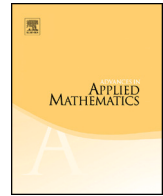




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R-hulloid of the vertices of a tetrahedron

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ABSTRACT

The R -hulloid, in the Euclidean space $\mathbb{R}^3$, of the set of vertices V of a tetrahedron T is the minimal closed set containing V such that its complement is the union of open balls of radius R . When R is greater than the circumradius of T , the boundary of the R -hulloid consists of V and possibly of four spherical subsets of well defined spheres of radius R through the vertices of T . The existence of a value R^* such that these subsets collapse into a point O^* , in the interior of T , is investigated; in such a case O^* belongs to four spheres of radius R^* , each one through three vertices of T and not containing the fourth one. As a consequence, the range of ρ such that V is a ρ -body is described completely. This work generalizes to dimension three previous results, proved in the planar case and related to the three circles Johnson's Theorem.

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1. Introduction

Let R be a positive real number. The R -hulloid of a closed set E in the Euclidean space $\mathbb{R}^d$, denoted by $\text{co}_R(E)$, is the minimal closed set containing E such that its complement $(\text{co}_R(E))^c$ is the union of open balls of radius R . More generally, if the complement of a body (*i.e.*, non empty closed set) E is the union of open balls of radius R , then E is called an R -body. This concept is a natural generalization of the convex hull $\text{co}(E)$ of a set E , in which the role of closed half-spaces is played by the closed complement of balls of equal radius.

The study of R -bodies and of R -huloids of finite sets or more generally, finite unions of compact sets has been studied before (for instance under the name of ϵ -convex sets [15]); random approximations by such sets have been considered in set estimation theory [14]. These sets find application in statistics, but also in other areas such as stereology or computational geometry (see [4] and references therein).

There has been much work regarding different generalizations of convexity that involve, instead of intersections of closed half-spaces, some different family $\mathcal{C}$ of subsets of the Euclidean space. The case when $\mathcal{C}$ is a family of complements to balls was investigated in [5]. The work [8] describes the case when $\mathcal{C}$ is a family of affine transformations of a given convex body. Finite intersections or unions of balls have been considered, for example, in the context of the Knese-Poulsen conjecture, see [2] and references therein.

Finally, when $\mathcal{C}$ is a family of unit balls, one obtains a special subclass of convex bodies, the so-called ball-bodies. Ball-bodies have been studied extensively, see, for example, [1] for a recent paper, and [17] for results on the class of floating bodies related to ball-bodies. A comprehensive survey and a list of references can be found in [2].

Main contributions and outline. In this paper we consider the situation where $E \subset \mathbb{R}^d$ is the set of vertices V of a simplex. The planar setting ($d = 2$) is by far more understood than the general d -dimensional case. When $E = V$ is the set of vertices of a triangle T , a complete description of $\text{co}_\rho(V)$ was proved in [9] as elementary consequence of the three circles Johnson's Theorem [6,7] (we recall this result from [9] in Proposition 2.12). Our main goal is to give a geometric description of $\text{co}_\rho(V)$, for all $\rho > 0$ in dimension $d \geq 3$ and in particular for $d = 3$.

The description of $\text{co}_\rho(V)$ looks similar to the planar case using special ρ -supporting spheres of V , of which a precise definition is given in Definition 2.6. An explicit representation of $\text{co}_\rho(V)$ is obtained in Theorem 3.5. The proof relies on non trivial geometric arguments that are valid in every dimension $d \geq 3$, where Johnson's Theorem cannot be applied.

The shape of $\text{co}_\rho(V)$ depends on a special value of a radius by $r_L(V)$ defined as the supremum of the set of ρ for which V is a ρ -body. This critical value is defined in Definition 4.5 as the infimum of the set of ρ such that $\text{co}_\rho(V)$ has not empty interior.

For the regular simplex $T = \text{co}(V)$ in dimension $d \geq 2$, of circumradius $r(V)$, it is proved in [9, Theorem 5.6] that $r_L(V) = \frac{d}{2}r(V)$; in such a case there exist $d + 1$ spheres

of radius $r_L(V)$, supporting V and intersecting at the center of T . The definitions and the results of Section 3 are obtained in any dimensions $d \geq 3$; however the aim of the paper is to restrict to the three dimensional case since there is a connection with a solved conjecture for unit-sphere systems in $\mathbb{R}^3$, see [12,13]. This fact gives us a sharp bound for $r_L(V)$ for well-centered tetrahedra in Theorem 4.10.

For this reason in Section 4.1 we restrict the attention to configurations of four spheres S_j in $\mathbb{R}^3$, of same radius R^L , so that S_j contains three vertices $V \setminus \{v_j\}$ of T , and such that the spheres intersect at a single point O^L . If O^L does not lie on the circumsphere of T , it is called a *four crossing point* of T ; when $R^L = r(V)$ this configuration of spheres including the circumsphere of T would be a unit-sphere system in $\mathbb{R}^3$ if they are all distinct, Definition 4.2, see [12] and Remark 4.3.

In Theorem 4.8, it is proved that there exist four special spheres R^* -supporting V and with a common intersection at one point $O^* \in \text{int}(T)$ if and only if $R^* := r_L(V) > r(V)$; moreover in this case $\text{co}_{R^*}(V)$ reduces to $V \cup \{O^*\}$. As a consequence of this fact, a complete description of $\text{co}_\rho(V)$ for all $\rho > 0$ is given in Corollary 4.9 for all tetrahedra T .

Interestingly, the fifth point O^* in the hulloid might not coincide with special central points of T , see Remark 4.11.

A condition for the uniqueness of the four-crossing radius and four crossing point is given in Theorem 4.12. Conversely, examples where R^L and O^L are not unique are obtained in [11] using methods from symbolic computation, for the family of triangular pyramids, see Remark 4.13.

2. Definitions and preliminaries on R-bodies

$\mathbb{R}^d$ is the euclidean space of dimension $d \geq 2$. The elements of $\mathbb{R}^d$ are called vectors. The origin of $\mathbb{R}^d$ is denoted by $o = (0, \dots, 0)$. The open ball of center $x \in \mathbb{R}^d$ and radius $r > 0$ is denoted by $B(x, r)$, its boundary is the sphere $\partial B(x, r)$; the unit sphere $\partial B(o, 1) \subset \mathbb{R}^d$ is often denoted by S^{d-1} . In case R is a fixed positive real number, let us simply denote by $B(x)$ the open ball of radius R and center x , and by B any open ball of radius R . A body is a closed non empty set. The closure of a set A is $\text{cl}(A)$, its interior is $\text{int}(A)$ and its complement A^c . The usual scalar product between vectors $u, v \in \mathbb{R}^d$ will be denoted by $\langle u, v \rangle$. A cone C , with vertex o , is a subset of $\mathbb{R}^d$ with the following property: $\forall x \in C$ and $\forall \lambda \geq 0$, then $\lambda x \in C$. A cone C is pointed if $C \cap (-C) = \{o\}$. The apex set of a closed cone C is $C \cap (-C)$ and it contains o .

Let A be a body and $q \in A$. The *tangent cone* of A at q is defined as:

$$\text{Tan}(A, q) = \{v \in \mathbb{R}^d : \forall \varepsilon > 0, \exists x \in A \cap B(q, \varepsilon), \exists r > 0 \text{ s.t. } |r(x - q) - v| < \varepsilon\}.$$

Let us recall that if $\text{Tan}(A, q) \neq \{o\}$ then

$$S^{d-1} \cap \text{Tan}(A, q) = \bigcap_{\varepsilon > 0} \text{cl} \left\{ \frac{x - q}{|x - q|} : x \in A \cap B(q, \varepsilon) \text{ and } x \neq q \right\}.$$

The *dual cone* of a cone K is $K^* = \{y \in \mathbb{R}^d : \langle y, x \rangle \geq 0, \forall x \in K\}$. The *normal cone* at q to A is the non empty closed convex cone, given by:

$$\text{Nor}(A, q) := \{u \in \mathbb{R}^d : \langle u, v \rangle \leq 0, \forall v \in \text{Tan}(A, q)\} = -\{\text{Tan}(A, q)\}^*.$$

Definition 2.1. A body $A \subset \mathbb{R}^d$ will be called an *R-body* if $\forall y \in A^c$ there exists an open ball $B \subset \mathbb{R}^d$ of radius R satisfying $y \in B \subset A^c$, that is if:

$$A = \bigcap_{B \cap A = \emptyset} B^c.$$

Definition 2.2. Let $E \subset \mathbb{R}^d$ be a body and let R be a positive real number. The set

$$\text{co}_R(E) := \bigcap_{B \cap E = \emptyset} B^c,$$

will be called the *R-hulloid* of E .

If the family of all open balls of radius R not intersecting E is empty, we assume $\text{co}_R(E) = \mathbb{R}^d$. The *R-hulloid* always exists and $\text{co}_R(E)$ is the minimal *R-body* containing E . Clearly every convex body E is an *R-body* (for all positive R), in such a case $E = \text{co}(E) = \text{co}_R(E)$ for every R . A body A is an *R-body* if and only if $A = \text{co}_R(A)$. Moreover, see [15, formula (8)]:

$$R_1 \leq R_2 \Rightarrow \text{co}_{R_1}(E) \subset \text{co}_{R_2}(E). \quad (1)$$

Proposition 2.3. [9, Thm 3.10] *The closed subsets of a sphere of radius r are R -bodies for all $R \leq r$.*

Let us look at special sets $E = V$, where V is the set of vertices of a simplex $T = \text{co}(V)$ in $\mathbb{R}^d$.

Definition 2.4. The following fact is well known: there exists a unique open ball $B(c(V), r(V))$ such that $V \subset \partial B(c(V), r(V))$; it is called the *circumball* to $T = \text{co}(V)$. Its radius $r(V)$ is the *circumradius* of T and its center is the *circumcenter* $c(V)$, denoted below simply by c . Let us recall that its closure $D(V) := \text{cl}(B(c, r(V)))$ does not coincide (in general) with the closed ball of minimum radius containing V .

The following proposition is a consequence of Proposition 2.3, with $r = r(V)$ and $R = \rho$:

Proposition 2.5. *Let $T = \text{co}(V)$ be a simplex in $\mathbb{R}^d$. Then $\text{co}_\rho(V) = V$ for all $\rho \leq r(V)$.*

Definition 2.6. ([10]) Let A be a closed set in $\mathbb{R}^d$ and $a \in \partial A$. Let $v \in \partial B(o, 1) = S^{d-1}$. We say that the ball $B(a + Rv)$ (of radius R) is R -supporting A at a if $A \subset (B(a + Rv))^c$. The sphere $\partial B(a + Rv)$ is called R -supporting A at a .

2.1. R -cones

Definition 2.7. Let $\rho > 0$ and $S = \partial B(o, \rho)$. There is a one-to-one correspondence between closed cones $K \subset \mathbb{R}^d$ and closed subsets $\mathcal{K} \subset S \subset \mathbb{R}^d$: for any closed cone K of $\mathbb{R}^d$, let $\mathcal{K} = K \cap S$; conversely, for any closed set $\mathcal{K} \subset S$, let $K = \{\lambda v : v \in \mathcal{K}, \lambda \geq 0\}$ be the related cone in $\mathbb{R}^d$.

In the previous definition $\partial B(o, \rho)$ can be replaced by a sphere S centered at any point $p \in \mathbb{R}^d$ and K by a cone with vertex p . Let us recall a family of R -bodies, called R -cones, see [9, §5]. The R -cones are a generalization of convex cones, as the R -bodies are a generalization of convex sets.

Definition 2.8. Let $\mathcal{K}$ be a body in S^{d-1} . An R -cone with vertex o (or simply R -cone) is the R -body:

$$C_{\mathcal{K}} := \bigcap_{v \in \mathcal{K}} (B(Rv, R))^c.$$

Similarly, for $x \in \mathbb{R}^d$, an R -cone with vertex x is the R -body: $C_{\mathcal{K}}^x := x + C_{\mathcal{K}} = \bigcap_{v \in \mathcal{K}} (B(x + Rv, R))^c$.

Proposition 2.9. [9, Theorem 5.6] Let $\mathcal{K}$ be a body in S^{d-1} and K its related cone in $\mathbb{R}^d$. Then

- a) $\text{Tan}(C_{\mathcal{K}}, o) = -K^*$;
- b) $\text{Tan}(C_{\mathcal{K}}) \setminus \{o\} \subset \text{int}(C_{\mathcal{K}})$.

Lemma 2.10. Let $B_j \subset \mathbb{R}^d, j = 1, \dots, d$, be open balls of radius ρ , not necessarily distinct. Let $p^* \in \bigcap_{j=1}^d \partial B_j$; then for every neighborhood $\mathcal{U}$ of p^* , one has $\text{int}(\mathcal{U} \setminus \bigcup_{j=1}^d B_j) \neq \emptyset$.

Proof. Let u_j be the unit inner normal at p^* to B_j . Let $\mathcal{K} = \{u_1, u_2, \dots, u_d\} \subset S^{d-1}$. The closed set $\bigcap_{j=1}^d B_j^c$ is called ρ -cone with vertex p^* in [10]. The ρ -cone $C_{\mathcal{K}} = \bigcap_{j=1}^d B_j^c - p^*$ with vertex at $\{o\}$ is a generalization of the usual convex cone. Let $K = \{\lambda u_i \mid u_i \in \mathcal{K}, \lambda \geq 0\}$ be the related cone to $\mathcal{K}$. By properties a) and b) of Proposition 2.9

$$-K^* - \{o\} = \text{Tan}(C_{\mathcal{K}}) - \{o\} \subset \text{int}(C_{\mathcal{K}}) \quad (2)$$

where $\text{Tan}(C_{\mathcal{K}})$ is the tangent cone to the ρ -cone $C_{\mathcal{K}}$ at o . Since $\mathcal{K}$ is contained in a closed hemisphere, the dual cone K^* is not empty and the tangent cone is non empty too.

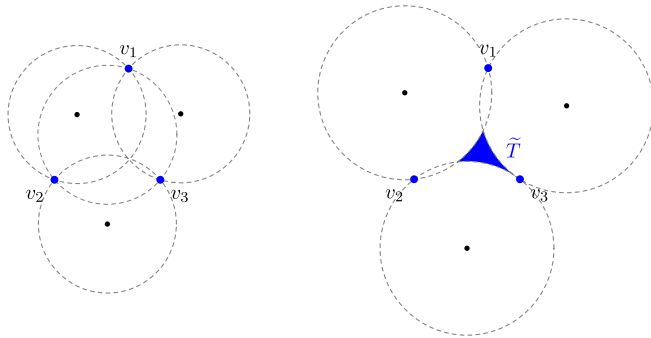


Fig. 1. ρ -hull of three points in $\mathbb{R}^2$, $\rho = r(V)$ (left) and $\rho > r(V)$ (right).

Then, by inclusion (2), for every $v \neq o$ in the tangent cone: $v \in \text{int}(C_{\mathcal{K}})$. Let $y = p^* + v$, then $y \in \text{int}((\cup_{j=1}^d B_j)^c) \neq \emptyset$ and

$$\text{dist}(y, \cup_{j=1}^d B_j) > 0; \quad (3)$$

the thesis follows. $\square$

Lemma 2.11. *Let $B_j \subset \mathbb{R}^d$, $j = 1, \dots, d$, be open balls of radius ρ . Let $p^* \in \cap_{j=1}^d \partial B_j$; let $C_{\mathcal{K}}^{p^*} = \cap_{j=1}^d B_j^c$ be the related ρ -cone with vertex p^* and let $\mathcal{T}$ be a closed convex cone with vertex p^* . Then*

- a) *if $\text{Tan}(\mathcal{T}, p^*) \cap \text{Tan}(C_{\mathcal{K}}^{p^*}, p^*) \setminus \{o\} \neq \emptyset$ then, for every neighborhood $\mathcal{U}$ of p^* , there exists a point $y \in \mathcal{T} \cap \mathcal{U}$ such that*

$$\text{dist}(y, (C_{\mathcal{K}}^{p^*})^c) > 0; \quad (4)$$

- b) *if $\text{Tan}(\mathcal{T}, p^*) \cap \text{Tan}(C_{\mathcal{K}}^{p^*}, p^*) = \{o\}$, then p^* is an isolated point of $\mathcal{T} \cap C_{\mathcal{K}}^{p^*}$.*

Proof. It can be assumed that $p^* = o$. In case a), let r be a ray starting from o in the cone $\mathcal{T} \cap C_{\mathcal{K}}$. From (2) for every $y \in r$, $y \neq o$, the inequality (3) holds. Then for every neighborhood $\mathcal{U}$ of o and for every $y \in r \cap \mathcal{U}$, $y \neq o$ formula (4) holds.

In case b), by contradiction let $y_n \rightarrow o$, a sequence of points $y_n \in \mathcal{T} \cap C_{\mathcal{K}}$, $y_n \neq o$. Then up to a subsequence $v = \lim_n y_n / |y_n| \neq o$ belongs both to $\text{Tan}(\mathcal{T}) = \mathcal{T}$ and to $\text{Tan}(C_{\mathcal{K}})$; this is a contradiction. $\square$

2.2. Planar case

Let us recall the following result as consequence of Johnson's Theorem [6] (cf. Fig. 1).

Proposition 2.12. [9, Theorem 4.2] *Let $V \subset \mathbb{R}^2$ be the set of vertices of a triangle T with circumradius $r(V)$. If $\rho > r(V)$, then*

$$\text{co}_\rho(V) = V \cup \tilde{T},$$

where $\tilde{T} \subset T$ is the curvilinear triangle bordered by three arcs of circles of radius ρ , each one through two vertices of T . If T is right-angled or obtuse-angled then the vertex of the major angle of T is also a vertex of $\tilde{T}$.

3. R -huloids of the vertices of a simplex

Let $V = \{v_i, i = 1, \dots, d+1\} \subset \mathbb{R}^d$ be the set of vertices of a simplex $T = \text{co}(V) \subset \mathbb{R}^d$.

Let us notice that, by Proposition 2.5, $\text{co}_\rho(V) = V$ for $\rho \leq r(V)$.

From now on, let us assume $\rho > r(V)$ and let us denote by $c = c(V)$ the circumcenter of T , so that $B(c, r(V))$ is the circumball of T .

Let us denote by $V_i = V \setminus \{v_i\}$, for $1 \leq i \leq d+1$, and by $T_i = \text{co}(V_i)$ the facets of T , by $H_i = \text{span}(T_i)$ and let c_i, r_i be the circumcenter and the circumradius respectively of T_i in H_i .

Definition 3.1. The simplex $T = \text{co}(V)$ will be called *well-centered* if $c(V) \in \text{int}(T)$ (in dimension two, the well-centered simplexes are the acute-angled triangles).

Let $D(V)$ be the closed circumball of T . The set $D(V) \cap H_i$ is a closed circle of radius r_i and center c_i . Let Ω_i be its relative interior.

As the vertices in V are affinely independent, then $v_i \notin H_i$, for all i . Let H_{i+} be the open half space bounded by H_i containing v_i , $H_i^+ = \text{cl}(H_{i+})$ and l_i the line orthogonal to H_i containing c_i .

Lemma 3.2. For every $\rho > r(V)$ and for any face T_i of $T = \text{co}(V)$ and line l_i orthogonal to H_i at the circumcenter c_i of T_i , there exists a unique point $o_i(\rho)$ such that $V_i \subset \partial B(o_i(\rho), \rho) \cap H_i$ and $v_i \notin B(o_i(\rho), \rho)$. Moreover $\rho \mapsto H_i^+ \cap (B(o_i(\rho), \rho))^c$ is a continuous map of strictly nested ρ -bodies for $\rho > r(V)$.

Proof. Let us denote $B_i(\rho) := B(o_i(\rho), \rho)$, $B(V) := \text{int}(D(V)) = B(c, r(V))$ and $B_i^+(r(V)) := \lim_{\rho \rightarrow r(V)^+} B_i(\rho)$. Let us notice that two different cases can occur:

- a) T is well-centered. In this case $\forall i$, $c \in H_{i+}$; $o_i(\rho)$ lies on the half line l_i^* of l_i , with origin in $\tilde{c}_i$, the symmetric point of c with respect to H_i , not intersecting H_i^+ . The half line l_i^* is oriented as the outwards unit vectors n_i to the facet T_i of T . In this case $B(V) \neq B_i^+(r(V)) = B(\tilde{c}_i, r(V))$ and

$$\lim_{\rho \rightarrow r(V)^+} H_i^+ \cap (B_i(\rho))^c = H_i^+ \cap (B_i^+(r(V)))^c;$$

- b) T is not well-centered: then, there are indices i such that the plane H_i separates c from v_i ; for these indices, $o_i(\rho)$ lies on the half line l_i^* of l_i starting from c and not

intersecting H_i^+ ; the half line l_i^* is again oriented as the outwards unit vectors n_i to the facet T_i of T . For such indices i it turns out that $B(V) = B_i^+(r(V))$, then

$$\lim_{\rho \rightarrow r(V)^+} H_i^+ \cap (B_i(\rho))^c = H_i^+ \cap (B(V))^c. \quad \square$$

Remark 3.3.

$$\bigcap_i \left(H_i^+ \cap (B(c, r(V)))^c \right) = \left(\bigcap_i H_i^+ \right) \cap \left(B(c, r(V)) \right)^c = T \cap \left(B(c, r(V)) \right)^c = V.$$

Lemma 3.4. *If $\rho > r(V)$, then every open ball $B(z, \rho)$, not intersecting V and intersecting $\text{co}(V)$, has the following properties:*

- a) $\text{dist}(z, c) > \rho - r(V)$;
- b) *the radical plane H through $\partial B(c, r(V)) \cap \partial B(z, \rho)$ separates z from T ;*
- c) $\text{dist}(z, T) > (\rho - r(V))/2$.

Proof. Since $\text{int}(T) \subset B(c, r(V))$, the spheres $\partial B(c, r(V))$ and $\partial B(z, \rho)$ have non empty intersection; thus they have a real radical plane H . Since every vertex v_i belongs to $\partial B(c, r(V)) \setminus B(z, \rho)$ then $\text{dist}(z, c) > \text{dist}(z, v_i) - \text{dist}(v_i, c) > \rho - r(V)$. Moreover H separates $D(V) \setminus B(z, \rho)$ from z , then it separates V from z , this proves b). Since $\rho > r(V)$, then H has distance from c less than from z , therefore $\text{dist}(z, T) > \text{dist}(z, H) > \text{dist}(z, (c+z)/2) > (\rho - r(V))/2$, and c) is proved. $\square$

Theorem 3.5. *Let $T = \text{co}(V)$ be a simplex in $\mathbb{R}^d$ and let $\rho > r(V)$, and $B_i(\rho)$ as in Lemma 3.2. Then*

$$\text{co}_\rho(V) = \text{co}(V) \setminus \left(\bigcup_{i=1}^{d+1} B_i(\rho) \right) \quad (5)$$

and $r(V) < \rho \mapsto \text{co}_\rho(V)$ is a not decreasing family of closed sets, with

$$\lim_{\rho \rightarrow +\infty} \text{co}_\rho(V) = V \cup \text{int}(T). \quad (6)$$

Proof. First let us notice that

$$G := \text{co}(V) \setminus \left(\bigcup_{i=1}^{d+1} B_i(\rho) \right) = \bigcap_i H_i^+ \cap \bigcap_i B_i^c(\rho).$$

Then G is a ρ -body as intersection of ρ -bodies, and $G \supset V$, thus $F := \text{co}_\rho(V) \subset G$. Let us prove now that $G \subset F$. This is equivalent to prove that

$$F^c \subset G^c. \quad (7)$$

First $V \subset F$, by definition of ρ -hulloid. Let $y \notin F$, then $y \notin V$. If $y \notin T$, then it is obvious that $y \notin G$; if $y \in \partial T \setminus V$, there exists a face T_i such that $y \in T_i \setminus V \subset B_i(\rho)$. Then, $y \in G^c$.

If $y \in \text{int}(T)$ and $y \notin F$, then by definition of hulloid there exists an open ball $\tilde{B}(z, \rho) \ni y$, with $\tilde{B}(z, \rho) \cap V = \emptyset$. Let us consider the family

$$\Gamma_y = \{\tilde{B}(x, \rho) : \tilde{B}(x, \rho) \ni y, \tilde{B}(x, \rho) \cap V = \emptyset\}.$$

Since Γ_y is not empty let

$$r_y = \inf_z \{|z - y| : y \in B(z, \rho) \in \Gamma_y\}.$$

Obviously $r_y < \rho$. From Lemma 3.4, $r_y > 0$; by continuity argument, r_y is a minimum and there exists $B(z^*, \rho) \in \Gamma_y$ such that $|z^* - y| = r_y$. Assume for the moment that $d = 3$.

Claim A: $\partial B(z^*, \rho) \cap V \neq \emptyset$.

By contradiction if $\partial B(z^*, \rho) \cap V = \emptyset$, moving z^* towards y , it is possible to get another ball $\tilde{B}(\tilde{x}, \rho) \in \Gamma_y$ such that

$$|\tilde{x} - y| < |z^* - y| = r_y,$$

which is impossible. Then Claim A is proved and there exists $v_3 \in \partial B(z^*, \rho) \cap V$. With a similar argument there exists at least another vertex $v_2 \neq v_3$, $v_2 \in \partial B(z^*, \rho)$, otherwise a rotation of $B(z^*, \rho)$ around v_3 towards y would decrease the value of r_y , which would contradict the definition of r_y .

Claim B: $\{v_1, v_2, v_3\} \subset \partial B(z^*, \rho)$.

First let us notice that, since $y \in \text{int}(T)$, it does not belong to the line through v_2, v_3 ; moreover $|z^* - v_3| = |z^* - v_2|$. Let $\mathcal{C}$ be the circle with center $(v_2 + v_3)/2$, through z^* , in the plane orthogonal to the axis $v_2 v_3$. Let Λ be the plane through y, v_2, v_3 and let z^+ be the closest point of $\mathcal{C} \cap \Lambda$ to y .

The function $z \rightarrow |z - y|$, such that $y \in B(z, \rho) \in \Gamma_y$, has minimum at z^* , then its restriction to $\mathcal{C}$ has the same minimum value at z^* . Since the function $\mathcal{C} \ni z \rightarrow |z - y|$, with no restriction on z , attains its minimum on z^+ and decreases its value moving z on $\mathcal{C}$ towards z^* ; then two cases need to be considered:

- a) $z^* = z^+$;
- b) there exists $v_1 \in \partial B(z^*, \rho) \cap V$.

Case a) cannot hold. Indeed, by item c) of Lemma 3.4, the radical plane of the two spheres $\partial B(z^+, \rho)$, $\partial B(c, r(V))$ (containing v_2, v_3) separates z^+ from y .

Then item b) holds: in such a case $B(z^*, \rho)$ cannot rotate towards z^+ , to get another $B(z, \rho) \in \Gamma_y$, $B(z, \rho) \ni y$, to decrease further the value of r_y . Then y belongs to the open

ball $B(z^*, \rho)$, which coincides with $B_4(\rho)$ of the family of balls defined in Lemma 3.2. Therefore $y \in G^c$. This proves (7) for $d = 3$.

A similar argument proves that (7) holds for any $d > 3$ and (5) is proved.

The monotone property of $\text{co}_\rho(V)$ follows from (1). Since, see [15],

$$\bigcup_{\rho > r(V)} \text{int}(\text{co}_\rho(V)) = \text{int}(T),$$

then the limit property (6) holds. $\square$

Corollary 3.6. *Let $B_i(\rho)$ and $B_i^+(r(V))$ be as in Lemma 3.2. If T is not well-centered, then*

$$\lim_{\rho \rightarrow r(V)^+} \text{co}_\rho(V) = V; \quad (8)$$

if T is well-centered, then

$$\lim_{\rho \rightarrow r(V)^+} \text{co}(V) \setminus \bigcup_{i=1}^{d+1} B_i(\rho) = \text{co}(V) \setminus \bigcup_{i=1}^{d+1} B_i^+(r(V)). \quad (9)$$

Proof. In case T is not well-centered, then there exists i such that H_i separates c from v_i , then

$$\lim_{\rho \rightarrow r(V)^+} B_i(\rho) = B(c, r(V)) \supset \text{co}(V) \setminus V$$

and (8) holds; the limit (9) follows from continuity and monotone properties of the map $\rho \mapsto \text{co}_\rho(V)$. $\square$

4. Critical R-hulloid of the vertices of a tetrahedron

In this section we focus on the three-dimensional case since there is a connection with a solved conjecture about unit-sphere-systems in $\mathbb{R}^3$ [12,13]. Let V be the set of vertices of a tetrahedron $T = \text{co}(V) \subset \mathbb{R}^3$.

First let us consider a more general question considered also in [11].

4.1. The four-crossing point of a tetrahedron

Definition 4.1. Given a tetrahedron T of vertices $V = \{v_i, 1 \leq i \leq 4\}$, let R^L be the radius of four distinct spheres $S_j, 1 \leq j \leq 4$, satisfying the following conditions, see Fig. 2:

- a) S_j contains the vertices v_i , for all $i \neq j$;

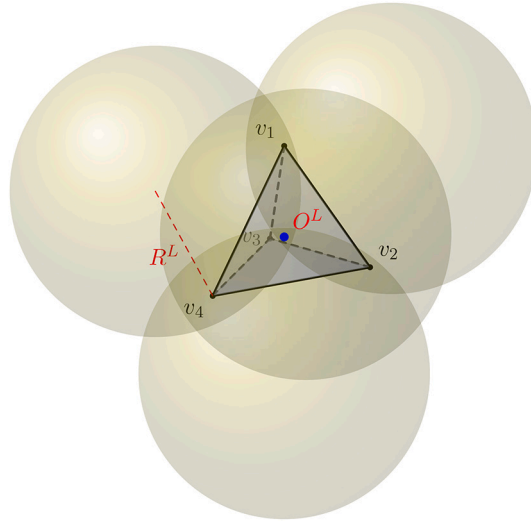


Fig. 2. A four-crossing radius and a four-crossing point of the regular tetrahedron.

- b) the intersection of the four spheres is one point $\{O^L\}$;
- c) O^L does not belong to the circumsphere of T .

If there exist R^L and O^L satisfying the previous conditions, they are called the *four-crossing radius* and the *four-crossing point of T* , respectively.

Note that the spheres S_j in Definition 4.1 are not necessarily supporting spheres of V . The method developed in [11], based on symbolic computation, allows to determine the values of R^L and O^L for the family of triangular pyramids.

The following example describes a special pyramid in $\mathbb{R}^3$ with a configuration of supporting spheres similar to that of the rectangular triangle in the plane.

Example 1 ([11, Example 5]). Let $T = \text{co}(V)$ be the triangular pyramid with apex v_1 , equidistant 1 from the vertices $V \setminus \{v_1\}$ of an equilateral base $T_1 = \text{co}(V \setminus \{v_1\})$, with sides length $\sqrt{12/5}$. Then there exists a unique set of four distinct spheres S_j of radius $\rho = r(V) = \sqrt{5/4}$, the circumradius of T , where S_1 is the circumsphere of V and S_2, S_3, S_4 satisfy the following properties:

1. S_j contains the vertices v_i , for all $i \neq j$;
2. $\cap_{j=2}^4 S_j = \{v_1\}$.

Let us notice that in this case property c) of Definition 4.1 is not satisfied and therefore, this configuration does not correspond to a four-crossing point of T . The existence of the four-crossing radius R^L and the four-crossing point O^L has some connections with the unit-sphere-systems in $\mathbb{R}^3$.

Definition 4.2 ([12]). A *unit-sphere-system* in $\mathbb{R}^d$ is a set of $d + 2$ distinct spheres in $\mathbb{R}^d$ of equal radius, with empty intersection, and such that each subgroup of $d + 1$ spheres has a common intersection.

Remark 4.3. It was proved in [13, Thm. 8.1] that there are no unit-sphere-systems for $d = 3$. This implies that R^L does not coincide with the circumradius $r(V)$ of a tetrahedron $T = \text{co}(V)$: indeed, if this was the case, the set of four spheres S_j supporting V , together with the circumsphere of T , would form a unit-sphere-system in $\mathbb{R}^3$.

4.2. R^* -hulloid of the vertices of a tetrahedron

Definition 4.4. Let $\rho > 0$. The ρ -hulloid of V is called *full* if $\text{int}(\text{co}_\rho(V)) \neq \emptyset$.

Definition 4.5. Let us define

$$r_L(V) = \inf\{\rho > r(V) : \text{co}_\rho(V) \text{ is full}\}.$$

Let us notice that, for $d = 2$ and $\rho > 0$, Propositions 2.5 and 2.12 imply that the set $\text{co}_\rho(V)$ is full if and only if $\rho > r(V)$.

Lemma 4.6. For $\rho > r(V)$, the points $o_i(\rho)$, $1 \leq i \leq 4$, defined by Lemma 3.2, are the vertices of a simplex W .

Proof. The points $o_i(\rho)$ are distinct. Otherwise if $o_i = o_j$ then $B_i(\rho) = B_j(\rho)$ would be an open ball of radius ρ containing all the vertices V on the boundary in contradiction with assumption $r(V) < \rho$. Moreover, see cases a) and b) in the proof of Lemma 3.2, by construction, each point o_i belongs to the corresponding half lines l_i^* contained in the lines l_i through the circumcenter c and orthogonal to the facet T_i . All distances of $o_i(\rho)$ from c are positive; the vectors from c to c_i are outwards normal vectors to the facet T_i of T . Then the simplex W , with vertex $o_i(\rho)$, contains, up to a translation and dilatation, the simplex T' whose vertices are the contact points of the inscribed sphere to T ; see also [3] for this simplicial vertex-normal duality. $\square$

Definition 4.7. Let us denote $r(\rho)$ the circumradius of the simplex W with circumcenter $c(W)$ and vertices $o_i(\rho)$, defined in Lemma 4.6.

Let us notice that, by definition of $c(W)$ and $r(\rho)$, the intersection of the spherical surfaces $\partial B(o_i(\rho), r(\rho))$ is $c(W)$.

Theorem 4.8. Let V be the set of vertices of a tetrahedron T . There exist $R^* > r(V)$ and a configuration of four distinct R^* -supporting spheres of V intersecting at some $O^* \in \text{int}(T)$ if and only if

$$r_L(V) > r(V). \quad (10)$$

In such a case $R^* = r_L(V)$ and

$$\text{co}_{R^*}(V) = V \cup \{O^*\}. \quad (11)$$

Proof. If (10) holds, let us consider, for all $\rho > r(V)$, the centers $o_i(\rho)$ of the balls $B_i(\rho)$, $i = 1, \dots, 4$, as in (5); let $r = r(\rho)$ be the circumradius of $W = \text{co}\{o_1(\rho), \dots, o_4(\rho)\}$, and let $c(W)$ be its circumcenter. Then for every i , the map $\rho \mapsto r(\rho) = |c(W) - o_i(\rho)|$ is continuous.

Let $R^* = r_L(V)$. Then one has

$$r(R^*) = \lim_{\rho \rightarrow R^*} r(\rho) \quad \text{and} \quad \text{co}_{R^*}(V) = \lim_{\rho \rightarrow (R^*)^+} \text{co}_\rho(V).$$

Let us prove now that

$$r(R^*) = R^*. \quad (12)$$

The set $T \setminus \cup_{i=1}^4 B_i(\rho)$ has non empty interior for $\rho > R^*$, and it is contained in a bounded set. Therefore there exists a sequence $\rho_n \rightarrow (R^*)^+$ and a point p^* such that

$$p^* \in \lim_n \text{int}(T \setminus \cup_{i=1}^4 B_i(\rho_n)). \quad (13)$$

Three cases could happen:

- i) $p^* \in \text{int}(T)$;
- ii) $p^* \in \partial T \setminus V$;
- iii) $p^* \in V$;

First, let us consider case i). Then there exists a neighborhood $\mathcal{U} \subset \text{int}(T)$ of p^* . In case $p^* \in \partial B_i(R^*)$ for every i , then p^* is the circumcenter O^* of W and (12) holds. Otherwise, by contradiction, if all four spheres do not cross at p^* , then there exists i and at most three spheres $\partial B_j(R^*)$, $j \neq i$, intersecting at p^* . Then, by Lemma 2.10, $\tilde{\mathcal{U}} = \mathcal{U} \cap \cap_{j \neq i} (B_j(R^*))^c$ has non empty interior. Since $\text{dist}(p^*, B_i(R^*)) > 0$ then $B_i(R^*)^c \cap \tilde{\mathcal{U}}$ has non empty interior. Then $T \setminus \cup_{j=1}^4 B_j(R^*)$ has non empty interior too. By continuity, for $\varepsilon > 0$ small enough, $T \setminus \cup_{j=1}^4 B_j(R_\varepsilon^*)$ has non empty interior, with $R_\varepsilon^* = R^* - \varepsilon < R^*$. Then $\text{co}_{R_\varepsilon^*}(V)$ is full, in contradiction with the definition of $r_L(V) = R^*$.

Let us prove now that case ii) does not occur. Since the distance of $T \setminus \cup_{i=1}^4 B_i(\rho_n)$ from every $o_i(\rho_n)$ is ρ_n , then, by (13), $\text{dist}(p^*, o_i(R^*)) \geq R^*$, $\forall i$. Obviously $p^* \in T$. Since for every q on the face $T_i \setminus V$, $\text{dist}(q, o_i(R^*)) < R^*$, then p^* does not belong to $T_i \setminus V$, for every i .

Let us prove that case *iii*) does not occur. By contradiction let $p^* = v_4 \in V$. If $p^* \in \partial B_4(R^*)$ then $\partial B_4(R^*)$ would be the circumsphere of T and $R^* = r(V)$, in contradiction with the assumption (10). Then $p^* \notin \partial B_4(R^*)$, therefore $\text{dist}(p^*, B_4(R^*)) > 0$.

Let us consider the R^* -hull of V , namely $\text{co}_{R^*}(V) = T \cap \bigcap_{i=1}^4 (B_i(R^*))^c$. Since $\text{dist}(p^*, B_4(R^*)) > 0$, there exists a neighborhood $\mathcal{U}$ of p^* , such that

$$\text{co}_{R^*}(V) \cap \mathcal{U} = \mathcal{U} \cap (T \cap \bigcap_{i=1}^3 B_i(R^*))^c = \mathcal{U} \cap (T \cap (C_{\mathcal{K}}^{p^*})),$$

where $C_{\mathcal{K}}^{p^*}$ is the R^* -cone of Lemma 2.11, with $\rho = R^*$.

Let us consider Lemma 2.11 with $\mathcal{T} := \text{Tan}(T, p^*)$.

In case *a*) of Lemma 2.11, for every neighborhood $\mathcal{U}$ of p^* , small enough, there exists $y \in \mathcal{T} \cap \mathcal{U} \subset T$, $y \in \text{int}(C_{\mathcal{K}}^{p^*})$. By continuity argument there exists R_ε , $r(V) < R_\varepsilon < R^*$, such that $y \in \text{int}(\bigcap_{i=1}^4 (B_i(R_\varepsilon))^c)$. Since every neighborhood of $y \in T$ has non empty intersection with $\text{int}(T)$, then

$$\text{int}(T) \cap \text{int}(\bigcap_{i=1}^4 (B_i(R_\varepsilon))^c) \neq \emptyset.$$

Then $\text{co}_{R_\varepsilon}(V) = T \cap \bigcap_{i=1}^4 (B_i(R_\varepsilon))^c$ has non empty interior and it is full. This is impossible by definition of $r_L(V) = R^*$.

In case *b*) of Lemma 2.11 the vertex p^* is an isolated point of $T \setminus \bigcup_{i=1}^4 B_i(R^*) = \text{co}_{R^*}(V)$. This is in contradiction with (13).

Then, only case *i*) holds and $p^* = O^* \in \text{int}(T)$ is the circumcenter of W . Let us prove now (11).

Since $O^* \in (\bigcup_{i=1}^4 B_i(R^*))^c \cap T$, by (5), it follows that $O^* \in \text{co}_{R^*}(V)$: then

$$V \cup \{O^*\} \subset \text{co}_{R^*}(V). \quad (14)$$

Let us consider the pyramid T_i^* with apex O^* and opposite face T_i . As $O^* \in \partial B(o_i(R^*), R^*)$, then $T_i^* \subset B(o_i(R^*), R^*) \cup V \cup \{O^*\}$; thus

$$\text{co}_{R^*}(V) = T \setminus \bigcup_i B(o_i(R^*), R^*) \subset \left(T \setminus \bigcup_i T_i^*\right) \cup V \cup \{O^*\} = V \cup \{O^*\}.$$

This and (14) prove (11) under the condition (10).

In case (10) does not hold, that is if $r_L(V) = r(V)$, then a configuration of four supporting spheres S_i of radius $r(V)$ intersecting at a point $O^* \in \text{int}(T)$, $O^* \notin S$, the circumsphere of T , cannot exist. By contradiction the collection $\{S, S_i, i = 1, \dots, 4\}$ would be a unit-sphere-system in $\mathbb{R}^3$. This is in contradiction with [13, Thm. 8.1], see Remark 4.3. $\square$

Corollary 4.9. *Let V be the set of vertices of a tetrahedron $T = T(V)$ in $\mathbb{R}^3$, and let $R^* = r_L(V)$. Then*

$$\begin{aligned}\mathrm{co}_\rho(V) &= V & \text{for } \rho < R^*; \\ \mathrm{co}_\rho(V) &= V \cup \tilde{\Gamma} & \text{for } R^* < \rho,\end{aligned}$$

where

$$\tilde{\Gamma} = \mathrm{int}(T) \setminus \bigcup_i B_i(\rho)$$

is a connected, not empty set, with $\partial\tilde{\Gamma}$ the union of connected subsets of $\partial B_i(\rho)$.

For $\rho = R^*$ it holds

$$\begin{aligned}\mathrm{co}_{R^*}(V) &= V \cup \{O^*\} & \text{if } r_L(V) > r(V), \\ \mathrm{co}_{R^*}(V) &= V & \text{if } r_L(V) = r(V).\end{aligned}$$

Theorem 4.10. *If T is well-centered then (10) and (11) hold.*

Proof. By contradiction, if (10) does not hold, then $r_L(V) = r(V)$ and for all $\rho_n > r(V)$ the set $T \setminus \bigcup_{i=1}^4 B_i(\rho)$ has non empty interior. Arguing as in the proof of Theorem 4.8, there exists a point p^* satisfying (13). Since (10) does not hold, Theorem 4.8 implies that *i*) can not occur; the same argument in the proof of Theorem 4.8 proves that case *ii*) cannot occur; since T is well-centered, every vertex v_i has a positive distance greater than $\delta(T) > 0$ from all balls $B(o_i(\rho_n), \rho_n)$. Then, p^* cannot be a vertex of V . Therefore *iii*) cannot occur, too. This is impossible. $\square$

Remark 4.11. For the trirectangular pyramid, see [11, Example 4], O^* differs in general from the orthocenter, and hence from the Monge point, of some triangular pyramid. It differs also from the circumcenter. In this example, condition (10) holds; therefore since the trirectangular pyramid is not well-centered this implies that condition (10) is not equivalent to the condition that T is well-centered. Note that in the plane the two conditions are equivalent.

Let us notice that the value $R^* = r_L(V)$, satisfying (10) and the point O^* , satisfying (11) in Theorem 4.8, are a four-crossing radius and a four-crossing point respectively of T , since $O^* \in \mathrm{int}(T) \subset B(V)$.

Theorem 4.12. *Let (R^L, O^L) be a four-crossing radius and a four-crossing point for a simplex $T = \mathrm{co}(V)$. If $O^L \in \mathrm{int}(T)$, then R^L and O^L are uniquely determined.*

Proof. Since $O^L \in \mathrm{int}(T)$ then $v_j \notin \mathrm{cl}(B_j)$; by Lemma 3.2 the family $\rho \mapsto H_j^+ \cap (B(o_j(\rho), \rho))^c$ is strictly nested. Then, B_j has radius R^L greater than $r(V)$. Therefore S_j is the boundary of an open ball B_j , which is R^L -supporting V at all vertices v_i with $i \neq j$. Then $R^L = r_L(V)$ of Definition 4.5 and satisfies (10) of Theorem 4.8. This implies that $R^L = R^*$ and $O^L = O^*$ defined by (11). $\square$

Remark 4.13. Without the condition $O^L \in \text{int}(T)$, the four-crossing radius and point of a tetrahedron are not uniquely determined, as proved in [11] for the class of triangular pyramids. For a regular tetrahedron there are seven different four-crossing points, one of which is the center of the tetrahedron, see Fig. 2 and [11, Example 1].

Conclusions

The ρ -hulloid $\text{co}_\rho(E)$ of a body E is a generalization of the convex hull of E , in which the support hyperplanes of E (resp. open half spaces not intersecting E), important tools of convex analysis, are replaced by spheres of equal radius ρ supporting E (resp. by open balls not intersecting E).

This paper provides a representation formula of $\text{co}_\rho(V)$ for V the set of vertices of a tetrahedron $T \subset \mathbb{R}^3$. This result generalizes a similar formula in $\mathbb{R}^2$ that derives from the three circles Johnson's Theorem [6], that cannot be applied in dimension $d > 2$. It is explicitly proved here in $\mathbb{R}^3$ for connections with some applications (for instance in graphical simulation of molecular models [16] or in image analysis [4]) and with a recent result on the non-existence of unit-sphere-systems in $\mathbb{R}^3$ [12,13].

Based on continuity arguments on the family of spheres of radius ρ supporting V , the construction of four spheres of radius R^* , supporting V and intersecting at a point O^* , is obtained for all tetrahedra T .

In this paper it is also proved that for every T , there exists R^* , greater or equal to the circumradius of T , such that the set $\text{co}_\rho(V)$ has non empty interior if and only if $\rho > R^*$; for $\rho > R^*$ the boundary of $\text{co}_\rho(V)$ consists of four spherical subsets of four spheres of radius R^* . For $\rho = R^*$ the set $\text{co}_\rho(V) \setminus V$ collapses in a point O^* of T . The value of R^* cannot attain the value of the circumradius of T when T is a well-centered tetrahedron. This is related to the above-mentioned results [12,13] and to explicit algebraic computations of R^* and O^* obtained for the class of triangular pyramids in [11].

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References

- [1] S. Artstein-Avidan, A. Chor, D. Florentin, A full classification of the isometries of the class of ball-bodies, *Bull. Lond. Math. Soc.* 57 (12) (2025) 3691–3698.
- [2] K. Bezdek, Z. Lángi, M. Naszódi, Selected topics from the theory of intersections of balls, *Discrete Appl. Math.* 382 (2026) 60–82.
- [3] J. Brandts, M. Krizek, Simplicial vertex-normal duality with applications to well-centered simplices, in: F.A. Radu, et al. (Eds.), *Proceedings of the 12th European Conference on Numerical Mathematics and Advanced Applications, ENUMATH 2017*, vols, Springer, Berlin, 2019, pp. 761–768.
- [4] A. Cuevas, R. Fraiman, B. Pateiro-López, On statistical properties of sets fulfilling rolling-type conditions, *Adv. App. Probab. (S.G.S.A.)* 44 (2012) 311–329.

- [5] V.P. Golubyatnikov, V.Yu. Rovenski, Some extensions of the class of k -convex bodies, *Sib. Math. J.* 60 (5) (2009) 820–829.
- [6] R.A. Johnson, A circle theorem, *Am. Math. Mon.* 23 (1916) 161–162.
- [7] R.A. Johnson, *Advanced Euclidean Geometry*, Dover Publication, 1960.
- [8] Z. Kabluchko, A. Marynych, I. Molchanov, Generalized convexity with respect to families of affine maps, *Isr. J. Math.* (2025) 1–45.
- [9] M. Longinetti, P. Manselli, A. Venturi, On the complements of union of open balls of fixed radius in the Euclidean space, *Matematiche* 1 (2023) 3–22.
- [10] M. Longinetti, P. Manselli, A. Venturi, On the complements of union of open ball and related set classes, arXiv:2406.15427, *Matematiche* (2026), in press.
- [11] M. Longinetti, S. Naldi, On the configurations of four spheres supporting the vertices of a tetrahedron, arXiv:2405.16167, submitted for publication.
- [12] H. Maehara, N. Tokushige, On a special arrangement of spheres, *Ryukyu Math. J.* 19 (2006) 15–24.
- [13] H. Maehara, N. Tokushige, From line-systems to sphere-systems – Schläfli’s double six, Lie’s line-sphere transformation, and Grace’s theorem, *Eur. J. Comb.* 30 (5) (2009) 1337–1351.
- [14] B. Pateiro López, Set estimation under convexity type restrictions, Doctoral thesis, University of Santiago de Compostela, 2008.
- [15] J. Perkal, Sur les ensembles ϵ -convexes, *Colloq. Math.* IV (1) (1956) 1–10.
- [16] C. Quan, B. Stamm, Mathematical analysis and calculation of molecular surfaces, *J. Comput. Phys.* 322 (2016) 760–782.
- [17] C. Schütt, E.M. Werner, D. Yalikum, Floating bodies for ball-convex bodies, *J. Lond. Math. Soc.* 112 (2025) 6.