \subseteq L(a^{bb})$ and $L(U(a, b), a^b) = LU(a, b) \cap L(a^b) \subseteq L(a^{bb}) \cap L(a^b) = L(a^{bb}, a^b) = L(b)$, by (BP3). (iii) Straightforward. □

COROLLARY 4.21
The class of strong skew Hilbert algebras and $s\mathcal{PSB}$ are term equivalent.

COROLLARY 4.22
The class of lattice skew Hilbert algebras and the class of lattice-ordered $s\mathcal{PSB}$s are term equivalent.

Upon recalling that a lattice with sectional antitone involutions is a structure $\mathbf{A} = (A, \vee, \wedge, (^p : p \in A), 0, 1)$ such that $(A, \vee, \wedge, 0, 1)$ is a bounded lattice and, for any $p \in A$, $([p, 1], \vee, \wedge, ^p, p, 1)$ is a lattice with antitone involution, the following corollary easily follows.

COROLLARY 4.23
The class of lattices with sectional antitone involutions is term equivalent to the variety of lattice skew Hilbert algebras with bottom element 0 satisfying

$$(x * y) * y \approx (y * x) * x.$$

We close this section by discussing the relationship between skew Hilbert algebras and BCK-algebras. More specifically, we will show that the two classes of structures intersect exactly at Hilbert algebras. Therefore, they must be regarded as 'alternative' generalizations of the structures they abstract from.

The notion of a BCK-algebra was introduced by Imai and Iséki [24, 25] in 1966. This concept has motivations ranging from set theory to classical and non-classical propositional calculi. Over the past years, it has attracted the attention of practitioners in group theory, functional analysis, probability theory, topology, fuzzy set theory and so on. See [17] for an account. A BCK-algebra $\mathbf{A}$ is an algebra $(A, \circ, 0)$ of type $(2, 0)$ such that, for any $x, y, z \in A$,

(BCK-1) $((x \circ y) \circ (x \circ z)) \circ (z \circ y) = 0$,
(BCK-2) $(x \circ (x \circ y)) \circ y = 0$,
(BCK-3) $x \circ x = 0$,
(BCK-4) $x \circ y = 0$ and $y \circ x = 0$ imply $x = y$,
(BCK-5) $0 \circ x = 0$.

It can be proven that, for any BCK-algebra $\mathbf{A}$, the binary relation $\leq$ such that $x \leq y$ if and only if $x \circ y = 0$ is a partial order having 0 as the minimal element [17, Proposition 5.1.1]. Moreover, it is easily seen that, for any $x \in A$, $x \circ 0 = x$ [17, Proposition 5.1.3]. Now, let $\mathbf{A}$ be a BCK-algebra, and let $\mathbf{A}^\partial = (A, \leq^\partial, *, 0)$ be the structure where $*$ is a binary operation such that, for any $x, y \in A$, $x * y = y \circ x$, and $\leq^\partial$ is a binary relation such that $x \leq^\partial y$ iff $x * y = 0$. Clearly, $\leq^\partial$ is a partial order dualizing $\leq$. We have the following lemma.

LEMMA 4.24
Let $\mathbf{A}$ be a BCK-algebra. Then $\mathbf{A}^\partial$ is a strong skew Hilbert algebra if and only if the following quasi-equation holds:

$$z * (z * y) = 0 \text{ entails } z * y = 0. \tag{h}$$

PROOF. Suppose that $\mathbf{A}^\partial$ is a skew Hilbert algebra. Then one has that $z \leq z * y$ implies that $z \in L(U(z,y), z*y) = L(y)$. So $z \leq y$, i.e. $z*y = 0$. Conversely, it is easily seen that (S1),(S2'),(S3) hold. Concerning (S4), by $y \leq x*y$, one has that $L(y) \subseteq L(U(x,y), x*y)$. Moreover, if $z \in L(U(x,y), x*y)$, then by properties of BCK-algebras $0 = z*(x*y) = x*(z*y)$. Therefore, $x \leq z*y$. So $z*y \in U(x,y)$ and we conclude $z \leq z * y$. By (h), we have $z \leq y$. □

In the proof of the next theorem, we will take advantage of the next identities which hold in any BCK-algebra:

$$(y \circ z) \circ x = (y \circ x) \circ z. \tag{ex}$$

THEOREM 4.25
Let $\mathbf{A}$ be a BCK-algebra. Then $\mathbf{A}^\partial$ satisfies (h) if and only if it is a Hilbert algebra.

PROOF. Concerning the non-trivial direction, let us first observe that $y * (y * ((y * (y * x)) * x)) = y*((y*(y*x))*(y*x)) = 0$, by (ex) and (BCK-2). Therefore, by (h), one has $y*((y*(y*x))*x) = 0$ and so, by (ex), $(y * (y * x)) * (y * x) = 0$, i.e. $y * (y * x) = y * x$. Moreover, by (ex) and (h), one has

$$x * (y * (x * z)) = y * (x * z) \tag{a}$$

and, by several applications of (ex) and (BCK-1),

$$\begin{aligned}(x * y) * (z * ((z * x) * y)) &= (x * y) * ((z * x) * (z * y)) \\ &= (z * x) * ((x * y) * (z * y)) = 0.\end{aligned}$$

Therefore, we have also

$$(x * y) * (z * ((z * x) * y)) = 0. \tag{b}$$

Finally, we compute

$$\begin{aligned}x * ((y * (x * z)) * ((x * y) * z)) &= (y * (x * z)) * (x * ((x * y) * z)) && \text{(ex)} \\ &= (y * (x * z)) * ((x * y) * (x * z)) && \text{(ex)} \\ &= (y * (x * z)) * (x * ((x * y) * (x * z))) && \text{(a)} \\ &= 0. && \text{(b)}\end{aligned}$$

By several applications of (ex) to the left-hand side of $x * ((y * (x * z)) * ((x * y) * z)) = 0$, one has

$$(x * (y * z)) * ((x * y) * (x * z)) = 0.$$

Since it is well known that the latter condition implies that $\mathbf{A}^\partial$ is a Hilbert algebra, our result follows. □

Note that Theorem 4.25 obviously holds also for BCI-algebras [24]. Indeed, any BCI-algebra $\mathbf{A}$ such that $\mathbf{A}^\partial$ is a skew Hilbert algebra must satisfy $0 \circ x = 0$, namely it is a BCK-algebra.

5 Special subsets of skew Hilbert algebras

In this section, we will describe basic properties of some special subsets of skew Hilbert algebras, and so generalizing some well-known properties enjoyed by Hilbert algebras. In particular, we will highlight the connections between the set of 'closed' elements of a bounded skew Hilbert algebra and orthoposets. Then, we will investigate the relationships between the set of dense and weakly dense elements in a skew Hilbert algebra.

The above results are motivated by generalizing analogous obtainments for Hilbert algebras. Indeed, it is well known that closed (regular) elements in Hilbert algebras form a Boolean algebra. Moreover, we will single out some properties of dense elements in a skew Hilbert algebra **A** which, like in Hilbert algebras framework, form a subalgebra of **A**.

It can be noticed that, in the skew Hilbert algebras from Examples 3.5 and 3.2, the elements of the form $x * 0$ form a Boolean algebra. In what follows, we show that, in general, this is not the case.

For any skew Hilbert algebra $\mathbf{S} = (S, \leq, *, 1)$ with bottom element 0, put $x' := x * 0$ for all $x \in S$ and $S' := \{x' \mid x \in S\}$, and let $\mathbb{O}(\mathbf{S})$ denote the bounded poset $(S', \leq, ', 0, 1)$. (Observe that $0 = 1' \in S'$ and $1 = 0' \in S'$.)

The next theorem describes the connections between skew Hilbert algebras and orthoposets. Recall the definition of $\mathbb{S}$ before Theorem 4.15.

THEOREM 5.1

(i) Let $\mathbf{S} = (S, \leq, *, 1)$ be a skew Hilbert algebra with bottom element 0. Then
  (a) $\mathbb{O}(\mathbf{S})$ is an orthoposet;
  (b) $\mathbb{S}(\mathbb{O}(\mathbf{S})) = \mathbf{S}$ if and only if $S' = S$ and $x * y = y$ for all $x, y \in S$ with $x \not\leq y \neq 0$.

(ii) Let $\mathbf{P} = (P, \leq, ', 0, 1)$ be an orthoposet and $\mathbb{S}(\mathbf{P}) = (P, \leq, *, 1)$. Then
  (c) $\mathbb{S}(\mathbf{P})$ is a skew Hilbert algebra with bottom element 0 satisfying $P' = P$ and $x * y = y$ for all $x, y \in P$ with $x \not\leq y \neq 0$ (wherefrom we conclude that it is strong);
  (d) $\mathbb{O}(\mathbb{S}(\mathbf{P})) = \mathbf{P}$.

PROOF.

(i) Let $\mathbb{S}(\mathbb{O}(\mathbf{S})) = (S', \leq, \circ, 1)$ and $a, b \in S$.
  (a) Since $0, 1 \in S'$, $(S', \leq, 0, 1)$ is a bounded poset. Moreover, $'$ is a unary operation on S'. Because of (S3), $'$ is antitone. By (S2), we have $a \leq a''$. From this, we conclude $a' \leq a'''$ and by (S3) also $a''' \leq a'$. Together, we have $a''' = a'$ showing that $'$ is an involution on S'. Finally, because of (S4), we conclude $L(a, a') = L(U(a, 0), a') = L(0) = \{0\}$ and, since $'$ is an antitone involution, $U(a, a') = (L(a', a))' = \{0'\} = \{1\}$ showing that $'$ is a complementation on $(S', \leq, 0, 1)$.
  (b) This follows from

$$a \circ b = \begin{cases} 1 = a * b & \text{if } a \leq b, \\ a' = a * b & \text{if } b = 0. \end{cases}$$

  (c) This follows from Theorem 4.15.
  (d) Let $\mathbb{O}(\mathbb{S}(\mathbf{P})) = (P', \leq, ^+, 0, 1)$ and $a \in P$. According to Theorem 4.15, $\mathbb{S}(\mathbf{P})$ is a (strong) skew Hilbert algebra. Moreover, $P' = P$ and $a^+ = a'$. □

REMARK 5.2

Theorem 5.1 shows that the mappings $\mathbb{O}$ and $\mathbb{S}$ establish a one-to-one correspondence between the skew Hilbert algebras $(S, \leq, *, 1)$ satisfying $S' = S$ and $x * y = y$ for all $x, y \in S$ with $x \not\leq y \neq 0$ (which are automatically strong) on the one hand and orthoposets on the other.

EXAMPLE 5.3
If $O_6 := \{0, a, b, a', b', 1\}$ and $\mathbf{O}_6 = (O_6, \vee, \wedge, ', 0, 1)$ denotes the (non-modular) ortholattice shown in Figure 4,

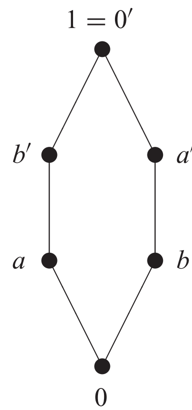

FIGURE 4

and $*$ the binary operation on O_6 defined by

$*$	0	a	a'	b	b'	1
0	1	1	1	1	1	1
a	a'	1	a'	b	1	1
a'	b'	a	1	b	b'	1
b	b'	a	1	1	b'	1
b'	a'	a	a'	b	1	1
1	0	a	a'	b	b'	1

then $\mathbb{S}(\mathbf{O}_6) = (O_6, \vee, \wedge, *, 1)$ is a lattice skew Hilbert algebra and, by Theorem 5.1, $\mathbb{O}(\mathbb{S}(\mathbf{O}_6)) = \mathbf{O}_6$.

Recall that a Boolean poset, in the sense of Tkadlec [30], is an orthoposet $\mathbf{P} = (P, \leq, ', 0, 1)$ such that, for any $x, y \in P$,

$$x \wedge y = 0 \quad \text{if and only if} \quad x \leq y'.$$

It can be shown that an orthoposet $\mathbf{P}$ is Boolean if and only if the following LU-identity holds (see [9]):

$$U(L(x,y),z) \approx U(L(U(x,z),U(y,z))).$$

PROPOSITION 5.4
Let $\mathbf{S} = (S, \leq, *, 1)$ be a sectionally pseudocomplemented skew Hilbert algebra with bottom element 0. Then the orthoposet $\mathbb{O}(\mathbf{S})$ is Boolean.

PROOF. Let $a, b \in S'$ with $a \wedge b = 0$ (in S'), and consider $L(a,b)$ (in S). If $c \in L(a,b)$, then $c'' \in \mathbb{O}(\mathbf{S})$ and $c'' \leq a, b$. So $c \leq c'' \leq 0 = a \wedge b$. Therefore, $L(U(b,0),a) = \{0\}$ and, by Remark 3.4, $a \leq b'$. □

LEMMA 5.5
Let $\mathbf{S} = (S, \leq, *, 1)$ be a skew Hilbert algebra with bottom element 0 and $a \in S$. Then $a = a'' \wedge (a'' * a)$.

PROOF. We have $a'' \geq a$ by (S2). From Remark 3.3 we obtain $a = a'' \wedge (a'' * a)$. □

Let $\mathbf{S} = (S, \leq, *, 1)$ be a skew Hilbert algebra with bottom element 0 and $a \in S$, and define

$$F_a := \{x \in S \mid x'' = a\},$$
$$D(\mathbf{S}) := \{x \in S \mid x' = 0\},$$
$$W(\mathbf{S}) := \{x \in S \mid \text{there exists some } y \in S \text{ with } y'' * y = x\}.$$

The elements of $D(\mathbf{S})$ and $W(\mathbf{S})$ are called *dense* and *weakly dense*, respectively. Note that (see [12]), for a Hilbert algebra $\mathbf{S} = (S, \leq, *, 1)$ with bottom element 0,

- $S' \cap D(\mathbf{S}) = \{1\}$;
- $D(\mathbf{S})$ is an upper subset of $\mathbf{S}$;
- $(D(\mathbf{S}), \leq, *, 1)$ is a Hilbert subalgebra of $\mathbf{S}$.

LEMMA 5.6
Let $\mathbf{S} = (S, \leq, *, 1)$ be a skew Hilbert algebra with bottom element 0. Then,

(i) $D(\mathbf{S}) \subseteq W(\mathbf{S})$;
(ii) $S' \cap W(\mathbf{S}) = \{1\}$;
(iii) $S' \cap D(\mathbf{S}) = \{1\}$;
(iv) $D(\mathbf{S})$ is an upper subset of $\mathbf{S}$;
(v) $(D(\mathbf{S}), \leq, *, 1)$ is a skew Hilbert subalgebra of $\mathbf{S}$.

PROOF. Suppose that $\mathbf{S} = (S, \leq, *, 1)$ is a skew Hilbert algebra with bottom element 0.

(i) If $a \in D(\mathbf{S})$, then $a' = 0$ and hence $a = 1 * a = a'' * a \in W(\mathbf{S})$, i.e. $D(\mathbf{S}) \subseteq W(\mathbf{S})$.
(ii) Let $a \in S' \cap W(\mathbf{S})$. Then $a = c'' * c$ for some $c \in S$ by definition, and $a'' = a$ by Theorem 5.1, item (i)(a). Correspondingly, we have $(c'' * c)'' = c'' * c$ and also $c \leq c''$. Moreover, since $c \leq c'' * c$, we have $c'' \leq (c'' * c)'' = c'' * c$. By (S4), we get

$$L(c) = L(U(c'', c), c'' * c) = L(U(c''), c'' * c) = L(c'', c'' * c) = L(c'').$$

Thus, $c = c''$ and consequently $a = 1$. Hence, we obtain $S' \cap W(\mathbf{S}) = \{1\}$.
(iii) Follows from the inclusion given in (i) and (ii).
(iv) Let $a \in D(\mathbf{S})$ and $a \leq b$. This yields $a * b = 1$, and from (S3), we also obtain $(b*0)*(a*0) = 1$. Since $a \in D(\mathbf{S})$, then $(b * 0) * 0 = 1$ which implies $b * 0 = 0$. Hence, we get $b \in D(\mathbf{S})$.
(v) This follows from the fact that $D(\mathbf{S})$ is an upper subset of $\mathbf{S}$ and hence $a, b \in D(\mathbf{S})$ implies $b \leq a * b \in D(\mathbf{S})$. □

EXAMPLE 5.7
Consider the lattice $(S, \leq)$ shown in Figure 5

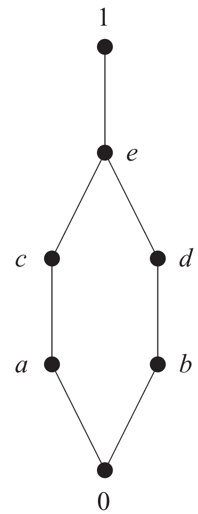

FIGURE 5

which is sectionally pseudocomplemented with the following binary operation:

$*$	0	a	b	c	d	e	1
0	1	1	1	1	1	1	1
a	d	1	b	1	d	1	1
b	c	a	1	c	1	1	1
c	d	a	b	1	d	1	1
d	c	a	b	c	1	1	1
e	0	a	b	c	d	1	1
1	0	a	b	c	d	e	1

Automatically, $\mathbf{S} = (S, \leq, *, 1)$ is a skew Hilbert algebra with bottom element 0. We have

$$D(\mathbf{S}) = \{e, 1\},$$
$$W(\mathbf{S}) = \{a, b, e, 1\},$$

and hence $D(\mathbf{S}) \neq W(\mathbf{S})$. Moreover, $W(\mathbf{S})$ is not an upper set of $\mathbf{S}$ since $b \in W(\mathbf{S})$ and $b \leq d$, but $d \notin W(\mathbf{S})$.

REMARK 5.8
However, in the bounded Hilbert algebra case, the sets of dense and weakly dense elements coincide (see [12,Lemma 3.7] for the inclusion $W(\mathbf{S}) \subseteq D(\mathbf{S})$).

6 Congruences in skew Hilbert algebras

In this last section, we will investigate the structure theory for the variety $\mathcal{V}$ of lattice skew Hilbert algebras (see, e.g. [8]). In particular, we will show that $\mathcal{V}$ is arithmetical and weakly regular. Moreover, since any congruence on a lattice skew Hilbert algebra is determined by its 1-coset, a further task will be characterizing these sets by introducing a suitable notion of *filter* and then proving that the complete lattice of filters on a lattice skew Hilbert algebra $\mathbf{L}$ is isomorphic to the complete lattice of congruences on $\mathbf{L}$. Furthermore, by extending an analogous notion for Hilbert algebras (see, e.g. [4]), we will introduce the concept of a *deductive system*. A full characterization thereof will follow. Finally, we will consider a notion of 'order-compatible' congruence for (strong) skew Hilbert algebras which need not be lattice-ordered. In turn, we show that, under minimal requirements, many of the aforementioned results hold.

First, let us recall the following concepts.

Let $\mathcal{C}$ be a class of algebras of the same type and $\mathcal{W}$ a variety with equationally definable constant 1. Then the class $\mathcal{C}$ is called

- *congruence permutable* if $\Theta \circ \Phi = \Phi \circ \Theta$ for all $\mathbf{A} \in \mathcal{C}$ and $\Theta, \Phi \in \operatorname{Con}\mathbf{A}$;
- *congruence distributive* if $(\Theta \vee \Phi) \wedge \Psi = (\Theta \wedge \Psi) \vee (\Phi \wedge \Psi)$ for all $\mathbf{A} \in \mathcal{C}$ and $\Theta, \Phi, \Psi \in \operatorname{Con}\mathbf{A}$;
- *arithmetical* if it is both congruence permutable and congruence distributive;
- *weakly regular* if for each $\mathbf{A} = (A, F) \in \mathcal{C}$ and all $\Theta, \Phi \in \operatorname{Con}\mathbf{A}$ with $[1]\Theta = [1]\Phi$, we have $\Theta = \Phi$.

The following is well known (cf. [8], Theorem 3.1.8, Corollary 3.2.4 and Theorem 6.4.3).

- The class $\mathcal{C}$ is congruence permutable if there exists a so-called *Maltsev term*, i.e. a ternary term p satisfying
$$p(x,x,y) \approx p(y,x,x) \approx y.$$
- The class $\mathcal{C}$ is congruence distributive if there exists a so-called *majority term*, i.e. a ternary term m satisfying
$$m(x,x,y) \approx m(x,y,x) \approx m(y,x,x) \approx x.$$
- The variety $\mathcal{W}$ is weakly regular if and only if there exists a positive integer n and binary terms $t_1, \dots, t_n$ such that
$$t_1(x,y) = \cdots = t_n(x,y) = 1 \text{ if and only if } x = y.$$

We are going to show that the variety $\mathcal{V}$ of lattice skew Hilbert algebras satisfies very strong congruence properties.

THEOREM 6.1
The variety $\mathcal{V}$ of lattice skew Hilbert algebras is arithmetical and weakly regular.

PROOF. Since the underlying posets are lattices, $\mathcal{V}$ is congruence distributive. Now, put

$$p(x,y,z) := ((x * y) * z) \wedge ((z * y) * x).$$

By (L2), we have $z \leq (z * x) * x$ and $x \leq (x * z) * z$ and hence

$$p(x,x,z) \approx ((x * x) * z) \wedge ((z * x) * x) \approx (1 * z) \wedge ((z * x) * x) \approx z \wedge ((z * x) * x) \approx z,$$

$$p(x,z,z) \approx ((x * z) * z) \wedge ((z * z) * x) \approx ((x * z) * z) \wedge (1 * x) \approx ((x * z) * z) \wedge x \approx x,$$

i.e. p is a Maltsev term which means that $\mathcal{V}$ is also congruence permutable and therefore arithmetical. For weak regularity, consider the binary terms

$$t_1(x,y) := x * y,$$
$$t_2(x,y) := y * x.$$

Clearly, $t_1(x,x) \approx t_2(x,x) \approx 1$. Conversely, $t_1(x,y) = t_2(x,y) = 1$ implies $x \leq y \leq x$ and therefore $x = y$. □

Weak regularity means that every congruence Θ on a lattice skew Hilbert algebra $\mathbf{L}$ is fully determined by its *kernel*, i.e. the congruence class $[1]\Theta$. Since Θ is also a lattice congruence, every class of it is a convex subset of $\mathbf{L}$. Hence, our first task is to describe these classes. For this purpose, we introduce the following concept.

DEFINITION 6.2
Let $\mathbf{L} = (L, \vee, \wedge, *, 1)$ be a lattice skew Hilbert algebra. A *filter* of $\mathbf{L}$ is a subset F of L containing 1 such that $x * y, y * x, z * v, v * z \in F$ implies

$$(x \vee z) * (y \vee v), (x \wedge z) * (y \wedge v), (x * z) * (y * v) \in F.$$

Let Fil$\mathbf{L}$ denote the set of all filters of $\mathbf{L}$. For any subset M of L, define a binary relation $\Phi(M)$ on L as follows:

$$\Phi(M) := \{(x,y) \in L^2 \mid x * y, y * x \in M\}.$$

The relationship between congruences and filters in lattice skew Hilbert algebras is illuminated in the next two theorems.

THEOREM 6.3
Let $\mathbf{L} = (L, \vee, \wedge, *, 1)$ be a lattice skew Hilbert algebra and $\Theta \in$ Con$\mathbf{L}$. Then $[1]\Theta \in$ Fil$\mathbf{L}$ and for any $x, y \in L$,

$$(x,y) \in \Theta \text{ if and only if } x * y, y * x \in [1]\Theta,$$

i.e. $\Phi([1]\Theta) = \Theta$.

PROOF. Let $a, b \in L$. If $(a,b) \in \Theta$, then $a * b, b * a \in [a * a]\Theta = [1]\Theta$, i.e. $(a,b) \in \Phi([1]\Theta)$. Conversely, if $(a,b) \in \Phi([1]\Theta)$, then $a * b, b * a \in [1]\Theta$ and hence, using (3) and (L2),

$$a = a \wedge ((a * b) * b) \;\Theta\; (1 * a) \wedge (1 * b) \;\Theta\; ((b * a) * a) \wedge b = b,$$

i.e. $(a,b) \in \Theta$. This shows $\Phi([1]\Theta) = \Theta$. Due to the substitution property of Θ with respect to $\vee$, $\wedge$ and $*$ we see that $[1]\Theta$ satisfies the conditions from Definition 6.2 and hence $[1]\Theta \in$ Fil$\mathbf{L}$. □

Theorem 6.3 witnesses that lattice skew Hilbert algebras are weakly regular.
We can also prove the converse.

THEOREM 6.4
Let $\mathbf{L} = (L, \vee, \wedge, *, 1)$ be a lattice skew Hilbert algebra and $F \in$ Fil$\mathbf{L}$. Then $\Phi(F) \in$ Con$\mathbf{L}$ and $[1](\Phi(F)) = F$.

PROOF. Let $a, b, c, d \in L$. Evidently, $\Phi(F)$ is symmetric and since $1 \in F$ and $x * x \approx 1$ by (L1), it is also reflexive. Assume $a * b, b * a, c * d, d * c \in F$. Then, by Definition 6.2,

$$(a \vee c) * (b \vee d), (b \vee d) * (a \vee c), (a \wedge c) * (b \wedge d), (b \wedge d) * (a \wedge c), (a * c) * (b * d), (b * d) * (a * c) \in F,$$

whence

$$(a \vee c, b \vee d), (a \wedge c, b \wedge d), (a * c, b * d) \in \Phi(F).$$

Hence, $\Phi(F)$ has the substitution property with respect to all basic operations of $\mathbf{L}$. Since the variety $\mathcal{V}$ is congruence permutable, $\Phi(F)$ is also transitive; see, e.g. Werner's theorem [33] or Corollary 3.1.13 in [8], and hence $\Phi(F) \in \mathrm{Con}\mathbf{L}$. Finally, the following are equivalent:

$$\begin{aligned} a \in [1](\Phi(F)) &\Leftrightarrow (a, 1) \in \Phi(F) \\ &\Leftrightarrow a * 1, 1 * a \in F \\ &\Leftrightarrow 1, a \in F \\ &\Leftrightarrow a \in F, \end{aligned}$$

and hence $[1](\Phi(F)) = F$. □

It is elementary to check that for every lattice skew Hilbert algebra $\mathbf{L}$, $(\mathrm{Fil}\mathbf{L}, \subseteq)$ is a complete lattice.

The following corollary follows from Theorems 6.3 and 6.4.

COROLLARY 6.5
For every lattice skew Hilbert algebra $\mathbf{L}$, the mappings $\Phi \mapsto [1]\Phi$ and $F \mapsto \Phi(F)$ are mutually inverse isomorphisms between the complete lattices $(\mathrm{Con}\mathbf{L}, \subseteq)$ and $(\mathrm{Fil}\mathbf{L}, \subseteq)$.

Since the operation $*$ may serve as implication in the logic based on a skew Hilbert algebra, we can consider also corresponding deductions. For this reason, we introduce the following concept.

A *deductive system* of a skew Hilbert algebra $(S, \leq, *, 1)$ is a subset D of S containing 1 and satisfying the following condition:

$$\text{if } a \in D, b \in S \text{ and } a * b \in D \text{ then } b \in D.$$

In the sequel, we use the following convention: if $(G, *)$ is a groupoid, $A, B \subseteq G$ and $a, b \in G$, then

$$\begin{aligned} A * B &:= \{x * y \mid x \in A, y \in B\}, \\ A * b &:= \{x * b \mid x \in A\}, \\ a * B &:= \{a * y \mid y \in B\}. \end{aligned}$$

THEOREM 6.6
Let $\mathbf{L} = (L, \vee, \wedge, *, 1)$ be a lattice skew Hilbert algebra, $\Theta \in \mathrm{Con}\mathbf{L}$, $F \in \mathrm{Fil}\mathbf{L}$ and $a, b \in L$. Then,

(i) every class of Θ is a convex subset of $(L, \leq)$;
(ii) F is a deductive system of $\mathbf{L}$;
(iii) F is a lattice filter of $\mathbf{L}$;
(iv) $a * (F \wedge a) \subseteq F$ and $(F * (F * a)) * a \subseteq F$.

PROOF.

(i) If $c, d \in [a]\Theta$ and $c \leq b \leq d$, then

$$b = c \vee b \in [d \vee b]\Theta = [d]\Theta = [a]\Theta.$$

(ii) If $a, a * b \in F$, then

$$b = 1 * b \in [a * b](\Phi(F)) = [1](\Phi(F)) = F.$$

(iii) If $a \in F$, then

$$a \vee b \in [1 \vee b](\Phi(F)) = [1](\Phi(F)) = F.$$

Moreover, if $a, b \in F$, then

$$a \wedge b \in [1 \wedge 1](\Phi(F)) = [1](\Phi(F)) = F.$$

(iv)

$$a * (F \wedge a) \subseteq [a * (1 \wedge a)](\Phi(F)) = [a * a](\Phi(F)) = [1](\Phi(F)) = F,$$

$$(F * (F * a)) * a \subseteq [(1 * (1 * a)) * a](\Phi(F)) = [(1 * a) * a](\Phi(F)) = [a * a](\Phi(F)) = \\ = [1](\Phi(F)) = F.$$

□

In what follows, we consider congruences in non-lattice skew Hilbert algebras.

Non-lattice skew Hilbert algebras have only one everywhere defined operation, namely $*$. However, the concept of a congruence should respect also the partial order relation. Hence, we present the following definition.

DEFINITION 6.7

A binary relation ρ on a poset $(P, \leq)$ is called min-*stable* if whenever $(a, b), (c, d) \in \rho$, a and c are comparable with each other and b and d are comparable with each other, then

$$(\min(a, c), \min(b, d)) \in \rho.$$

Now, we define a congruence on a skew Hilbert algebra as follows.

DEFINITION 6.8

Let $\mathbf{S} = (S, \leq, *, 1)$ be a skew Hilbert algebra. Then,

- an *algebraic congruence* on $\mathbf{S}$ is a congruence on its reduct $(S, *)$;
- a *congruence* on $\mathbf{S}$ is a min-stable algebraic congruence on $\mathbf{S}$.

Let Con$\mathbf{S}$ denote the set of all congruences on $\mathbf{S}$.

REMARK 6.9

Note that any congruence on a lattice skew Hilbert algebra $\mathbf{L} = (L, \vee, \wedge, *, 1)$ is automatically a congruence on the underlying skew Hilbert algebra $\mathbf{L} = (L, \leq, *, 1)$.

However, a congruence on a skew Hilbert algebra that is a lattice may not be a congruence on the underlying lattice (see the following counter-example).

EXAMPLE 6.10
Let $(L, \vee, \wedge)$ denote the lattice considered in Example 3.5 (a), and define a binary operation $*$ on L by

$*$	0	a	b	c	d	e	1
0	1	1	1	1	1	1	1
a	0	1	b	1	d	e	1
b	0	a	1	1	1	1	1
c	0	a	b	1	d	e	1
d	0	a	e	c	1	e	1
e	0	a	d	c	d	1	1
1	0	a	b	c	d	e	1

If we set

$$\Theta := \{0\}^2 \cup \{a\}^2 \cup \{b, e\}^2 \cup \{c, d, 1\}^2,$$

then $\mathbf{S} := (L, \leq, *, 1)$ is a lattice skew Hilbert algebra, $\Theta \in \operatorname{Con}\mathbf{S}$, but $\Theta \notin \operatorname{Con}(L, \vee, \wedge)$ since

$$(c, d) \in \Theta, \text{ but } (c \wedge a, d \wedge a) = (a, 0) \notin \Theta.$$

Using the min-stability property of congruences in skew Hilbert algebras we can prove the following theorem.

THEOREM 6.11
Let $\mathbf{S} = (S, \leq, *, 1)$ be a skew Hilbert algebra and $\Theta \in \operatorname{Con}\mathbf{S}$. Then every class of Θ is a convex subset of $(S, \leq)$.

PROOF. If $a, c \in S$, $b, d \in [a]\Theta$ and $b \leq c \leq d$, then by (S2), we obtain

$$(c * d) * b = 1 * b = b \leq c \leq (c * b) * b.$$

So $((c * d) * b, (c * b) * b) \in \Theta$ and hence by min-stability of Θ, we have

$$(b, c) = (\min((c * d) * b, c), \min((c * b) * b, c)) \in \Theta$$

which implies $c \in [b]\Theta = [a]\Theta$. □

We now investigate quotients of skew Hilbert algebras and strong skew Hilbert algebras with respect to their congruences.

Let $\mathbf{S} = (S, \leq, *, 1)$ be a skew Hilbert algebra and Θ be an algebraic congruence on $\mathbf{S}$. We define a binary relation $\leq'$ on S/Θ by

$$[a]\Theta \leq' [b]\Theta \text{ if and only if } [a]\Theta * [b]\Theta = [1]\Theta \quad (a, b \in S).$$

Recall that a *poset* $(P, \leq)$ is called up-directed if for any $x, y \in P$, there exists some $z \in P$ with $x, y \leq z$. Hence, every poset that has a top element is up-directed.

DEFINITION 6.12

An algebraic congruence Θ on a skew Hilbert algebra $\mathbf{S} = (S, \leq, *, 1)$ is called *strong* if it satisfies the following condition for all $a, b \in S$:

$$[a]\Theta \leq' [b]\Theta \text{ if and only if there exists some } c \in [b]\Theta \text{ with } a \leq c \text{ and } b \leq c.$$

We naturally define the term *strong congruence* as being a strong algebraic congruence which is min-stable.

THEOREM 6.13

Let $\mathbf{S} = (S, \leq, *, 1)$ be a skew Hilbert algebra with $a, a_1, \ldots, a_n, b \in S$ for some $n \in \mathbb{Z}^+$. Further, let Θ be an algebraic congruence on $\mathbf{S}$. Then,

(i) $a \leq b$ implies $[a]\Theta \leq' [b]\Theta$;
(ii) if $\mathbf{S}$ is strong, then Θ is strong.

If Θ is a strong algebraic congruence, then

(iii) every class of Θ is up-directed;
(iv) $U([a_1]\Theta, \ldots, [a_n]\Theta) = \{[x]\Theta \mid x \in U(a_1 \ldots, a_n)\}$ in $(S/\Theta, \leq')$.

If Θ is moreover a strong congruence, then

(v) $(S/\Theta, \leq')$ is a poset.

PROOF.

(i) If $a \leq b$, then $a * b = 1$ whence $a * b \;\Theta\; 1$, i.e. $[a]\Theta * [b]\Theta = [a * b]\Theta = [1]\Theta$, thus $[a]\Theta \leq' [b]\Theta$.
(ii) Suppose that $\mathbf{S}$ is a strong skew Hilbert algebra. If $[a]\Theta \leq' [b]\Theta$, then $a * b \;\Theta\; 1$ and hence $a \leq (a * b) * b \in [1 * b]\Theta = [b]\Theta$. So one can put $c := (a * b) * b$ and we have also $b \leq c$. If, conversely, there exists some $c \in [b]\Theta$ with $a \leq c$, then according to (i), we have $[a]\Theta \leq' [c]\Theta = [b]\Theta$.
(ii) For the following items, we have to assume that Θ is a strong algebraic congruence.
(iii) Let $b, c \in [a]\Theta$. Then $[b]\Theta \leq' [c]\Theta$. Hence, there exists some $d \in [c]\Theta = [a]\Theta$ such that $b \leq d$ and $c \leq d$.
(iv) Assume $[a]\Theta \in U([a_1]\Theta, \ldots, [a_n]\Theta)$. Since Θ is strong, for all $i \in \{1, \ldots, n\}$, there exists some $b_i \in [a]\Theta$ with $a_i \leq b_i$. Because of (iii), $([a]\Theta, \leq)$ is up-directed and hence there exists some $c \in [a]\Theta$ with $b_1, \ldots, b_n \leq c$. This shows

$$[a]\Theta = [c]\Theta \in \{[x]\Theta \mid x \in U(a_1, \ldots, a_n)\}.$$

The reverse inclusion follows from (i).
(iv) Now, assume that Θ is moreover min-stable (i.e. Θ turns into a strong congruence).
(v) Obviously, $\leq'$ is reflexive. Now, assume $[a]\Theta \leq' [b]\Theta$ and $[b]\Theta \leq' [a]\Theta$. Since Θ is strong, there exists some $c \in [b]\Theta$ with $a \leq c$. Because of $[c]\Theta = [b]\Theta \leq' [a]\Theta$, there exists some $d \in [a]\Theta$ with $c \leq d$. Since $a \leq c \leq d$, $a, d \in [a]\Theta$ and $([a]\Theta, \leq)$ is convex, we conclude $c \in [a]\Theta$. Therefore, $[a]\Theta = [c]\Theta = [b]\Theta$ which proves the antisymmetry of $\leq'$. Finally, let $c \in S$, and assume $[a]\Theta \leq' [b]\Theta$ and $[b]\Theta \leq' [c]\Theta$. Then, from the fact that Θ is strong, we can find some $e \in [b]\Theta$ with $a \leq e$ and because of $[e]\Theta = [b]\Theta \leq' [c]\Theta$ some $f \in [c]\Theta$ with $e \leq f$. From $a \leq e \leq f$, we have $a \leq f$ which implies $[a]\Theta \leq' [f]\Theta = [c]\Theta$ by (i), proving the transitivity of $\leq'$. □

Note that we have used the expression $U(M)$ for a subset M of S/Θ though $(S/\Theta, \leq')$ need not be a poset. But the meaning of $U(M)$ is clear.

From (iii), we conclude that if $(S, \leq)$ satisfies the ascending chain condition (in particular, if S is finite), then every class of a strong algebraic congruence Θ has a greatest element. However, this is not true in general. To see this, consider the following example.

EXAMPLE 6.14
Let $\mathbf{S} = (S, \leq, 1)$ denote the poset with top element 1 shown in Figure 6

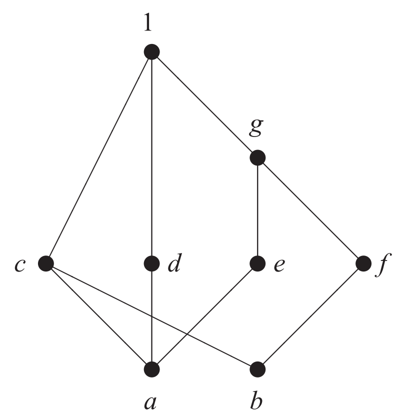

FIGURE 6

and $*$ the binary operation on S defined by

$*$	a	b	c	d	e	f	g	1
a	1	f	1	1	1	f	1	1
b	e	1	1	d	e	1	1	1
c	a	b	1	d	e	f	g	1
d	a	b	c	1	e	f	g	1
e	a	b	c	d	1	f	1	1
f	a	b	c	d	e	1	1	1
g	a	b	c	d	e	f	1	1
1	a	b	c	d	e	f	g	1

Then $(S, \leq, *, 1)$ is a skew Hilbert algebra which is not strong since

$$a \not\leq b = f * b = (a * b) * b.$$

Moreover,

$$\Theta := \{a, b\}^2 \cup \{c\}^2 \cup \{d, e, f, g, 1\}^2$$

is a congruence on $\mathbf{S}$ and $([a]\Theta, \leq) = (\{a, b\}, \leq)$ has no greatest element. From (iii) of Theorem 6.13, we conclude that Θ is not strong.

THEOREM 6.15
Let $\mathbf{S} = (S, \leq, *, 1)$ be a skew Hilbert algebra and Θ an algebraic congruence on $\mathbf{S}$ such that every class of Θ satisfies the ascending chain condition. Then $\Theta \in \mathrm{Con}\mathbf{S}$.

PROOF. Suppose $(a,b),(c,d) \in \Theta$ where a and c are comparable with each other and b and d are comparable with each other. We have the following four possibilities:

(a) $a \leq c$ and $b \leq d$,
(b) $c \leq a$ and $d \leq b$,
(c) $a \leq c$ and $d \leq b$,
(d) $c \leq a$ and $b \leq d$.

It is evident that in the first two cases $(\min(a,c), \min(b,d)) \in \Theta$. Now, consider case (c). We have $a \leq c$ and $d \leq b$. Put $a_0 := a$ and $c_0 := c$. Then $a_0 \; \Theta \; a$, $c_0 \; \Theta \; c$ and $c_0 \geq a_0$. If $c_0 = a_0$ then $a = a_0 = c_0 = c \; \Theta \; d$ and we are done. Otherwise, we have $c_0 > a_0$. Put $a_1 := (c_0 * a_0) * a_0$. Then,

$$a_1 \; \Theta \; (c * a) * a \; \Theta \; (d * b) * a = 1 * a = a$$

and $a_1 \geq c_0$ according to (S2). If $a_1 = c_0$, then $a \; \Theta \; a_1 = c_0 = c \; \Theta \; d$ and we are done. Otherwise, we have $a_1 > c_0$. Put $c_1 := (a_1 * c_0) * c_0$. Then,

$$c_1 \; \Theta \; (a * c) * c = 1 * c = c$$

and $c_1 \geq a_1$ according to (S2). If $c_1 = a_1$, then $a \; \Theta \; a_1 = c_1 \; \Theta \; c \; \Theta \; d$ and we are done. Otherwise, we have $c_1 > a_1$. Put $a_2 := (c_1 * a_1) * a_1$. Then,

$$a_2 \; \Theta \; (c * a) * a \; \Theta \; (d * b) * a = 1 * a = a$$

and $a_2 \geq c_1$ according to (S2). By going on in this way, we get a chain of the form

$$(*) \quad a_0 < c_0 < a_1 < c_1 < a_2 < \cdots,$$

where $a_0 := a$, $c_0 := c$ and

$$a_k := (c_{k-1} * a_{k-1}) * a_{k-1},$$
$$c_k := (a_k * c_{k-1}) * c_{k-1}$$

for $k > 0$. Moreover, $a_k \; \Theta \; a$ and $c_k \; \Theta \; c$ for $k \geq 0$. If the chain (*) would be infinite, then

$$a_0 < a_1 < a_2 < \cdots$$

would be an infinite ascending chain in $([a]\Theta, \leq)$ contradicting the assumption that this poset satisfies the ascending chain condition. Hence, there exists some $m \geq 0$ such that either $c_m = a_m$ or $a_{m+1} = c_m$. In the first case, we have

$$a \; \Theta \; a_m = c_m \; \Theta \; c \; \Theta \; d,$$

whereas in the second case,

$$a \; \Theta \; a_{m+1} = c_m \; \Theta \; c \; \Theta \; d.$$

This shows $a \; \Theta \; d$ in case (c). Case (d) is symmetric to case (c). □

From the preceding theorem, we obtain: if $(S, \leq)$ satisfies the ascending chain condition (in particular, if S is finite), then every algebraic congruence on $\mathbf{S}$ is a congruence on $\mathbf{S}$. Moreover, if **S** is in addition a strong skew Hilbert algebra, then every algebraic congruence on **S** is a strong congruence on **S**.

We are now going to show that although the class of strong skew Hilbert algebras does not form a variety, each of its members is weakly regular.

Analogous to the lattice case, we define the following.

DEFINITION 6.16
Let $\mathbf{S} = (S, \leq, *, 1)$ be a skew Hilbert algebra and F a subset of S containing 1. We say that

(a) F is a *$*$-filter* of **S** if it satisfies the following condition for all $x, y, z, v \in S$:
  (F1) if $x * y, y * x, z * v, v * z \in F$ then $(x * z) * (y * v) \in F$.
(b) F is a *filter* of **S** if it is a $*$-filter of **S** and satisfies the following condition for all $x, y, z, v \in S$:
  (F2) if $x * y, y * x, z * v, v * z \in F$, x and z are comparable with each other, and y and v are comparable with each other then $\min(x, z) * \min(y, v) \in F$.
(c) F is a *strong $*$-filter* of **S** if it is a $*$-filter of **S** and satisfies the following condition for all $x, y \in S$:
  (F3) if $x * y \in F$ then there exists some $z \in S$ such that $x, y \leq z$ and $z * y \in F$.
(d) F is a *strong filter* of **S** if it is a filter satisfying (F3).

Let Fil**S** denote the set of all filters of **S**. It is elementary to check that for every skew Hilbert algebra **S**, (Con**S**, $\subseteq$) and (Fil**S**, $\subseteq$) are complete lattices. For any subset M of S, put

$$\Phi(M) := \{(x, y) \in S^2 \mid x * y, y * x \in M\}.$$

The relationship between congruences and filters in strong skew Hilbert algebras is illustrated by the following theorems and corollaries.

THEOREM 6.17
Let $\mathbf{S} = (S, \leq, *, 1)$ be a skew Hilbert algebra and Θ a strong congruence on **S**. Then,

$$(x, y) \in \Theta \text{ if and only if } x * y, y * x \in [1]\Theta,$$

i.e. $\Phi([1]\Theta) = \Theta$.

PROOF. For $a, b \in S$, the following are equivalent:

$$\begin{aligned}(a, b) \in \Phi([1]\Theta) &\Leftrightarrow a * b, b * a \in [1]\Theta \\ &\Leftrightarrow [a]\Theta \leq' [b]\Theta \leq' [a]\Theta \\ &\Leftrightarrow [a]\Theta = [b]\Theta \\ &\Leftrightarrow (a, b) \in \Theta.\end{aligned}$$

□

We have shown that every strong congruence Θ on a skew Hilbert algebra is fully determined by its 1-class $[1]\Theta$. Hence, we can draw the following conclusion.

COROLLARY 6.18
Strong skew Hilbert algebras are weakly regular.

PROOF. Let $\mathbf{S} = (S, \leq, *, 1)$ be a strong skew Hilbert algebra and $\Theta, \Phi \in \mathrm{Con}\mathbf{S}$. Thus, Θ, Φ are strong from Theorem 6.13. Suppose that $[1]\Theta = [1]\Phi$. Then, by Theorem 6.17, we obtain

$$(a,b) \in \Theta \Leftrightarrow a*b, b*a \in [1]\Theta \Leftrightarrow a*b, b*a \in [1]\Phi \Leftrightarrow (a,b) \in \Phi.$$

□

The preceding corollary is not true in the more general case of skew Hilbert algebras (see the following example).

EXAMPLE 6.19
Let $\mathbf{S} = (S, \leq, 1)$ denote the poset with top element 1 shown in Figure 7

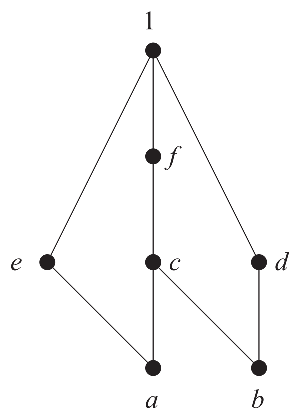

FIGURE 7

and $*$ the binary operation on S defined by

$*$	a	b	c	d	e	f	1
a	1	d	1	d	1	1	1
b	e	1	1	1	e	1	1
c	a	b	1	d	e	1	1
d	a	b	c	1	e	f	1
e	a	b	c	d	1	f	1
f	a	b	c	d	e	1	1
1	a	b	c	d	e	f	1

Then $(S, \leq, *, 1)$ is a skew Hilbert algebra which is not strong since

$$a \not\leq b = d*b = (a*b)*b.$$

We have a congruence given by

$$\Theta := \{a\}^2 \cup \{b\}^2 \cup \{c\}^2 \cup \{d,e,f,1\}^2.$$

It is readily checked that $[1]\Theta$ is a (strong) filter. However, $a*b, b*a \in [1]\Theta$ but $(a,b) \notin \Theta$.

We can prove also the converse of Theorem 6.17.

THEOREM 6.20
Let $\mathbf{S} = (S, \leq, *, 1)$ be a skew Hilbert algebra and F a $*$-filter of $\mathbf{S}$. Then $\Phi(F)$ is an algebraic congruence on $\mathbf{S}$ and $[1](\Phi(F)) = F$.

PROOF. Let $a, b, c, d \in S$. Evidently, $\Phi(F)$ is symmetric and since $1 \in F$ and $x * x \approx 1$, it is also reflexive. Let us show that $(a, b), (c, d) \in \Phi(F)$ implies $(a * c, b * d) \in \Phi(F)$. We have $a * b, b * a, c * d, d * c \in F$ by definition. From (F1), we get

$$(a * c) * (b * d) \in F,$$
$$(b * d) * (a * c) \in F$$

which yields $(a * c, b * d) \in \Phi(F)$ by definition. Hence, $\Phi(F)$ has the substitution property with respect to $*$. If $(a, b), (b, c) \in \Phi(F)$, then

$$a * c = 1 * (a * c) = (b * b) * (a * c) \in F,$$
$$c * a = 1 * (c * a) = (b * b) * (c * a) \in F,$$

and hence $(a, c) \in \Phi(F)$. This shows the transitivity of $\Phi(F)$. Finally, the following are equivalent:

$$\begin{aligned} a \in [1](\Phi(F)) &\Leftrightarrow (a, 1) \in \Phi(F) \\ &\Leftrightarrow a * 1, 1 * a \in F \\ &\Leftrightarrow 1, a \in F \\ &\Leftrightarrow a \in F, \end{aligned}$$

showing $[1](\Phi(F)) = F$. □

COROLLARY 6.21
Let $\mathbf{S} = (S, \leq, *, 1)$ be a skew Hilbert algebra and F a $*$-filter of $\mathbf{S}$.

(a) If $F \in \mathrm{Fil}\mathbf{S}$, then $\Phi(F) \in \mathrm{Con}\mathbf{S}$.
(b) If F is strong, then $\Phi(F)$ is strong.

PROOF. Let $a, b, c, d \in S$.

(a) If $(a, b), (c, d) \in \Phi(F)$, a and c are comparable with each other and b and d are comparable with each other then from (F2), we get

$$\min(a, c) * \min(b, d), \min(b, d) * \min(a, c) \in F,$$

i.e. $(\min(a, c), \min(b, d)) \in \Phi(F)$. This shows that $\Phi(F)$ is min-stable.
(b) Assume $[a]\Phi(F) \leq' [b]\Phi(F)$. Then $(a * b, 1) \in \Phi(F)$, i.e. $a * b \in F$. From (F3), we get that there exists some $c \in S$ such that $a, b \leq c$ and $c * b \in F$. Since $1 = b * c \in F$, we obtain $c \in [b]\Phi(F)$, i.e. $\Phi(F)$ is strong. □

The following corollary follows from the above theorems and corollaries.

COROLLARY 6.22
For every strong skew Hilbert algebra $\mathbf{S}$, the mappings $\Phi \mapsto [1]\Phi$ and $F \mapsto \Phi(F)$ are mutually inverse isomorphisms between the complete lattices $(\mathrm{Con}\mathbf{S}, \subseteq)$ and $(\mathrm{Fil}\mathbf{S}, \subseteq)$.

The following result is analogous to the corresponding result for lattice skew Hilbert algebras.

THEOREM 6.23
Let $\mathbf{S} = (S, \leq, *, 1)$ be a skew Hilbert algebra, $F \in \mathrm{Fil}\mathbf{S}$, and $c \in S$. Then,

(i) F is a deductive system of $\mathbf{S}$;
(ii) F is an order filter of $\mathbf{S}$;
(iii) $S * F \subseteq F$;
(iv) $c * (F \wedge c), (F * (F * c)) * c \subseteq F$.

PROOF. We use the fact that the filter F is the 1-class of the congruence $\Phi(F)$.

(i) If $a \in F, b \in S$ and $a * b \in F$, then

$$b = 1 * b \in [a * b](\Phi(F)) = [1](\Phi(F)) = F.$$

(ii) If $a \in F, b \in S$ and $a \leq b$, then $a * b = 1 \in F$ and hence $b \in F$ by (i).
(iii) If $a \in S$ and $b \in F$, then $a * b \in [a * 1](\Phi(F)) = [1](\Phi(F)) = F$.
(iv)

$$c * (F \wedge c) \subseteq [c * (1 \wedge c)](\Phi(F)) = [c * c](\Phi(F)) = [1](\Phi(F)) = F,$$
$$(F * (F * c)) * c \subseteq [(1 * (1 * c)) * c](\Phi(F)) = [(1 * c) * c](\Phi(F)) = [c * c](\Phi(F)) =$$
$$= [1](\Phi(F)) = F.$$

□

According to Theorem 6.23, every filter of a strong skew Hilbert algebra $\mathbf{S} = (S, \leq; *, 1)$ is a deductive system. However, to prove that a subset M of S containing 1 is a deductive system, we need not assume that M is a filter but we can suppose a simpler condition; see the following result.

PROPOSITION 6.24
Let $\mathbf{S} = (S, \leq, *, 1)$ be a strong skew Hilbert algebra and M a subset of S containing 1 and satisfying $(M * (M * x)) * x \subseteq M$ for all $x \in S$. Then M is a deductive system of $\mathbf{S}$.

PROOF. Let $a \in M$ and $b \in S$. We have $1 \in M$. If $a \leq b$, then

$$b = 1 * b = (a * b) * b = (a * (1 * b)) * b \in (M * (M * b)) * b \subseteq M.$$

Hence, if $a*b \in M$, then, because of $a \leq (a*b)*b$, we have $(a*b)*b \in M$. Therefore, we conclude

$$b = 1 * b = (((a * b) * b) * ((a * b) * b)) * b \in (M * (M * b)) * b \subseteq M.$$

□

Observe that the condition mentioned in Proposition 6.24 is just the second one from (iv) of Theorem 6.23.

For the concept of an ideal of a universal algebra which corresponds to our concept of a filter and for the concept of ideal terms, the reader is referred to [31]. In particular, for ideals (alias filters) in permutable and weakly regular varieties see also [8] for details.

DEFINITION 6.25
An *ideal term* for lattice skew Hilbert algebras is a term $t(x_1, \ldots, x_n, y_1, \ldots, y_m)$ in the language of lattice skew Hilbert algebras satisfying the identity

$$t(x_1, \ldots, x_n, 1, \ldots, 1) \approx 1.$$

Of course, there exists an infinite number of ideal terms in lattice skew Hilbert algebras. The following list including four ideal terms is a so-called *basis for filters* in lattice skew Hilbert algebras, i.e. filters can be characterized by this short list of ideal terms.

Consider the following terms for lattice skew Hilbert algebras $(L, \vee, \wedge, *, 1)$:

$$\begin{aligned} t(x,y,z,u) &:= (x \vee y) \wedge (z * y) \wedge u, \\ t_1 &:= 1, \\ t_2(x_1,x_2,x_3,x_4,y_1,y_2,y_3,y_4) &:= (t(x_1,x_2,y_1,y_2) \vee t(x_3,x_4,y_3,y_4)) * (x_2 \vee x_4), \\ t_3(x_1,x_2,x_3,x_4,y_1,y_2,y_3,y_4) &:= (t(x_1,x_2,y_1,y_2) \wedge t(x_3,x_4,y_3,y_4)) * (x_2 \wedge x_4), \\ t_4(x_1,x_2,x_3,x_4,y_1,y_2,y_3,y_4) &:= (t(x_1,x_2,y_1,y_2) * t(x_3,x_4,y_3,y_4)) * (x_2 * x_4). \end{aligned}$$

LEMMA 6.26
We have

$$\begin{aligned} t(x,y,1,1) &\approx y, \\ t(x,y,x*y,y*x) &\approx x. \end{aligned}$$

PROOF. We have

$$t(x,y,1,1) = (x \vee y) \wedge (1 * y) \wedge 1 = (x \vee y) \wedge y = y$$

according to (3) and

$$\begin{aligned} t(x,y,x*y,y*x) &= (x \vee y) \wedge ((x*y)*y) \wedge (y*x) = \\ &= ((y \vee x) \wedge (y*x)) \wedge ((x*y)*y) = x \wedge ((x*y)*y) = x \end{aligned}$$

according to (L2) and (L4). □

LEMMA 6.27
The terms $t_1, \ldots, t_4$ are ideal terms for lattice skew Hilbert algebras.

PROOF. We have

$$\begin{aligned} t_1 &\approx 1, \\ t_2(x_1,x_2,x_3,x_4,1,1,1,1) &\approx (t(x_1,x_2,1,1) \vee t(x_3,x_4,1,1)) * (x_2 \vee x_4) \approx \\ &\approx (x_2 \vee x_4) * (x_2 \vee x_4) \approx 1, \\ t_3(x_1,x_2,x_3,x_4,1,1,1,1) &\approx (t(x_1,x_2,1,1) \wedge t(x_3,x_4,1,1)) * (x_2 \wedge x_4) \approx \\ &\approx (x_2 \wedge x_4) * (x_2 \wedge x_4) \approx 1, \\ t_4(x_1,x_2,x_3,x_4,1,1,1,1) &\approx (t(x_1,x_2,1,1) * t(x_3,x_4,1,1)) * (x_2 * x_4) \approx \\ &\approx (x_2 * x_4) * (x_2 * x_4) \approx 1. \end{aligned}$$

□

The closedness with respect to ideal terms was also introduced by Ursini [31].

DEFINITION 6.28
A *subset* A of a lattice skew Hilbert algebra $\mathbf{L} = (L, \vee, \wedge, *, 1)$ is said to be *closed* with respect to the ideal terms $t_i(x_1, \dots, x_n, y_1, \dots, y_m)$, $i \in I$, if for every $i \in I$, all $x_1, \dots, x_n \in L$ and all $y_1, \dots, y_m \in A$ we have $t_i(x_1, \dots, x_n, y_1, \dots, y_m) \in A$.

Now, we prove that the ideal terms listed before Lemma 6.26 form a basis for filters, i.e. filters are characterized as those subsets which are closed with respect to these ideal terms.

THEOREM 6.29
Let $\mathbf{L} = (L, \vee, \wedge, *, 1)$ be a lattice skew Hilbert algebra and $F \subseteq L$. Then $F \in \mathrm{Fil}\mathbf{L}$ if and only if F is closed with respect to the ideal terms $t_1, \dots, t_4$ listed before Lemma 6.26.

PROOF. If $F \in \mathrm{Fil}\mathbf{L}$, then $F = [1](\Phi(F))$ according to Theorem 6.4, and if

$$t_i(x_1, \dots, x_n, y_1, \dots, y_m), \quad i \in \{1, \dots, 4\},$$

are the ideal terms listed in Lemma 6.27, $a_1, \dots, a_n \in L$ and $b_1, \dots, b_m \in F$, then

$$t_i(a_1, \dots, a_n, b_1, \dots, b_m) \in [t_i(a_1, \dots, a_n, 1, \dots, 1)](\Phi(F)) = [1](\Phi(F)) = F$$

according to Lemma 6.27 and hence F is closed with respect to the ideal terms $t_1, \dots, t_4$. Conversely, assume F to be closed with respect to the ideal terms $t_1, \dots, t_4$. Then $1 = t_1 \in F$ and if $a, b, c, d \in L$ and $a * b, b * a, c * d, d * c \in F$, then

$$\begin{aligned}
(a \vee c) * (b \vee d) &= (t(a, b, a * b, b * a) \vee t(c, d, c * d, d * c)) * (b \vee d) = \\
&= t_2(a, b, c, d, a * b, b * a, c * d, d * c) \in F, \\
(a \wedge c) * (b \wedge d) &= (t(a, b, a * b, b * a) \wedge t(c, d, c * d, d * c)) * (b \wedge d) = \\
&= t_3(a, b, c, d, a * b, b * a, c * d, d * c) \in F, \\
(a * c) * (b * d) &= (t(a, b, a * b, b * a) * t(c, d, c * d, d * c)) * (b * d) = \\
&= t_4(a, b, c, d, a * b, b * a, c * d, d * c) \in F
\end{aligned}$$

showing $F \in \mathrm{Fil}\mathbf{L}$. □

REMARK 6.30
Let us note that the term t from the proof of Theorem 6.29 gives rise to a Maltsev term. Namely, if

$$\begin{aligned}
t(x, y, z, u) &:= (x \vee y) \wedge (z * y) \wedge u \text{ and} \\
q(x, y, z) &:= t(x, z, x * y, y * x),
\end{aligned}$$

then

$$\begin{aligned}
q(x, y, z) &= (x \vee z) \wedge ((x * y) * z) \wedge (y * x), \\
q(x, x, z) &= (x \vee z) \wedge ((x * x) * z) \wedge (x * x) = (x \vee z) \wedge (1 * z) \wedge 1 = (x \vee z) \wedge z = z, \\
q(x, z, z) &= (x \vee z) \wedge ((x * z) * z) \wedge (z * x) = ((z \vee x) \wedge (z * x)) \wedge ((x * z) * z) = \\
&= x \wedge ((x * z) * z) = x.
\end{aligned}$$

Observe that the Maltsev term $q(x,y,z)$ is different from that in Theorem 6.1.

In the following, we write $a \wedge b \wedge c$ instead of $\inf(a,b,c)$.

Now, we introduce a certain modification of the notion an ideal term (for posets) which need not be defined everywhere. This will be used in the sequel.

DEFINITION 6.31
A *partial ideal term* for skew Hilbert algebras is a partially defined term $T(x_1,\dots,x_n,y_1,\dots,y_m)$ in the language of skew Hilbert algebras satisfying the identity

$$T(x_1,\dots,x_n,1,\dots,1) \approx 1.$$

This language contains also a binary operator $U(x,y)$.

Note that since a skew Hilbert algebra has a top element 1, the set $U(x,y)$ will always be non-empty.

Using the concept of partial ideal terms, we will also try to describe filters in strong skew Hilbert algebras. Similar to Lemma 6.27, we first get a list of three partial ideal terms which will be shown to suffice.

Consider the following partial terms for skew Hilbert algebras $(S,\leq,*,1)$:

$$T(x,y,z,u) := U(x,y) \wedge (z*y) \wedge u,$$

$$T_1 := 1,$$

$$T_2(x_1,x_2,x_3,x_4,y_1,y_2,y_3,y_4) := (T(x_1,x_2,y_1,y_2) * T(x_3,x_4,y_3,y_4)) * (x_2 * x_4),$$

$$T_3(x_1,x_2,x_3,x_4,y_1,y_2,y_3,y_4) := (T(x_1,x_2,y_1,y_2) \wedge T(x_3,x_4,y_3,y_4)) * (x_2 \wedge x_4).$$

LEMMA 6.32
We have

$$T(x,y,1,1) \approx y,$$

$$T(x,y,x*y,y*x) \approx x.$$

PROOF. We have

$$T(x,y,1,1) \approx U(x,y) \wedge (1*y) \wedge 1 \approx U(x,y) \wedge y \approx y$$

according to (3) and Remark 3.3 and

$$T(x,y,x*y,y*x) \approx U(x,y) \wedge ((x*y)*y) \wedge (y*x) \approx$$
$$\approx (U(y,x) \wedge (y*x)) \wedge ((x*y)*y) \approx x \wedge ((x*y)*y) \approx x$$

according to (S2') and Remark 3.3. □

LEMMA 6.33
The partial terms T_1, T_2, T_3 are partial ideal terms for skew Hilbert algebras.

PROOF. We have

$$T_1 \approx 1,$$
$$T_2(x_1,x_2,x_3,x_4,1,1,1,1) \approx (T(x_1,x_2,1,1) * T(x_3,x_4,1,1)) * (x_2 * x_4) \approx$$
$$\approx (x_2 * x_4) * (x_2 * x_4) \approx 1,$$
$$T_3(x_1,x_2,x_3,x_4,1,1,1,1) \approx (T(x_1,x_2,1,1) \wedge T(x_3,x_4,1,1)) * (x_2 \wedge x_4) \approx$$
$$\approx (x_2 \wedge x_4) * (x_2 \wedge x_4) \approx 1.$$

□

Now, we define closedness with respect to partial ideal terms.

DEFINITION 6.34
A *subset* A of a skew Hilbert algebra $\mathbf{S} = (S, \leq, *, 1)$ is said to be *closed* with respect to the partial ideal terms $T_i(x_1,\dots,x_n,y_1,\dots\dots,y_m)$, $i \in I$, if for every $i \in I$, all $x_1,\dots,x_n \in S$ and all $y_1,\dots,y_m \in A$ we have that $T_i(x_1,\dots,x_n,y_1,\dots,y_m)$ is defined and $T_i(x_1,\dots,x_n,y_1,\dots,y_m) \in A$.

Although our ideal terms are only partial, we can prove that every subset of a strong skew Hilbert algebra $\mathbf{S}$ closed with respect to them is really a filter of $\mathbf{S}$.

THEOREM 6.35
Let $\mathbf{S} = (S, \leq, *, 1)$ be a strong skew Hilbert algebra and F a subset of S that is closed with respect to the partial ideal terms T_1, T_2, T_3 listed in Lemma 6.33. Then $F \in \mathrm{Fil}\mathbf{S}$.

PROOF. We have $1 = T_1 \in F$. Now, assume $a,b,c,d \in S$ and $a * b, b * a, c * d, d * c \in F$. Then,

$$(a * c) * (b * d) = (T(a,b,a * b,b * a) * T(c,d,c * d,d * c)) * (b * d) =$$
$$= T_2(a,b,c,d,a * b,b * a,c * d,d * c) \in F.$$

Moreover, if a and c are comparable with each other and b and d are comparable with each other then we apply the partial term T_3 to derive

$$\min(a,c) * \min(b,d) = (T(a,b,a * b,b * a) \wedge T(c,d,c * d,d * c)) * (b \wedge d) =$$
$$= T_3(a,b,c,d,a * b,b * a,c * d,d * c) \in F.$$

This shows $F \in \mathrm{Fil}\mathbf{S}$. □

REMARK 6.36
Let us consider the partial term $T(x,y,z,u) := U(x,y) \wedge (z * y) \wedge u$ defined before Lemma 6.32 and put

$$T_4(x,y,z) := T(x,z,x * y,y * x),$$

i.e.

$$T_4(x,y,z) = U(x,z) \wedge ((x * y) * z) \wedge (y * x).$$

Of course, this is only a partial term because the infimum in T_4 need not exist for some elements from a skew Hilbert algebra $(S, \leq, *, 1)$. It is of some interest that in the case of strong skew Hilbert algebras this partial term behaves like a Maltsev term. Namely, we can easily compute

$$T_4(x,x,z) = U(x,z) \wedge ((x*x)*z) \wedge (x*x) = U(x,z) \wedge (1*z) \wedge 1 = U(x,z) \wedge z = z,$$

$$T_4(x,z,z) = U(x,z) \wedge ((x*z)*z) \wedge (z*x) = (U(z,x) \wedge (z*x)) \wedge ((x*z)*z) =$$
$$= x \wedge ((x*z)*z) = x.$$

Moreover, these expressions $T_4(x,x,z)$ and $T_4(x,z,z)$ are defined for all $x,z \in S$.

For every lattice skew Hilbert algebra $\mathbf{L} = (L, \vee, \wedge, *, 1)$ and every $M \subseteq L$, let $F(M)$ denote the filter of $\mathbf{L}$ generated by M.

The connection between filters generated by a certain subset and congruences on lattice skew Hilbert algebras is described in the following proposition.

PROPOSITION 6.37
Let $(L, \vee, \wedge, *, 1)$ be a lattice skew Hilbert algebra, $M \subseteq L$ and $a \in L$. Then,

$$\Phi(F(M)) = \Theta(M \times \{1\}),$$
$$[1](\Theta(M \times \{1\})) = F(M).$$

In particular,

$$\Phi(F(a)) = \Theta(a, 1),$$
$$[1](\Theta(a, 1)) = F(a).$$

PROOF. Since $M \times \{1\} \subseteq \Phi(F(M))$, we have

$$\Theta(M \times \{1\}) \subseteq \Phi(F(M)).$$

Hence, one has

$$[1](\Theta(M \times \{1\})) \subseteq [1](\Phi(F(M))) = F(M),$$

by Corollary 6.5. Because of $M \subseteq [1](\Theta(M \times \{1\}))$, we have

$$F(M) \subseteq [1](\Theta(M \times \{1\})),$$

and hence,

$$\Phi(F(M)) \subseteq \Phi([1](\Theta(M \times \{1\}))) = \Theta(M \times \{1\}),$$

again by Corollary 6.5. □

An analogous result holds for strong skew Hilbert algebras.

7 Conclusion and future research

In this paper, the class of skew Hilbert algebras has been introduced with the aim of including within a unified landscape of several classes of structures arising as algebras of (even non-distributive) logics. In particular, we have shown that orthomodular implication algebras, generalized orthomodular lattices, lattices with sectional antitone involutions (basic algebras) and their subvarieties can be regarded as (lattice) skew Hilbert algebras once a subreduct containing → and 1 is taken into account. Several results concerning special subsets of elements in Hilbert algebras have been generalized to our framework. Finally, the structure theory of skew Hilbert algebras has been outlined showing that many characterizing features of the aforementioned algebras rest on very minimal requirements. However, as it happens, this paper raises more problems than it solves.

Recall that a *skew Heyting algebra* [14] is a co-strongly distributive skew lattice $(A, \wedge, \vee, 1)$ whose principal sections $\uparrow u$ can be equipped with a binary operation $\rightarrow_u$ such that $(\uparrow u, \wedge, \vee, \rightarrow_u, u, 1)$ is a Heyting algebra. The natural question arises if connections between skew Heyting algebras and skew Hilbert algebras exist. Indeed, it is easily seen that any skew Heyting algebra $\mathbf{A}$ forms a poset with sectional Brouwerian pseudocomplements by setting $x^y = (y \vee x \vee y) \rightarrow_y y$, for any $x, y \in A$. Moreover, for any skew Heyting algebra $\mathbf{A}$, the quotient $\mathbf{A}/\mathcal{D}$ (where $\mathcal{D}$ is Green's congruence over $\mathbf{A}$) is a generalized Heyting algebra and so it can be *a fortiori* regarded as a lattice skew Hilbert algebra. However, there are skew Heyting algebras $\mathbf{A}$ whose $\rightarrow$ operation does not induce a skew Hilbert algebra as the next example shows.

EXAMPLE 7.1
Let us consider the skew Heyting algebra $\mathbf{A}$ described by the following Cayley tables and Hasse diagram:

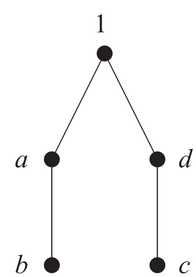

FIGURE 8

$\rightarrow$	a	1	b	c	d
a	1	1	b	b	a
1	a	1	b	c	d
b	1	1	1	a	a
c	a	1	a	1	1
d	a	1	b	b	1

$\wedge$	a	1	b	c	d
a	a	a	b	b	a
1	a	1	b	c	d
b	b	b	b	b	b
c	c	c	c	c	c
d	d	d	c	c	d

$\vee$	a	1	b	c	d
a	a	1	a	d	d
1	1	1	1	1	1
b	a	1	b	c	d
c	a	1	b	c	d
d	a	1	a	d	d

Note that $L(U(a,d), a \rightarrow d) = \{a, b\}$ but $L(d) = \{d, c\}$.

Therefore, a further research task will be investigating the relationship between skew Heyting algebras and skew Hilbert algebras.

Finally, as it has been pointed out in Section 3, skew Hilbert algebras need not be lattice ordered. Therefore, a natural question rises: is any (strong) skew Hilbert algebra embeddable into a lattice ordered one? If not, under which condition(s) does it hold? Moreover, can any skew Hilbert algebra be *join*- and *meet*-densely embedded into a complete lattice skew Hilbert algebra? Put another way, does any skew Hilbert algebra have a MacNeille completion? If not, under which conditions does it hold? We leave the development of this stream of research to future inquiries.

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