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This is the Authors' [*accepted*] manuscript version of the following contribution:

[Andrea Loi and Stefano Matta, Robustness of Statistical Manifolds, Topology and Its Applications, volume 327, year 2023, 108438]

The publisher's version is available at:

<https://doi.org/10.1016/j.topol.2023.108438>

When citing, please refer to the published version.

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ROBUSTNESS OF STATISTICAL MANIFOLDS

ANDREA LOI AND STEFANO MATTA

ABSTRACT. A statistical structure (g, T) on a smooth manifold M induced by $(\tilde{M}, \tilde{g}, \tilde{T})$ is said to be *robust* if there exists an open neighborhood of (g, T) in the fine C^∞ -topology consisting of statistical structures induced by $(\tilde{M}, \tilde{g}, \tilde{T})$. Using Nash–Gromov implicit function theorem, we show robustness of the generic statistical structure induced on M by the standard linear statistical structure on \mathbb{R}^N , for N sufficiently large.

Keywords: Statistical manifolds, statistical models, isostatistical maps, free statistical maps, robustness, Nash-Gromov implicit function theorem.

Subj. Class: 53B12, 53C05, 53C42, 58C15.

1. INTRODUCTION

The concept of statistical manifold [10] provides an intrinsic approach and a useful abstraction to encompass various concepts and results in information geometry. A statistical manifold is a manifold M endowed with a statistical structure (g, T) , where g is a Riemannian metric and T is a 3-symmetric tensor, which generalize the Fisher metric and the Amari-Chentsov tensor, respectively [10].

Recently, [11] has positively addressed a question raised by [10] on whether a statistical manifold (M, g, T) is a statistical model, i.e. a smoothly parametrized family of probability measures on some sample space Ω , $\mathcal{P}(\Omega)$, whose parameters belong to M . The answer has been provided by [11] showing the existence of an immersion of any statistical manifold in some $\mathcal{P}(\Omega)$, which preserves the statistical structure. More precisely (see [11, 4]), any statistical manifold admits an isostatistical embedding in $\mathcal{P}(\Omega)$ endowed with the statistical structure represented by the Fisher metric and the Amari-Chentsov tensor.

We recall that an immersion $h : (M, g, T) \rightarrow (\tilde{M}, \tilde{g}, \tilde{T})$ is isostatistical if it preserves the statistical structure, i.e. $f^*\tilde{g} = g$ and $f^*\tilde{T} = T$. The statistical structure (g, T) on M is then said to be *statistically induced* by $(\tilde{M}, \tilde{g}, \tilde{T})$. Hence it follows from this definition that a probability density for the structure (\tilde{g}, \tilde{T}) , $p : \Omega \times \tilde{M} \rightarrow \mathbb{R}$, induces a probability density for (g, T) . Observe that, as highlighted by [4], this immersion, being metric and tensor preserving, can be seen as an “intrinsic counterpart” of sufficient statistic.

In this paper we follow this intrinsic approach. Our aim is to study the robustness property of the class of statistical structures $\{(g, T)\}$ on a manifold M , which are statistically induced by (\tilde{g}, \tilde{T}) , the statistical structure of a manifold \tilde{M} . We provide the following definition of robustness.

Definition 1. *A statistical structure (g, T) on a smooth manifold M induced by $(\tilde{M}, \tilde{g}, \tilde{T})$ is said to be robust if there exists an open neighborhood of (g, T) in the fine C^∞ -topology consisting of statistical structures induced by $(\tilde{M}, \tilde{g}, \tilde{T})$.*

The first author was supported by INdAM. GNSAGA - Gruppo Nazionale per le Strutture Algebriche, Geometriche e le loro Applicazioni. Both authors were supported by STAGE - Funded by Fondazione di Sardegna.

We think that this investigation is either natural, since the function space $\{(g, T)\}$ can be equipped with the fine (Whitney) C^∞ -topology, which coincides with the ordinary C^∞ -topology if M is compact, either (hopefully) interesting.

The tool used in our analysis, Nash-Gromov implicit function theorem [12, 8, 9], highlights a case of special interest, i.e. when $(\tilde{M}, \tilde{g}, \tilde{T})$ is the standard linear statistical manifold, namely $\tilde{M} = \mathbb{R}^N$, $\tilde{g} = g_{can} = \sum_i^N dx_i^2$ and $\tilde{T} = T_{can} = \sum_i^N dx_i^3$. Hence, in the sequel, by a N -induced statistical structure we will mean a statistical structure on M induced by $(\mathbb{R}^N, g_{can}, T_{can})$.

The main result of the present paper is the following theorem, which shows that, for N -induced statistical structures, robustness generically holds true, if N is assumed to be sufficiently large. In other words, the space of robust N -induced statistical structures is dense in the space of N -induced statistical structures.

Theorem 1. *Let M be a smooth n -dimensional manifold and let (g_0, T_0) be a N -induced statistical structure on M . Assume $N \geq \frac{n(n^2+9n+20)}{6}$. Then (g_0, T_0) can be approximated in the Whitney C^∞ -topology by robust N -induced statistical structures.*

The reader may notice that our result can be interpreted as a (local) variant of the celebrated Nash's isometric immersion theorem [12], which says that every n -dimensional Riemannian manifold M^n can be isometrically embedded in some $\mathbb{R}^{N(n)}$ endowed with the flat metric. Indeed, the above statement is weaker than Nash's because there is an obstruction stemming from the invariance of the norm of the 3-symmetric tensor which prevents a straightforward generalization of Nash's theorem. For example, $(S(2)_+^n, g_+, T_+)$, the n -dimensional positive upper sphere of radius 2 endowed with the metric $g_{can|_{S(2)_+^n}}$ and the tensor $\sum_{i=1}^n \frac{dx_i^3}{x_i}$ on $|_{S(2)_+^n}$, which identifies the space of all positive probability measures on a sample space of $n+1$ elementary events endowed with the Fisher metric and the Amari-Chentsov tensor, does not admit any isostatistical immersion on $(\mathbb{R}^N, g_{can}, T_{can})$, even if T_{can} is multiplied by a positive constant (the reader is referred to [4, Sec 4.5.2] for obstructions for the existence of an isostatistical immersion between statistical manifolds). The fact that $S(2)_+^n$ is not compact plays a crucial role. In fact, [11] proves that any n -dimensional compact statistical manifold (M, g, T) , can be isostatistically embedded into $(\mathbb{R}^N, g_{can}, aT_{can})$, for a suitable $a > 0$ and a sufficiently large N .

In this paper we are not assuming any topological assumption on M and, moreover, we are dealing with the standard 3-symmetric tensor T_{can} and not with its multiples.

The proof of Theorem 1 is based on Nash's implicit function theorem for infinitesimally invertible differential operators. Roughly speaking, the idea of the proof of Theorem 1 is as follows. Since (g_0, T_0) is N -induced, then there exists a smooth immersion $f_0 : M \rightarrow \mathbb{R}^N$ such that $f_0^*g_{can} = g_0$ and $f_0^*T_{can} = T_0$. The strategy is to show that the linearization L_{can} of the smooth operator \mathcal{D}_{can} , which assigns to each smooth immersion $f : M \rightarrow \mathbb{R}^N$ the induced statistical structure $(g, T) = (f^*g_{can}, f^*T_{can})$ on M , can be infinitesimally inverted.

This paper is organized as follows. In Section 2 we derive the linearization formula for the differential operator \mathcal{D}_{can} which corresponds to the statistical structures under study. In Section 3, after introducing and discussing the notion of free statistical maps, which is relevant to define the class of maps where Gromov's technique is applicable, namely where the linearization is invertible, we prove Theorem 1.

2. THE OPERATOR \mathcal{D}_{can} AND ITS LINEARIZATION L_{can}

Our study of statistically inducing maps follows the same approach and uses the same terminology as in [8], where the reader is referred to for a general discussion on induced geometric structures developed in the context of Nash's immersion theory. The key tool is a Nash-type implicit function theorem proved by Gromov for a special class of differential operators (see Section 2.3.1 in [8] for its various formulations and refinements). A general criterion for the validity of the Nash-Gromov implicit function theorem is the *infinitesimal invertibility* of the relevant differential operator and, in fact, we will work it out explicitly for the inducing differential operator \mathcal{D}_{can} , namely the operator which assigns to each smooth immersion the induced statistical structure $(g, T) = (f^*g_{can}, f^*T_{can})$ on M for the fixed pair (g_{can}, T_{can}) on \mathbb{R}^N . More precisely, the operator $\mathcal{D}_{can} : \{f\} \rightarrow \{(g, T)\}$ is a differential operator between the space $\{f\}$ of smooth immersions $M \rightarrow \mathbb{R}^N$ and the space of statistical structures $\{(g, T)\}$ on M , namely the set of pairs (g, T) where g is a Riemannian metric on M and T is a symmetric 3-tensor on M . We equip these spaces with the fine C^∞ -topology.

Observe that Riemannian metrics g (resp. symmetric 3-tensors T) on M are viewed as smooth sections $g : M \rightarrow S^2(M)$ (resp. $T : M \rightarrow S^3(M)$) where $S^2(M)$ (resp. $S^3(M)$) denotes the symmetric square (resp. the symmetric cube) of the cotangent bundle of M . This allows us to interpret our pair of structures (metric, 3-tensor) = (g, T) as elements of the space $\Gamma(S^2(M) \oplus S^3(M))$ of smooth sections of the smooth bundle $S^2(M) \oplus S^3(M)$. Thus, the space $\{(g, T)\}$ of statistical structures of M can be seen as a proper subspace of the space $\Gamma(S^2(M) \oplus S^3(M))$ (it is a proper subspace since a Riemannian metric g on M viewed as a smooth section of $S^2(M)$ is positive definite, while a generic section of $S^2(M)$ is just a symmetric 2-tensor).

The linearization of the operator \mathcal{D}_{can}

Here we construct the linearization of the operator \mathcal{D}_{can} . In easy terms, this linearization, denoted by L_{can} , is the differential of \mathcal{D}_{can} at $f \in \{f\}$ and so it is a linear operator from the tangent space $T_f\{f\}$ to the tangent space $T_{(f^*g_{can}, f^*T_{can})}\{(g, T)\}$. It is worth pointing out that these tangent spaces have to be considered as formal structures since the spaces involved are not smooth manifolds in the usual sense. More precisely, a map $f : (-\varepsilon, \varepsilon) \rightarrow \{(g, T)\} \subset \Gamma(S^2(M) \oplus S^3(M))$ is by definition a *smooth curve* if $f(t)(x) = F(x, t)$, for a smooth map $F : M \times (-\varepsilon, \varepsilon) \rightarrow S^2(M) \oplus S^3(M)$ and, for $(g_0, T_0) \in \{(g, T)\}$, the tangent space $T_{(g_0, T_0)}\{(g, T)\}$ is, by definition, the set of all velocity vectors $f'(0)$ of smooth curves $f : (-\varepsilon, \varepsilon) \rightarrow \{(g, T)\}$ starting at (g_0, T_0) , i.e. $f(0) = (g_0, T_0)$.

Observe that, due to the above splitting $S^2(M) \oplus S^3(M)$, one can decompose the operator \mathcal{D}_{can} into the sum of two operators,

$$\mathcal{D}_{can} = \mathcal{D}_{g_{can}} \oplus \mathcal{D}_{T_{can}} : \{f\} \rightarrow \{(g, T)\},$$

where, for a given smooth immersion $f : M \rightarrow \mathbb{R}^N$,

$$\mathcal{D}_{g_{can}}(f) := f^*g_{can} = g$$

and

$$\mathcal{D}_{T_{can}}(f) := f^*T_{can} = T.$$

We start by analyzing the linearization of $\mathcal{D}_{g_{can}}$ and $\mathcal{D}_{T_{can}}$. Although, by the previous decomposition, we can analyze the linearization of these two components separately, in the following, for the resolution of the system (4)+(5), we should consider them jointly as they depend on the same argument f .

The linearization of the operator $\mathcal{D}_{g_{can}}$

Our first operator $\mathcal{D}_{g_{can}}$ in a neighborhood of $x \in M$ equipped with local coordinates x_1, \dots, x_n , can be expressed by

$$\mathcal{D}_{g_{can}}(f) = \{g_{ij} = g_{can}(f_i, f_j)\}, \quad i, j = 1, \dots, n,$$

where $f_i = df(\frac{\partial}{\partial x_i})$, $i = 1, \dots, n$, denote the images of the vector fields $\frac{\partial}{\partial x_i}$ on M under the differential of f and where g_{ij} are the components of the metric $g = f^*g_{can}$ in our local coordinates.

The linearization of the operator $\mathcal{D}_{g_{can}}$ at f is the linear operator

$$L_{g_{can}} : C^\infty(M, \mathbb{R}^N) \rightarrow S^2(M),$$

assigning to each vector field y on \mathbb{R}^N along $f(M)$ a quadratic form g on M . We take a smooth one-parameter family of smooth maps $f_t : M \rightarrow \mathbb{R}^N$, $t \in (-\varepsilon, \varepsilon)$, such that $f_0 = f$ and $\frac{df_t}{dt}|_{t=0} = y$ for a given $y : M \rightarrow \mathbb{R}^N$ (here $f_t(x) = F(x, t)$, where $F : M \times (-\varepsilon, \varepsilon) \rightarrow \mathbb{R}^N$ is a smooth map).

Set $y_i = \frac{\partial y}{\partial x_i}$, $i = 1, \dots, n$. Then (compare either [8, 2.3.1] or [12]) the expression for $L_{g_{can}}(y) = \frac{d}{dt}\mathcal{D}_{g_{can}}(f_t)|_{t=0}$ in local coordinates x_1, \dots, x_n is as follows:

$$y \mapsto g_{can}(f_i, y_j) + g_{can}(f_j, y_i), \quad i, j = 1, \dots, n. \quad (1)$$

The linearization of the operator $\mathcal{D}_{T_{can}}$

The second operator $\mathcal{D}_{T_{can}}$ reads, in local coordinates, x_1, \dots, x_n , as

$$\mathcal{D}_{T_{can}}(f) = \{T_{ijk} = T_{can}(f_i, f_j, f_k)\}, \quad i, j, k = 1, \dots, n,$$

where T_{ijk} are the components of the 3-symmetric tensor $T = f^*T_{can}$ in our local coordinates. The linearization of the operator $\mathcal{D}_{T_{can}}$ at f is the linear operator

$$L_{T_{can}} : C^\infty(M, \mathbb{R}^N) \rightarrow S^3(M).$$

As before, we take a smooth 1-parametric family of maps $f_t : M \rightarrow \mathbb{R}^N$, $t \in (-\varepsilon, \varepsilon)$ such that $f_0 = f$ and $\frac{df_t}{dt}|_{t=0} = y$ for a given $y : M \rightarrow \mathbb{R}^N$. Then (cf. [8, 3.1.4]) $L_{T_{can}}(y) = \frac{d}{dt}\mathcal{D}_{T_{can}}(f_t)|_{t=0}$ is given by:

$$y \mapsto T_{can}(y_i, f_j, f_k) + T_{can}(f_i, y_j, f_k) + T_{can}(f_i, f_j, y_k), \quad i, j, k = 1, \dots, n. \quad (2)$$

The inversion of the operator $L_{can} = (L_{g_{can}}, L_{T_{can}})$

To (locally) invert the operator \mathcal{D}_{can} , we invert its linearization $L_{can} = (L_{g_{can}}, L_{T_{can}})$. This amounts to solving the equation

$$L_{can}(y) = (L_{g_{can}}(y), L_{T_{can}}(y)) = (g', T') \quad (3)$$

where the right-hand side (g', T') consists of an arbitrary quadratic 2-tensor g' on M and an arbitrary 3-tensor T' on M , respectively. In view of (1) and (2), we express (3) by the following system of p.d.e. in the unknowns y :

$$g_{can}(f_i, y_j) + g_{can}(f_j, y_i) = g'_{ij} \quad (4)$$

$$T_{can}(y_i, f_j, f_k) + T_{can}(f_i, y_j, f_k) + T_{can}(f_i, f_j, y_k) = T'_{ijk}, \quad (5)$$

where g'_{ij} and T'_{ijk} , $i, j, k = 1, \dots, n$, are smooth functions on M representing, in the local coordinates x_i , the components of g' and T' , respectively.

Next, we impose the following two technical conditions for the field y (see [5] and [12]), namely

$$g_{can}(f_i, y) = 0, \quad i = 1, \dots, n, \quad (6)$$

and

$$T_{can}(f_j, f_k, y) = 0, \quad j, k = 1, \dots, n, \quad (7)$$

These conditions are imposed in order to transform the system (4)-(5) into the system (12)-(13) which does not involve the derivatives y_i of the unknown vector field y .

By differentiating (6) and alternating the index i and j , the system (4) together with the extra-condition (6) becomes equivalent to:

$$g_{can}(f_{ij}, y) = -\frac{1}{2}g'_{ij}, \quad g_{can}(f_i, y) = 0, \quad i, j = 1, \dots, n \quad (8)$$

where $f_{ij} = \partial_i \partial_j f$. On the other hand, if we differentiate (7), we get

$$T_{can}(f_j, f_k, y_i) = -T_{can}(f_{ij}, f_k, y) - T_{can}(f_j, f_{ik}, y), \quad i, j, k = 1, \dots, n.$$

Therefore the system (5) with the conditions (7) is equivalent to

$$\begin{aligned} T_{can}(f_i, f_{jk}, y) + T_{can}(f_j, f_{ik}, y) + T_{can}(f_k, f_{ij}, y) &= \\ = -\frac{1}{2}T'_{ijk}, \quad T_{can}(f_j, f_k, y) = 0, \quad i, j, k = 1, \dots, n \end{aligned} \quad (9)$$

Notice now that since $T_{can} = \sum_{i=1}^n dx_i^3$ and $g_{can} = \sum_{i=1}^n dx_i^2$, one gets

$$T_{can}(u, v, w) = g_{can}(u \odot v, w), \quad \forall u, v, w \in \mathbb{R}^N, \quad (10)$$

where

$$u \odot v := (u_1 v_1, \dots, u_N v_N), \quad (11)$$

for $u = (u_1, \dots, u_N)$ and $v = (v_1, \dots, v_N)$. Notice that (11) is well-defined because it is expressed in terms of the global basis of vector fields on \mathbb{R}^N associated with Cartesian coordinates.

Therefore, by (8) and (9), the issue of infinitesimally inverting the operator \mathcal{D}_{can} is reduced to find the solution y of the following system (12)+(13):

$$g_{can}(f_{ij}, y) = -\frac{1}{2}g'_{ij}, \quad g_{can}(f_i, y) = 0, \quad i \leq j, \quad (12)$$

$$g_{can}(f_i \odot f_{jk} + f_j \odot f_{ik} + f_k \odot f_{ij}, y) = -\frac{1}{2}T'_{ijk}, \quad g_{can}(f_j \odot f_k, y) = 0, \quad i \leq j \leq k, \quad (13)$$

in the unknown field y . For each $x \in M$, this system is an *algebraic* system consisting of

$$m_n := n + 2s_n + \binom{n+2}{3} = \frac{n(n^2 + 9n + 14)}{6} \quad (14)$$

equations, where $s_n := \frac{n(n+1)}{2}$. Notice that every solution of the system (12)+(13) also gives a solution of the original linearized system (4)+(5) with the extra conditions (6) and (7).

3. FREE STATISTICAL MAPS AND THE PROOF OF THEOREM 1

The previous discussion enables us to see how the linearization of the operator L_{can} , expressed by the system (12)+(13) (and the consequent infinitesimal invertibility of the differential operator \mathcal{D}_{can}) can be used for obtaining our desired result (Theorem 1). The key step is to show that the operator \mathcal{D}_{can} , which associates to each immersion $f : M \rightarrow \mathbb{R}^N$ the induced statistical structure (f^*g_{can}, f^*T_{can}) , is an open map on a dense subset in the space of maps. We call these maps, which satisfy a certain regularity condition, *free statistical maps* (see Definition 2 below). Our proof follows the line of reasoning of Theorem 0.4.A in [6] and Theorem 1.1 in [7]. In fact, both papers follow

the same pattern of the case of Riemannian isometric immersions (see [12] and also [8]), where the relevant regularity condition is *freedom* of the involved map $f : M \rightarrow \mathbb{R}^N$, i.e. linear independence of the $n + \frac{n(n+1)}{2}$ vectors of the first and second partial derivatives of f (see Remark 3 below).

Definition 2 (Free statistical maps). *Let $f : M \rightarrow \mathbb{R}^N$ be a smooth map and fix local coordinates x_1, \dots, x_n around a point $x \in M$ and denote by f_i and f_{ij} , $i, j = 1, \dots, n$ the first and second derivatives of the map f with respect to these coordinates. The map $f : M \rightarrow \mathbb{R}^N$ is called a free statistical map if, for all $x \in M$, the m_n (see (14)) vectors*

$$\{f_i(x), f_{ij}(x), f_j(x) \odot f_k(x), f_i(x) \odot f_{jk}(x) + f_j(x) \odot f_{ik}(x) + f_k(x) \odot f_{ij}(x)\} \quad (15)$$

are linear independent, for every $x \in M$ and for all $i \leq j \leq k$.

REMARK 2. It is not hard to see that Definition 2 does not depend on the choice of local coordinates.

REMARK 3. Let $f : M \rightarrow \mathbb{R}^N$ be a smooth map. Denote by $T_f^1(x) \subset T_f^2(x) \subset \mathbb{R}^N$ the first and second osculating space respectively of the map f at the given point $x \in M$. Namely, $T_f^1(x) = df_x(T_x M)$ and $T_f^2(x) \subset \mathbb{R}^N$ is the subspace spanned by $f_i(x)$ and $f_{ij}(x)$, $i, j = 1, \dots, n$, at x . Then the dimension of $T_f^2(x)$ can vary between 0 and $\min(N, n + s_n)$, for $s_n = \frac{n(n+1)}{2}$ and the map f is *free* in the sense of Nash if $\dim T_f^1(x) = n = \dim M$, $\dim T_f^2(x) = \frac{n(n+3)}{2} = n + s_n$ or, equivalently, the $n + s_n$ vectors $\{f_i(x), f_{ij}(x)\}$ are linear independent, for every $x \in M$ and for all $i \leq j$.

In Gromov's terminology (see [8, 3.1.4]), a smooth map $f : M \rightarrow \mathbb{R}^N$ is called T_{can} -free if the $s_n + \binom{n+2}{3}$ vectors

$$\{f_i(x) \odot f_{jk}(x) + f_j(x) \odot f_{ik}(x) + f_k(x) \odot f_{ij}(x), f_j(x) \odot f_k(x)\}$$

are linear independent, for every $x \in M$. and for all $i \leq j \leq k$. Hence our definition of free statistical map extends both Nash's freedom and Gromov's T_{can} -freedom conditions.

EXAMPLE 4. When $M = \mathbb{R}^n$ it is not hard to see that the map $f : \mathbb{R}^n \rightarrow \mathbb{R}^{m_n}$ given by

$$(x_1, 2x_1, \dots, x_n, 2x_n, \{x_j x_k\}_{j \leq k}, \{x_j + x_k\}_{j < k}, \{x_p + x_q^2\}_{p, q=1, \dots, n}, \{x_a + x_b x_c\}_{a < b < c})$$

(where the strings are ordered in lexicographic order) is a free statistical map.

In the following proposition we prove that the operator

$$(d\mathcal{D}_{can})_f = L_{can} : T_f\{f\} \rightarrow T_{(f^*g_{can}, f^*T_{can})}\{(g, T)\}$$

is invertible if f is free statistical.

Proposition 5. *Let $f : M \rightarrow \mathbb{R}^N$ be a free statistical map. Then the linear operator L_{can} is invertible over all of M by some differential operator M_f , i.e. $L_{can} \circ M_f = id$.*

Proof. It follows from Section 2 that we need to find a solution y of the system of equations (12)+(13). Since the map $f : M \rightarrow \mathbb{R}^N$ is free statistical, it follows that the solution of (12)+(13) forms an affine bundle over M of rank $N - m_n$. Now, every affine bundle admits a section over M . To choose it in a canonical way, one may use any fixed auxiliary Riemannian metric on \mathbb{R}^N (e.g., we can use g_{can}) and then take as canonical solution, say y_0 , the solution y of (12)+(13) which has the minimal norm with respect to this metric at every point $f(x) \in \mathbb{R}^N$ (see, e.g., [8], [9], [12]). Finally, we define the inverse M_f of L_{can} by $M_f(g', T') = y_0$. \square

To make sure that the results we get are non-empty, we show the following:

Proposition 6. For $N \geq \frac{n(n^2+9n+20)}{6}$, generic maps $f : M \rightarrow \mathbb{R}^N$ are free statistical.

Proof. We shall interpret *non free statistical condition* as a singularity in the space $J^2(M, \mathbb{R}^N)$ of 2-jets of our maps $M \rightarrow \mathbb{R}^N$, so that we can use an argument based on *Thom's transversality theorem*. Recall that the 2-jet, $J_f^2(x)$, of a given smooth map $f : M \rightarrow \mathbb{R}^N$ at the point x is given by:

$$J_f^2(x) = (x, f(x), Df_x, D^2f_x),$$

where $Df_x : T_x M \rightarrow T_{f(x)} \mathbb{R}^N = \mathbb{R}^N$ (resp. $D^2f_x : S^2(T_x M) \rightarrow T_{f(x)} \mathbb{R}^N = \mathbb{R}^N$) is the first (resp. second) derivative of f at x , and where $S^2(T_x M)$ denotes the symmetric square of $T_x M$. For fixed $x \in M$ consider the set

$$J_x^2 = \{(x, y, \alpha, \beta) \mid y \in \mathbb{R}^N, \alpha \in \text{Hom}(T_x M, T_y \mathbb{R}^N), \beta \in \text{Hom}(S^2(T_x M), T_y \mathbb{R}^N)\}$$

and the 2-jet bundle $J^2(M, \mathbb{R}^N) = \bigsqcup_{x \in M} J_x^2$. Then $J^2(M, \mathbb{R}^N)$ inherits the structure of smooth bundle over M with fibers J_x^2 and natural projection

$$J^2(M, \mathbb{R}^N) \rightarrow M, (x, y, \alpha, \beta) \mapsto x.$$

Thus, using the 2-jets of a smooth function $f : M \rightarrow \mathbb{R}^N$ one can construct the smooth section of this bundle, namely the smooth map

$$J_f^2 : M \rightarrow J^2(M, \mathbb{R}^N), x \mapsto J_f^2(x).$$

If we fix local coordinates x_1, \dots, x_n around $x \in M$, then the 2-jet $J_f^2(x)$ of a given map $f : M \rightarrow \mathbb{R}^N$ at the point x is given by the first and second derivatives

$$J_f^2(x) = (x, f(x), f_i(x), f_{ij}(x)), \quad i, j = 1, \dots, n.$$

We also notice that the *non free statistical regularity* at $x \in M$ depends on $J_f^2(x)$ and hence we can define the set $\Sigma_x \subset J_x^2$ consisting of 2-jets of *non free statistical maps*. Let $M(m_n, N)$ be the set of $m_n \times N$ matrices with real entries, where m_n is defined by (14). Then it follows by Definition 2 that Σ_x can be identified with the matrices of $M(m_n, N)$ of rank strictly less than m_n . Thus (cf., e.g., [2] or [8, Section 1.3.1]) $\Sigma_x \subset M(m_n, N)$ is a Whitney stratified space of codimension $N - m_n + 1$. Therefore the set $\Sigma = \cup_{x \in M} \Sigma_x \subset J^2(M, \mathbb{R}^N)$, which fibers over M , is a Whitney stratified space of codimension $N - m_n + 1$. Now, by the very definition of Σ , it follows that a map $f : M \rightarrow \mathbb{R}^N$ is free statistical iff $J_f^2(M) \subset J^2(M, \mathbb{R}^N)$ does not meet Σ . Finally, (the special case of) *Thom's transversality theorem* (see, e.g. [8] Corollary D' , p. 33) tells us that generic maps do have the property $J_f^2(M) \cap \Sigma = \emptyset$ if $N - m_n + 1 \geq n + 1$ or equivalently $N \geq \frac{n(n^2+9n+20)}{6}$. \square

Proof of Theorem 1 By assumption, $(g_0 = f_0^* g_{can}, T_0 = f_0^* T_{can})$ for a smooth map $f_0 : M \rightarrow \mathbb{R}^N$, $N \geq \frac{n(n^2+9n+20)}{6}$. Then, by Proposition 6, there exists a free statistical map, say $f_1 : M \rightarrow \mathbb{R}^N$, which is arbitrarily C^∞ -close to the map f_0 . It follows that the induced statistical structure $\mathcal{D}_{can}(f_1) = (g_1 = f_1^* g_{can}, T_1 = f_1^* T_{can})$ is C^∞ -close to (g_0, T_0) . It remains to prove that (g_1, T_1) is robust. We know by Proposition 5 that the linearization of the operator \mathcal{D}_{can} at f_1 admits an inverse (or, using the terminology in [8], that the operator \mathcal{D}_{can} is infinitesimally invertible at f_1). This allows us to apply the Nash-Gromov's implicit function theorem to deduce that \mathcal{D}_{can} is an open operator from a neighborhood of f_1 to a neighborhood \mathcal{U} of $\mathcal{D}_{can}(f_1)$. Therefore all the statistical structures (g, T) in \mathcal{U} are N -induced and this concludes the proof of Theorem 1. \square

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