PROSPECTIVE UNIVERSITY STUDENTS IN MATHEMATICS REFLECTING ON UNCERTAINTY: RESULTS AND COMPARISONS

Francesco Beccuti

Università di Pavia

This paper investigates the written reflections of prospective undergraduate students in mathematics on a simple problem involving coin tosses. I analyze these in terms of recency and equiprobability effects understood non-normatively, observing in particular a major tendency of this type of students to equiprobability answers. The comparison of the present results with analogous results obtained in previous studies points to the fact that the influence of a university education in mathematics on the equiprobability effect is overall limited. It thus follows that the tendency to equiprobability answers of university students in mathematics is most likely acquired during previous compulsory education.

INTRODUCTION

The study of the recency and equiprobability effects constitutes a common ground of research for the psychology of mathematics and for mathematics education. These effects have been usually problematized when the respondents' answers to questionnaires or problems were deemed to be undesirable or wrong and thus often termed "biases" or "fallacies" (cf., e.g., Morsanyi et al., 2009; Chernoff & Sriraman, 2020; Batanero, 2020). For instance, in a seminal paper, Fischbein and Schnarch considered the following question.

> When tossing a coin, there are two possible outcomes: either heads or tails. Ronni flipped a coin three times and in all cases heads came up. Ronni intends to flip the coin again. What is the chance of getting heads the fourth time? (Fischbein & Schnarch, 1997, p. 98)

These authors regarded the answer "equal to the chance of getting tails" to be correct. They further regarded the answer "smaller than the chance of getting tails" to be evidence of a negative recency bias, while they deemed the answer "greater than the chance of getting tails" to be evidence of a positive recency bias. As Beccuti and Robutti noticed, however,

> nothing in the statement of Fischbein and Schnarch's word problem as reported suggests that the hypothetical coin tossed by Ronni has to be considered a *fair* coin or that the way in which Ronni tosses the coin is not biased towards heads. As Gigerenzer (1991, 1996) has argued, probabilistic word-problems usually do not have only one correct answer over which there exists unquestioned consensus. It is true that often people's answers deviate problematically from the generally accepted norm. However, this discrepancy could be caused by the respondents' divergent interpretation of the situation presented to them (Chiesi & Primi, 2009, p. 152). (Beccuti & Robutti, 2022, pp. 1-2, emphasis in the original)

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Indeed, among others, Sharma (2008) already evidenced the crucial role of the wording of these types of problems as well as the connected plausibility of students' realistic considerations when discussing them. In view of this, Beccuti and Robutti, building on Rubel (2007), evidenced the need to go beyond usual research methods based on multiple-choice questionnaires when investigating recency and equiprobability. They thus suggested a method to classify people's reflections which allows to understand possible divergent interpretations of such problems as well as to evidence the problematicity of answers which are usually deemed to be normatively correct. In order to do this, these authors pointed out the need to employ the following non-normative definitions of recency and equiprobability.

> The *positive recency effect* is the tendency to interpret the manifestation of some event as evidence that the same event is likely to happen again in the future. On the other hand, the *negative recency effect* is the tendency to interpret the manifestation of some event as evidence that the same event is less likely to happen again in the future. Moreover, the *equiprobability effect* is the tendency to judge a set of events as all equally likely to happen. (Beccuti & Robutti, 2022, p. 1, emphasis in the original)

Notice that these definitions deviate from the usual definitions of these effects given by researchers solely in the fact that they do not depend on whether the behaviors they describe are at odds with the usual normative interpretation of the involved problems: i.e., these definitions are non-normative. Hence, they are simply more general than the usual normative definitions, in the sense that the category of phenomena or behaviors they subsume includes as a subgroup the category of phenomena usually subsumed by normative definitions (cf., e.g., Chiesi & Primi, 2009; Gauvrit & Morsanyi, 2014; Morsanyi & Szucs, 2014).

These non-normative definitions were employed by Beccuti and Robutti to analyze the reflections of master's students in mathematics on a problem involving coin tosses analogous to the one employed Fischbein and Schnarch. In the present paper, I will thus employ the same methodology in order to analyze the reflections of a comparable population of prospective undergraduate students in mathematics on the same problem. I thus aim at investigating the following first research question with respect to such problem: *how do prospective undergraduate students in mathematics reason about uncertainty?* Moreover, a comparison of the results of the present study with those obtained by Beccuti and Robutti will also allow me to investigate the following related second research question: *what is the influence of a university education in mathematics on students' reasoning about uncertainty?*

SUMMARY OF PREVIOUS RESEARCH

Batanero and colleagues (1996) tested secondary school students with problems involving the throw of dice and spinners, finding that non-normative equiprobability answers are given less by younger students, while normative equiprobability answers are given more by older children. These authors argued that their results could be related to the participants' exposure to formal education. The aforementioned study of Fischbein and Schnarch found that the negative recency bias decreases with age (and thus with degree of formal education), while overall the positive recency bias is almost negligible after a certain age. Furthermore, these authors observed an increase with age of the equiprobability bias, which they conjectured could be explained by increased exposure to formal education in probability. Rubel (2007) investigated secondary students' responses on problems involving coin tosses and further analysed the participants' justifications for their answers. She did not observe a correlation between age and the equiprobability bias or the recency biases. This fact, together with the fact that the students involved in her study were subjected only to a limited exposure to instruction in probability, appears to evidence that the prevalence of equiprobability answers is caused by formal education in probability rather than age.

Chiesi and Primi (2009) found (testing a problem involving drawing from a bag with replacement on primary school children and on undergraduate university students) that the positive recency bias decreases with age whereas they found no age-related differences for the negative recency bias. As to university students in particular, Chiesi and Primi observed a noteworthy manifestation of the negative recency bias as well as a less prominent manifestation of the equiprobability bias. A seminal cross-educational and cross-national study (concerning the answers of university students on various problems involving uncertainty) by Morsanyi and colleagues (2009) established a correlation between the equiprobability bias and formal education in probability and statistics. More recently, as mentioned, Beccuti and Robutti (2022) studied the reflections of master's students in mathematics on a problem involving coin tosses. By employing the non-normative definitions quoted in the previous section, these authors found manifestations of the positive recency effect while they observed an almost negligible negative recency effect. Most importantly, they observed a remarkable predominance of the equiprobability effect (which they further nuanced according to whether their participants problematized equiprobability or assumed it without questioning). These authors also argued that their participants' answers were connected to their formal education in probability and statistics.

The present study thus aims to extend the research on the equiprobability and recency effects on a type of students which appears to not have been studied before by previous literature: prospective undergraduate students in mathematics (first research question). Furthermore, the present study aims to put the results evidenced by Beccuti and Robutti under a new light and establish by comparison whether these were ascribable to their participants' university education in mathematics (second research question).

THE PRESENT STUDY

Participants

The participants of the present study are one cohort of 81 students (38 males and 43 females of median age 19) enrolled in the course "Introduction to mathematics" at the Department of Mathematics of the University of Turin, Italy (i.e., the same university and department where the aforementioned study of Beccuti and Robutti took place). The course is a preliminary non-compulsory course (which was lectured in presence by the author of the present paper) taking place before the formal beginning of the undergraduate program in pure and applied mathematics. The course aims at reviewing basic mathematical concepts and techniques that are deemed by the faculty to be important to be mastered by the students before entering the program. Typically, the students had just completed their secondary education and were thus enrolled for the first time in a university degree.

Procedure

The following experiment was performed by testing the students with a procedure involving a short computer-based questionnaire administered in the first day of the course. The questionnaire was divided into two tasks which the students were instructed to address individually as part of a larger written assignment which included a selection of mathematical problems and tasks (aimed to evaluate the students' preliminary mathematical competences). As to the present questionnaire, each participant was presented with the following multiple-choice question.

> **Task 1.** Sara tosses for ten times a coin and for ten times she obtains heads. Sara then asks Luca to bet on the outcome of the next toss. Luca bets on the next toss resulting in heads again. Do you agree with Luca's choice?
> [Possible answers:] *Yes*; *No*; *In part*; *I am not sure*.

As soon as the students submitted their answers to the first task, the following related second task was immediately presented to them.

> **Task 2.** Explain your reasoning.

To complete this task, the students could type in the computer a text possibly containing mathematical symbols.

Explanation of the choices

Following Beccuti and Robutti (2022), I concentrate here on the students' written responses to Task 2, leaving an analysis of the responses to Task 1 (and of the relations between answers to Task 1 and 2) to a subsequent article. Except from minor details (e.g., the names of the characters of the fictional situation presented in Task 1), the wording of the tasks, the mode of administration as well as the procedure of analysis were chosen to be identical to that of Beccuti and Robutti (2022), in order to favor comparison in view of the second research question. Following these authors, the wordings of both tasks were selected in order to stimulate in the students the possibility of ample articulation of their reflections, in view of the first research question. In particular, as to Task 1, the range of possibilities of answers presented to the participants were selected with the aim to stimulate reflection over a decision problem rather than simply to confine the participants within the limited constraints of a yes/no or most-likely/least-likely answer (cf. Rubel, 2007). Similarly, the wording of Task 2 (and, crucially, the possibility to answer by submitting an open text) was chosen in order to give to the participants the amplest possibility to express their view over the fictional situation presented to them.

Method of analysis

The submitted answers to Task 2 were classified according to the deductive coding procedure (cf. Braun & Clarke, 2006) elaborated by Beccuti and Robutti (2022). Indeed, in view of the aforementioned definitions of recency and equiprobability, I classified each of the participants' submitted texts in relation to the conclusion of the argument that these texts presented as connected to the possibility of predicting the outcome of a hypothetical 11th coin toss in the fictional situation presented.

More specifically, a submitted text was classified as *Equiprobable tout court (Group A)* if it argued that the possible outcomes of a further hypothetical coin toss are equiprobable without any explicit doubt or reservation. Moreover, a text was classified as *Equiprobable with reservation (Group B)* if it argued that the outcomes are equiprobable but explicitly contained some form of doubt or reservation about this fact. Furthermore, a text was classified as *Heads is more likely (Group C)* or *Tails is more likely (Group D)* if it argued that outcome of an 11th coin toss is more probable or likely to result in heads or tails respectively. Finally, a text was classified as *Mixed (Group E)*, if the text was not unilaterally classifiable within any of the above groups (because it contained contrasting remarks without favoring any explicit conclusion).

RESULTS

Table 1 summarizes the participants' answers to Task 1 and Task 2.

	Group A	Group B	Group C	Group D	Group E	Empty	Total
Yes	2	1	5	0	2	2	12
No	12	13	0	4	0	0	29
In part	14	13	4	1	4	0	36
Not sure	0	3	1	0	0	0	4
Total	28	30	10	5	6	2	81

Table 1: Summary of the participants' answers to Task 1 and Task 2.

As said, I concentrate in the present paper on the students' answers to Task 2 (shown in the last row of Table 1). In the following subsections, I thus present a summary of each of the aforementioned groups of answers to Task 2 together with exemplifying extracts from the students' texts (translated as literally as possible from Italian).

Group A

Many of the students (28 participants) either simply affirm that heads or tails are equiprobable without doubt or reservation, or else they care to specify that the results of the previous tosses do not affect the outcome of the next toss.

> The coin landed 10 times on heads […] but this fact does not change the fact that, by performing a new toss, the coin may land on tails, since the probability is always 0.5.

Logically speaking, the independence of the events of heads and tails is assumed by these participants as the starting point of their reasoning. This is argued with reference to typical descriptions of sample spaces of idealized games involving coin tosses, or else it is simply stated as an unquestioned assumption.

Group B

A slightly more numerous group of students (30 participants) affirms the same conclusion of the previous group, but at the same time cares to specify that such conclusion depends on an unproven assumption: that the coin or the game is not biased or rigged.

> Getting tails or getting head should be the same […] if the coin is not biased.

Group C

Some of the students (10 participants) state that heads is more likely or probable in relation to the fact that there is evidence for deeming the coin to be loaded or the game to be unfair.

> Since the flip resulted ten times in heads, then probably Sara is cheating.

Group D

Few students (5 participants) argue that tails is more likely to be the outcome of a further hypothetical coin toss. These students argue by referring to mathematical principles (e.g., the law of large numbers) or else to arguments of a mathematical form (ultimately unsound).

> Tails is more probable, since the coin landed ten times on heads […] the law of large numbers make it so that at some point the coin has to land on tails.

Group E

A small number of students' answers (6 participants) were uncategorizable because they contained mixed conflicting statements and did not reach a conclusive decision on the fictional situation presented. 2 of these texts also contained reference to unsound mathematical arguments.

DISCUSSION

Overall, a comparatively small number of students manifests the positive recency effect (Group C), and an even smaller amount manifests negative recency (Group D). The largest portion of the participants manifests the equiprobability effect (Group A and Group B). With respect to Group A, in particular, unquestioned equiprobability answers appear to be stated in a similar fashion as those found in textbooks' formal presentations of idealized games involving idealized coins. Such answers can be deemed to be problematic and at odds with a full understanding of situations or decision problems in conditions of uncertainty (cf. Batanero, 2020, p. 685).

Beccuti and Robutti (2022), by using the same mode of administration and procedure of analysis of responses to Task 2, obtained comparable results from 84 master's students in mathematics enrolled at the same university (all holding a bachelor's degree in mathematics comprising at least a course in probability and statistics). These are reported in terms of percentages in Table 2 together with the present results (already appearing in terms of absolute numbers in the last row of Table 1).

	Group A	Group B	Group C	Group D	Group E	Empty
prospective bachelor's students (present study)	34.57%	37.04%	12.35%	6.17%	7.41%	2.47%
master's students (Beccuti & Robutti, 2022)	40.48%	35.71%	13.10%	3.57%	5.95%	1.19%

Table 2: Comparison of the results of Task 2 in the present study with those obtained by Beccuti and Robutti (2022, p. 5).

As we can see from Table 2, older and more mathematically-educated students are overall slightly more subject to the equiprobability effect. In particular, answers of equiprobability without reservation are more prevalent in older students, while equiprobability answers without reservation are slightly less prevalent. We can further observe a very small relative difference in positive recency answers and a relatively consistent difference in negative recency answers (as well as in mixed or empty answers).

Globally, however, the present results are not substantially dissimilar from those found by Beccuti and Robutti, especially in terms of the overall prevalence of the equiprobability effect. This points to the fact that the tendency to equiprobability answers is likely not acquired during a university education in mathematics but possibly during previous compulsory education. Nevertheless, the equiprobability effect (in its unrestricted form) may possibly be slightly exacerbated by a university education in mathematics. Similarly, this type of education appears to reduce the negative recency effect, while positive recency appears to remain almost unaffected.

Further studies involving different types of students as well as involving the same type of students tested with different problems (or with the same problem presented in altered situations) may serve to further elucidate the relationship of recency and equiprobability with previous education and thus contribute both to mathematics education research as well as to research in the psychology of mathematics.

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