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Properties of given and detected unbounded solutions to a class of chemotaxis models

Alessandro Columbu | **Silvia Frassu** | **Giuseppe Viglialoro**

Dipartimento di Matematica e Informatica, Università di Cagliari, Via Ospedale, Cagliari, Italy

Correspondence
Silvia Frassu, Dipartimento di Matematica e Informatica, Università di Cagliari, Via Ospedale 72, 09124. Cagliari, Italy.
Email: silvia.frassu@unica.it

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Abstract

This paper deals with unbounded solutions to a class of chemotaxis systems. In particular, for a rather general attraction–repulsion model, with nonlinear productions, diffusion, sensitivities, and logistic term, we detect Lebesgue spaces where given unbounded solutions also blow up in the corresponding norms of those spaces; subsequently, estimates for the blow-up time are established. Finally, for a simplified version of the model, some blow-up criteria are proved.

More precisely, we analyze a zero-flux chemotaxis system essentially described as

$$\begin{cases} u_t = \nabla\cdot((u+1)^{m_1-1}\nabla u - \chi u(u+1)^{m_2-1}\nabla v \\ \qquad + \xi u(u+1)^{m_3-1}\nabla w) + \lambda u - \mu u^k & \text{in } \Omega\times(0,T_{max}), \\ 0 = \Delta v - \frac{1}{|\Omega|}\int_\Omega u^\alpha + u^\alpha = \Delta w - \frac{1}{|\Omega|}\int_\Omega u^\beta + u^\beta & \text{in } \Omega\times(0,T_{max}). \end{cases} \tag{$\diamond$}$$

The problem is formulated in a bounded and smooth domain Ω of $\mathbb{R}^n$, with $n \geq 1$, for some $m_1, m_2, m_3 \in \mathbb{R}$, $\chi, \xi, \alpha, \beta, \lambda, \mu > 0$, $k > 1$, and with $T_{max} \in (0, \infty]$. A sufficiently regular initial data $u_0 \geq 0$ is also fixed.

Under specific relations involving the above parameters, one of these always requiring some largeness conditions on $m_2 + \alpha$,

(i) we prove that any given solution to ($\diamondsuit$), blowing up at some finite time T_{max} becomes also unbounded in $L^{\mathfrak{p}}(\Omega)$-norm, for all $\mathfrak{p} > \frac{n}{2}(m_2 - m_1 + \alpha)$;
(ii) we give lower bounds T (depending on $\int_\Omega u_0^{\bar{p}}$) of T_{max} for the aforementioned solutions in some $L^{\bar{p}}(\Omega)$-norm, being $\bar{p} = \bar{p}(n, m_1, m_2, m_3, \alpha, \beta) \geq \mathfrak{p}$;
(iii) whenever $m_2 = m_3$, we establish sufficient conditions on the parameters ensuring that for some u_0 solutions to ($\diamondsuit$) effectively are unbounded at some finite time.

Within the context of blow-up phenomena connected to problem ($\diamondsuit$), this research partially improves the analysis in Wang et al. (*J Math Anal Appl*. 2023;518(1):126679) and, moreover, contributes to enrich the level of knowledge on the topic.

KEYWORDS
attraction–repulsion, blow-up time, chemotaxis, lower bound, nonlinear production

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1 | INTRODUCTION, MOTIVATIONS, AND STATE OF THE ART

1.1 | The continuity equation: The initial-boundary value problem

The well-known continuity equation

$$u_t = \nabla \cdot F + h \tag{1}$$

describes the transport of some quantity $u = u(x, t)$, at the position x and at the time $t > 0$. In this equation, the flux F models the motion of such a quantity, whereas h is an additional source idealizing some external action by means of which u itself may be created or destroyed throughout the time.

In this paper, we are interested in the analysis of equation (1) in the context of self-organization mechanisms for biological populations, that is, phenomena for which organisms or entities direct their trajectory in response to one or more chemical stimuli. More precisely, we want to deal with the motion of a certain cell density $u = u(x,t)$ whose flux has a smooth diffusive part and another contrasting this spread. In the specific, this counterpart accounts of an attractive and a repulsive effect, associated with two chemical signals and indicated, respectively, with $v = v(x,t)$ and $w = w(x,t)$; v (the *chemoattractant*), tends to gather the cells, w (the *chemorepellent*) to scatter them. Additionally, an external source with an increasing and decreasing effect on the cell density is also included.

For our purposes, wanting to formulate what is said above in terms of the continuity equation (1), it appears meaningful (and convenient) defining for $\chi,\xi,\lambda,\mu > 0$, $m_1,m_2,m_3 \in \mathbb{R}$, and $k > 1$ the fluxes $F = F_{m_1,m_2,m_3} = F_{m_1,m_2,m_3}(u,v,w)$, $G_{m_1} = G_{m_1}(u)$, $H_{m_2} = H_{m_2}(u,v)$, $I_{m_3} = I_{m_3}(u,w)$, and the source $h = h_k = h_k(u)$:

$$\begin{cases} F = F_{m_1,m_2,m_3} = (u+1)^{m_1-1}\nabla u - \chi u(u+1)^{m_2-1}\nabla v + \xi u(u+1)^{m_3-1}\nabla w =: G_{m_1} + H_{m_2} + I_{m_3} \\ h = h_k = \lambda u - \mu u^k. \end{cases} \tag{2}$$

In this way, the diffusion is smoother and smoother for higher and higher values of m_1, the aggregation/repulsion effects $-\chi u(u+1)^{m_2-1}\nabla v/\xi u(u+1)^{m_3-1}\nabla w$ increase for larger sizes of χ and ξ and m_2 and m_3, and the cell density may increment with rate λu and may attenuate with rate $-\mu u^k$. Naturally, since the flux is influenced by the two signals v and w, we will have to consider two more equations (which for the time being we indicate with $P(v) = 0$ and $Q(w) = 0$, but that will be specified later), one for the chemoattractant v and another for the chemorepellent w, to be coupled with the continuity equation. Furthermore, the analysis is studied in impenetrable domains (so homogeneous Neumann or zero flux boundary conditions are imposed) and some initial configurations for the cell and chemical densities are assigned: essentially, with position (2) in mind, we are concerned with this initial boundary value problem:

$$\begin{cases} u_t = \nabla \cdot F_{m_1,m_2,m_3} + h_k & \text{in } \Omega \times (0,T_{max}), \\ P(v) = Q(w) = 0 & \text{in } \Omega \times (0,T_{max}), \\ u_0(x) = u(x,0) \geq 0;\ v_0(x) = v(x,0) \geq 0;\ w_0(x) = w(x,0) \geq 0 & x \in \bar{\Omega}, \\ u_\nu = v_\nu = w_\nu = 0 & \text{on } \partial\Omega \times (0,T_{max}). \end{cases} \tag{3}$$

The problem is formulated in a bounded and smooth domain Ω of $\mathbb{R}^n$, with $n \geq 1$, u_ν (and similarly for v_ν and w_ν) indicates the outward normal derivative of u on $\partial\Omega$. Moreover, $T_{max} \in (0,\infty]$ identifies the maximum time up to which solutions to the system can be extended.

1.2 | A view on the state of the art: The attractive and the repulsive models and the attraction–repulsion model

The aforementioned discussion finds, of course, its roots in the well-known Keller–Segel models idealizing chemotaxis phenomena (see the celebrated papers[1–3]), that since the last 50 years have been attracting the interest of the mathematical community.

In particular, if we refer to chemotaxis models with a single proliferation signal, taking in mind (2), problem (3) is a (more general) combination of this aggregative signal-production mechanism

$$\begin{aligned} u_t = \nabla \cdot (G_1 + H_1) = \Delta u - \chi \nabla \cdot (u\nabla v) \quad \text{and} \quad P(v) \\ = P_1^\tau(v) = \tau v_t - \Delta v + v - u = 0, \qquad \text{in } \Omega \times (0,T_{max}), \end{aligned} \tag{4}$$

and this repulsive signal-production one

$$u_t = \nabla \cdot (G_1 + I_1) = \Delta u + \xi \nabla \cdot (u \nabla w) \quad \text{and} \quad Q(w) = Q_1^{\tau}(w) = \tau w_t - \Delta w + w - u = 0, \qquad \text{in } \Omega \times (0, T_{max}). \tag{5}$$

(Here $\tau \in \{0, 1\}$, and it distinguishes between a stationary and evolutive equation for the chemical.) The above models present linear diffusion and linear production rates; specifically, v and w are linearly produced by the cells themselves, and their mechanism is opposite when in $P_1^{\tau}(v)$ the term $v - u$ (or in $Q_1(w)$ the term $w - u$) is replaced by uv (or uw); in this case, the particle density consumes the chemical. (We will elaborate on models with absorption.) As far as problem (4) is concerned, since the attractive signal v increases with u, the natural spreading process of the cells' density could interrupt and very high and spatially concentrated spikes formations (*chemotactic collapse* or *blow-up at finite time*) may appear; this is, generally, due to the size of the chemosensitivity χ, the initial mass of the particle distribution, that is, $m = \int_{\Omega} u_0(x)dx$, and the space dimension n. In this direction, the reader interested in learning more can find in Refs. [4–8] analyses dealing with existence and properties of global, uniformly bounded or blow-up (local) solutions to models connected to (4).

On the other hand, for nonlinear segregation chemotaxis models like those we are interested in, when in problem (4) one has that $P(v) = P_{\alpha}^1(v) = v_t - \Delta v + v - u^{\alpha} = 0$, with $0 < \alpha < \frac{2}{n}$ $(n \geq 1)$, the uniform boundedness of all its solutions is proved in Liu and Tao.[9]

Concerning the literature about problem (5), it seems rather poor and general (see, for instance, Mock[10,11] for analyses on similar contexts). In particular, no result on the blow-up scenario is available; this is meaningful due to the repulsive nature of the phenomenon.

Contrarily, the level of understanding for attraction–repulsion chemotaxis problems involving both (4) and (5) is sensitively rich; more specifically, if we refer to the linear diffusion and sensitivities version of model (3), for which $F = F_{1,1,1}$ and $P_{\alpha}^{\tau}(v) = \tau v_t - \Delta v + bv - au^{\alpha}$ and $Q_{\beta}^{\tau}(w) = \tau w_t - \Delta w + dw - cu^{\beta}$, $a, b, c, d, \alpha, \beta > 0$, equipped with regular initial data $u_0(x), \tau v_0(x), \tau w_0(x) \geq 0$, we can recollect the following outcomes. In the absence of logistics ($h_k \equiv 0$), when linear growths of the chemoattractant and the chemorepellent are taken into consideration, and for elliptic equations for the chemicals (i.e., when $P_1^0(v) = Q_1^0(w) = 0$), the value $\Theta := \chi a - \xi c$ measures the difference between the attraction and repulsion impacts, and it is such that whenever $\Theta < 0$ (repulsion-dominated regime), in any dimension all solutions to the model are globally bounded, whereas for $\Theta > 0$ (attraction-dominated regime) and $n = 2$ unbounded solutions can be detected (see Refs. [12–16] for some details on the issue). Indeed, for more general expressions of the proliferation laws, modeled by the equations $P_{\alpha}^0(v) = Q_{\beta}^0(w) = 0$, to the best of our knowledge,[17] is the most recent result in this direction; herein, some interplay between α and β and some technical conditions on ξ and u_0 are established so to ensure globality and boundedness of classical solutions. (See also Chiyo and Yokota[18] for blow-up results in the frame of nonlinear attraction–repulsion models with logistics as those formulated in (3) with $F = F_{m_1,m_2,m_3}$ and $h = h_k$, and with linear segregation for the stimuli, that is, with equations for v, w reading as $P_1^0(v) = Q_1^0(w) = 0$.)

Putting our attention on evolutive equations for chemoattractant and chemorepellent, $P_1^1(v) = Q_1^1(w) = 0$, in Tao and Wang[14] it is proved that in two-dimensional domains sufficiently smooth initial data emanate global-in-time bounded solutions whenever

$$\Theta < 0 \quad \text{and} \quad b = d \quad \text{or} \quad \Theta < 0 \quad \text{and} \quad -\frac{\chi^2 \alpha^2 (b-d)^2}{2\Theta b^2 C} \int_{\Omega} u_0(x)dx \leq 1, \quad \text{for some } C > 0.$$

(As to blow-up results, we are only aware of Lankeit,[19] where unbounded solutions in three-dimensional domains are constructed.) When $h = h_k \not\equiv 0$, for both linear and nonlinear productions scenarios, and stationary or evolutive equations (formally, $P^{\tau}_{\alpha}(v) = Q^{\tau}_{\beta}(w) = 0$), criteria toward boundedness, long time behaviors, and blow-up issues for related solutions are studied in Refs. [20–23].

1.3 | The nonlocal case

In this work, we are mainly interested in the so-called nonlocal models tied to (3), for which, specifically, $P(v) = P_\alpha(v) := \Delta v - \frac{1}{|\Omega|}\int_\Omega u^\alpha + u^\alpha = 0$, and $Q(w) = Q_\beta(w) := \Delta w - \frac{1}{|\Omega|}\int_\Omega u^\beta + u^\beta$. In particular, recalling the position in (2), we herein mention the most recent researches we are aware of and inspiring our study; for this purpose, we refer to the only attraction version

$$u_t = \nabla \cdot \left(G_{m_1} + H_{m_2}\right) + h_k \quad \text{and} \quad P_\alpha(v) = 0 \quad \text{in } \Omega \times (0, T_{max}), \tag{6}$$

and the attraction–repulsion one

$$u_t = \nabla \cdot F_{m_1,m_2,m_3} + h_k \quad \text{and} \quad P_\alpha(v) = Q_\beta(w) = 0 \quad \text{in } \Omega \times (0, T_{max}). \tag{7}$$

For the linear diffusion and sensitivity version of model (6), that is, with flux $G_1 + H_1$, if $h_k \equiv 0$, it is known that boundedness of solutions is achieved for any $n \geq 1$ and $0 < \alpha < \frac{2}{n}$, whereas for $\alpha > \frac{2}{n}$ blow-up phenomena may be observed (see Winkler[24]). Some of these results have been generalized in Cieślak and Winkler,[25], where the situation with nonlinear fluxes of the type $G_{m_1} + H_1$ but linear production ($P_1(v) = 0$) are discussed.

On the other hand, in the same flavor of the Cauchy problem $u' = \lambda u - \mu u^k$, with $k > 1$ and $u(0) = u_0 > 0$, whose solution is bounded, intuitively the presence of logistic sources involving superlinear damping effects in chemotaxis models should provide smoothness to the mechanism and henceforth boundedness of related solutions. Nevertheless, this occurs for sufficiently large values of μ and k; precisely, for $G_1 + H_1$ in (6) and $P^0_1(v) = 0/P^1_1(v) = 0$ we refer to Tello and Winkler[26] or Winkler.[27] Oppositely, for k close to 1, the destabilizing action coming from the drift term may overcome the stabilizing one from the logistics and appearances of δ-formations at finite time may be detected. In the specific (remaining in the same context of (6)), for the linear flux $G_1 + H_1$, $P_1(v) = 0$ and some values of $k > 1$, first (Winkler[28]) in domains Ω of $\mathbb{R}^n$ with $n \geq 5$, but later (Winkler[29]) also in three-dimensional domains, being $k < \frac{7}{6}$ and $P^0_1(v) = 0$, coalescence phenomena are constructed. In addition, similar situations have been seen to exist for some sub-quadratic growth of h_k, and precisely for h_k with $1 < k < \frac{n(\alpha+1)}{n+2} < 2$ (see Yi et al.[30]). But there is more; for the limit linear production scenario, corresponding to $P_1(v) = 0$, some unbounded solutions have been constructed in Fuest[31] even for quadratic sources $h = h_2$, whenever $n \geq 5$ and $\mu \in (0, \frac{n-4}{n})$.

In nonlinear models without dampening logistic effects (i.e., general flux $G_{m_1} + H_{m_2}$ and $h_k \equiv 0$) and linear production (i.e., $P_1(v) = 0$), in Winkler and Djie,[32] it is shown, among other things, that for $m_1 \leq 1, m_2 > 0, m_2 > m_1 + \frac{2}{n} - 1$ situations with unbounded solutions at some finite time T_{max} can be found. (See also Marras et al.[33] for questions connected to estimates of T_{max}.) Some results have also been extended in Tanaka[34] when $h_k \not\equiv 0$ and for nonlinear segregation contexts,

$P_\alpha(v)=0$; in particular, inter alia, for $m_1\in\mathbb{R}$, $m_2>0$, blow-up phenomena are seen to appear if $m_2+\alpha>\max\{m_1+\frac{2}{n}k,k\}$, whenever $m_1\geq 0$ or $m_2+\alpha>\max\{\frac{2}{n}k,k\}$, provided $m_1<0$.

On the other hand, for the attraction–repulsion models, in Liu and Li[35] it is proved, together with other results, that if $P_\alpha(v)=Q_\beta(w)=0$, $\alpha>\frac{2}{n}$ and $\alpha>\beta$ ensure the existence of unbounded solutions to (7) for the linear flux $F=F_{1,1,1}$, without logistic ($h_k\equiv 0$). Conversely, detecting gathering mechanisms for the nonlinear situation is more complex and we are only aware of Wang et al.[36] ; indeed, this issue is therein addressed only for $F=F_{m_1,1,1}$, with $m_1\in\mathbb{R}$, but even in the presence of dampening logistics. Since in our research, we will show the existence of blow-up solutions for a larger class of fluxes, precisely for $F=F_{m_1,m_2,m_2}$ with $m_1\in\mathbb{R}$ and any $m_2=m_3>0$, we will analyze details of Wang et al.[36] below, precisely in Section 3.2.

2 | PRESENTATION OF THE MODEL AND OF THE MAIN RESULTS: AIMS OF THE PAPER

2.1 | The formulation of the mathematical problem

In light of what we presented so far, and under the aforementioned main positions, basically in this paper we are interested in properties of unbounded classical solutions $(u,v,w)=(u(x,t),v(x,t),w(x,t))$ to this problem

$$\begin{cases} u_t=\nabla\cdot((u+1)^{m_1-1}\nabla u-\chi u(u+1)^{m_2-1}\nabla v+\xi u(u+1)^{m_3-1}\nabla w)+\lambda u-\mu u^k & \text{in } \Omega\times(0,T_{max}),\\ 0=\Delta v-m_1(t)+f_1(u) & \text{in } \Omega\times(0,T_{max}),\\ 0=\Delta w-m_2(t)+f_2(u) & \text{in } \Omega\times(0,T_{max}),\\ u_\nu=v_\nu=w_\nu=0 & \text{on } \partial\Omega\times(0,T_{max}),\\ u(x,0)=u_0(x) & x\in\bar{\Omega},\\ \int_\Omega v(x,t)dx=\int_\Omega w(x,t)dx=0 & \text{for all } t\in(0,T_{max}). \end{cases} \tag{8}$$

Additionally, the initial cell distribution $u_0:=u_0(x)$ and the production laws $f_i=f_i(u)$ (for $i\in\{1,2\}$) are supposed to be nonnegative and sufficiently regular, and the functions $m_i(t)$ are defined for compatibility in the second and third equations (by integrating these over Ω) in terms of $f_i(u)$ themselves, exactly as

$$m_1(t)=\frac{1}{|\Omega|}\int_\Omega f_1(u) \quad\text{and}\quad m_2(t)=\frac{1}{|\Omega|}\int_\Omega f_2(u). \tag{9}$$

Herein, we assume

$$0\leq f_i\in\bigcup_{\theta\in(0,1)} C^\theta_{loc}([0,\infty))\cap C^1((0,\infty)) \quad\text{and } 0\leq u_0\in\bigcup_{\theta\in(0,1)} C^\theta(\bar{\Omega}); \tag{10}$$

we also might need that for all $s\geq 0$, $\alpha,\beta>0$, and some $k_1,k_2,k_3>0$,

$$f_1(s)\leq k_1(s+1)^\alpha \quad\text{and}\quad f_2(s)\leq k_2(s+1)^\beta, \tag{11}$$

or

$$f_1, f_2 \text{ nondecreasing}, f_1(s) \geq k_3(s+1)^{\alpha}, \quad f_2(s) \leq k_2(s+1)^{\beta} \text{ and}$$
$$u_0 = u_0(|x|) \text{ radially symmetric and nonincreasing.} \tag{12}$$

Remark 1. Let us clarify that in nonlocal models, and henceforth in this work, v stands for the deviation of the chemoattractant; the deviation is the difference between the signal concentration and its mean, and that it changes sign in contrast to what happens with the cell and signal densities (which are nonnegative). In particular, it follows from the definition of v itself that its mean is zero (as specified in the last positions of the problem (8)), which in turn ensures the uniqueness of the solution of the Poisson equation under homogeneous Neumann boundary conditions. The same comments apply for the chemorepellent w. (We did not introduce different symbols to indicate the chemicals and their deviations since it is clear from the context.)

2.2 | Presentation of the theorems: Overall aims of the paper

Our project finds its motivations in the observation that there is no automatic connection between the occurrence of blow-up for solutions to model (8) in the $L^\infty(\Omega)$-norm and that in $L^p(\Omega)$-norm ($p > 1$). Indeed, for a bounded domain Ω, it is seen that

$$\|u(\cdot,t)\|_{L^p(\Omega)} \leq |\Omega|^{\frac{1}{p}} \|u(\cdot,t)\|_{L^\infty(\Omega)},$$

so that unboundedness in $L^p(\Omega)$-norm implies that in $L^\infty(\Omega)$-norm, but oppositely $\int_\Omega u^p$ might even remain bounded in a neighborhood of T_{max} when $\max_\Omega u$ uncontrollably increases at some finite time T_{max}.

In light of this, in order to bridge the gap between the analysis of the blow-up time T_{max} in the two different mentioned norms, we aim at

(i) detecting suitable $L^p(\Omega)$ spaces, for certain p depending on n, m_1, m_2, m_3, α and β, such that given unbounded solutions also blow up in the associated $L^p(\Omega)$-norms;
(ii) providing lower bounds for the blow-up time of the aforementioned solutions in these $L^p(\Omega)$-norms.

Additionally, another objective of our work is

(iii) giving sufficient conditions on the data of the model such that related solutions are actually unbounded at finite time.

In order to deal with issues (i)–(iii), we fix the following relations, determining some precise interplay involving constants defining system (8):

Assumptions 1. Let $n \in \mathbb{N}$, $m_1, m_2, m_3 \in \mathbb{R}$ and $\alpha, \beta, \chi, \xi, \lambda, \mu > 0$ and $k > 1$ be such that

$(\mathcal{H}_1)$ $m_2 + \alpha > m_1 + \frac{2}{n}$.

Moreover, for $\beta > 0$, let either

$(\mathcal{H}_2)$ $m_2 + \alpha > \max\{1, m_3 + \beta\}$ or $(\mathcal{H}_3)$ $m_2 + \alpha \geq m_3 + \beta$ and $m_1 > 1 - \frac{2}{n}$,

whereas, for $\beta \in (0, 1]$, let either

$(\mathcal{H}_4)$ $m_3 \leq 1$ and $m_1 > 1 - \frac{2}{n}$ or $(\mathcal{H}_5)$ $m_2 + \alpha \geq m_3$ and $m_1 > 1 - \frac{2}{n}$.

Finally, for $m_2, \beta > 0$, let also

$$(\mathcal{H}_6)\quad \alpha > \beta \quad \text{and} \quad \begin{cases} m_2 + \alpha > \max\left\{m_1 + \frac{2}{n}k, k\right\} & \text{if } m_1 \geq 0, \\ m_2 + \alpha > \max\left\{\frac{2}{n}k, k\right\} & \text{if } m_1 < 0. \end{cases}$$

With the support of the above position, and as far as the analysis of (i) is concerned, in the spirit of Theorem 2.2 of Freitag[37], we will prove this first result, dealing with properties of *given unbounded solutions* to model (8). Specifically, the proof is based on the analysis of the functional $\varphi(t) = \frac{1}{p}\int_\Omega (u+1)^p$, defined for local solutions to problem (8) on $(0, T_{max})$. We will show that if for some p sufficiently large $\varphi(t)$ is uniformly bounded in time, then it also is for any arbitrarily large $p > 1$, so contrasting with the unboundedness of u itself.

Theorem 1. *Let Ω be a bounded and smooth domain of $\mathbb{R}^n$, and condition $(\mathcal{H}_1)$ as well as one of $(\mathcal{H}_2), (\mathcal{H}_3), (\mathcal{H}_4), (\mathcal{H}_5)$ in Assumptions 1 hold true. Moreover, for f_i and u_0 complying with (10) and (11), let (u, v, w), with*

$$u \in C^0(\bar{\Omega} \times [0, T_{max})) \cap C^{2,1}(\bar{\Omega} \times (0, T_{max})) \text{ and } v, w \in \bigcap_{q>n} L^\infty((0, T_{max}); W^{1,q}(\Omega)) \cap C^{2,0}(\bar{\Omega} \times (0, T_{max})),$$

be a solution to problem (8) which blows up at some finite time T_{max} in the sense that

$$\limsup_{t \to T_{max}} \|u(\cdot, t)\|_{L^\infty(\Omega)} = +\infty. \tag{13}$$

Then, for any $\mathfrak{p} > \frac{n}{2}(m_2 - m_1 + \alpha)$, we also have

$$\limsup_{t \to T_{max}} \|u(\cdot, t)\|_{L^{\mathfrak{p}}(\Omega)} = +\infty.$$

Successively aim (ii) is achieved by establishing for the same functional $\varphi(t)$ a first-order differential inequality (ODI) of the type $\varphi'(t) \leq \Psi(\varphi(t))$ on $(0, T_{max})$. In particular, for any $\tau_0 > 0$ the function $\Psi(\tau)$ obeys the Osgood criterion,[38]

$$\int_{\tau_0}^{\infty} \frac{d\tau}{\Psi(\tau)} < \infty \quad \text{with } \tau_0 > 0, \tag{14}$$

so that an integration on $(0, T_{max})$ of the ODI implies, whenever $\limsup_{t\to T_{max}} \varphi(t) = \infty$, the following lower bound for T_{max}

$$T_{max} \geq \int_{\varphi(0)}^{\infty} \frac{d\varphi}{\Psi(\varphi)} := T,$$

and thereby a safe interval of existence $[0, T)$ for solutions to the model itself.

Theorem 2. *Let the hypotheses of Theorem 1 be satisfied. Then there exists $\bar{p} > 1$ and positive constants $\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}$, as well as $\gamma > \delta > 1$, such that the blow-up time T_{max} complies with both the implicit estimate*

$$T_{max} \geq \int_{\varphi(0)}^{\infty} \frac{d\tau}{\mathcal{A}\tau^{\gamma} + \mathcal{B}\tau^{\delta} + \mathcal{C}}, \tag{15}$$

and the explicit one

$$T_{max} \geq \frac{\varphi(0)^{1-\gamma}}{\mathcal{D}(\gamma - 1)},$$

where $\varphi(0) = \frac{1}{p}\int_{\Omega}(u_0 + 1)^p$, for any $p \geq \bar{p}$.

Finally, the last theorem (connected to item (iii)) establishes (at least in a particular case) the *existence of unbounded solutions* to system (8), so as to make the two previous statements meaningful. The basic idea consists of the analysis of the temporal evolution of the functional $\phi(t) := \int_0^{s_0} s^{-\gamma}(s_0 - s)U(s,t)\,ds$ for $t \in [0, T_{max})$, being $U(s,t)$ the so-called mass accumulation function of u, obeying a superlinear ODI.

Theorem 3. *Let $\chi, \xi, \lambda, \mu, m_2 = m_3 > 0$ and $k > 1$. Additionally, for some $R > 0$, let $\Omega = B_R(0) \subset \mathbb{R}^n$ be a ball and let f_1 and f_2 satisfy (10) and (12). Finally, let hypotheses $(\mathcal{H}_6)$ be satisfied. Then for any $M_0 \geq C$, with $C = \left(\frac{\lambda}{\mu}|\Omega|^{k-1}\right)^{\frac{1}{k-1}}$, there exist $\epsilon_0 \in (0, M_0)$ and $r_* \in (0, R)$ with the property that whenever u_0 complies with (10) and (12) and it is also chosen such that*

$$\int_{\Omega} u_0(x)dx = M_0 \quad \text{and} \quad \int_{B_{r_*}(0)} u_0(x)dx \geq M_0 - \epsilon_0, \tag{16}$$

the corresponding classical solution (u, v, w) to model (8) blows up at some finite time T_{max}, in the sense of relation (13).

3 | MISCELLANEOUS AND GENERAL COMMENTS

3.1 | On the parameters $\mathfrak{p}$ and $\bar{p}$

We herein want to discuss on the role of the parameters $\mathfrak{p}$ and $\bar{p}$ appearing in Theorems 1 and 2. In particular, as we will observe in Lemma 2 below, $\mathfrak{p}$ and $\bar{p}$ depend on n, m_1, m_2, m_3, α, and β, and Figure 1 presents some of their values on the p axis. In the specific, for a blowing up solution to (8),

FIGURE 1 Some infimum of $\mathfrak{p}$ and $\bar{p}$, taken from their definitions in Lemma 2. From top to bottom: ➢ Under assumption $(\mathcal{H}_3)$, and $n=1$, $m_1=\frac{81}{50}$, $m_2=-\frac{149}{100}$, $m_3=\frac{33}{20}$, $\alpha=\frac{587}{100}$ and $\beta=\frac{63}{25}$, we have $\mathfrak{p}=\frac{n}{2}(m_2-m_1+\alpha)=\frac{69}{50}$ and $\bar{p}=m_3(n+2)(n+1)=\frac{99}{10}$; ➢ Under the assumption $(\mathcal{H}_2)$, and $n=5$, $m_1=-\frac{3}{50}$, $m_2=\frac{6}{5}$, $m_3=-\frac{143}{100}$, $\alpha=\frac{163}{100}$ and $\beta=\frac{169}{100}$, we have $\mathfrak{p}=\frac{n}{2}(m_2-m_1+\alpha)=\frac{289}{40}$ and $\bar{p}=m_2(n+2)(n+1)=\frac{252}{5}$; ➢ Under the assumption $(\mathcal{H}_2)$, and $n=4$, $m_1=-\frac{187}{100}$, $m_2=-\frac{89}{100}$, $m_3=-\frac{181}{100}$, $\alpha=\frac{353}{100}$ and $\beta=\frac{19}{50}$, we have $\mathfrak{p}=\frac{n}{2}(m_2-m_1+\alpha)=\frac{451}{50}$ and $\bar{p}=n\alpha=\frac{353}{25}$; ➢ Under the assumption $(\mathcal{H}_3)$, and $n=4$, $m_1=\frac{47}{50}$, $m_2=\frac{6}{25}$, $m_3=-\frac{63}{50}$, $\alpha=\frac{179}{100}$ and $\beta=\frac{119}{50}$, we have $\mathfrak{p}=\frac{n}{2}(m_2-m_1+\alpha)=\frac{109}{50}$ and $\bar{p}=n\beta=\frac{238}{25}$.

we will have that $\int_\Omega u^p$ is bounded on $(0,T_{max})$ for $p=1$, while it blows up for $p\geq\mathfrak{p}$; in general, the behavior of $\int_\Omega u^p$ on $(0,T_{max})$ for $p\in(1,\mathfrak{p})$ is unknown. On the other hand, an estimate for T_{max} is given in terms of $\frac{1}{p}\int_\Omega(u_0+1)^p$, for $p\geq\bar{p}$, but not when $p\in(\mathfrak{p},\bar{p})$.

3.2 | Improving Theorem 1.1 in Wang et al.[36] and addressing an open question in Remark 1.2 of Wang et al.[36]

Herein we want to compare Theorem 3 and Theorem 1.1 in Wang et al.,[36] both dealing with blow-up solutions to attraction–repulsion models with logistics. First, in order to have consistency between these results, we have to fix $m_2=1$ in $(\mathcal{H}_6)$. Additionally, for the ease of the reader, let us also rephrase the related blow-up assumptions:

$$\text{Blow-up conditions in Theorem 3:}\qquad \alpha>\beta \quad\text{and}\quad \begin{cases}\alpha>\max\left\{m_1+\frac{2}{n}k-1,k-1\right\} & \text{if } m_1\geq 0,\\ \alpha>\max\left\{\frac{2}{n}k-1,k-1\right\} & \text{if } m_1<0,\end{cases} \tag{17}$$

and

$$\text{Blow-up conditions in Theorem 1.1 in Wang et al. [36]:}\qquad \alpha>\beta \quad\text{and}\quad \begin{cases}\alpha>\max\left\{m_1+\frac{2}{n}k-1,k-1\right\} & \text{if } m_1>1,\\ \alpha>\max\left\{\frac{2}{n}k,k-1\right\} & \text{if } m_1\leq 1.\end{cases} \tag{18}$$

We easily note that if $m_1\geq 1$ the two conditions (17) and (18) coincide. Now, we analyze the cases $m_1\leq 0$ and $0<m_1<1$, separately.

- ➢ Case $m_1\leq 0$. By comparing $\alpha>\max\{\frac{2}{n}k-1,k-1\}$ in (17) and $\alpha>\max\{\frac{2}{n}k,k-1\}$ in (18), we observe that (17) provides a larger range of values of α for which blow-up occurs than (18) does whenever $n\in\{1,2\}$ or $n\geq 3$ provided $1<k<\frac{n}{n-2}$.

- Case $0 < m_1 < 1$. The conditions above become $\alpha > \max\{m_1 + \frac{2}{n}k - 1, k-1\}$ and $\alpha > \max\{\frac{2}{n}k, k-1\}$, respectively. In particular, thanks to the fact that $m_1 < 1$, also in this situation a sharper condition is achieved for $n \in \{1,2\}$ or $n \geq 3$, under the assumptions $1 < k < \frac{n}{n-2}$ and $0 < m_1 < \frac{k(n-2)}{n}$ or $1 < k < \frac{n}{n-2}$ and $\frac{k(n-2)}{n} < m_1 < 1$.

From the above analysis, it is seen that Theorem 3 improves Theorem 1.1 in Wang et al.[36]; additionally, it establishes that (1.15) in Theorem 1.1 in Wang et al.[36] is not optimal, so giving an answer to an open question in Remark 1.2 in Wang et al.[36].

3.3 | On the automatic applicability of Theorems 1 and 2 in some related chemotaxis contexts

We can observe what follows:

- Once the blow-up constrains in $(\mathcal{H}_6)$ are accomplished, and taking into account $m_2 = m_3 > 0$, assumptions $(\mathcal{H}_1)$ and $(\mathcal{H}_2)$ are immediately satisfied; subsequently, Theorems 1 and 2 are applicable to unbounded solutions to model (8).
- Theorems 1 and 2 can also be used in models close to (8), for which unbounded solutions can be detected (see Section 1); in particular, for attraction–repulsion (linear and nonlinear) models with or without logistic and general production laws (Refs. [35, 36]) or for only attraction ones (Refs. [24, 30, 34, 39, 40]).

3.4 | An open problem

As far as we know, establishing conditions ensuring blow-up solutions for the case $m_2 \neq m_3$ in model (8), is still an open problem; in particular, we will give some details on related technical difficulties connected to this issue in Remark 3.

4 | LOCAL EXISTENCE AND NECESSARY PARAMETERS

By an adaption of standard reasoning in the frame of the fixed-point theorem, we can show the following result on the local existence and extensibility of classical solutions to (8).

Lemma 1. *Let Ω be a bounded and smooth domain of $\mathbb{R}^n$, with $n \geq 1$, $\chi, \xi, \lambda, \mu > 0$, $m_1, m_2, m_3 \in \mathbb{R}$, $k > 1$, and let f_i and u_0 comply with (10). Then there exist $T_{max} \in (0, \infty]$ and a unique solution (u, v, w) to problem (8), defined in $\Omega \times (0, T_{max})$ and such that*

$$u \in C^0(\bar{\Omega} \times [0, T_{max})) \cap C^{2,1}(\bar{\Omega} \times (0, T_{max})) \text{ and } v, w \in \bigcap_{q>n} L^\infty_{loc}((0, T_{max}); W^{1,q}(\Omega)) \cap C^{2,0}(\bar{\Omega} \times (0, T_{max})).$$

Additionally, one has $u \geq 0$ in $\Omega \times (0, T_{max})$,

$$\int_\Omega u \leq M := \max\{M_0, C\} \quad \text{for all } t \in [0, T_{max}), \tag{19}$$

where $M_0 = \int_\Omega u_0(x)\,dx$ *and* $C := \left(\frac{\lambda}{\mu}|\Omega|^{k-1}\right)^{\frac{1}{k-1}}$, *and*

$$\text{if}\quad T_{max} < \infty, \qquad \text{then}\qquad \limsup_{t\to T_{max}} \|u(\cdot,t)\|_{L^\infty(\Omega)} = \infty.$$

Furthermore, if u_0 *also satisfies the symmetrical assumptions in (12) and* $\Omega = B_R(0)$, *with some* $R > 0$, *then* u, v, *and* w *are radially symmetric with respect to* $|x|$ *in* $\Omega \times (0, T_{max})$.

Proof. *The proof can be achieved by following well-established results: for instance, we refer the interested reader to Refs. [7, 25, 32, 41]. In particular, an integration of the first equation in (8) and an application of the Hölder inequality give bound (19).*

□

Let us start with the following technical

Lemma 2. *Let* $n \in \mathbb{N}$, $m_1, m_2, m_3, \alpha, \beta$ *be as in Assumptions 1. Moreover, for* $\mathfrak{p} > \frac{n}{2}(m_2 - m_1 + \alpha)$ *let*

$$\bar{p} > \max\left\{\begin{array}{c} \mathfrak{p} \\ 1 - m_1(n+2) \\ 1 - m_2 \\ 1 - m_3 \\ 2 - m_1 - \frac{2}{n} \\ n\alpha \\ n\beta \\ m_2(n+2)(n+1) \\ m_3(n+2)(n+1) \end{array}\right\},$$

$$\sigma := \frac{2(p + m_2 + \alpha - 1)}{p + m_1 - 1}, \quad \hat{\sigma} := \frac{2p}{p + m_1 - 1}, \quad \gamma := \frac{\frac{m_1 - m_2 - \alpha}{2} + \frac{p}{n} + \frac{m_2 + \alpha - 1}{n}}{\frac{m_1 - m_2 - \alpha}{2} + \frac{p}{n}}, \quad \delta := \frac{p + m_2 + \alpha - 1}{p}.$$

Then for all $p \geq \bar{p}$ *these relations hold*

$$p > \frac{n}{2}(1 - m_1), \tag{20a}$$

$$0 < \theta := \frac{\frac{p+m_1-1}{2\mathfrak{p}} - \frac{p+m_1-1}{2(p+m_2+\alpha-1)}}{\frac{p+m_1-1}{2\mathfrak{p}} + \frac{1}{n} - \frac{1}{2}} < 1, \tag{20b}$$

$$0 < \frac{\sigma\theta}{2} < 1, \tag{20c}$$

$$0 < \hat{\theta} = \frac{\frac{p+m_1-1}{2} - \frac{p+m_1-1}{2p}}{\frac{p+m_1-1}{2} + \frac{1}{n} - \frac{1}{2}} < 1, \tag{20d}$$

$$0 < \frac{\hat{\sigma}\hat{\theta}}{2} < 1, \tag{20e}$$

$$0 < \frac{p+m_3-1}{p+m_2+\alpha-1} < 1, \tag{20f}$$

$$0 < \frac{\beta}{p+m_2+\alpha-1} < 1, \tag{20g}$$

$$\frac{p+m_3-1}{p+m_2+\alpha-1} + \frac{\beta}{p+m_2+\alpha-1} < 1, \tag{20h}$$

$$0 < \frac{p}{p+m_2+\alpha-1} < 1, \tag{20i}$$

$$\gamma > \delta > 1, \tag{20j}$$

$$0 < \bar{\theta} := \frac{\frac{p+m_1-1}{2p} - \frac{p+m_1-1}{2(p+m_2+\alpha-1)}}{\frac{p+m_1-1}{2p} + \frac{1}{n} - \frac{1}{2}} < 1, \tag{20k}$$

$$0 < \frac{\sigma\bar{\theta}}{2} < 1, \tag{20l}$$

$$0 < \frac{p+m_3-1}{p} < 1. \tag{20m}$$

Proof. *To show our relations, we will need $p > 1 - m_1$ and $p > 1 + \beta - m_2 - \alpha$. As to the first inequality, due to $(\mathcal{H}_1)$ and the restriction on $\mathfrak{p}$, we have that $p \geq \bar{p} > \mathfrak{p} > 1 \geq 1 - m_1$ for $m_1 \geq 0$, while for $m_1 < 0$ it suddenly derives from $p \geq \bar{p} > 1 - m_1(n+2)$. For the second lower bound, assumptions $(\mathcal{H}_2)$ or $(\mathcal{H}_3)$, together with the definition of $\bar{p}$, give $p \geq \bar{p} > 1 - m_3 > 1 + \beta - m_2 - \alpha$. Besides, $m_2 + \alpha > 1$ is automatically true through $(\mathcal{H}_2)$, or alternatively by means of assumption $(\mathcal{H}_1)$ in conjunction with one among $(\mathcal{H}_3)$–$(\mathcal{H}_5)$. The same condition $m_2 + \alpha > 1$, $p \geq \bar{p}$ and the restriction on $\mathfrak{p}$ and $p > 1 - m_1$ ensure relations (20a), (20b), (20c), (20k), and (20l). Moreover, from (20a), $p > 1 - m_1$ and $p > 2 - m_1 - \frac{2}{n}$, we obtain (20d) and by using also $m_1 > 1 - \frac{2}{n}$, we get (20e). On the other hand, restrictions (20f), (20g), (20h), (20i), and (20m) come from the definition of $\bar{p}$ in conjunction with $m_2 + \alpha > m_3$, $p > 1 + \beta - m_2 - \alpha$, $m_2 + \alpha > m_3 + \beta$, and $m_2 + \alpha > 1$, respectively. Finally, from $(\mathcal{H}_1)$ it follows that $m_2 + \alpha > m_1$, which combined with $m_2 + \alpha > 1$ gives (20j).* □

Remark 2. For reasons which will be exploited later on, and precisely in Lemma 3, it appears important to point out that assumptions $(\mathcal{H}_3)$–$(\mathcal{H}_5)$ imply that the ratios (20f), (20h), and (20m) in Lemma 2 can also be taken equal to 1.

5 | A PRIORI ESTIMATES AND PROOF OF THEOREMS 1 AND 2

In this section, we will use $(\mathcal{H}_1)$–$(\mathcal{H}_5)$. Moreover, without explicitly computing their values, we underline that the constants c_i appearing below and throughout the paper depend inter alia on p, are positive and their subscripts i start anew in each new proof.

Lemma 3. *Under the hypotheses of Lemma 2, let $p = \bar{p}$ and $\mathfrak{p}$ be any of the constants therein defined. If (u, v, w) is a classical solution to problem (8), $u \in L^\infty((0, T_{max}); L^{\mathfrak{p}}(\Omega))$ and $\varphi(t)$ is the energy function*

$$\varphi(t) := \frac{1}{p}\int_\Omega (u+1)^p \quad \text{on } (0, T_{max}),$$

then there exist c_1, c_2 such that

$$\varphi'(t) \leq -c_1 \int_\Omega |\nabla (u+1)^{\frac{p+m_1-1}{2}}|^2 + c_2 \quad \text{for all } t \in (0, T_{max}). \tag{21}$$

Proof. Let us differentiate the functional $\varphi(t) = \frac{1}{p}\int_\Omega (u+1)^p$. Using the first equation of (8) and the divergence theorem, we have for every $t \in (0, T_{max})$

$$\begin{aligned}
\varphi'(t) = \int_\Omega (u+1)^{p-1}u_t &= \int_\Omega (u+1)^{p-1}\nabla\cdot((u+1)^{m_1-1}\nabla u) - \chi\int_\Omega (u+1)^{p-1}\nabla\cdot(u(u+1)^{m_2-1}\nabla v)\\
&\quad + \xi\int_\Omega (u+1)^{p-1}\nabla\cdot(u(u+1)^{m_3-1}\nabla w) + \lambda\int_\Omega (u+1)^{p-1}u - \mu\int_\Omega (u+1)^{p-1}u^k\\
&= -(p-1)\int_\Omega (u+1)^{p+m_1-3}|\nabla u|^2 + (p-1)\chi\int_\Omega u(u+1)^{p+m_2-3}\nabla u\cdot\nabla v\\
&\quad - (p-1)\xi\int_\Omega u(u+1)^{p+m_3-3}\nabla u\cdot\nabla w + \lambda\int_\Omega (u+1)^{p-1}u - \mu\int_\Omega (u+1)^{p-1}u^k.
\end{aligned}$$

For $j \in \{m_2, m_3\}$, we now define

$$F_j(u) = \int_0^u \hat{u}(\hat{u}+1)^{p+j-3}d\hat{u},$$

so observing that

$$0 \leq F_j(u) \leq \frac{1}{p+j-1}\left[(u+1)^{p+j-1} - 1\right]. \tag{22}$$

By considering the definition of $F_j(u)$ above again the divergence theorem, we have for every $t \in (0, T_{max})$,

$$\begin{aligned}
\varphi'(t) \leq& -c_3\int_\Omega |\nabla(u+1)^{\frac{p+m_1-1}{2}}|^2 + c_4\int_\Omega \nabla F_{m_2}(u)\cdot\nabla v - c_5\int_\Omega \nabla F_{m_3}(u)\cdot\nabla w + c_6\int_\Omega (u+1)^p - c_7\int_\Omega (u+1)^{p-1}u^k\\
=& -c_3\int_\Omega |\nabla(u+1)^{\frac{p+m_1-1}{2}}|^2 - c_4\int_\Omega F_{m_2}(u)\Delta v + c_5\int_\Omega F_{m_3}(u)\Delta w + c_6\int_\Omega (u+1)^p - c_7\int_\Omega (u+1)^{p-1}u^k.
\end{aligned} \tag{23}$$

Let us now specify how each of the constrains in Assumptions 1 takes part in our computation.

First, from $(\mathcal{H}_1)$, we have $\mathfrak{p} > 1$; this makes meaningful our assumption $u \in L^\infty((0, T_{max}); L^{\mathfrak{p}}(\Omega))$. (Recall that $u \in L^\infty((0, T_{max}); L^1(\Omega))$ is always met by (19).)

Now, by exploiting $(\mathcal{H}_2)$, the second and third equation of (8) and (22), we can see that from (23), if we neglect the nonpositive terms we get on $(0, T_{max})$

$$\varphi'(t) \leq -c_3 \int_\Omega |\nabla (u+1)^{\frac{p+m_1-1}{2}}|^2 + c_8 \int_\Omega (u+1)^{p+m_2+\alpha-1} + c_9 \int_\Omega (u+1)^{p+m_3-1} \int_\Omega (u+1)^\beta + c_6 \int_\Omega (u+1)^p. \tag{24}$$

As to the third term, by using twice Hölder's inequality (recall (20f) and (20g)), we obtain, for every $t \in (0, T_{max})$,

$$c_9 \int_\Omega (u+1)^{p+m_3-1} \int_\Omega (u+1)^\beta \leq c_{10} \left(\int_\Omega (u+1)^{p+m_2+\alpha-1}\right)^{\frac{p+m_3-1}{p+m_2+\alpha-1}} \left(\int_\Omega (u+1)^{p+m_2+\alpha-1}\right)^{\frac{\beta}{p+m_2+\alpha-1}}. \tag{25}$$

Moreover, since for any $\varepsilon > 0$ there is $d(\varepsilon) > 0$ such that this inequality (see Lemma 4.3 in Frassu and Viglialoro[42])

$$A^{d_1} B^{d_2} \leq \varepsilon(A+B) + d(\varepsilon), \quad A, B \geq 0,\ d_1, d_2 > 0,\ d_1 + d_2 < 1,$$

is true, by virtue of (20h), we have that

$$c_9 \int_\Omega (u+1)^{p+m_3-1} \int_\Omega (u+1)^\beta \leq \int_\Omega (u+1)^{p+m_2+\alpha-1} + c_{11} \quad \textit{for all } t \in (0, T_{max}). \tag{26}$$

Through (20i), an application of Young's inequality provides

$$c_6 \int_\Omega (u+1)^p \leq \int_\Omega (u+1)^{p+m_2+\alpha-1} + c_{12} \quad \textit{on } (0, T_{max}). \tag{27}$$

To control the term $\int_\Omega (u+1)^{p+m_2+\alpha-1}$, we invoke the Gagliardo–Nirenberg and Young's inequalities, so to bound the mentioned integral with $\int_\Omega |\nabla(u+1)^{\frac{p+m_1-1}{2}}|^2$. More exactly by relying on relations (20b) and (20c), boundedness of $\int_\Omega (u+1)^{\mathfrak{p}}$ and of $\int_\Omega u$ provides for any $L_1 > 0$

$$\begin{aligned}
L_1 \int_\Omega (u+1)^{p+m_2+\alpha-1} &= L_1 \|(u+1)^{\frac{p+m_1-1}{2}}\|^{\frac{2(p+m_2+\alpha-1)}{p+m_1-1}}_{L^{\frac{2(p+m_2+\alpha-1)}{p+m_1-1}}(\Omega)} \\
&\leq c_{13} \left(\|\nabla (u+1)^{\frac{p+m_1-1}{2}}\|^{\sigma\theta}_{L^2(\Omega)} \|(u+1)^{\frac{p+m_1-1}{2}}\|^{\sigma(1-\theta)}_{L^{\frac{2\mathfrak{p}}{p+m_1-1}}(\Omega)} + \|(u+1)^{\frac{p+m_1-1}{2}}\|^{\sigma}_{L^{\frac{2}{p+m_1-1}}(\Omega)} \right) \\
&\leq c_{14} \left(\int_\Omega |\nabla (u+1)^{\frac{p+m_1-1}{2}}|^2 \right)^{\frac{\sigma\theta}{2}} + c_{15} \leq \frac{c_3}{2} \int_\Omega |\nabla (u+1)^{\frac{p+m_1-1}{2}}|^2 + c_{16} \quad \textit{for all } t \in (0, T_{max}).
\end{aligned} \tag{28}$$

(We underline that herein we have used the elementary inequality

$$C_1(\tau)(A^\tau + B^\tau) \leq (A+B)^\tau \leq C_2(\tau)(A^\tau + B^\tau) \quad \textit{for all } A, B \geq 0, \tau > 0 \textit{ and proper } C_1, C_2 > 0, \tag{29}$$

which might tacitly be used in the next lines.) Putting together (24) and (26)–(28), we have the claim.

If assumption $(\mathcal{H}_3)$ is complied, bound (27) can be replaced for any $L_2 > 0$ with

$$
\begin{aligned}
L_2 \int_\Omega (u+1)^p &= L_2 \|(u+1)^{\frac{p+m_1-1}{2}}\|_{L^{\frac{2p}{p+m_1-1}}(\Omega)}^{\frac{2p}{p+m_1-1}} \\
&\leq c_{17}\left(\|\nabla (u+1)^{\frac{p+m_1-1}{2}}\|_{L^2(\Omega)}^{\hat{\sigma}\hat{\theta}} \|(u+1)^{\frac{p+m_1-1}{2}}\|_{L^{\frac{2}{p+m_1-1}}(\Omega)}^{\hat{\sigma}(1-\hat{\theta})} + \|(u+1)^{\frac{p+m_1-1}{2}}\|_{L^{\frac{2}{p+m_1-1}}(\Omega)}^{\hat{\sigma}} \right) \qquad (30)\\
&\leq c_{17}\left(\int_\Omega |\nabla (u+1)^{\frac{p+m_1-1}{2}}|^2 \right)^{\frac{\hat{\sigma}\hat{\theta}}{2}} + c_{18} \leq \frac{c_3}{4}\int_\Omega |\nabla (u+1)^{\frac{p+m_1-1}{2}}|^2 + c_{19} \quad on\ (0, T_{max}),
\end{aligned}
$$

where in this last step we used Young's inequality in conjunction with (20d) and (20e). Moreover, by taking $m_2 + \alpha = m_3 + \beta$ in $(\mathcal{H}_3)$ (recall Remark 2), in estimate (25) the powers at the right-hand side have 1 as sum; subsequently, up to constants, it becomes (26). So, by considering (24), (26), (28), and (30) we have the claim when either $(\mathcal{H}_1)$ and $(\mathcal{H}_2)$ or $(\mathcal{H}_1)$ and $(\mathcal{H}_3)$ are used.

Let us show how to achieve the same conclusion by applying either $(\mathcal{H}_1)$ and $(\mathcal{H}_4)$ or $(\mathcal{H}_1)$ and $(\mathcal{H}_5)$. The main idea is alternatively treating in relation (24) the term

$$
\int_\Omega (u+1)^{p+m_3-1} \int_\Omega (u+1)^\beta \quad on\ (0, T_{max}).
$$

More specifically, if $\beta \in (0,1]$ and we take into account the boundedness of $\int_\Omega u$ on $(0, T_{max})$, we have that

$$
\int_\Omega (u+1)^{p+m_3-1} \int_\Omega (u+1)^\beta \leq c_{20} \int_\Omega (u+1)^{p+m_3-1} \quad for\ all\ t \in (0, T_{max}). \tag{31}
$$

In turn, by relying on, respectively, assumption $(\mathcal{H}_4)$ or $(\mathcal{H}_5)$ (both with strict inequality), by means of Young's inequality, thanks to (20m) and (20f), it is also possible to see that

$$
c_{20}\int_\Omega (u+1)^{p+m_3-1} \leq \int_\Omega (u+1)^p + c_{21} \quad on\ (0, T_{max}), \tag{32}
$$

or

$$
c_{20}\int_\Omega (u+1)^{p+m_3-1} \leq \int_\Omega (u+1)^{p+m_2-1+\alpha} + c_{22} \quad for\ all\ t \in (0, T_{max}). \tag{33}
$$

As before, bounds (24), (31), (32), or alternatively (33), (30), and (28), lead to the same conclusion. (For the limit cases in $(\mathcal{H}_4)$ or $(\mathcal{H}_5)$, namely $m_3 = 1$ or $m_2 + \alpha = m_3$, by relying again on Remark 2, we can directly exploit estimate (31), without using Young's inequality, and conclude as above.)

□

The next step consists of ensuring some time independent estimate of u in the $L^{\bar{p}}(\Omega)$-norm.

Lemma 4. *Let Lemma 3 be true. Then $u \in L^\infty((0, T_{max}); L^{\bar{p}}(\Omega))$.*

Proof. With a view to inequality (21), as already done in (30), a further application of the Gagliardo–Nirenberg inequality, supported by (20d), leads also thanks to (29) to

$$\int_\Omega (u+1)^p \le c_1\left(\|\nabla(u+1)^{\frac{p+m_1-1}{2}}\|^{\hat\sigma\hat\theta}_{L^2(\Omega)}+1\right) \le c_2\left(\int_\Omega |\nabla(u+1)^{\frac{p+m_1-1}{2}}|^2+1\right)^{\frac{\hat\sigma\hat\theta}{2}} \quad \text{on } (0,T_{max}),$$

or, equivalently

$$-c_3\int_\Omega |\nabla(u+1)^{\frac{p+m_1-1}{2}}|^2 \le -\left(\int_\Omega (u+1)^p\right)^{\frac{2}{\hat\sigma\hat\theta}}+c_3 \quad \text{for every } t\in(0,T_{max}). \tag{34}$$

Finally, by using relations (21) and (34), we arrive at this initial problem

$$\begin{cases}\varphi'(t)\le c_4-c_5\varphi(t)^{\frac{2}{\hat\sigma\hat\theta}} & \text{for every } t\in(0,T_{max}),\\ \varphi(0)=\frac{1}{p}\int_\Omega (u_0+1)^p, & \end{cases}$$

which ensures that $\int_\Omega u^{\bar p}=\int_\Omega u^p\le\int_\Omega (u+1)^p\le p\max\{\varphi(0),(\frac{c_4}{c_5})^{\frac{\hat\sigma\hat\theta}{2}}\}$ for all $t<T_{max}$. □

By taking advantage of the previous lemma, let us show the uniform-in-time boundedness of u.

Lemma 5. *Under the hypotheses of Lemma 4, $u\in L^\infty((0,T_{max});L^\infty(\Omega))$.*

Proof. With the same nomenclature used by Tao and Winkler, u also classically solves in $\Omega\times(0,T_{max})$ problem (A.1) in Appendix A of Tao and Winkler[43] for

$$D(x,t,u)=(u+1)^{m_1-1},\quad f(x,t)=\chi u(u+1)^{m_2-1}\nabla v-\xi u(u+1)^{m_3-1}\nabla w,\quad g(x,t)=\lambda u-\mu u^k.$$

In particular, also taking into account the boundary condition on v and w, we can see that (A.2)–(A.5) are met. On the other hand, for any $\lambda,\mu>0$ and $k>1$, it holds that $\lambda u-\mu u^k$ has a positive maximum L at $u_M=(\frac{\lambda}{k\mu})^{\frac{1}{k-1}}$, so that from $g(x,t)\le L$ in $\Omega\times(0,T_{max})$ the second inclusion of (A.6) is accomplished for any choice of q_2. As to (A.7)–(A.10), let us first define the quantities

$$l_1(q_1)=1-m_1\frac{(n+1)q_1-(n+2)}{q_1-(n+2)},\quad l_2(q_2)=1-m_1\frac{1}{1-\frac{nq_2}{(n+2)(q_2-1)}},\quad \text{and}\quad l_3=\frac{n}{2}(1-m_1).$$

Recalling the definition and properties of $p=\bar p$, we have $p>1-m_1(n+2)$ and henceforth for any $m_1\in\mathbb{R}$, it holds $1-m_1(n+2)\ge l_1((n+2)(n+1))$ and for all $q_2\le n+1$ we also have $1-m_1(n+2)\ge l_2(q_2)$. Subsequently, (A.7) in Tao and Winkler[43] (i.e., $u\in L^\infty((0,T_{max});L^p(\Omega))$, that is, $\int_\Omega u^p\le c_1$) is obviously accomplished, (A.8) in Lemma A.1 in Tao and Winkler[43] is fulfilled for $q_1=(n+2)(n+1)$, (A.9) in Lemma A.1 in Tao and Winkler[43] for $\frac{n+2}{2}<q_2\le n+1$, whereas (A.10) in Lemma A.1 in Tao and Winkler[43] is directly true thanks to (20a). Let us dedicate to the first inclusion of (A.6) in Tao and Winkler[43]. Starting from the gained bound of u, let us exploit elliptic regularity results applied to the second and third equations of system (8); in particular, we only analyze $-\Delta v=f_1(u)-\frac{1}{|\Omega|}\int_\Omega f_1(u)$, being the case for w equivalent. Let us observe that from relation

(11) on f_1, and $p = \bar{p} > n\alpha$ naturally implying $(1+u)^\alpha \le (1+u)^p$, we have for all $t \in (0, T_{max})$

$$\int_\Omega |f_1(u) - \frac{1}{|\Omega|}\int_\Omega f_1(u)|^{\frac{p}{\alpha}} \le c_2 \int_\Omega (1+u)^p + c_3 \int_\Omega \left(\int_\Omega (1+u)^\alpha\right)^{\frac{p}{\alpha}} \le c_2 \int_\Omega (1+u)^p + c_3 \int_\Omega \left(\int_\Omega (1+u)^p\right)^{\frac{p}{\alpha}} \le c_4,$$

which gives $f_1(u) - \frac{1}{|\Omega|}\int_\Omega f_1(u) \in L^\infty((0,T_{max}); L^{\frac{p}{\alpha}}(\Omega))$, in turn $v \in L^\infty((0,T_{max}); W^{2,\frac{p}{\alpha}}(\Omega))$ and, hence, through the Sobolev embeddings $\nabla v \in L^\infty((0,T_{max}); W^{1,\frac{p}{\alpha}}(\Omega)) \hookrightarrow L^\infty((0,T_{max}); L^\infty(\Omega))$. Consequently, thanks to the Hölder inequality (recall that from Lemma 2 one has that $p > m_2(n+2)(n+1)$), by using the uniform-in-time boundedness of u in $L^{\bar{p}}(\Omega)$ we can write on $(0, T_{max})$

$$\int_\Omega |u(u+1)^{m_2-1}\nabla v|^{(n+2)(n+1)} \le \|\nabla v(\cdot,t)\|^{(n+2)(n+1)}_{L^\infty(\Omega)} |\Omega|^{\frac{p-m_2(n+2)(n+1)}{p}} \left(\int_\Omega (u+1)^p\right)^{\frac{m_2(n+2)(n+1)}{p}} \le c_5.$$

Reasoning in a similar way on the third equation of (8), we have $\nabla w \in L^\infty((0,T_{max}); L^\infty(\Omega))$,

$$\int_\Omega |u(u+1)^{m_3-1}\nabla w|^{(n+2)(n+1)} \le c_6 \quad \textit{for all } t \in (0, T_{max}),$$

and as a consequence

$$f = \chi u(u+1)^{m_2-1}\nabla v - \xi u(u+1)^{m_3-1}\nabla w \in L^\infty((0,T_{max}); L^{(n+2)(n+1)}(\Omega)).$$

Since all the hypotheses of Lemma A.1 in Tao and Winkler[43] are fulfilled, we have the claim. □

Now we are in a position to prove our first results.

Proof of Theorem 1

Let (u, v, w) be a given blow-up solution at some finite time T_{max} to problem (8). If u was not unbounded in some $L^{\mathfrak{p}}(\Omega)$-norm, Lemma 5 would imply the uniform-in-time boundedness of u, contradicting hypothesis (13).

Proof of Theorem 2

By making use of (26) and (27) (up to a constant) or altogether (31)–(33) and again (27), we observe that bound (24) can essentially be reorganized as

$$\varphi'(t) \le -c_1 \int_\Omega |\nabla (u+1)^{\frac{p+m_1-1}{2}}|^2 + c_2 \int_\Omega (u+1)^{p+m_2+\alpha-1} + c_3 \quad \text{on } (0, T_{max}). \tag{35}$$

Moreover, by recalling (20k) and (20l), we can derive again through the Gagliardo–Nirenberg and Young's inequalities this bound

$$\begin{aligned} c_2\int_\Omega (u+1)^{p+m_2+\alpha-1} &= c_2\|(u+1)^{\frac{p+m_1-1}{2}}\|_{L^{\frac{2(p+m_2+\alpha-1)}{p+m_1-1}}(\Omega)}^{\frac{2(p+m_2+\alpha-1)}{p+m_1-1}} \\ &\leq c_4\left(\|\nabla(u+1)^{\frac{p+m_1-1}{2}}\|_{L^2(\Omega)}^{\sigma\bar\theta}\|(u+1)^{\frac{p+m_1-1}{2}}\|_{L^{\frac{2p}{p+m_1-1}}(\Omega)}^{\sigma(1-\bar\theta)}+\|(u+1)^{\frac{p+m_1-1}{2}}\|_{L^{\frac{2p}{p+m_1-1}}(\Omega)}^{\sigma}\right) \\ &\leq c_5\left(\int_\Omega|\nabla(u+1)^{\frac{p+m_1-1}{2}}|^2\right)^{\frac{\sigma\bar\theta}{2}}\left(\int_\Omega(u+1)^p\right)^{\frac{p+m_1-1}{2p}\sigma(1-\bar\theta)}+c_6\left(\int_\Omega(u+1)^p\right)^{\frac{p+m_1-1}{2p}\sigma} \\ &\leq c_1\int_\Omega|\nabla(u+1)^{\frac{p+m_1-1}{2}}|^2+c_7\left(\int_\Omega(u+1)^p\right)^{\gamma}+c_8\left(\int_\Omega(u+1)^p\right)^{\delta} \quad \text{on } (0,T_{max}). \end{aligned} \tag{36}$$

Finally, by inserting relation (36) into (35), we obtain for proper positive $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$

$$\varphi'(t)\leq \mathcal{A}\varphi^\gamma(t)+\mathcal{B}\varphi^\delta(t)+\mathcal{C}\quad \text{on } (0,T_{max}). \tag{37}$$

Since $p=\bar p>\mathfrak{p}$, from Theorem 1 we know that $\limsup_{t\to T_{max}}\frac{1}{p}\int_\Omega(u+1)^p=\infty$. On the other hand, since $\varphi(t)$ satisfies relation (37) for any $0<t<T_{max}$, the function $\Psi(\xi)=\mathcal{A}\xi^\gamma+\mathcal{B}\xi^\delta+\mathcal{C}$ obeys the Osgood criterion (14). Thereafter, by integrating (37) between 0 and T_{max}, we obtain estimate (15), and the first conclusion is achieved.

As to the derivation of the explicit expression for the lower bound T, let us reduce (37) as follows: from $|\Omega|\leq\int_\Omega(u+1)^p=p\varphi(t)$, we can estimate $\mathcal{C}$ in relation (37) as

$$\mathcal{C}\leq p\frac{\mathcal{C}}{|\Omega|}\varphi(t)=:\bar{\mathcal{C}}\varphi(t),$$

so that (37) can be rewritten in this form:

$$\varphi'(t)\leq \mathcal{A}\varphi^\gamma(t)+\mathcal{B}\varphi^\delta(t)+\bar{\mathcal{C}}\varphi(t)\quad \text{on } (0,T_{max}). \tag{38}$$

Now, since φ blows up at finite time T_{max}, there exists a time $t_1\in[0,T_{max})$ such that

$$\varphi(t)\geq\varphi(0)\quad \text{for all } t\geq t_1.$$

From $\gamma>\delta>1$ (recall (20j)), we can estimate the second and third terms on the right-hand side of (38) by means of φ^γ:

$$\varphi^\delta(t)\leq\varphi(0)^{\delta-\gamma}\varphi^\gamma(t)\quad\text{and}\quad\varphi(t)\leq\varphi(0)^{1-\gamma}\varphi^\gamma(t)\quad\text{for all } t\geq t_1. \tag{39}$$

By plugging expressions (39) into (38), we obtain for

$$\mathcal{D}=\mathcal{A}+\mathcal{B}\varphi(0)^{\delta-\gamma}+\bar{\mathcal{C}}\varphi(0)^{1-\gamma},$$

$$\varphi'(t)\leq\mathcal{D}\varphi^\gamma(t)\quad\text{for all } t\geq t_1, \tag{40}$$

so that an integration of (40) on (t_1, T_{max}) yields this explicit lower bound for T_{max}:

$$T = \frac{\varphi(0)^{1-\gamma}}{\mathcal{D}(\gamma-1)} = \int_{\varphi(0)}^{\infty} \frac{d\tau}{\mathcal{D}\tau^{\gamma}} \le \int_{t_1}^{T_{max}} d\tau \le \int_0^{T_{max}} d\tau = T_{max}.$$

6 | FINITE-TIME BLOW-UP TO A SIMPLIFIED VERSION OF PROBLEM (8)

This section is dedicated to prove finite time blow-up for solutions to problem (8) in a more specific case; in our computations, we will partially extend,[34,36] where adapting the method in Winkler[24] blow-up is, respectively, established in a model with only attraction, nonlinear diffusion and sensitivity, and logistic term, and in an attraction–repulsion one, with nonlinear diffusion but linear sensitivities and logistics.

6.1 | Detecting unbounded solutions to problem (8) for $m_2 = m_3 > 0$

Let us fix $m_2 = m_3 > 0$ in model (8), and in turn let us set $z = \chi v - \xi w$, $m(t) = \chi m_1(t) - \xi m_2(t)$ and $f(u) = \chi f_1(u) - \xi f_2(u)$, being $m_1(t)$ and $m_2(t)$ defined in (9); in this way, problem (8) itself is reduced into

$$\begin{cases} u_t = \nabla \cdot ((u+1)^{m_1-1}\nabla u - u(u+1)^{m_2-1}\nabla z) + \lambda u - \mu u^k & \text{in } \Omega \times (0, T_{max}),\\ 0 = \Delta z - m(t) + f(u) & \text{in } \Omega \times (0, T_{max}),\\ u_\nu = z_\nu = 0 & \text{on } \partial\Omega \times (0, T_{max}),\\ u(x,0) = u_0(x) & x \in \bar{\Omega},\\ \int_\Omega z(x,t)dx = 0 & \text{for all } t \in (0, T_{max}). \end{cases} \tag{41}$$

In particular, if we confine our study to radially symmetric cases, by setting $r := |x|$ and by considering $\Omega = B_R(0) \subset \mathbb{R}^n$, $n \ge 1$ and some $R > 0$, the radially symmetric local solution

$$(u, z) = (u(r,t), z(r,t))$$

to model (41) solves the following scalar problem:

$$\begin{cases} r^{n-1}u_t = (r^{n-1}(u+1)^{m_1-1}u_r)_r - (r^{n-1}u(u+1)^{m_2-1}z_r)_r + \lambda r^{n-1}u - \mu r^{n-1}u^k & r \in (0,R), t \in (0, T_{max}),\\ 0 = (r^{n-1}z_r)_r - r^{n-1}m(t) + r^{n-1}f(u) & r \in (0,R), t \in (0, T_{max}),\\ u_r = z_r = 0 & r = R, t \in (0, T_{max}),\\ u(r,0) = u_0(r) & r \in (0,R),\\ \int_0^R r^{n-1}z(r,t)\,dr = 0 & \text{for all } t \in (0, T_{max}). \end{cases} \tag{42}$$

In the same spirit of Jäger and Luckhaus,[5] we introduce the mass accumulation function

$$U(s,t) := \int_0^{s^{\frac{1}{n}}} \rho^{n-1}u(\rho,t)\,d\rho \quad \text{for } s \in [0, R^n] \text{ and } t \in [0, T_{max}), \tag{43}$$

which implies that

$$U_s(s,t) = \frac{1}{n}u(s^{\frac{1}{n}},t) \quad \text{and} \quad U_{ss}(s,t) = \frac{1}{n^2}s^{\frac{1}{n}-1}u_r(s^{\frac{1}{n}},t) \quad \text{for } s \in (0,R^n) \text{ and } t \in (0,T_{max}). \tag{44}$$

By the definition of U and by exploiting (44), we obtain

$$\begin{aligned} U_t(s,t) &= s^{1-\frac{1}{n}}(u(s^{\frac{1}{n}},t)+1)^{m_1-1}u_r(s^{\frac{1}{n}},t) - s^{1-\frac{1}{n}}u(s^{\frac{1}{n}},t)(u(s^{\frac{1}{n}},t)+1)^{m_2-1}z_r(s^{\frac{1}{n}},t) \\ &\quad + \lambda\int_0^{s^{\frac{1}{n}}}\rho^{n-1}u(\rho,t)\,d\rho - \mu\int_0^{s^{\frac{1}{n}}}\rho^{n-1}u^k(\rho,t)\,d\rho \\ &= n^2s^{2-\frac{2}{n}}(nU_s(s,t)+1)^{m_1-1}U_{ss}(s,t) - ns^{1-\frac{1}{n}}U_s(s,t)(nU_s(s,t)+1)^{m_2-1}z_r(s^{\frac{1}{n}},t) \\ &\quad + \lambda U(s,t) - \mu n^{k-1}\int_0^s U_s^k(\sigma,t)\,d\sigma \quad \text{for } s \in (0,R^n) \text{ and } t \in (0,T_{max}). \end{aligned} \tag{45}$$

Besides, with an integration over $(0,r)$ of the second equation in (42) and the substitution $r = s^{\frac{1}{n}}$, we arrive at

$$z_r = \frac{m(t)}{n}s^{\frac{1}{n}} - \frac{1}{n}s^{\frac{1}{n}-1}\int_0^s f(nU_s(\sigma,t))\,d\sigma \quad \text{for } s \in (0,R^n) \text{ and } t \in (0,T_{max}),$$

which inserted into relation (45) gives (observe $U \geq 0$)

$$\begin{aligned} U_t(s,t) &\geq n^2s^{2-\frac{2}{n}}(nU_s+1)^{m_1-1}U_{ss} - sU_s(nU_s+1)^{m_2-1}m(t) + U_s(nU_s+1)^{m_2-1}\int_0^s f(nU_s(\sigma,t))\,d\sigma \\ &\quad - \mu n^{k-1}\int_0^s U_s^k(\sigma,t)\,d\sigma \quad \text{for all } s \in (0,R^n) \text{ and } t \in (0,T_{max}). \end{aligned} \tag{46}$$

In addition, given $s_0 \in (0,R^n)$, $\gamma \in (-\infty,1)$, and U as in (43), we introduce the moment-type functional

$$\phi(t) := \int_0^{s_0} s^{-\gamma}(s_0-s)U(s,t)\,ds \quad \text{for } t \in [0,T_{max}), \tag{47}$$

which is well defined and belongs to $C^0([0,T_{max}))\cap C^1((0,T_{max}))$. Moreover, we define

$$\psi(t) := \int_0^{s_0} s^{1-\gamma}(s_0-s)U_s^{m_2+\alpha}(s,t)\,ds \quad \text{for } t \in (0,T_{max}), \tag{48}$$

and the set

$$S_\phi = \left\{ t \in (0,T_{max}) : \phi(t) \geq \frac{M-s_0}{(1-\gamma)(2-\gamma)\omega_n}s_0^{2-\gamma} \right\}, \tag{49}$$

where M is the bound of the $L^1(\Omega)$-norm of u established in (19) and $\omega_n = n|B_1(0)|$. With these preparations in our hands, let us give a series of necessary lemmas, some of which are not new.

We start with a result dealing with the concavity of U and some estimate for $m(t)$.

Lemma 6. *Let f_1, f_2, and u_0 satisfy (10) and (12), $\alpha > \beta$ and $\gamma \in (-\infty, 1)$. Then the following relations hold:*

$$U_{ss}(s,t) \leq 0 \quad \text{for all } s \in (0, R^n) \text{ and } t \in (0, T_{max}),$$

and

$$m(t) \leq c_0 + \frac{1}{2s}\int_0^s f(nU_s(\sigma,t))\, d\sigma \quad \text{for } s_0 \in \left(0, \frac{R^n}{6}\right] \text{and for all } s \in (0, s_0), \tag{50}$$

for all $t \in S_\phi$, and with some c_0.

Proof. *As to the concavity property, the proof is based on minor adjustments of Lemma 3.2 in Wang et al.*[36] *or Lemma 2.2 in Winkler*[24]*. The remaining conclusion follows by closely reasoning as in Lemma 2.5 in Liu and Li*[35] *, but herein we have to underline some specifics. First of all, the derivation of bound (50) requires a previous estimate from above of $z_r(r,t)$; such an estimate involves, inter alia, the integral*

$$-\int_0^r \rho^{n-1} f(u(\rho,t))\, d\rho. \tag{51}$$

We observe that in model (41) the function $f(u) = \chi f_1(u) - \xi f_2(u)$, $u \geq 0$, is not automatically nonnegative, and henceforth the term (51) cannot be neglected (conversely to what done in Lemma 2.1 in Winkler[24] *). In particular, we can see from $\alpha > \beta$ and (12) that*

$$f(u) \geq 0, \quad \text{for } u \geq C^* := \max\left\{1, \left(\frac{\xi k_2}{\chi k_3}\right)^{\frac{1}{\alpha-\beta}}\right\}, \quad \text{and} \quad f(u) < 0 \quad \text{for } u < C^*.$$

Consequently, by splitting (51) in the sets where $u \geq C^*$ and $u < C^*$, dropping the nonpositive contribution tied to the first set, only the nonnegative integral

$$-\int_0^r \chi_{\{u(\cdot,t)<C^*\}}(\rho)\rho^{n-1} f(u(\rho,t))\, d\rho$$

has to be controlled; to this aim, we use assumption (10), from which it is seen that the value $\max_{s\in[0,C^*]} |f(s)|$ is finite and u_0-independent. In this way, we have

$$-\int_0^r \chi_{\{u(\cdot,t)<C^*\}}(\rho)\rho^{n-1} f(u(\rho,t))\, d\rho \leq \max_{s\in[0,C^*]} |f(s)| \int_0^r \chi_{\{u(\cdot,t)<C^*\}}(\rho)\rho^{n-1}\, d\rho \leq \max_{s\in[0,C^*]} |f(s)| \int_0^r \rho^{n-1}\, d\rho.$$

The remaining technical details can be found in Liu and Li,[35] *where similar splitting procedures on integrals involving f_1 are as well performed; in particular, for a further finite and u_0-independent $L > 0$ one can achieve this expression for c_0:*

$$c_0 = \chi f_1\left(\frac{8n}{2^\gamma(3-\gamma)\omega_n}\right) + \frac{1}{6}\left(\frac{\chi k_3(\alpha-\beta)}{\beta}\left(\frac{2\xi k_2 \beta}{\chi k_3 \alpha}\right)^{\frac{\alpha}{\alpha-\beta}} + L(\chi+2)\right) > 0.$$

(We point out that c_0 will be used in some other places below.) □

Let us now start with the analysis of the temporal evolution of ϕ.

Lemma 7. *Under the same assumptions of Lemma 6, let $s_0 \in (0, \frac{R^n}{6}]$. Then*

$$\begin{aligned}
\phi'(t) \geq{}& n^2 \int_0^{s_0} s^{2-\frac{2}{n}-\gamma}(s_0-s)(nU_s+1)^{m_1-1}U_{ss}\,ds - c_0\int_0^{s_0} s^{1-\gamma}(s_0-s)U_s(nU_s+1)^{m_2-1}\,ds\\
&+\frac{1}{2}\int_0^{s_0} s^{-\gamma}(s_0-s)U_s(nU_s+1)^{m_2-1}\left[\int_0^s f(nU_s(\sigma,t))\,d\sigma\right]ds\\
&-\mu n^{k-1}\int_0^{s_0} s^{-\gamma}(s_0-s)\left[\int_0^s U_s^k(\sigma,t)\,d\sigma\right]ds\\
=:{}& I_1+I_2+I_3+I_4 \quad \textit{for all } t\in S_\phi.
\end{aligned} \tag{52}$$

Proof. *By the definition of ϕ (recall (47)) and exploiting (46), we get this estimate*

$$\begin{aligned}
\phi'(t) ={}& \int_0^{s_0} s^{-\gamma}(s_0-s)U_t(s,t)\,ds\\
\geq{}& n^2\int_0^{s_0} s^{2-\frac{2}{n}-\gamma}(s_0-s)(nU_s+1)^{m_1-1}U_{ss}\,ds - \int_0^{s_0} s^{1-\gamma}(s_0-s)U_s(nU_s+1)^{m_2-1}m(t)\,ds\\
&+\int_0^{s_0} s^{-\gamma}(s_0-s)U_s(nU_s+1)^{m_2-1}\left[\int_0^s f(nU_s(\sigma,t))\,d\sigma\right]ds\\
&-\mu n^{k-1}\int_0^{s_0} s^{-\gamma}(s_0-s)\left[\int_0^s U_s^k(\sigma,t)\,d\sigma\right]ds \quad \textit{for all } t\in S_\phi.
\end{aligned}$$

Finally, by applying (50), we obtain the thesis. □

The next results provide some lower bounds of I_1, I_2, I_3, and I_4 in respect of $\psi(t)$ defined in (48).

Lemma 8. *Let u_0 satisfy the related assumptions in (10) and (12), $m_1 \in \mathbb{R}$, $m_2, \alpha > 0$, $k > 1$, and $\gamma \in (-\infty, 1)$.*

➢ *If m_1, m_2, α and γ comply with*

$$m_2+\alpha > m_1 \quad \textit{and} \quad \gamma > 2-\frac{2}{n}\frac{(m_2+\alpha)}{(m_2+\alpha-m_1)} \quad \textit{if} \quad m_1 \geq 0,$$
$$\gamma < 2-\frac{2}{n} \quad \textit{if} \quad m_1 < 0,$$

then there exist $\varepsilon > 0$ (sufficiently small), and c_1, c_2, c_3, c_4, c_5 such that for any $s_0 \in (0, \frac{R^n}{6}]$

$$I_1 \geq \begin{cases} -c_1 s_0^{(3-\gamma)\frac{(m_2+\alpha-m_1)}{m_2+\alpha}-\frac{2}{n}}\psi^{\frac{m_1}{m_2+\alpha}}(t) - c_2 s_0^{3-\gamma-\frac{2}{n}} & \textit{if } m_1 > 0,\\ -c_3 s_0^{(3-\gamma)\frac{(m_2+\alpha-\varepsilon)}{m_2+\alpha}-\frac{2}{n}}\psi^{\frac{\varepsilon}{m_2+\alpha}}(t) - c_4 s_0^{3-\gamma-\frac{2}{n}} & \textit{if } m_1 = 0,\\ -c_5 s_0^{3-\gamma-\frac{2}{n}} & \textit{if } m_1 < 0 \end{cases}$$

for all $t \in S_\phi$.

➢ *If m_2, α, k, and γ are such that*

$$m_2 + \alpha > k \quad and \quad 2 - \frac{(m_2+\alpha)}{k} < \gamma < 1,$$

then there exists c_6 such that for any $s_0 \in (0, \frac{R^n}{6}]$

$$I_4 \geq -c_6 \psi^{\frac{k}{m_2+\alpha}}(t) s_0^{(3-\gamma)\frac{(m_2+\alpha-k)}{m_2+\alpha}} \quad for\ all\ t \in S_\phi.$$

Proof. *The proof can be found in Lemmas 3.6 and 3.7 in Tanaka,[34] where the related details are given for a model in which the expressions of the diffusion and the sensitivity are more general than $(u+1)^{m_1-1}$ and $u(u+1)^{m_2-1}$ appearing in (41).* □

As to the estimate of $I_2 + I_3$, we need to rearrange some computations; this is exactly where we have to go beyond the analysis in Tanaka[34] and Wang et al.[36]

Lemma 9. *Under the same assumptions of Lemma 6, let, moreover, $m_2, \alpha > 0$ comply with $m_2 + \alpha > 1$. Then for some c_1, c_2, and c_3, we have*

$$I_2 + I_3 \geq c_1\psi(t) - c_2\psi^{\frac{m_2}{m_2+\alpha}}(t) s_0^{(3-\gamma)\frac{\alpha}{m_2+\alpha}} - c_3 s_0^{3-\gamma} \quad for\ any\ s_0 \in \left(0, \frac{R^n}{6}\right] \ and\ for\ all\ t \in S_\phi. \tag{53}$$

Proof. *Since $\alpha > \beta$, by applying the Young inequality and (12), we get*

$$\xi f_2(u) \leq \xi k_2 (u+1)^\beta \leq \frac{\chi k_3}{2}(u+1)^\alpha + c_4 \leq \frac{\chi}{2} f_1(u) + c_4,$$

with $c_4 = (\frac{2\beta\xi k_2}{\chi k_3 \alpha})^{\frac{\alpha}{\alpha-\beta}} \frac{\chi k_3(\alpha-\beta)}{2\beta} > 0$, which implies

$$\frac{\chi}{2} f_1(u) - c_4 \leq f(u) \leq \chi f_1(u). \tag{54}$$

From the concavity of U in Lemma 6, it is seen that U_s is nonincreasing, namely $U_s(\sigma,t) \geq U_s(s,t)$ for any $\sigma \in (0,s)$. Henceforth, since f_1 is nondecreasing (recall (12)), we have

$$\int_0^s f_1(nU_s(\sigma,t))\,d\sigma \geq \int_0^s f_1(nU_s(s,t))\,d\sigma = s f_1(nU_s(s,t)) \quad for\ all\ s \in (0,s_0)\ and\ t \in (0,T_{max}). \tag{55}$$

Therefore, from (54) and (55) we derive the estimate

$$\begin{aligned}
I_3 &= \frac{1}{2}\int_0^{s_0} s^{-\gamma}(s_0 - s) U_s (nU_s+1)^{m_2-1}\left[\int_0^s f(nU_s(\sigma,t))\,d\sigma\right] ds \\
&\geq \frac{1}{2}\int_0^{s_0} s^{-\gamma}(s_0 - s) U_s (nU_s+1)^{m_2-1}\left(\frac{\chi}{2} s f_1(nU_s) - c_4 s\right) ds \\
&\geq \frac{\chi k_3}{4}\int_0^{s_0} s^{1-\gamma}(s_0-s) U_s (nU_s+1)^{m_2-1+\alpha}\,ds - \frac{c_4}{2}\int_0^{s_0} s^{1-\gamma}(s_0-s)U_s(nU_s+1)^{m_2-1}\,ds \quad for\ all\ t \in S_\phi.
\end{aligned}$$

Then, for c_0 as in Lemma 6,

$$I_2 + I_3 \geq \frac{\chi k_3}{4}\int_0^{s_0} s^{1-\gamma}(s_0-s)U_s(nU_s+1)^{m_2-1+\alpha}\,ds - \left(c_0+\frac{c_4}{2}\right)\int_0^{s_0} s^{1-\gamma}(s_0-s)U_s(nU_s+1)^{m_2-1}\,ds \quad \text{for all } t\in S_\phi. \tag{56}$$

Now, we focus on the first integral in (56). Since $m_2+\alpha-1>0$, clearly we get that $(nU_s+1)^{m_2+\alpha-1} > (nU_s)^{m_2+\alpha-1}$, and by exploiting this latter we have

$$\frac{\chi k_3}{4}\int_0^{s_0} s^{1-\gamma}(s_0-s)U_s(nU_s+1)^{m_2-1+\alpha}\,ds \geq \frac{\chi k_3 n^{m_2+\alpha-1}}{4}\int_0^{s_0} s^{1-\gamma}(s_0-s)U_s^{m_2+\alpha}\,ds = \frac{\chi k_3 n^{m_2+\alpha-1}}{4}\psi(t) \quad \text{on } S_\phi. \tag{57}$$

In order to treat the second integral in (56), we use inequality (29): since $m_2>0$, we obtain for $c_5 = (c_0+\frac{c_4}{2})$ the following estimate:

$$\begin{aligned}
&-c_5\int_0^{s_0} s^{1-\gamma}(s_0-s)U_s(nU_s+1)^{m_2-1}\,ds = -\frac{1}{n}c_5\int_0^{s_0} s^{1-\gamma}(s_0-s)\frac{nU_s}{nU_s+1}(nU_s+1)^{m_2}\,ds\\
&\geq -\frac{1}{n}c_5\int_0^{s_0} s^{1-\gamma}(s_0-s)(nU_s+1)^{m_2}\,ds\\
&\geq -c_6\int_0^{s_0} s^{1-\gamma}(s_0-s)U_s^{m_2}\,ds - c_7\int_0^{s_0} s^{1-\gamma}(s_0-s)\,ds \quad \text{for all } t\in S_\phi.
\end{aligned} \tag{58}$$

Additionally, from $\frac{m_2}{m_2+\alpha}<1$ we can apply the Hölder inequality to the first integral in (58), which leads to

$$\begin{aligned}
\int_0^{s_0} s^{1-\gamma}(s_0-s)U_s^{m_2}\,ds &= \int_0^{s_0}\left(s^{1-\gamma}(s_0-s)U_s^{m_2+\alpha}\right)^{\frac{m_2}{m_2+\alpha}} s^{\frac{(1-\gamma)\alpha}{m_2+\alpha}}(s_0-s)^{\frac{\alpha}{m_2+\alpha}}\,ds\\
&\leq \left(\int_0^{s_0} s^{1-\gamma}(s_0-s)U_s^{m_2+\alpha}\,ds\right)^{\frac{m_2}{m_2+\alpha}}\left(\int_0^{s_0} s^{1-\gamma}(s_0-s)\,ds\right)^{\frac{\alpha}{m_2+\alpha}}\\
&= \psi^{\frac{m_2}{m_2+\alpha}}(t)\,((2-\gamma)(3-\gamma))^{-\frac{\alpha}{m_2+\alpha}} s_0^{\frac{(3-\gamma)\alpha}{m_2+\alpha}} \quad \text{for all } t\in S_\phi.
\end{aligned} \tag{59}$$

By putting (59) into (58), we obtain for every $t\in S_\phi$

$$-c_5\int_0^{s_0} s^{1-\gamma}(s_0-s)U_s(nU_s+1)^{m_2-1}\,ds \geq -c_6\psi^{\frac{m_2}{m_2+\alpha}}(t)\,((2-\gamma)(3-\gamma))^{-\frac{\alpha}{m_2+\alpha}} s_0^{\frac{(3-\gamma)\alpha}{m_2+\alpha}} - \frac{c_7}{(2-\gamma)(3-\gamma)}s_0^{3-\gamma}, \tag{60}$$

so by invoking (56) and taking into account (57) and (60), we can conclude. □

In order to obtain the desired superlinear ODI for ϕ, we have to rely on some relations involving U and ψ and ϕ.

Lemma 10. *Let u_0 satisfy its related assumptions in (10) and (12), $m_2,\alpha>0$ and $\gamma\in(-\infty,1)$ be such that*

$$m_2+\alpha>1 \quad \text{and} \quad 2-(m_2+\alpha)<\gamma<1.$$

Then there exist c_1, c_2 *such that for any* $s_0 \in (0, \frac{R^n}{6}]$

$$U(s,t) \le c_1 s^{\frac{m_2+\alpha+\gamma-2}{m_2+\alpha}} (s_0 - s)^{-\frac{1}{m_2+\alpha}} \psi^{\frac{1}{m_2+\alpha}}(t) \quad \text{for all } s \in (0, s_0) \text{ and } t \in S_\phi, \tag{61}$$

and

$$\psi(t) \ge c_2 s_0^{-(3-\gamma)(m_2+\alpha-1)} \phi^{m_2+\alpha}(t) \quad \text{for all } t \in S_\phi. \tag{62}$$

Proof. *The proof of (61) and (62) follows by Lemma 3.8 in Tanaka*[34] *and Lemma 3.7 in Wang and Li*[39], *respectively.* ☐

The following is precisely the lemma relying on assumption $(\mathcal{H}_6)$.

Lemma 11. *Under the same assumptions of Lemma 6, let* $m_1 \in \mathbb{R}$, $m_2, \alpha, \beta > 0$ *and* $k > 1$ *be such that constrains* $(\mathcal{H}_6)$ *in Assumptions 1 are satisfied. Then, there exist* $\varepsilon > 0$ *small enough,* $\gamma = \gamma(m_1, m_2, \alpha, k) \in (-\infty, 1)$ *and* c_1, c_2, c_3, c_4, c_5 *and* c_6 *such that for* $s_0 \in (0, \frac{R^n}{6}]$ *one has*

$$\phi'(t) \ge \begin{cases} c_1 s_0^{-(3-\gamma)(m_2+\alpha-1)} \phi^{m_2+\alpha}(t) - c_2 s_0^{3-\gamma-\frac{2}{n}\frac{(m_2+\alpha)}{m_2+\alpha-m_1}} & \text{if } m_1 > 0, \\ c_3 s_0^{-(3-\gamma)(m_2+\alpha-1)} \phi^{m_2+\alpha}(t) - c_4 s_0^{3-\gamma-\frac{2}{n}\frac{(m_2+\alpha)}{m_2+\alpha-\varepsilon}} & \text{if } m_1 = 0, \\ c_5 s_0^{-(3-\gamma)(m_2+\alpha-1)} \phi^{m_2+\alpha}(t) - c_6 s_0^{3-\gamma-\frac{2}{n}} & \text{if } m_1 < 0, \end{cases} \quad \text{for all } t \in S_\phi.$$

Proof. *By substituting the estimates of* I_1, I_2, I_3, I_4 *given in Lemmas 8–10 into Lemma 7, we can adapt Lemma 3.10 in Tanaka*[34] *taking into account that (3.12) in Lemma 3.5 in Tanaka*[34] *is replaced by (53), so being necessary manipulating the term involving* $\psi^{\frac{m_2}{m_2+\alpha}}(t) s_0^{(3-\gamma)\frac{\alpha}{m_2+\alpha}}$ *by the Young inequality and relation (62).* ☐

The previous lemmas allows us to conclude.

Proof of Theorem 3

We focus only on the situation where $m_1 > 0$, the cases $m_1 = 0$ and $m_1 < 0$ being similar. For $C = \left(\frac{\lambda}{\mu}|\Omega|^{k-1}\right)^{\frac{1}{k-1}}$, let us fix M_0 according to our hypothesis $M_0 \ge C$. Next, since $(\mathcal{H}_6)$ holds, we pick $s_0 \le \frac{R^n}{6}$ small enough such that

$$s_0 \le \frac{M_0}{2} \tag{63}$$

and for all $\gamma \in (-\infty, 1)$

$$s_0^{m_2+\alpha-\frac{2}{n}\frac{(m_2+\alpha)}{m_2+\alpha-m_1}} \le \frac{c_7}{2c_8}\left(\frac{M_0}{2(1-\gamma)(2-\gamma)\omega_n}\right)^{m_2+\alpha}. \tag{64}$$

Moreover, from

$$\frac{M_0-\epsilon_0}{\omega_n}\int_{s_*}^{s_0} s^{-\gamma}(s_0-s)\,ds=\frac{M_0-\epsilon_0}{\omega_n}\left[\frac{s_0^{2-\gamma}}{(1-\gamma)(2-\gamma)}-s_*^{1-\gamma}\frac{s_0}{1-\gamma}+\frac{s_*^{2-\gamma}}{2-\gamma}\right],$$

we can take $\epsilon_0\in(0,\frac{s_0}{2})$ and $s_*\in(0,s_0)$ so small and satisfying

$$\frac{M_0-\epsilon_0}{\omega_n}\int_{s_*}^{s_0} s^{-\gamma}(s_0-s)\,ds>\frac{M_0-s_0}{(1-\gamma)(2-\gamma)\omega_n}s_0^{2-\gamma}. \tag{65}$$

We set $r_*:=s_*^{\frac{1}{n}}\in(0,R)$ and require to u_0 to comply with (10), (12), and (16). With such u_0 in our hands, we recall the definition of ϕ in (47) and the assumption $M=\max\{M_0,C\}=M_0$; in order to show that $T_{max}<\infty$, we argue by establishing that if by contradiction $T_{max}=\infty$, then $\tilde{T}=\sup\tilde{S}\in(0,\infty]$, with

$$\tilde{S}:=\left\{T>0\text{ such that }\phi(t)>\frac{M_0-s_0}{(1-\gamma)(2-\gamma)\omega_n}s_0^{2-\gamma}\quad\text{for all }t\in[0,T]\right\}, \tag{66}$$

would be at the same time finite and infinite.

First, by observing that $\phi(0)>\frac{M_0-s_0}{(1-\gamma)(2-\gamma)\omega_n}s_0^{2-\gamma}$, the continuity of ϕ ensures that $\tilde{S}$ is not empty. Indeed, we have that for any $s\in(s_*,R^n)$

$$U(s,0)\geq U(s_*,0)=\frac{1}{\omega_n}\int_{B_{r_*}(0)}u_0\,dx\geq\frac{M_0-\epsilon_0}{\omega_n},$$

so we deduce from (65) that

$$\phi(0)\geq\int_{s_*}^{s_0}s^{-\gamma}(s_0-s)U(s,0)\,ds\geq\frac{M_0-\epsilon_0}{\omega_n}\int_{s_*}^{s_0}s^{-\gamma}(s_0-s)\,ds>\frac{M_0-s_0}{(1-\gamma)(2-\gamma)\omega_n}s_0^{2-\gamma}.$$

Now, by exploiting (66) and (63) leads to

$$\phi(t)\geq\frac{M_0}{2(1-\gamma)(2-\gamma)\omega_n}s_0^{2-\gamma}\quad\text{for all }t\in(0,\tilde{T}),$$

so that $(0,\tilde{T})\subset S_\phi$ (recall (49) with $M=M_0$).

On the other hand, under the assumption $(\mathcal{H}_6)$, we can apply Lemma 11 and find $\gamma\in(-\infty,1)$, c_7,c_8 such that for u_0 and s_0 as above, one deduces

$$\phi'(t)\geq c_7 s_0^{-(3-\gamma)(m_2+\alpha-1)}\phi^{m_2+\alpha}(t)-c_8 s_0^{3-\gamma-\frac{2}{n}\frac{(m_2+\alpha)}{m_2+\alpha-m_1}}\quad\text{for all }t\in S_\phi. \tag{67}$$

Second, condition (64) implies

$$\frac{\frac{c_7}{2}s_0^{-(3-\gamma)(m_2+\alpha-1)}\phi^{m_2+\alpha}(t)}{c_8 s_0^{3-\gamma-\frac{2}{n}\frac{(m_2+\alpha)}{m_2+\alpha-m_1}}}\geq\frac{c_7}{2c_8}\left(\frac{M_0}{2(1-\gamma)(2-\gamma)\omega_n}\right)^{m_2+\alpha}s_0^{-(m_2+\alpha)+\frac{2}{n}\frac{(m_2+\alpha)}{m_2+\alpha-m_1}}\geq 1\quad\text{on }(0,\tilde{T}),$$

which subsequently gives from (67)

$$\phi'(t) \geq \frac{c_7}{2} s_0^{-(3-\gamma)(m_2+\alpha-1)} \phi^{m_2+\alpha}(t) \geq 0 \quad \text{for all } t \in (0, \tilde{T}). \tag{68}$$

With these facts, let us now establish the inconsistency $\tilde{T} < \infty$ and $\tilde{T} = \infty$. By integrating inequality (68) on $(0, \tilde{T})$, we have that

$$\int_0^{\tilde{T}} \left(\frac{1}{1-(m_2+\alpha)} \phi^{1-(m_2+\alpha)}(t) \right)' dt \geq \int_0^{\tilde{T}} \frac{c_7}{2} s_0^{-(3-\gamma)(m_2+\alpha-1)} \, dt,$$

so that due to $m_2 + \alpha - 1 > 0$ and the nonnegative property of ϕ

$$\frac{c_7}{2} s_0^{-(3-\gamma)(m_2+\alpha-1)} \tilde{T} \leq \frac{\phi^{1-(m_2+\alpha)}(\tilde{T})}{1-(m_2+\alpha)} - \frac{\phi^{1-(m_2+\alpha)}(0)}{1-(m_2+\alpha)} \leq \frac{\phi^{1-(m_2+\alpha)}(0)}{m_2+\alpha-1},$$

or explicitly

$$\tilde{T} \leq \frac{2}{c_7(m_2+\alpha-1)\phi^{m_2+\alpha-1}(0)} s_0^{(3-\gamma)(m_2+\alpha-1)} < \infty.$$

Nevertheless, we have to exclude the finiteness of $\tilde{T} = \sup \tilde{S}$; in fact, by the definition of $\tilde{S}$ in (66), we should have

$$\phi(\tilde{T}) = \frac{M_0 - s_0}{(1-\gamma)(2-\gamma)\omega_n} s_0^{2-\gamma}, \tag{69}$$

because if

$$\phi(\tilde{T}) > \frac{M_0 - s_0}{(1-\gamma)(2-\gamma)\omega_n} s_0^{2-\gamma},$$

by continuity of ϕ we would have that $\tilde{T}$ cannot be the supremum of $\tilde{S}$. But (69) cannot be true since from the nondecreasing of ϕ in view of inequality (68), we would arrive at

$$\frac{M_0 - s_0}{(1-\gamma)(2-\gamma)\omega_n} s_0^{2-\gamma} = \phi(\tilde{T}) \geq \phi(0) > \frac{M_0 - s_0}{(1-\gamma)(2-\gamma)\omega_n} s_0^{2-\gamma}.$$

As a conclusion, the constructed u_0 implies that T_{max} has to be finite.

Remark 3 (Finite-time blow-up with $m_2 \neq m_3$). As already said in Section 3, the proof of the blow-up for problem (8) with $m_2 \neq m_3$ is still an open problem. Since the transformation $z = \chi v - \xi w$ used above to reorganize problem (8) with $m_2 = m_3$ into the simplified version (41) is no longer employable, by reasoning as in Lemma 7, the corresponding inequality (52) would read

$$\begin{aligned}
\phi'(t) \geq\ & n^2 \int_0^{s_0} s^{2-\frac{2}{n}-\gamma}(s_0 - s)(nU_s+1)^{m_1-1} U_{ss} \, ds - \chi f_{1_\gamma} \int_0^{s_0} s^{1-\gamma}(s_0-s) U_s (nU_s+1)^{m_2-1} \, ds \\
& + \frac{\chi}{2} \int_0^{s_0} s^{-\gamma}(s_0-s) U_s (nU_s+1)^{m_2-1} \left[\int_0^s f_1(nU_s(\sigma,t)) \, d\sigma \right] ds - \mu n^{k-1} \int_0^{s_0} s^{-\gamma}(s_0-s) \left[\int_0^s U_s^k(\sigma,t) \, d\sigma \right] ds \\
& + \xi \int_0^{s_0} s^{1-\gamma}(s_0-s) U_s (nU_s+1)^{m_3-1} m_2(t) \, ds - \xi \int_0^{s_0} s^{-\gamma}(s_0-s) U_s (nU_s+1)^{m_3-1} \left[\int_0^s f_2(nU_s(\sigma,t)) \, d\sigma \right] ds,
\end{aligned}$$

valid for all $t \in S_\phi$ and with $f_{1_\gamma} = f_1(\frac{8n}{2^\gamma(3-\gamma)\omega_n})$. In particular, the extra terms involving the repulsion coefficient ξ make the analysis more complex. This is also connected to the signs of such terms (exactly opposite to those of the integrals associated with the attraction coefficient χ), which do not allow to use the right inequalities tied to the hypotheses on f_i, and in turn on m_i.

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DATA AVAILABILITY STATEMENT

Data sharing is not applicable to this article as no new data were created or analyzed in this study.

ORCID

Giuseppe Viglialoro https://orcid.org/0000-0002-2994-4123

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