[Autori: M. Marras, F. Ragnedda, S. Vernier-Piro, , V. Vespri;
Titolo: Holder estimates of weak solutions to degenerate chemotaxis systems with a source term;
Rivista: Journal of Differential Equations; volume 366, 2023, pagg. 42-70
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https://doi.org/10.1016/j.jde.2023.04.013

Hölder estimates of weak solutions to degenerate chemotaxis systems with a source term

In memory of our friend Emmanuele DiBenedetto

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Abstract

In this note we consider degenerate chemotaxis systems with porous media type diffusion and a source term satisfying the Hadamard growth condition. We prove the Hölder regularity for bounded solutions to parabolic-parabolic as well as for parabolic-elliptic chemotaxis systems.

Keywords: Chemotaxis systems, degenerate parabolic equations, elliptic equations, Hölder regularity.
2020 MSC: 92C17, 35K65, 35J70, 35B65

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1. Introduction

Let us consider the following class of degenerate chemotaxis systems

$$\begin{cases} u_t = \mathrm{div}(\nabla u^m) - \chi \mathrm{div}(u^{\mathfrak{q}-1}\nabla v) + B(x,t,u,\nabla u), & \text{in } \mathbb{R}^N \times (t>0), \\ \tilde{\tau} v_t = \Delta v - av + u, & \text{in } \mathbb{R}^N \times (t>0), \\ u(x,0) = u_0(x) \geq 0, \quad v(x,0) = v_0(x) \geq 0, & \text{in } \mathbb{R}^N, \end{cases} \tag{1.1}$$

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with $N \geq 2, m \geq 1, \mathfrak{q} \geq \max\{\frac{m+1}{2}, 2\}$ and $a, \chi > 0$. The constant $\tilde{\tau}$ is taken nonnegative. When $\tilde{\tau} = 0$ we are in the parabolic-elliptic case and, when $\tilde{\tau} > 0$, we are in the parabolic-parabolic case. In the latter case, without loss of generality, we may assume $\tilde{\tau} = 1$. The initial data $(u_0(x), v_0(x))$ satisfies

$$\begin{cases} u_0(x) \geq 0,\ u_0(x) \in L^\infty(\mathbb{R}^N) \cap L^1(\mathbb{R}^N),\ u_0^m \in H^1(\mathbb{R}^N), \\ v_0(x) \geq 0,\ v_0(x) \in L^1(\mathbb{R}^N) \cap W^{1,p}(\mathbb{R}^N), \end{cases} \tag{1.2}$$

with $1 < p < \infty$.
A pair (u, v) of non negative measurable functions defined in $\mathbb{R}^N \times [0, T]$, $T > 0$ is a local weak solution to (1.1) if

$$u \in L^\infty(0, T; L^p(\mathbb{R}^N)),\ u^m \in L^2(0, T; H^1(\mathbb{R}^N)),\ v \in L^\infty(0, T; H^1(\mathbb{R}^N)),$$

and (u, v) satisfies (1.1) in the sense that, for every compact set $\mathcal{K} \subset \mathbb{R}^N$ and every time interval $[t_1, t_2] \subset [0, T]$, one has

$$\begin{aligned} &\int_{\mathcal{K}} u\psi dx\Big|_{t_1}^{t_2} + \int_{t_1}^{t_2}\int_{\mathcal{K}}\Big[-u\psi_t + (\nabla u^m, \nabla\psi) - \chi u^{\mathfrak{q}-1}(\nabla v, \nabla\psi)\Big]dx\ dt \\ &= \int_{t_1}^{t_2}\int_{\mathcal{K}} B(x, t, u, \nabla u)\psi\ dxdt; \end{aligned} \tag{1.3}$$

$$\int_{\mathcal{K}} \tilde{\tau} v\psi dx\Big|_{t_1}^{t_2} + \int_{t_1}^{t_2}\int_{\mathcal{K}}\Big[-\tilde{\tau} v\psi_t + (\nabla v, \nabla\psi)\Big]dxdt = \int_{t_1}^{t_2}\int_{\mathcal{K}}(-av + u)\psi dxdt, \tag{1.4}$$

for all locally bounded non negative testing function $\psi \in W^{1,2}_{loc}(0, T; L^2(\mathcal{K})) \cap L^p_{loc}(0, T; W^{1,p}_0(\mathcal{K}))$.

In the last years, there was a growing interest in the chemotaxis systems. We recall that Keller and Segel in the seminal paper [15] proposed a mathematical model describing the aggregation process of amoebae by chemotaxis. For such a reason, nowadays, such kind of systems are named Keller-Segel in their honour. Recently many authors studied systems with porous medium-type diffusion and with a power factor in the drift term (see [12], [13], [14], [16], [21] and the references therein). In this note, we consider a degenerate chemotaxis model with porous media type diffusion with $m > 1$. When

$m = 1,\ \mathfrak{q} = 2,\ B = 0$, the system (1.1) is reduced to the classical Keller-Segel system. In our model, the diffusion of the cells ($\operatorname{div}(\nabla u^m)$) depends only on own density and degenerates when $u = 0$. The number m denotes the intensity of diffusion and the exponent $\mathfrak{q}$ in the drift term takes in account the nonlinear aspects of the biological phenomenon. Moreover we assume $\chi > 0$ which means that the cells move toward the increasing signal concentration (chemoattractant). For sake of simplicity, we take $\chi = 1$.
This model relies on the presence of the the source term B which describes the growth of the cells. Some experimental evidences (see [1]) show that B is a nonlinear term and satisfies *natural* or Hadamard growth condition, more precisely it satisfies the inequality (see [5], [7])

$$|B(x, t, u, \nabla u)| \leq C|\nabla u^m|^2 + \phi(x, t), \quad C > 0, \tag{1.5}$$

with $\phi(x, t)$ in the parabolic space $L^{q,r}_{\mathbb{R}^N \times (t>0)} = L^r(0, \infty; L^q(\mathbb{R}^N))$. The presence of this term makes more challenging the math approach of this system (see the monograph by Giaquinta [9]).

In literature a great part of the results concerns the case $B = 0$ where, depending upon the choice of m and q, it is possible to find initial data for which we have global existence and initial data for which blow-up in finite time occurs (see [12], [13], [14], [22] and references therein). To our knowledge, in the more general case $B \neq 0$, the global existence of the solutions and the blow-up phenomenon are not still studied in fully detail. For the above reason, we will work in a bounded time interval $[0, T]$ i.e. before the eventual blow-up time, assuming, therefore, that the solution u remains bounded.
In the next future, we intend to investigate the existence, uniqueness, boundedness and blow up of the solutions.

For the existence, our idea is to approximate the system with regular ones whose existence of the solution is known. By the regularity results that we will prove in this paper, the approximate solutions are equi Hölder continuous and, by well known theorems, it is possible to find a subsequence converging to a couple of Hölder continuous functions (u, v). Then, we will try to apply a compactness result such as Minty's lemma ([18]) to prove that the couple (u, v) is a weak solution of the system.

In order to prove the existence, we will try to follow the Kim and Lee approach ([16]).

The estimates concerning the boundedness of the solution should be obtained modifying to our context the classical DiBenedetto estimates (see, for instance, Chapter 5 of ([5]).

For the global existence and the blow up issues, we intend to draw inspiration from the works by Winkler ([24], [25]).
For the solutions to (1.1) with $B = 0$ and $\tilde{\tau} = 1$, Ishida and Yokota in [12] proved that a weak solution exists globally when $\mathfrak{q} < m + \frac{2}{N}$ without restriction on the size of initial data, improving both Sugiyama ([20]) and Sugiyama and Kunii results ([21]) where $q \leq m$ was assumed. In [13], the authors established the global existence of weak solutions with small initial data when $\mathfrak{q} \geq m + \frac{2}{N}$, while in [23] Winkler proved that there are initial data such that if $\mathfrak{q} > m + \frac{2}{N}$ the solution blows up in finite time. Moreover, in [14] uniform boundedness of nonnegative solutions was derived assuming $\mathfrak{q} < m + \frac{2}{N}$.
If $B = 0$ and $\tilde{\tau} = 0$, the existence in large of the solutions was proved in the case of $m > \mathfrak{q} - \frac{2}{N}$ without any restrictions on initial data and in the case $1 \leq m \leq \mathfrak{q} - \frac{2}{N}$ only for small initial data ([21]). We refer to [1] for (local and global) existence and nonexistence of solutions to different classes of Keller-Segel type system.
If $B \neq 0$ and $\tilde{\tau} = 0, m = 1, q = 2$, in [17] the authors investigated blow-up phenomena and obtained a safe time interval of existence for the solutions by deriving a lower bound of the blow-up time.
In [16], following the De Giorgi approach, Kim and Lee proved regularity and uniqueness results for solutions to degenerate chemotaxis parabolic-parabolic system assuming that the source term is vanishing.

In this paper we focus our attention only on the local Hölder regularity of the solution (u, v). More precisely, we give a unitary and more organic proof that allows us to treat in the same framework a more general equation (with source term) either in the parabolic-parabolic case and in the parabolic-elliptic case.
Our approach is based on suitable a-priori estimates on the function v that solves the second equation of (1.1) and on De Giorgi-DiBenedetto approach ([4], [5], [6]) for proving regularity of u^m. The regularity of v, either for $\tilde{\tau} = 1$ or $\tilde{\tau} = 0$, follows in a straightforward way from classical results theory (see, for instance, [5], [10], [19]). The proof has however some remarkable differences from the classical approach by DiBenedetto. In this paper we focus our attention only on the real novelties. When the modifications are based only on technicalities, we quote the corresponding papers by DiBenedetto and when it is a structural modification we give a detailed proof. Our main

result is

Theorem 1.1. *(Regularity)*
Let u be a locally bounded local weak solution of (1.1) *and let B satisfy* (1.5)*. Then* $(x,t) \to u(x,t)$ *is Hölder continuous in* $\mathbb{R}^N \times (0,T)$ *and there exists* $\alpha_o \in (0,1)$ *such that, for every* $\varepsilon > 0$*, there exists a constant* $\gamma(\varepsilon) > 0$ *such that*

$$|u^m(x_1,t_1) - u^m(x_2,t_2)| \le \gamma(\varepsilon)(|x_1 - x_2|^{\alpha_o} + |t_1 - t_2|^{\frac{\alpha_o}{2}}),$$

for every pair of points $(x_1,t_1), (x_2,t_2) \in \mathbb{R}^N \times (\varepsilon, T)$.

The scheme of this paper is the following: in Section 2 we present some preliminary lemmas that will be used to prove our main results.

In Sections 3 and 4 we consider the parabolic-parabolic chemotaxis system. We study the behavior of u^m near the ess inf u^m(*First Alternative*), and near the ess sup u^m(*Second Alternative*) respectively and combining these estimates we have the Hölder continuity of the solution. The Section 5 is devoted to the study of the Hölder regularity for the parabolic-elliptic chemotaxis system.

2. Preliminary results

In this section we present some results we will use in the sequel.
In $\mathbb{R}^N$, define the N-dimensional cube centered at the origin and wedge $2R$: $K_R = \{x \in \mathbb{R}^N / \max_{1\le i\le N} |x_i| < R\}$ and let $|K_R|$ be its measure. Let $Q_R(T) := K_R \times [0,T]$. Consider the parabolic space $L^{q,r}(Q_R(T))$ with the norm

$$||w||_{L^{q,r}(Q_R(T))} \equiv \Big(\int_0^T \Big(\int_{K_R} |w|^q dx \Big)^{\frac{r}{q}} dt \Big)^{\frac{1}{r}} < \infty.$$

Moreover, for $p > 1$, w belongs to the space

$$V^p(Q_R(T)) \equiv L^\infty(0,T; L^p(K_R)) \cap L^p(0,T; W^{1,p}(K_R))$$

if $||w||_{V^p(Q_R(T))} \equiv \operatorname*{ess\,sup}_{(0,T)} ||w||_{L^p(K_R)} + ||\nabla w||_{L^p(Q_R(T))} < \infty$.
Define $V_0^p(Q_R(T)) \equiv L^\infty(0,T; L^p(K_R)) \cap L^p(0,T; W_0^{1,p}(K_R))$.

The proof of the following lemmata can be found in the monograph [5].

Lemma 2.1. *(Sobolev Lemma) Let $\bar{\zeta}(x,t)$ be a cut off function compactly supported in a cube K_R, $R>0$ and let $u(x,t)$ be defined in $\mathbb{R}^N\times(t_1,t_2)$, for any $t_2>t_1>0$. Then*

$$||\bar{\zeta}u||_{L^{\frac{2N}{N-2}}(\mathbb{R}^N)}\leq C||\nabla(\bar{\zeta}u)||_{L^2(\mathbb{R}^N)};$$

and, for some $C>0$,

$$||\bar{\zeta}u||^2_{L^2(t_1,t_2;L^2(\mathbb{R}^N))} \leq C\Big(\sup_{(t_1\leq t\leq t_2)}||\bar{\zeta}u||^2_{L^2(\mathbb{R}^N)}+\ ||\nabla(\bar{\zeta}u)||^2_{L^2(t_1,t_2;L^2(\mathbb{R}^N))}\Big)|\{\bar{\zeta}u>0\}|^{\frac{2}{N+2}}. \tag{2.1}$$

For the parabolic spaces, the following embedding inequality holds.

Lemma 2.2. *(Embedding Lemma) There exists a positive constant $\gamma_1=\gamma_1(N,p)$ such that for every function $w\in V_0^p(Q_R(T))$*

$$||w||_{L^{q,r}(Q_R(T))}\leq\gamma_1||w||_{V_0^p(Q_R(T))}, \tag{2.2}$$

where p,q,r satisfy the relation

$$\frac{1}{r}+\frac{N}{pq}=\frac{N}{p^2},$$

and in the case $1\leq p<N$, the admissible range is $q\in\Big[p,\frac{Np}{N-p}\Big]$, $r\in[p,\infty]$.

Lemma 2.3. *(Fast geometric convergence Lemma). Let (X_i) and (Y_i), $i=0,1,...$ be two sequences of positive numbers satisfying the recursive inequalities*

$$X_{i+1}\leq c\ \mathrm{b}^i(X_i^{1+\hat{\alpha}}+X_i^{\hat{\alpha}}\ Y_i^{1+\kappa});\qquad Y_{i+1}\leq c\ \mathrm{b}^i(X_i+\ Y_i^{1+\kappa})$$

with $c,\mathrm{b}>1$ and $\kappa,\hat{\alpha}>0$ given numbers. If

$$X_0+\ Y_0^{1+\kappa}\leq(2c)^{-\frac{1+\kappa}{\sigma}}\mathrm{b}^{-\frac{1+\kappa}{\sigma^2}},\quad \sigma=\min(\kappa,\hat{\alpha}),$$

then (X_i) and $(Y_i)\to 0$ as $i\to\infty$.

Steklov averages.

Since the solutions of (1.1) possess a modest degree of regularity in the time variable, we utilize the Steklov average u_h of the weak solution u, for $h>0$:

$$u_h(\cdot,t)=\frac{1}{h}\int_t^{t+h}u(\cdot,\tau)d\tau.$$

For a complete statement of Steklov averages and for their convergence to u, as $h \to 0$, we refer the reader to [5] and [6]. Then we point out an alternative formulation of weak solution to (1.1): fix $t \in (0,T)$, let $h > 0$ with $0 < t < t+h < T$ and replace in (1.3) with $t_1 = t$ and $t_2 = t+h$; choosing a test function ψ independent on $\tau \in (t, t+h)$, dividing by h, using the Steklov averages we get

$$\begin{aligned}&\int_{\mathcal{K}\times t} [(u_h)_t\psi + ((\nabla u^m)_h, \nabla\psi) - \chi(u^{\mathfrak{q}-1}\nabla v)_h \nabla\psi] dxd\tau \\ &= \int_{\mathcal{K}\times t} B(x,t,u,\nabla u)_h \psi \; dxd\tau,\end{aligned} \tag{2.3}$$

for all locally bounded non negative testing function $\psi \in W^{1,2}_{loc}(0,T; L^2(\mathcal{K})) \cap L^p_{loc}(0,T; W^{1,p}_0(\mathcal{K}))$.
Integrating over $[t_1, t_2]$ and letting $h \to 0$, (2.3) gives (1.3).

We introduce now apriori estimates on the $L^p - L^{p'}$ norm of the solutions of evolution equations with

$$1 \le p' \le p \le \infty, \qquad \frac{1}{p'} - \frac{1}{p} < \frac{1}{N}. \tag{2.4}$$

Consider the following Cauchy problem :

$$\begin{cases} v_t = \Delta v - av + w, & (x,t) \in \mathbb{R}^N \times (t > 0), \\ v(x,0) = v_0(x), & x \in \mathbb{R}^N, \end{cases} \tag{2.5}$$

then by classical L^p maximal regularity properties (see for instance [11], see also [21]) we have:

Lemma 2.4. *(Heat) Let v be the solution to* (2.5)*. If $v_0 \in W^{1,p}(\mathbb{R}^N)$ and $w \in L^\infty(0,\infty; L^{p'}(\mathbb{R}^N))$, with p, p' in* (2.4)*, then for $t \in [0,\infty)$, there exist positive constants $C_0, \bar{C}_0$, depending on p, p' and N such that*

$$\begin{cases} ||v(t)||_{L^p(\mathbb{R}^N)} \le \; ||v_0||_{L^p(\mathbb{R}^N)} + C_0 ||w(\tau)||_{L^\infty(0,T;L^{p'}(\mathbb{R}^N))}, \\ ||\nabla v(t)||_{L^p(\mathbb{R}^N)} \le ||\nabla v_0||_{L^p(\mathbb{R}^N)} + \bar{C}_0 ||w(\tau)||_{L^\infty(0,T;L^{p'}(\mathbb{R}^N))}. \end{cases} \tag{2.6}$$

An essential tool for the regularity are the energy estimates. Define $(k - u^m)_+$ ($(k-u^m)_-$, resp.) as $k - u^m$ if $k > u^m$ ($u^m - k$ if $k < u^m$, resp.)

and 0 otherwise. Here we state these estimates only for $(k-u^m)_+$, (omitting the sign +), being the other case specular. Let $q,\, r>1,\; 0<\kappa<\frac{2}{N}$ and introduce $\tilde{q},\tilde{r}$ related to $q, r,\; \kappa$ by the formulas:

$$1-\frac{1}{q}=\frac{2(1+\kappa)}{\tilde{q}},\quad 1-\frac{1}{r}=\frac{2(1+\kappa)}{\tilde{r}}. \tag{2.7}$$

Lemma 2.5. *(Local Energy Estimates) Let $t_1<t_2,\; \mathfrak{q}>2,\; m>1$ and let (u,v) be a locally bounded weak solution of problem* (1.1).
Then there exist constants $\gamma^*>0$ and $\mathbf{C}>0$ depending only upon the data and $||u||_{L^\infty(R^N\times(\varepsilon,T))}$, such that for a cut-off function η compactly supported in K_R and for every level k,

$$\begin{aligned}&\int_{K_R\times t_2}\eta^2\Big[\int_0^{(k-u^m)}\Phi(\xi)\; d\xi\Big]dx+\gamma^*\int_{t_1}^{t_2}\int_{K_R}|\nabla(\eta(k-u^m)|^2dxdt\\&\le \mathbf{C}m\Big(\int_{t_1}^{t_2}\int_{K_R}(k-u^m)^2|\nabla\eta|^2dxdt+\int_{t_1}^{t_2}\int_{K_R}\Big[\int_0^{(k-u^m)}\Phi(\xi)d\xi\Big]|\eta\eta_t|dxdt\\&+\int_{K_R\times t_1}\eta^2\Big[\int_0^{(k-u^m)}\Phi(\xi)d\xi\Big]dx+\Big[\int_{t_1}^{t_2}(|A_{k,R}|^{\frac{\tilde{r}}{\tilde{q}}}dt\Big]^{\frac{2(1+k)}{\tilde{r}}}\Big)\end{aligned} \tag{2.8}$$

with $\Phi(\xi):=(k-\xi)^{\frac{1}{m}-1}\xi$ and $A_{k,R}(t)=\{x\in K_R:(k-u^m)>0\}$.

Proof. The proof of this estimate follows an argument similar to the one by DiBenedetto ([5]) and for the reader's convenience we give here a detailed proof.
Starting from the definition of weak solution with the Steklov averages (2.3) and with $\psi=-(k-u^m)\eta^2$, integrating from t_1 to t_2 and letting $h\to 0$, we obtain

$$\begin{aligned}&-\int_{\tilde{Q}}u_t(k-u^m)\eta^2dxdt+\int_{\tilde{Q}}\nabla u^m\cdot\nabla(-(k-u^m)\eta^2)dxdt\\&=\int_{\tilde{Q}}u^{\mathfrak{q}-1}\nabla v\cdot\nabla(-(k-u^m)\eta^2\; dxdt+\int_{\tilde{Q}}B(-(k-u^m))\eta^2dxdt,\end{aligned} \tag{2.9}$$

where we have denoted by $\tilde{Q}:=K_R\times[t_1,t_2]$. We rewrite (2.9) as $M_1+M_2=M_3+M_4$.
For $k>u^m$ the following identity holds

$$-\int_{K_R}(k-u^m)\eta^2u_t\,dx=\frac{1}{m}\int_{K_R}\frac{d}{dt}\left(\int_0^{k-u^m}\Phi(\xi)d\xi\right)\eta^2dx, \tag{2.10}$$

with $\Phi(\xi)$ defined in (2.8). Integrating by parts the right side of (2.10) with respect to t leads to

$$M_1 = \frac{1}{m}\int_{K_R\times t_2}\Big(\int_0^{k-u^m}\Phi(\xi)d\xi\Big)\eta^2 dx - \frac{1}{m}\int_{K_R\times t_1}\Big(\int_0^{k-u^m}\Phi(\xi)d\xi\Big)\eta^2 dx$$
$$-\frac{2}{m}\int_{\tilde{Q}}\Big(\int_0^{k-u^m}\Phi(\xi)d\xi\Big)\eta\ \eta_t\ dxdt = M_{11} - M_{12} - M_{13}.$$

By standard calculations we derive

$$M_2 = \int_{\tilde{Q}}|\nabla((k-u^m)\eta)|^2 dxdt - \int_{\tilde{Q}}(k-u^m)^2|\nabla\eta|^2 dxdt = M_{21} - M_{22},$$

$$M_3 = \int_{\tilde{Q}} -\eta\nabla((k-u^m)\eta)(u^{\mathfrak{q}-1}\nabla v) - \int_{\tilde{Q}}(k-u^m)\eta\nabla\eta(u^{\mathfrak{q}-1}\nabla v) = M_{31} + M_{32}.$$

By using the Young inequality we have

$$M_{31} \le \frac{1}{2}\int_{\tilde{Q}}|\nabla(k-u^m)\eta|^2 dxdt +$$
$$\frac{1}{2}\int_{t_1}^{t_2}\int_{K_R\cap\{(k-u^m)>0\}} u^{2(\mathfrak{q}-1)}\eta^2|\nabla v|^2 dxdt = M_{311} + M_{312}.$$

In the same manner we obtain

$$M_{32} \le \frac{1}{2}\int_{\tilde{Q}}(k-u^m)^2|\nabla\eta|^2$$
$$+\frac{1}{2}\int_{t_1}^{t_2}\int_{K_R\cap\{(k-u^m)>0\}} u^{2(\mathfrak{q}-1)}\eta^2|\nabla v|^2 dxdt = M_{321} + M_{322}.$$

With $A_{k,R}(t) = \{x \in K_R : (k-u^m(x,t)) > 0\}$, applying the Hölder inequality first in the variable x and then in t we obtain

$$M_{312} + M_{322} \le \mu_+^{\frac{2(\mathfrak{q}-1)}{m}}\Big(\int_{t_1}^{t_2}\Big(\int_{K_R}|\nabla v|^{2q}dx\Big)^{\frac{r}{q}}dt\Big)^{\frac{1}{r}}\Big(\int_{t_1}^{t_2}\Big|A_{k,R}(t)\Big|^{\frac{r(q-1)}{q(r-1)}}dt\Big)^{\frac{r-1}{r}}.$$

We first observe that Lemma 2.4 is satisfied with $w = u$, $p = 2q$, $p' = 2, \frac{1}{2} - \frac{1}{2q} = \frac{\kappa}{N}$, $q = \frac{N}{N-2\kappa}, \kappa$ in (2.7), then for $(t_2 - t_1)$ small enough there exists a constant $I_u > 0$ such that

$$\Big(\int_{t_1}^{t_2}\Big(\int_{K_R}|\nabla v|^{2q}dx\Big)^{\frac{r}{q}}dt\Big)^{\frac{1}{r}}$$

$$\leq 2^{1-\frac{1}{r}}\left[\bar{C}_0\sup_{t_1<t<t_2}\Big(\int_{K_R}|u(x,t)|^2dx\Big)^{\frac{1}{2}}+\Big(\int_{K_R}|\nabla v_0(x)|^{2q}dx\Big)^{\frac{1}{2q}}\right]^2(t_2-t_1)^{\frac{1}{r}}:=I_u. \tag{2.11}$$

By inserting (2.11) in (2) and using (2.7) it follows that

$$M_{312}+M_{322}\ \ \leq I_u\ \mu_+^{\frac{2(q-1)}{m}}\ \left(\int_{t_1}^{t_2}\Big|A_{k,R}(t)\Big|^{\frac{\tilde{r}}{\tilde{q}}}dt\right)^{\frac{2}{\tilde{r}}(1+\kappa)}. \tag{2.12}$$

Note that I_u is a constant depending on $||u||_{L^\infty(t_1,t_2)L^2(K_R))}$ and $||\nabla v_0(x)||_{L^{2q}(K_R)}$. In order to estimate the term M_4 we recall that, since we are using the truncation $(k-u^m)$, we take $k=\mu^-+\frac{\omega}{2^{s_0}}$. By assumptions on B and taking into account that $(k-u^m)\leq\frac{\omega}{2^{s_0}}<1$

$$M_4\leq C\int_{\tilde{Q}}|\nabla(k-u^m)|^2\eta^2\ dxdt+\int_{\tilde{Q}}\phi(k-u^m)\eta^2dxdt=M_{41}+M_{42}.$$

An application of the Young Inequality leads, for a suitable $\gamma>1$, to

$$M_{41}=C\int_{\tilde{Q}}|\nabla(k-u^m)|^2\eta^2dxdt$$

$$\leq\frac{C}{2}\int_{\tilde{Q}}|\nabla(k-u^m)\eta|^2dxdt+C\int_{\tilde{Q}}(k-u^m)^2|\nabla\eta|^2dxdt=M_{411}+M_{412}.$$

$$M_{42}=\int_{\tilde{Q}}\phi(k-u^m)\eta^2dxdt\leq||\phi||_{L^{q,r}(\tilde{Q})}\left(\int_{t_1}^{t_2}\Big|A_{k,R}(t)\Big|^{\frac{\tilde{r}}{\tilde{q}}}dt\right)^{\frac{2}{\tilde{r}}(1+\kappa)}.$$

We observe that, choosing C small enough, we have for a suitable constant $\gamma^*>0$

$$M_{21}-M_{311}-M_{411}\geq\gamma^*\int_{\tilde{Q}}|\nabla(k-u^m)\eta|^2dxdt.$$

Collecting all the previous inequalities obtained for $M_i,\ i=1,...4$, the lemma is proved. □

We conclude this section recalling the preliminaries necessary for the regularity machinery.

Consider $0 < R < 1$ sufficiently small. Set $Q(2R, R^{2-\varepsilon}) := K_{2R} \times [-R^{2-\varepsilon}, 0]$, where ϵ is a positive number to be chosen later. We set

$$\mu_+ = \operatorname*{ess\,sup}_{Q(2R,R^{2-\varepsilon})} u^m, \;\; \mu_- = \operatorname*{ess\,inf}_{Q(2R,R^{2-\varepsilon})} u^m, \qquad \omega = \operatorname*{ess\,osc}_{Q(2R,R^{2-\varepsilon})} u^m = \mu_+ - \mu_-.$$

Let s_0 be the smallest integer such that

$$\theta_0 = \frac{\omega}{2^{s_0}} < 1, \quad \alpha = 1 - \frac{1}{m}, \quad a_0 = \frac{\omega}{A} \tag{2.13}$$

with $A > 2^{s_0}, A$ a positive constant to be determined later. We introduce sub cylinders with center at $(0, \bar{t}) : Q^{\bar{t}}(R, \theta_0^{-\alpha} R^2) = K_R \times [\bar{t} - \hat{\theta}, \bar{t}] := Q_R^{\bar{t}}(\hat{\theta})$, with $\hat{\theta} = \theta_0^{-\alpha} R^2$ and

$$Q(R, a_0^{-\alpha} R^2) := Q_R(\hat{a}), \quad \text{with} \quad Q_R^{\bar{t}}(\hat{\theta}) \subset Q_R(\hat{a}) \tag{2.14}$$

without loss of generality, assume that $(\frac{\omega}{A})^\alpha > R^\varepsilon$. If for any $R < 1$ this does not hold, we have $\omega \leq AR^{\frac{\varepsilon}{\alpha}}$ and there is nothing to prove since the oscillation is then comparable to the radius. Next, if $\omega > AR^{\frac{\varepsilon}{\alpha}}$, we prove that the oscillation of u^m is reduced by a fixed factor in the set $Q_R(\hat{\theta})$, by analyzing two complementary alternatives. In the first alternative, let us assume that there exists a subcylinder $Q_R^{\bar{t}}(\hat{\theta})$ where u^m is away from its infimum.
Under such hypothesis, we are able to prove that the oscillation decreases of a fixed factor in that sub cylinder. Then, by using the so-called expansion of positivity in time, we are able to transport that information to the top of the original cylinder, in the "right "sub cylinder.
In the second alternative we examine the case when in all the cylinders $Q_R^{\bar{t}}(\hat{\theta})$, u^m is essentially away from its supremum and we prove that the oscillation decreases by a fixed factor also in this case directly in the whole cylinder.

3. 1^{st} alternative in the parabolic-parabolic case

In this section we examine the first alternative with $\tilde{\tau} = 1$ in (1.1): i.e. there exists a cylinder $Q_R^{\bar{t}}(\hat{\theta})$ where

$$\frac{\left|\{(x,t) \in Q_R^{\bar{t}}(\hat{\theta}) : u^m < \mu_- + \frac{\omega}{2^{s_0+1}}\}\right|}{|Q_R^{\bar{t}}(\hat{\theta})|} \leq \nu \tag{3.1}$$

with the positive constant ν to be defined later.

To work in this context, we need the so-called *critical mass* and *expansion in time* lemmata. For more details about these definitions, see, for instance, the review paper [8].
We start with the folllowing lemma.

Lemma 3.1. *(Critical mass Lemma) Let us consider the cylinder* $Q_R^{\bar{t}}(\hat{\theta})$ *defined in* (2.14) *and let* s_0 *be defined in* (2.13)*. Then there exists a number* $\nu \in (0,1)$ *such that, if*

$$|\{(x,t) \in Q_R^{\bar{t}}(\hat{\theta}) : \ u^m < \mu_- + \frac{\omega}{2^{s_0}}\}| \leq \nu \ |Q_R^{\bar{t}}(\hat{\theta})|, \tag{3.2}$$

then

$$u^m > \mu_- + \frac{\omega}{2^{s_0+1}}, \quad a.e. \ \ in \ Q_{\frac{R}{2}}^{\bar{t}}(\hat{\theta}). \tag{3.3}$$

Proof. First we estimate $\int_0^{(k-u^m)} \Phi(\xi) d\xi$, with $\Phi(\xi) \ := (k-\xi)^{\frac{1}{m}-1}\xi$. We prove that there exist two positive constants $\widehat{c} \leq \frac{1}{2}$, and $\check{c} \geq \frac{m^2}{1+m}$, such that

$$\widehat{c} \ \frac{(k-u^m)^2}{(\theta_0+\mu_-)^\alpha} \ \leq \int_0^{(k-u^m)} \Phi(\xi) \ d\xi \ \leq \ \check{c} \ \frac{(k-u^m)^2}{(\theta_0+\mu_-)^\alpha}, \tag{3.4}$$

with $\alpha = 1 - \frac{1}{m}$. For $0 < u^m < k$, we derive

$$\begin{aligned}\int_0^{(k-u^m)} \Phi(\xi) \ d\xi &= \int_0^{(k-u^m)} [k(k-\xi)^{-\alpha} - (k-\xi)^{1-\alpha}] d\xi \\ &= k\frac{k^{1-\alpha} - (u^m)^{1-\alpha}}{1-\alpha} - \frac{k^{2-\alpha} - (u^m)^{2-\alpha}}{2-\alpha}.\end{aligned}$$

We introduce the function

$$r(u^m) := k\frac{k^{1-\alpha} - (u^m)^{1-\alpha}}{1-\alpha} - \frac{k^{2-\alpha} - (u^m)^{2-\alpha}}{2-\alpha} - \widehat{c} \ \frac{(k-u^m)^2}{(\theta_0+\mu_-)^\alpha},$$

with $r(0) = (k)^{2-\alpha}\Big(\frac{1}{(1-\alpha)(2-\alpha)} - \widehat{c}\Big)$ and $r(0) > 0$, if $\widehat{c} < \frac{1}{(1-\alpha)(2-\alpha)}$. Moreover $r(k) = 0$. Now we observe that $r'(u^m) < 0$ if $\widehat{c} \leq \frac{1}{2}$. As a consequence, the function $r(u^m)$, initially positive, is decreasing up to $(k, 0)$ and always non negative in $0 < u^m < k$. For the inequality on the right of (3.4) following the previous steps, we define the function

$$g(u^m) := k \ \frac{(k)^{1-\alpha} - (u^m)^{1-\alpha}}{1-\alpha} - \frac{k^{2-\alpha} - (u^m)^{2-\alpha}}{2-\alpha} - \check{c} \ \frac{(k-u^m)^2}{(\theta_0 + \mu_-)^\alpha}$$

and $g(k) = 0$ and $g(0) < 0$ if $\check{c} > \frac{1}{(1-\alpha)(2-\alpha)}$. Moreover g is decreasing up to its minimum reached at $u^m = \frac{\theta_0 + \mu_-}{(2\check{c})1/\alpha}$. Then g is negative and (3.4) is proved. Now we are ready to prove Lemma 3.1.

Let us introduce some technical tools. Let us construct a family of nested cylinders $Q_{R_i}(\hat{\theta})$ with $R_i = \frac{R}{2} + \frac{R}{2^{i+1}}, \ i = 0, 1, 2, ...$ for which the assumptions of Lemma 3.1 hold. After a translation we assume $(0, \bar{t}) = (0, 0)$. Let ζ_i be a piecewise cut-off function in $Q_{R_i}(\hat{\theta})$ such that

$$\begin{cases} 0 < \zeta_i(x,t) < 1, \quad (x,t) \in Q_{R_i}(\hat{\theta}), \quad \zeta_i = 1, \quad (x,t) \in Q_{R_{i+1}}(\hat{\theta}), \\ \zeta_i = 0, \quad \text{on the parabolic boundary of } Q_{R_i}(\hat{\theta}); \\ |\nabla \zeta_i| \le \dfrac{2^{i+1}}{R}, \qquad 0 \le (\zeta_i)_t \le \dfrac{2^{2(i+1)}}{\theta_0^{-\alpha} \ R^2}. \end{cases} \tag{3.5}$$

Apply the Energy Estimates (2.8) over the cylinders $Q_{R_i}(\hat{\theta})$ to the truncated functions $(k_i - u^m)$ with $k_i = \mu_- + \dfrac{\omega}{2^{s_0+1}} + \dfrac{\omega}{2^{s_0+1+i}}, \quad i = 0, 1, 2, ...$ In (2.8) the first term on the left, the second and the third on the right include the term $\int_0^{(k-u^m)} \Phi(\xi) \ d\xi$: then with k replaced by k_i, by using (3.4), we have

$$\int_{K_{R_i} \times \{t_2\}} \zeta_i^2 \Big[\int_0^{(k_i - u^m)} (k_i - \xi)^{\frac{1}{m}-1} \xi \ d\xi \Big] dx \ge \widehat{c} \ \ \frac{||\zeta_i(k_i - u^m)||^2_{L^2(K_{R_i})}}{(\mu_- + \theta_0)^\alpha}. \tag{3.6}$$

In (3.6) we take the supremum in time since t_2 is arbitrary. From (2.8) multiplied by $k_0^\alpha = (\mu_- + \theta_0)^\alpha$ and using (3.6), we get

$$\begin{aligned}
&\sup_{-\theta_0^\alpha R_i^2<t<0} ||\zeta_i(k_i - u^m)||^2_{L^2(K_{R_i})} + k_0^\alpha|||\nabla(\zeta_i(k_i-u^m))||^2_{L^2(R_i)} \\
&\le \mathbf{C}mk_0^\alpha\Big\{\int_{-\theta_0^{-\alpha}R_i^2}^0\int_{K_{R_i}}(k_i-u^m)^2|\nabla\zeta_i|^2\,dxdt \\
&+\int_{-\theta_0^{-\alpha}R_i^2}^0\int_{K_{R_i}}\Big[k_0^{-\alpha}(k_i-u^m)^2\Big]|\zeta_i(\zeta_i)_t|dxdt \\
&+\Big(\int_{-\theta_0^{-\alpha}R_i^2}^0|A_{k_i,R_i}(t)|^{\frac{\tilde r}{\tilde q}}dt\Big)^{\frac{2}{\tilde r}(1+\kappa)}\Big\} \qquad (3.7)\\
&\le \mathbf{C}mk_0^\alpha\Big\{(1+m)\frac{2^{2(i+1)}\,\omega^2}{R_i^2\;\,2^{2s_0}}\int_{-\theta_0^\alpha R_i^2}^0|A_{k_i,R_i}|(t)dt \\
&+\Big(\int_{-\theta_0^{-\alpha}R_i^2}^0|A_{k_i,R_i}(t)|^{\frac{\tilde r}{\tilde q}}dt\Big)^{\frac{2}{\tilde r}(1+\kappa)}\Big\},
\end{aligned}$$

with $A_{k_i,R_i}(t) = \{x\in K_{R_i} : u^m < k_i\}$. In order to simplify the computations we perform the following change of time variable in (3.7): $z = \theta_0^\alpha t$. As a consequence we have $Q_{R_i}(\hat\theta)\to Q_i = Q(R_i,R_i^2)$; $u(x,t)\to u(x,\theta_0^{-\alpha}z) = \mathrm{v}(x,z)$; $\zeta(x,t)\to\hat\zeta(x,z)$; $dt = \theta_0^{-\alpha}dz$. We obtain

$$\begin{aligned}
&\operatorname*{ess\,sup}_{(-R_i^2,0)}||\hat\zeta_i(k_i-\mathrm{v}^m)||^2_{L^2(K_{R_i})} + ||\nabla(\hat\zeta_i(k_i-\mathrm{v}^m))||^2_{L^2(Q_i)} \\
&\le \mathbf{C}m\Big\{(1+m)\frac{\omega^2}{2^{2s_0}}\frac{2^{2(i+1)}}{R_i^2}\frac{k_0^\alpha}{\theta_0^\alpha}|Z_i| + \frac{k_0^\alpha}{\theta_0^{\alpha(1-\frac{1}{r})}}\left(\int_{-R_i^2}^0|Z_i(z)|^{\frac{\tilde r}{\tilde q}}dz\right)^{\frac{2}{\tilde r}(1+\kappa)}\Big\}, \qquad (3.8)
\end{aligned}$$

with $Z_i(z) = \{x\in K_{R_i},\ : \mathrm{v}^m(x,z) < k_i\}$ and $|Z_i| = \int_{-R_i^2}^0|Z_i(z)|dz$.
The sequence (Z_i) is connected with two sequences (X_i) and (Y_i) so defined:

$$X_i = \frac{|Z_i|}{|Q_i|};\qquad Y_i = \frac{\left(\int_{-R_i^2}^0|Z_i(z)|^{\frac{\tilde r}{\tilde q}}dz\right)^{\frac{2}{\tilde r}}}{|K_{R_i}|},$$

which satisfy Lemma 2.3. In fact we prove that $X_{i+1}\le c\ 16^i(X_i^{1+\hat\alpha} + X_i^{\hat\alpha}Y_i^{1+\kappa})$, $\hat\alpha = \frac{2}{N+2}$. By inequality (2.1) in the Sobolev lemma (Lemma

2.1) applied to $(k_i - \mathrm{v}^m)$ we have

$$\begin{aligned}||\hat{\zeta}_i(k_i-\mathrm{v}^m)||^2_{L^2(Q_i)} \le \mathbf{C} m(\frac{k}{\theta_0})^\alpha \Bigg\{ (1{+}m) \frac{\omega^2 \; 2^{2(i+1)}}{2^{2s_0} R_i^2} |Z_i|^{1+\frac{1}{N+2}} \\ + |Z_i|^{\frac{2}{N+2}} \theta_0^{\frac{\alpha}{r}} \left(\int_{-R_i^2}^0 |Z_i(z)|^{\frac{\tilde{r}}{\tilde{q}}} dz \right)^{\frac{2}{\tilde{r}}(1+\kappa)} \Bigg\}.\end{aligned} \tag{3.9}$$

Moreover, we have

$$\begin{aligned}&\int_{Q_i} |\hat{\zeta}_i(k_i-\mathrm{v}^m)|^2 dxdz \\ &\ge |k_{i+1}-k_i|^2 \int_{-R_i^2}^0 |\{(x,z)\in R_{i+1} : \mathrm{v}^m < k_{i+1}\}| dz = \left(\frac{\omega}{2^{s_0+i+2}}\right)^2 |Z_{i+1}|,\end{aligned} \tag{3.10}$$

from (3.9) and (3.10) we derive the following upper bound of $|Z_{i+1}|$:

$$\begin{aligned}|Z_{i+1}| &\le \mathbf{C}\ m(\frac{k}{\theta_0})^\alpha \Bigg\{ (1+m)\ \frac{2^{4(i+2)}}{R_i^2} |Z_i|^{1+\frac{2}{N+2}} \\ &+ (\frac{\omega}{2^{s_0+i+2}})^{-2} |Z_i|^{\frac{2}{N+2}} \theta_0^{\frac{\alpha}{r}} \left(\int_{-R_i^2}^0 |Z_i(z)|^{\frac{\tilde{r}}{\tilde{q}}} dz \right)^{\frac{2}{\tilde{r}}(1+\kappa)} \Bigg\} \\ &\le c\ 16^i \Bigg\{ \frac{1}{R_i^2} |Z_i|^{1+\frac{2}{N+2}} + \theta_0^{\alpha(\frac{1}{r}-\frac{2}{\alpha})} |Z_i|^{\frac{2}{N+2}} \Big(\int_{-R_i^2}^0 |Z_i(z)|^{\frac{\tilde{r}}{\tilde{q}}} dz \Big)^{\frac{2}{\tilde{r}}(1+\kappa)} \Bigg\},\end{aligned} \tag{3.11}$$

where the constant c depends on the data and r and $\tilde{r}$ in (2.7). Divide (3.11) by $|Q_{i+1}|$: in the first term on the right we have

$$\frac{|Z_i|^{1+\frac{2}{N+2}}}{R_i^2 R_i^{N+2}} = \frac{|Z_i|}{R_i^{N+2}} \times \frac{|Z_i|^{\frac{2}{N+2}}}{R_i^{(N+2)\frac{2}{N+2}}}$$

and in the second term

$$\frac{|Z_i|^{\frac{2}{N+2}} \Big(\int_{-R_i^2}^0 |Z_i(z)|^{\frac{\tilde{r}}{\tilde{q}}} dz \Big)^{\frac{2}{\tilde{r}}(1+\kappa)}}{R_i^N \times R_i^2} = \frac{|Z_i|^{\frac{2}{N+2}}}{R_i^{(N+2)\frac{2}{N+2}}} \times R_i^{N\kappa} \frac{\Big(\int_{-R_i^2}^0 |Z_i(z)|^{\frac{\tilde{r}}{\tilde{q}}} dz \Big)^{\frac{2}{\tilde{r}}(1+\kappa)}}{(R_i^N)^{1+\kappa}}.$$

Then

$$X_{i+1} \le c\ 16^i \Big\{ X_i^{1+\frac{2}{n+2}} + \frac{1}{\theta_0^{\alpha(\frac{2}{\alpha}-\frac{1}{r})}} R_{i+1}^{(N\kappa)} X_i^{\frac{2}{N+2}} Y_i^{1+\kappa} \Big\}.$$

Recalling that $\frac{1}{\theta_0^{\alpha}} < \frac{1}{R^{\epsilon}}$ we have $\frac{1}{\theta_0^{\alpha(\frac{2}{\alpha}-\frac{1}{r})}} < \frac{1}{R^{\epsilon\alpha(\frac{2}{\alpha}-\frac{1}{r})}}$, choosing ϵ such that $\epsilon\alpha(\frac{2}{\alpha}-\frac{1}{r}) < Nk$, we obtain

$$X_{i+1} \leq c\ 16^i \Big\{ X_i^{1+\frac{2}{N+2}} + X_i^{\frac{2}{N+2}} Y_i^{1+\kappa} \Big\},$$

where c is a constant independent of $R,\ \omega,\ A$.
Now we prove that $Y_{i+1} \leq c\ 16^i (X_i + Y_i^{1+\kappa})$. We have

$$Y_{i+1}(k_i - k_{i+1})^2 = \frac{1}{|K_{R_{i+1}}|} \left[\int_{-R_{i+1}^2}^{0} (k_i - k_{i+1})^{\tilde{r}} \left(\int_{\{x \in R_{i+1} : \mathrm{v}^m(x,z) < k_{i+1}\}} dx \right)^{\frac{\tilde{r}}{\tilde{q}}} \right]^{\frac{2}{\tilde{r}}}$$

$$\leq \frac{1}{|K_{R_{i+1}}|} \left[\int_{-R_{i+1}^2}^{0} \left(\int_{R_{i+1}} (\hat{\zeta}_i (k_i - \mathrm{v}^m))^{\tilde{q}}\ dx \right)^{\frac{\tilde{r}}{\tilde{q}}} dz \right]^{\frac{2}{\tilde{r}}}.$$

Then (from k_{i+1} to k_i)

$$Y_{i+1}|k_{i+1} - k_i|^2 \leq |K_{R_{i+1}}|^{-1}\ ||\hat{\zeta}_i(k_i - \mathrm{v}^m)||^2_{\tilde{q},\tilde{r};Q_i}.$$

Moreover, we apply (2.2) in Lemma 2.2 to the truncated $\hat{\zeta}_i(k_i - \mathrm{v}^m)$, which is zero on the boundary from the definition (3.5) of $\hat{\zeta}_i$. Then from the inequality (3.8) we have

$$Y_{i+1}|k_{i+1} - k_i|^2 \leq |K_{R_{i+1}}|^{-1}||\hat{\zeta}_i(k_i - \mathrm{v}^m)||^2_{\tilde{q},\tilde{r};Q_i} \leq \gamma |K_{R_{i+1}}|^{-1}||\hat{\zeta}_i(k_i - \mathrm{v}^m)||^2_{V^2(Q_i)}$$
$$\leq |K_{R_{i+1}}|^{-1} \Big(c_1|Z_i| + c_2 \Big(\int_{-R_i^2}^{0} |Z_i(z)|^{\frac{\tilde{r}}{\tilde{q}}} dz \Big)^{\frac{2}{\tilde{r}}(1+\kappa)} \Big), \tag{3.12}$$

with $c_1 = Cm\gamma(1+m)(\frac{\omega}{2^{s_0}})^2\ \frac{2^{2(i+1)}}{R_i^2},\ c_2 = Cm\gamma\theta_0^{\frac{\hat{\alpha}}{r}}$. Then from (3.12)

$$Y_{i+1} \leq \tilde{c}_1 \Bigg\{ \frac{\omega^2 2^{2(i+1)}}{2^{2s_0} R_i^2} \frac{1}{\frac{\omega^2}{2^{2(s_0+i+2)}}} |K_{R_{i+1}}|^{-1}|Z_i|$$
$$+ \frac{(\frac{\omega}{2^{s_0}})^{\frac{\hat{\alpha}}{q}}}{(\frac{\omega}{2^{s_0}})^2} (2^{2(i+2)}) |K_{R_{i+1}}|^{-1} \Bigg(\int_{-R_i^2}^{0} |Z_i(z)|^{\frac{\tilde{r}}{\tilde{q}}} dz \Bigg)^{\frac{2}{\tilde{r}}(1+\kappa)} \Bigg\}.$$

We conclude that $Y_{i+1} \leq \bar{c}_1\ 16^i (X_i + Y_i^{1+\kappa})$. The sequences $X_i,\ Y_i$ satisfy Lemma 2.3 with $\mathrm{b} = 2^4$ and

$$X_0 + Y_0^{1+k} \leq (\frac{1}{2\bar{c}})^{\frac{1+k}{\sigma}} (\frac{1}{16})^{\frac{1+k}{\sigma^2}} := \nu < 1.$$

Then $X_i, Y_i \to 0$ as $i \to \infty$. With such ν the hypothesis (3.2) of Lemma 3.1 holds, then as a consequence, returning to the original coordinate in time, we obtain (3.3). $\square$

Now we exhibit the second lemma.

Lemma 3.2. *(Expansion in time Lemma) For every $\nu_1 \in (0,1)$, there exists a positive integer s_1, depending only upon the data and independent of ω, R such that for all $t \in (\bar{t} - \hat{\theta}, 0)$*

$$\left| \left\{ x \in K_{\frac{R}{4}} : u^m(x,t) < \mu_- + \frac{\omega}{2^{s_1}} \right\} \right| \leq \nu_1 \left| K_{\frac{R}{4}} \right|,$$

Proof. First we estimate $\left(\int_{t^*}^0 \left| A_{k,R}(t) \right|^{\frac{\tilde{r}}{\tilde{q}}} dt \right)^{\frac{2}{\tilde{r}}(1+\kappa)}$.

Since for any $t \in [-t^*, 0]$, $\left| A_{k,R}(t) \right| \leq |K_R|$ we have

$$\begin{aligned} \left(\int_{-t^*}^0 \left| A_{k,R}(t) \right|^{\frac{\tilde{r}}{\tilde{q}}} dt \right)^{\frac{2}{\tilde{r}}(1+\kappa)} &\leq \left(t^* \left(|K_R| \right)^{\tilde{r}/\tilde{q}} \right)^{\frac{2(1+\kappa)}{\tilde{r}}} \\ \leq \left(|K_R|^{\tilde{r}/\tilde{q}} \right)^{\frac{2(1+\kappa)}{\tilde{r}}} \left(\frac{R^2}{\theta_0^\alpha} \right)^{\frac{2(1+\kappa)}{\tilde{r}}} &= |K_R| \, R^{N\kappa} \frac{1}{\theta_0^{\alpha(\frac{2(1+k}{\tilde{r}})}}, \end{aligned} \tag{3.13}$$

where κ satisfies (2.7).

Moreover, we have $R^{N\kappa} \theta_0^{-\alpha \frac{2(1+\kappa)}{\tilde{r}}} < R^{Nk - \varepsilon \frac{2(1+\kappa)}{\tilde{r}}}$. Pick ε so small in order to have $\frac{N\kappa}{\frac{2(1+\kappa)}{\tilde{r}}} > 2\varepsilon$. Then $N\kappa - \varepsilon \frac{2(1+\kappa)}{\tilde{r}} > \varepsilon \frac{2(1+\kappa)}{\tilde{r}}$ and taking into account that $(\frac{\omega}{A})^\alpha > R^\varepsilon$, it follows $R^{Nk - \varepsilon \frac{2(1+\kappa)}{\tilde{r}}} < \left(\frac{\omega}{A} \right)^{\alpha \frac{2(1+\kappa)}{\tilde{r}}}$. Insert the last estimate in (3.13) to get

$$\left(\int_{-t^*}^0 \left| A_{k,R}(t) \right|^{\frac{\tilde{r}}{\tilde{q}}} dt \right)^{\frac{2}{\tilde{r}}(1+\kappa)} \leq |K_R| \left(\frac{\omega}{A} \right)^{\alpha \frac{2(1+k)}{\tilde{r}}}. \tag{3.14}$$

Now we prove Lemma 3.2. Let s_0 be the smallest positive integer such that $\frac{\omega}{2^{s_0}} \leq 1$. Let $s_1 > s_0 + 2$ an integer to be stated later. By Lemma 3.1, in $K_{\frac{R}{2}} \times (\bar{t} - \hat{\theta}, \bar{t})$, $u > (\mu_- + \frac{\omega}{2^{s_0+1}})^{\frac{1}{m}}$ a.e.

Set $\hat{k} = (\mu_- + \frac{\omega}{2^{s_0+1}})^{\frac{1}{m}}$, $\hat{c} = (\frac{\omega}{2^{s_1}})^{\frac{1}{m}}$. Define in $K_{\frac{R}{2}} \times (\bar{t} - \hat{\theta}, 0)$

$$\hat{\varphi}(u) = ln^+ \left(\frac{\hat{H}}{\hat{H} - (\hat{k} - u)_+ + \hat{c}} \right),$$

where $\hat{H} = \operatorname*{ess\,sup}_{K_{\frac{R}{2}}\times(\bar{t}-\hat{\theta},0)} (\hat{k}-u)_+$. We have $\hat{\varphi}(u) \leq ln^+\left(\frac{\hat{H}}{\hat{c}}\right)$.

Since $\hat{H} \leq (\mu_- + \frac{\omega}{2^{s_0+1}})^{\frac{1}{m}} - \mu_-^{\frac{1}{m}}$, neglecting $\mu_-^{\frac{1}{m}}$ and observing that $f(\mu_-) := (\mu_- \frac{\omega}{2^{s_0+1}})^{\frac{1}{m}} - \mu_-^{\frac{1}{m}}$ attains its max value at $\mu_- = 0$, we have

$$\hat{\varphi}(u) \leq \frac{1}{m}(s_1 - s_0)\ ln2. \tag{3.15}$$

Moreover, $\hat{\varphi}'(u) = \frac{\partial\hat{\varphi}(u)}{\partial u} = \frac{-1}{\hat{H}-(\hat{k}-u)_+ + \hat{c}}$, $\hat{\varphi}''(u) = \frac{\partial^2\hat{\varphi}(u)}{\partial u^2} = (\hat{\varphi}'(u))^2$ and by Lemma 3.1 at the level time $\bar{t}-\hat{\theta}$, $\hat{\varphi}(u) = 0$ in $K_{\frac{R}{2}} \times \{\bar{t}-\hat{\theta}\}$. To prove Lemma 3.2 we consider the definition of weak solution with the Steklov average. Pick as test function $\psi = (\hat{\varphi}^2(u_h))'\xi^2$, where $\xi = \xi(x)$ is a cut-off function with $\xi = 1$ in $K_{\frac{R}{4}}$, $\xi = 0$ on $\partial K_{\frac{R}{2}}, |\nabla\xi| \leq \frac{8}{R}$ For any $\bar{t}-\hat{\theta} \leq t \leq 0$ understanding $h \to 0$, we directly compute

$$\begin{aligned}
&\iint_{K_{\frac{R}{4}}\times(\bar{t}-\hat{\theta},t)} (u)_t(\hat{\varphi}^2(u))'\xi^2 dxd\tau \\
&= -\iint_{K_{\frac{R}{4}}\times(\bar{t}-\hat{\theta},t)} (\nabla u^m)\cdot\nabla\left((\hat{\varphi}^2((u)'\xi^2\right) dxd\tau \\
&- \iint_{K_{\frac{R}{4}}\times(\bar{t}-\hat{\theta},t)} (u^{q-1}\nabla v)\cdot\nabla\left((\hat{\varphi}^2((u))'\xi^2\right) dxd\tau \\
&+ \iint_{K_{\frac{R}{4}}\times(\bar{t}-\hat{\theta},t)} (C(|\nabla u^m)|^2 + \phi)(\hat{\varphi}^2(u))'\xi^2 dxd\tau \\
&\leq -\int_{\bar{t}-\hat{\theta}}^0 \hat{I}dt + \int_{\bar{t}-\hat{\theta}}^0 \hat{J}dt + \int_{\bar{t}-\hat{\theta}}^0 \hat{L}_1 dt + \int_{\bar{t}-\hat{\theta}}^0 \hat{L}_2 dt.
\end{aligned} \tag{3.16}$$

For the first term on the left hand side

$$\iint_{K_{\frac{R}{4}}\times(\bar{t}-\hat{\theta},t)} ((\hat{\varphi}(u)\xi)^2)_t dxdt = \int_{K_{\frac{R}{4}}\times\{t\}} ((\hat{\varphi}(u)\xi))^2 dx. \tag{3.17}$$

To obtain an estimate from below of the term on the right in (3.17), let us integrate in the smaller set

$$\hat{P} := \{x \in K_{R/4} : u(x,t) < (\mu_- + \frac{\omega}{2^{s_1}})^{\frac{1}{m}}\}, \quad t \in (\bar{t}-\hat{\theta}, 0).$$

In such set $\hat{P}$, we have

$$\hat{\varphi}(u) \geq ln^{+} \frac{\hat{H}}{\hat{H} - \hat{k} + (\mu_{-} + \frac{\omega}{2^{s_1}})^{\frac{1}{m}} + \hat{c}}.$$

In this inequality, we apply $(\mu_{-} + \frac{\omega}{2^{s_1}})^{\frac{1}{m}} \leq (\mu_{-})^{\frac{1}{m}} + (\frac{\omega}{2^{s_1}})^{\frac{1}{m}}$.
Moreover, the right hand side is a decreasing function in $\hat{H}$. Then for $t \in (\bar{t} - \hat{\theta}, 0)$, with $\hat{H} \leq \hat{k} - \mu_{-}^{1/m}$, we have

$$\hat{\varphi}(u) \geq \ln^{+} \frac{\hat{k} - \mu_{-}^{\frac{1}{m}}}{2\hat{c}}.$$

Then

$$\int_{\hat{P}} \hat{\varphi}^2(u)dx \geq \frac{\hat{C}}{m^2}(s_1 - s_0 - 1)^2 \ln^2 2 \; |\hat{P}|, \tag{3.18}$$

with $\hat{C}$ is a constant depending on $||u||_{L^\infty}$. Let us estimate the terms on the right hand side of (3.16).

$$\begin{aligned}
-\hat{I} = &- \int_{K_{\frac{R}{4}}} mu^{m-1}\nabla u \cdot \left[(2((\nabla\hat{\varphi})\hat{\varphi}'\xi^2 + 2\hat{\varphi}(\nabla\hat{\varphi}')\xi^2 \right. \\
&\left. +4\hat{\varphi}\ \hat{\varphi}'\xi\nabla\xi\right] dx \leq -2m \int_{K_{\frac{R}{4}}} u^{m-1}(1+\hat{\varphi})\hat{\varphi}'^2\xi^2 \,|\nabla u|^2 dx \\
&+ 4m \int_{K_{\frac{R}{4}}} u^{m-1}|\nabla u||\hat{\varphi}\hat{\varphi}'|\xi \,|\nabla\xi| dx = -\hat{I}_1 + \hat{I}_2.
\end{aligned}$$

Moreover, by Young's inequality we have

$$\begin{aligned}
\hat{I}_2 &= 4m \int_{K_{\frac{R}{4}}} u^{m-1}\varphi(\frac{1}{\sqrt{2}}\varphi'|\nabla u|)(\sqrt{2}\nabla\xi) \leq 4m \int_{K_{\frac{R}{4}}} u^{m-1}\varphi \left[\frac{1}{4}\varphi'^2\xi^2\,|\nabla u|^2| + |\nabla\xi|^2\right] dx \\
&= m \int_{K_{\frac{R}{4}}} u^{m-1}\varphi\varphi'^2\xi^2|\nabla u|^2 dx + 4m \int_{K_{\frac{R}{4}}} u^{m-1}\varphi\,|\nabla\xi|^2 dx = \hat{I}_{21} + \hat{I}_{22}
\end{aligned}$$

$$\begin{aligned}
\hat{J} &= \int_{K_{\frac{R}{4}}} u^{\mathfrak{q}-1}\nabla v \cdot \left[2((\nabla\varphi)\varphi'\xi^2 + 2\varphi(\nabla\varphi')\xi^2 + 4\varphi\varphi'\xi\nabla\xi\right] dx \\
&\leq 2\int_{K_{\frac{R}{4}}} u^{\mathfrak{q}-1}(1+\varphi)\varphi'^2\xi^2|\nabla u||\nabla v| dx + 4\int_{K_{\frac{R}{4}}} u^{\mathfrak{q}-1}\varphi|\varphi'|\xi|\nabla v||\nabla\xi| dx \\
&= \hat{J}_1 + \hat{J}_2.
\end{aligned}$$

By hypothesis on $\mathfrak{q}$, we can split $\mathfrak{q}-1$ in the sum of two positive term of the form $\mathfrak{q}-1=\frac{m-1}{2}+(\mathfrak{q}-\frac{m+1}{2})$. Using again Young's inequality in $\hat{J}_1$ we obtain

$$\hat{J}_1 \leq \int_{K_{\frac{R}{4}}}(1+\varphi)\varphi'^2 u^{m-1}|\xi^2|\nabla u|^2 dx + \int_{K_{\frac{R}{4}}}(1+\varphi)\varphi'^2\xi^2 u^{2q-m-1}|\nabla v|^2 dx$$
$$= \hat{J}_{11} + \hat{J}_{12},$$

$$\hat{J}_2 = 4\int_{K_{\frac{R}{4}}} \varphi\left[(|\varphi'|u^{\mathfrak{q}-1}\xi|\nabla v|)(|\nabla\xi|)\right] dx$$
$$\leq 2\int_{K_{\frac{R}{4}}} \varphi\varphi'^2 u^{2\mathfrak{q}-2}\xi^2|\nabla v|^2 dx + 2\int_{K_{\frac{R}{4}}} \varphi|\nabla\xi|^2 dx = \hat{J}_{21} + \hat{J}_{22}$$

Note that $-\hat{I}+\hat{J}+\hat{L}_1+\hat{L}_2 \leq (-\hat{I}_1+\hat{J}_{11}+\hat{I}_{21}+\hat{L}_1)+(\hat{I}_{22}+\hat{J}_{22})$ $+(\hat{J}_{12}+\hat{J}_{21}+\hat{L}_2)$. Since $\hat{L}_1 \leq 0$, we have

$$-\hat{I}_1+\hat{J}_{11}+\hat{I}_{21}+\hat{L}_1 \leq -(m-1)\int_{K_{\frac{R}{4}}} u^{m-1}\varphi'^2\xi^2\,|\nabla u|^2 dx < 0,$$

a negative term that can be neglected.

$$\hat{I}_{22}+\hat{J}_{22} = 4m\int_{K_{\frac{R}{4}}} u^{m-1}\hat{\varphi}|\nabla\xi|^2 dx \;+\; 2\int_{K_{\frac{R}{4}}} \hat{\varphi}|\nabla\xi|^2$$
$$< C_1\int_{K_{\frac{R}{4}}} \hat{\varphi}\,|\nabla\xi|^2 dx \leq C_1(s_1-s_0-1)\ln 2\frac{2^6}{R^4}\,\left(\int_{-\theta}^{0}(|A_{k,\frac{R}{4}}|^{\frac{\hat{r}}{\hat{q}}}dt\right)^{\frac{2(1+k)}{\hat{r}}}, \tag{3.19}$$

with $C_1 = 4m\mu_+^{m-1}+2$. Now we can estimate

$$\begin{aligned}&\hat{J}_{12}+\hat{J}_{21}\\ &= \int_{K_{\frac{R}{4}}}(1+\hat{\varphi})\hat{\varphi}'^2 u^{2\mathfrak{q}-2}\xi^2|\nabla v|^2 dx + 2\int_{K_{\frac{R}{4}}} \hat{\varphi}\hat{\varphi}'^2 u^{2\mathfrak{q}-2}\xi^2|\nabla v|^2 dx\\ &< 3\int_{K_{\frac{R}{4}}} u^{2\mathfrak{q}-2}(1+\hat{\varphi})\hat{\varphi}'^2\xi^2|\nabla v|^2 dx.\end{aligned} \tag{3.20}$$

An integration in time of (3.20) yields

$$\begin{aligned}&\int_{-\hat{\theta}}^{0}(\hat{J}_{12}+\hat{J}_{21})dt\\&\leq \mu_{+}^{\frac{2q-2}{m}}\frac{3}{m}(s_1-s_0)\ln 2\left(\frac{2^{s_1}}{\omega}\right)^2\iint_{Q_{\frac{R}{4}}(\hat{\theta})}\xi^2|\nabla v|^2dxdt,\end{aligned}\tag{3.21}$$

where the estimate $1+\varphi<\frac{1}{m}(s_1-s_0)\ln 2$ is used, thanks to (3.15). Following the details in computing (2) and (2.11) there exists a positive constant I_u such that

$$\iint_{Q_{\frac{R}{4}}(\hat{\theta})}\xi^2|\nabla v|^2dxdt\leq I_u\Big(\int_{-\theta}^{0}(|A_{k,\frac{R}{4}}|^{\frac{\tilde{r}}{\tilde{q}}}dt\Big)^{\frac{2(1+k)}{\tilde{r}}}.\tag{3.22}$$

Thus, inserting (3.22) in (3.21) , we have

$$\begin{aligned}&\int_{-\hat{\theta}}^{0}(\hat{J}_{12}+\hat{J}_{21})dt\\&\leq \mu_{+}^{\frac{2q-2}{m}}\frac{3}{m}(s_1-s_0)\ln 2\ \left(\frac{2^{s_1}}{\omega}\right)^2 I_u\ \Big(\int_{-\hat{\theta}}^{0}(|A_{k,\frac{R}{4}}|^{\frac{\tilde{r}}{\tilde{q}}}dt\Big)^{\frac{2(1+k)}{\tilde{r}}}.\end{aligned}\tag{3.23}$$

Now we estimate $\hat{L}_2$. Since $\phi\in L^{q,r}_{\mathbb{R}^N\times(t>0)}$, applying Hölder inequality, the following estimate holds

$$\int_{-\hat{\theta}}^{0}\hat{L}_2dt\leq 2\big[(s_1-s_0)\ln 2\big(\frac{2^{s_1}}{\omega}\big)\big]||\phi||_{L^{q,r}_{Q_{\frac{R}{4}}(\hat{\theta})}}\Big(\int_{-\hat{\theta}}^{0}(|A_{k,\frac{R}{4}}|^{\frac{\tilde{r}}{\tilde{q}}}dt\Big)^{\frac{2(1+k)}{\tilde{r}}}.\tag{3.24}$$

Using the estimate (3.14) applied to $K_{\frac{R}{4}}$ and adding (3.23) with (3.24) , we obtain

$$\begin{aligned}&\int_{-\hat{\theta}}^{0}(\hat{J}_{12}+\hat{J}_{21}+L_2)dt\\&\leq (s_1-s_0)\ln 2\frac{2^{s_1}}{\omega}\Big(\mu_{+}^{\frac{2q-2}{m}}\frac{3}{m}\frac{2^{s_1}}{\omega}I_u+2||\phi||_{L^{q,r}_{Q_{\frac{R}{4}}}}\Big)|K_{\frac{R}{4}}|\left(\frac{\omega}{A}\right)^{\frac{2\alpha(1+k)}{\tilde{r}}}\end{aligned}\tag{3.25}$$

Inserting (3.18), (3.19) and (3.25) into (3.16), we obtain

$$\begin{aligned}
&\left(\frac{1}{2m}\right)^2 (s_1-s_0)^2 \ln^2 2 \ |\hat{P}| \\
&\leq 2mC_1(s_1-s_0)\ln 2 \frac{2^6}{R^2}|K_{\frac{R}{4}}| \left(\frac{\omega}{A}\right)^{\frac{2\alpha(1+k)}{\tilde{r}}} \\
&+ (s_1-s_0)\ln 2 \frac{2^{s_1}}{\omega}\left(\mu_+^{\frac{2q-2}{m}} \frac{3}{m}\frac{2^{s_0}}{\omega} I_u + 2||\phi||_{L^{q,r}_{Q_{\frac{R}{2}}}}\right)|K_{\frac{R}{4}}| \left(\frac{\omega}{A}\right)^{\frac{2\alpha(1+k)}{\tilde{r}}}.
\end{aligned} \tag{3.26}$$

Dividing by $\frac{\hat{C}}{m^2}(s_1-s_0)^2 \ ln^2 2$, choosing A and s_1 sufficiently large, we conclude

$$\begin{aligned}
|\hat{P}| \leq &\Bigg\{ \frac{2m^3 C_1 \frac{2^6}{R^2}}{\hat{C}(s_1-s_0)\ln 2} \\
&+ \frac{(m)^2 \frac{2^{s_1}}{\omega}}{\hat{C}(s_1-s_0)\ln 2}\left((\mu_+)^{\frac{2q-2}{m}} \frac{3}{m}\frac{2^{s_1}}{\omega} I_u + 2||\phi||_{L^{q,r}_{Q_{\frac{R}{2}}}}\right)\Bigg\}|K_{\frac{R}{4}}| \left(\frac{\omega}{A}\right)^{\frac{2\alpha(1+k)}{\tilde{r}}} \\
&= \nu_1 |K_{\frac{R}{4}}|.
\end{aligned}$$

□

Now, using Lemma 3.2 and following the same argument developed in Chapter III, Section 6 of [5], in the subcylinder $K_{\frac{R}{4}} \times (\bar{t} - \theta_0^{-\alpha}(\frac{R}{4})^2, \ 0)$, one can prove that the numbers ν_1 and s_1 of Lemma 3.2 can be chosen a priori depending only upon the data and independent of ω and R, such that we have

$$u^m > \mu_- + \frac{\omega}{2^{s_1+1}} \quad a.e. \ (x,t) \in K_{\frac{R}{8}} \times \left(\bar{t} - \theta_0^{-\alpha}\left(\frac{R}{8}\right)^2, \ 0\right).$$

This concludes the 1^{st} alternative, because we have proved the reduction of the oscillation in that sub cylinder. Thanks to the expansion of positivity in time (for more detail about this now a classic tool, see [5]) we are able to transport this information to the top of the original cylinder. In fact, we have shown that in the sub cylinder located at the top, there exist numbers $\eta_1 = \left(1 - \frac{1}{2^{s_1+1}}\right)$, and $\mathcal{A}_1 > A$ that can be determined a priori in terms of the data, such that

$$either \qquad \operatorname*{ess\ osc}_{Q_{\frac{R}{8}}(\hat{\theta})} u^m \leq \eta_1 \, \omega \qquad or \quad \omega \leq \mathcal{A}_1 R^{\frac{\epsilon}{\alpha}}.$$

4. 2^{nd} alternative for the parabolic-parabolic case

Suppose that the assumption of the first alternative is violated, i.e. for all subcylinders $Q_R^{\bar{t}}(\hat{\theta}) \subset Q_R(\hat{a})$

$$\Big| \Big\{ (x,t) \in\, Q_R^{\bar{t}}(\hat{\theta}) : u^m(x,t) < \mu_- + \frac{\omega}{2^{s_o}} \Big\} \Big| >\, \nu \big|Q_R(\hat{\theta})\big|. \qquad (4.1)$$

We can rewrite (4.1) as

$$\Big| \Big\{ (x,t) \in Q_R^{\bar{t}}(\hat{\theta}) : u^m(x,t) > \mu_+ - \frac{\omega}{2^{s_o}} \Big\} \Big| \leq (1-\nu) \big|Q_R(\hat{\theta})\big|. \qquad (4.2)$$

In order to estimate the measure of the set where $u^m(x,t) > \mu_+ - \frac{\omega}{2^{s_o}}$ within K_R, we use the following lemma proven in [5].

Lemma 4.1. *(t^* Lemma). Fix $Q_R^{\bar{t}}(\hat{\theta}) \subset Q_R(\hat{a})$ and assume that (4.2) holds. There exists a time level $t^* \in [\bar{t} - \theta_0^{-\alpha} R^2, \bar{t} - \frac{\nu}{2}\theta_0^{-\alpha} R^2]$ such that*

$$\Big| \Big\{ x \in K_R \colon u^m(x,t^*) > \mu_+ - \frac{\omega}{2^{s_o}} \Big\} \Big| \leq \frac{1-\nu}{1-\frac{\nu}{2}}\ |K_R|.$$

Now let us evaluate the measure of the set $\big\{ x \in\, K_R : u^m(x,t) > \mu_+ - \frac{\omega}{2^{s_1}} \big\}$, $t \in [t^*, 0]$, where $s_1 > s_o$ is an integer to be fixed later on.
At this end, pick $H = \operatorname*{ess\,sup}_{K_R \times [t^*,0]} \big(u^m - (\mu_+ - \frac{\omega}{2^{s_o}})\big)$. In $K_R \times [t^*, 0]$, consider the function

$$\varphi(H) = \varphi(u^m) = ln^+ \left(\frac{H}{H - (u^m - k) + c} \right),\ k = \mu_+ - \frac{\omega}{2^{s_o}},\ c = \frac{\omega}{2^{s_1}}.$$

Note that

$$\varphi'(u^m) = \frac{1}{H - (u^m - k) + c} > \frac{H}{H - (u^m - k) + c} \geq 1,$$
$$\varphi''(u^m) = (\varphi'(u^m))^2, \quad \nabla\varphi = \varphi' \nabla u^m, \ \ \nabla\varphi' = \varphi'^2 \nabla u^m.$$

Lemma 4.2. *(Logarithmic Lemma) There exists an integer $s_1 > s_0$, such that if $H > \frac{\omega}{2^{s_1}}$, then*

$$\Big| \Big\{ x \in\, K_R : u^m(x,t) > \mu_+ - \frac{\omega}{2^{s_1}} \Big\} \Big| \ \leq \Big(1 - \ \frac{\nu^2}{4}\Big)\ |K_R|,\ \ \forall t \in [t^*, 0].$$

Proof. Split $K_R = K_{(1-\sigma)R} + K_{\sigma R}, 0 < \sigma < 1$. Making use of the definition of weak solution, via the Steklov average, pick as test function $\psi(u_h^m) = m(u_h^m)^\alpha(\varphi^2(u_h^m))'\xi^2$, where $\xi = \xi(x)$ is a cut-off function, $\xi = 1$ in the cube $K_{(1-\sigma)R}$, $\xi = 0$ on ∂K_R, $|\nabla\xi| \le \frac{2}{\sigma R}$.
For any $t^* \le t \le 0$, letting $h \to 0$, we get

$$\iint_{K_R\times(t^*,t)} (\varphi^2(u^m)\xi^2)_t dx d\tau = \int_{K_R\times\{t\}} \varphi^2(u^m)\xi^2 dx - \int_{K_R\times\{t^*\}} \varphi^2(u^m)\xi^2 dx.$$

Define

$$\begin{aligned} &-\int_{t^*}^t \int_{K_R} (\nabla u^m)\cdot\nabla\psi dx d\tau + \int_{t^*}^t \int_{K_R} (u^{q-1}\nabla v)\cdot\nabla\psi dx d\tau \\ &+ \int_{t^*}^t \int_{K_R} C|\nabla u^m|^2\psi \; dx d\tau + \int_{t^*}^t \int_{K_R} \phi\psi dx d\tau \\ &:= -\int_{t^*}^t I d\tau + \int_{t^*}^t J d\tau + \int_{t^*}^t L_1 d\tau + \int_{t^*}^t L_2 d\tau \end{aligned}$$

and, from the definition of weak solution, get the following inequality

$$\begin{aligned} &\int_{K_{(1-\sigma)R}\times\{0\}} \varphi^2(u^m)\xi^2 \; dx \le \int_{K_R\times\{0\}} \varphi^2(u^m)\xi^2 \; dx \\ &= \int_{K_R\times\{t^*\}} \varphi^2((u^m)\xi^2 \; dx + \int_{t^*}^0 (-I + J + L_1 + L_2) \; dt. \end{aligned} \tag{4.3}$$

To estimate the term $\int_{K_{(1-\sigma)R}\times\{t\}} \varphi^2(u^m)\xi^2 \; dx$ define
$\bar{P} = \{x \in K_{(1-\sigma)R} : u^m(x,t) > \mu_+ - \frac{\omega}{2^{s_1}}\}$, with $t^* < t < 0$.
Using the notation $\varphi(u^m) = \varphi(H)$, in $\bar{P}$ we have $\varphi(H) \ge (s_1 - s_0 - 1)\ln 2$ (to prove this estimate we refer the reader to Chapter II, Section 3-(ii) in [5]). Hence by Lemma 4.1

$$\int_{K_R\times\{t^*\}} \varphi^2(u^m)\xi^2 \; dx \le \ln^2 2(s_1 - s_0)^2 \left(\frac{1-\nu}{1-\frac{\nu}{2}}\right) |K_R| \tag{4.4}$$

and

$$\begin{aligned} &\int_{\bar{P}\times\{t\}} \varphi^2(u^m)dx \\ &> (s_1 - s_0 - 1)^2 \ln^2 2 \left|\left\{x \in K_{(1-\sigma)R} : u^m(x,t) > \mu_+ - \frac{\omega}{2^{s_1}}\right\}\right|. \end{aligned} \tag{4.5}$$

Our aim is to estimate $\int_{t^*}^0(-I+J+L_1+L_2)dt$.

$$-I=-\int_{K_R} mu^{m-1}\nabla u\cdot\big[(2m(m-1)u^{m-2}(\nabla u)\varphi\varphi'\xi^2\,+\,2mu^{m-1}((\nabla\varphi)\varphi'\xi^2$$
$$+2mu^{m-1}\varphi(\nabla\varphi')\xi^2+4mu^{m-1}|\nabla u|\varphi\varphi'\xi\nabla\xi\big]\,dx\le -2(m-1)\int_{K_R}u^{-1}(\nabla u^m|)^2\varphi\varphi'\xi^2dx$$
$$-2m\int_{K_R}u^{m-1}(1+\varphi)\varphi'^2\xi^2\,|\nabla u^m|^2dx+4m\int_{K_R}u^{m-1}|\nabla u^m|\varphi\varphi'\xi\,|\nabla\xi|dx=-I_1-I_2+I_3.$$

Moreover, by Young's inequality we have

$$I_3=4m\int_{K_R}u^{m-1}\varphi(\frac{1}{\sqrt{2}}\varphi'\nabla u^m)(\sqrt{2}\nabla\xi)dx\le 4m\int_{K_R}u^{m-1}\varphi\left[\frac14\varphi'^2\xi^2\,|\nabla u^m|^2+|\nabla\xi|^2\right]dx$$
$$=m\int_{K_R}u^{m-1}\varphi\varphi'^2\xi^2|\nabla u^m|^2dx+4m\int_{K_R}u^{m-1}\varphi\,|\nabla\xi|^2dx=I_{31}+I_{32}$$

$$J=\int_{K_R}u^{q-1}\nabla v\cdot\big[2m(m-1)u^{m-2}(\nabla u)\varphi\varphi'\xi^2+2mu^{m-1}(\nabla\varphi)\varphi'\xi^2$$
$$+2mu^{m-1}\varphi(\nabla\varphi')\xi^2+4mu^{m-1}\varphi\varphi'\xi\nabla\xi\big]\,dx\le 2(m-1)\int_{K_R}u^{\mathfrak{q}-2}\varphi\varphi'\xi^2|\nabla u^m||\nabla v|dx$$
$$+2m\int_{K_R}u^{\mathfrak{q}+m-2}(1+\varphi)\varphi'^2\xi^2|\nabla u^m||\nabla v|dx+4m\int_{K_R}u^{\mathfrak{q}+m-2}\varphi\varphi'\xi|\nabla v||\nabla\xi|dx$$
$$=J_1+J_2+J_3.$$

Using Young's inequality and the fact that $\varphi'>1$ we have

$$J_1\le(m-1)\int_{K_R}u^{-1}\varphi\varphi'\xi^2|\nabla u^m|^2dx+(m-1)\int_{K_R}\varphi\varphi'^2\xi^2|\nabla v|^2u^{2\mathfrak{q}-3}dx$$
$$=J_{11}+J_{12}.$$

Using Young's inequality in J_2 we obtain

$$J_2\le 2m\int_{K_R}u^{m-1}(1+\varphi)(\varphi^2)'\xi^2\left[\frac12|\nabla u^m|^2+\frac12u^{2\mathfrak{q}-2}|\nabla v|^2\right]$$
$$=m\int_{K_R}u^{m-1}(1+\varphi)(\varphi^2)'\xi^2|\nabla u^m|^2+m\int_{K_R}u^m(1+\varphi)\varphi'^2u^{2\mathfrak{q}-3}\xi^2|\nabla v|^2$$
$$=J_{21}+J_{22},$$

$$J_3 = 4m \int_{K_R} u^{m-1}\varphi \left[(\varphi' u^{\mathfrak{q}-1}\xi|\nabla v|)(|\nabla \xi|)\right]$$
$$\leq 2m \int_{K_R} u^m \varphi\varphi'^2 u^{2\mathfrak{q}-3}\xi^2|\nabla v|^2 + 2m\int_{K_R} u^{m-1}\varphi|\nabla\xi|^2 = J_{31} + J_{32}.$$

Note that $-I + J + L_1 + L_2 \leq (-I_1 + J_{11}) + (-I_2 + J_{21} + I_{31} + L_1)$ $+(I_{32} + J_{32}) + (J_{12} + J_{22} + J_{31}) + L_2$.

Moreover

$$-I_1 + J_{11} = -(m-1)\int_{K_R} u^{-1}|\nabla u^m|^2 \varphi\varphi'\xi^2 dx < 0$$

and

$$-I_2 + J_{21} + I_{31} + L_1 \leq -m(1-2C)\int_{K_R} u^{m-1}\varphi'^2\xi^2\,|\nabla u^m|^2 dx;$$

taking $C < 1/2$, we obtain a negative term that can be neglected.

$$I_{32} + J_{32} = 4m \int_{K_R} u^{m-1}\varphi\,|\nabla\xi|^2 dx + 2m\int_{K_R} u^{m-1}\varphi|\nabla\xi|^2$$
$$= 6m\int_{K_R} u^{m-1}\varphi\,|\nabla\xi|^2 dx. \tag{4.6}$$

It follows that

$$J_{12} + J_{22} + J_{31}$$
$$= (m-1)\int_{K_R}\varphi\varphi'^2 u^{2\mathfrak{q}-3}\xi^2|\nabla v|^2 dx + m\int_{K_R} u^m(1+\varphi)\varphi'^2 u^{2\mathfrak{q}-3}\xi^2|\nabla v|^2 dx$$
$$+ 2m\int_{K_R} u^m\varphi\varphi'^2 u^{2\mathfrak{q}-3}\xi^2|\nabla v|^2 dx < 3m\int_{K_R}\varphi\varphi'^2 u^{2\mathfrak{q}-3}\xi^2|\nabla v|^2 dx$$
$$+ 3m\int_{K_R} u^m(1+\varphi)\varphi'^2 u^{2\mathfrak{q}-3}\xi^2|\nabla v|^2 dx \leq 3m\int_{K_R}(1+u^m)(1+\varphi)\varphi'^2 u^{2\mathfrak{q}-3}\xi^2|\nabla v|^2 dx.$$

From the last inequalities, taking in account (3.3) and (3.15) we deduce

$$\int_{Q_R(t^*)}(1+u^m)(1+\varphi)\varphi'^2 u^{2\mathfrak{q}-3}\xi^2|\nabla v|^2 dxdt$$
$$\leq (1+\mu_+)(1+(s_1-s_0)\ln 2)\left(\frac{2^{s_1}}{\omega}\right)^2(\mu_+)^{\frac{2\mathfrak{q}-3}{m}}\int_{Q_R(t^*)}\xi^2|\nabla v|^2 dxdt.$$

Following the details in computing (2.11), there exists I_u such that

$$\int_{Q_R(t^*)} \xi^2 |\nabla v|^2 dxdt \le I_u \left(\int_{t^*}^0 \Big| A_{k,R}(t) \Big|^{\frac{\tilde r}{\tilde q}} dt \right)^{\frac{2}{\tilde r}(1+\kappa)}$$

where $\Big| A_{k,R}(t) \Big| = \Big| \left\{ x \in K_R : u^m > \mu_+ \ - \frac{\omega}{2^{s_1}} \right\} \Big|$.
Thus by using (3.13) and (2.7) we have

$$\begin{aligned} &\int_{t^*}^0 (J_{12} + J_{22} + J_{31}) dt \\ &\le 3m I_u (1+\mu_+)(1 + (s_1 - s_0)\ln 2) \frac{2^{2s_1}}{\omega^2} \mu_+^{\frac{2q-3}{m}} |K_R| \, \left(\frac{\omega}{A} \right)^{\alpha(1-\frac{1}{r})}. \end{aligned}$$

Also we have by (4.6)

$$\begin{aligned} &\int_{t^*}^0 (I_{32} + J_{32}) dt \le 6m(s_1 - s_0) \ln 2 \, (\mu_+)^{m-1} \gamma \, |K_R| \, \theta_0^{-\alpha} \\ &\le 6\tilde\gamma m \, \frac{(2^{s_0})^\alpha (s_1 - s_0) \ln 2 \, \mu_+^{m-1}}{\omega^\alpha} \, \Big| K_R \Big|. \end{aligned} \tag{4.7}$$

Now we estimate L_2. Since $\phi \in L^{q,r}(\mathbb{R}^N \times (t > 0))$, applying Hölder inequality we have

$$\int_{t^*}^0 L_2 dt \le 2m\mu_+^{m-1}(s_1 - s_0) ln 2 \frac{2^{s_1}}{\omega} ||\phi||_{q,r,Q_R(\hat\theta)} \Big(\int_{-\theta}^0 (|A_{k,R}|^{\frac{\tilde r}{\tilde q}} dt \Big)^{\frac{2(1+k)}{\tilde r}}. \tag{4.8}$$

Adding (4.8) to (4.7), we have

$$\begin{aligned} &\int_{t^*}^0 (J_{12} + J_{22} + J_{31} + L_2) dt \\ &\le 3m \, I_u (1+\mu_+)(1 + (s_1 - s_0)\ln 2) \Big(\frac{2^{s_1}}{\omega} \Big)^2 \mu_+^{\frac{2q-3}{m}} |K_R| \, \Big(\frac{2^{s_0}}{A} \Big)^{\frac{2\alpha(1+k)}{\tilde r}} \\ &+ 2m \Big[\mu_+^{m-1} (s_1 - s_0) \ln 2 \Big(\frac{2^{s_1}}{\omega} \Big) \Big] \, ||\phi||_{q,r,Q_R(\hat\theta)} \Big(\frac{2^{s_0}}{A} \Big)^{\frac{2\alpha(1+k)}{\tilde r}} |K_R|. \end{aligned}$$

At the end we obtain the final estimate

$$\begin{aligned}
&\int_{t^*}^{0}(-I+J+L_1+L_2)dt\\
&\le 3m\, I_u(1+\mu_+)(1+(s_1-s_0)\ln 2)\Big(\frac{2^{s_1}}{\omega}\Big)^2\mu_+^{\frac{2q-3}{m}}\Big(\frac{2^{s_0}}{A}\Big)^{\frac{2\alpha(1+k)}{\tilde r}}|K_R|\\
&+2m\Big[\mu_+^{m-1}(s_1-s_0)\ln 2\Big(\frac{2^{s_1}}{\omega}\Big)\Big]||\phi||_{q,r,Q_R(\hat\theta)}\Big(\frac{2^{s_0}}{A}\Big)^{\frac{2\alpha(1+k)}{\tilde r}}|K_R|\\
&+6m\gamma\,\frac{(2^{s_0})^\alpha}{\omega^\alpha}\frac{(s_1-s_0)\ln 2\ \mu_+^{m-1}}{\sigma^2}\,|K_R|.
\end{aligned} \tag{4.9}$$

By inserting (4.4), (4.5) and (4.9) in (4.3) and dividing by $(s_1-s_0-1)^2\ln^2 2$, we obtain

$$\begin{aligned}
&|\{x\in K_R: u^m(x,t)>\mu_+-\frac{\omega}{2^{s_1}}\}\le\Big(\frac{s_1-s_0}{s_1-s_0-1}\Big)^2\Big(\frac{1-\nu}{1-\frac{\nu}{2}}\Big)|K_R|\\
&+3m\, I_u(1+\mu^+)\frac{(1+(s_1-s_0)\ln 2)}{(s_1-s_0-1)^2\ln^2 2}\Big(\frac{2^{s_1}}{\omega}\Big)^2\ \mu_+^{\frac{2q-3}{m}}\Big(\frac{2^{s_0}}{A}\Big)^{\frac{2\alpha(1+k)}{\tilde r}}|K_R|\\
&+2m\frac{\mu_+^{m-1}(s_1-s_0)}{(s_1-s_0-1)^2\ln 2}\frac{2^{s_1}}{\omega}||\phi||\Big(\frac{2^{s_0}}{A}\Big)^{\frac{2\alpha(1+k)}{\tilde r}}|K_R|\\
&+6\gamma m\frac{(2^{s_0})^\alpha}{\omega^\alpha}\frac{(s_1-s_0)\,\mu_+^{m-1}}{\sigma^2\ (s_1-s_0-1)^2\ln 2}|K_R|+N\sigma|K_R|\\
&\equiv(\mathsf{A}+\mathsf{B}_1+\mathsf{B}_2+\mathsf{C}+N\sigma)\ |K_R|,
\end{aligned}$$

where we used the fact that

$$\Big|\Big\{x\in K_R\colon u^m>\mu_+-\frac{\omega}{2^{s_1}}\Big\}\Big|\le\Big|\Big\{x\in K_{(1-\sigma)R}\colon u^m>\mu_+-\frac{\omega}{2^{s_1}}\Big\}\Big|+N\sigma|K_R|.$$

Choosing σ such that $N\sigma\le\frac{1}{4}\nu^2$, s_1 such that $\Big(\frac{s_1-s_0}{s_1-s_0-1}\Big)^2\le(1-\frac{1}{2}\nu)(1+\nu)$, we have $\mathsf{A}\le 1-\nu^2$ and $\mathsf{C}\le\frac{1}{4}\nu^2$ and for such σ and s_1, let A be such that $\mathsf{B}_1\le\frac{1}{8}\nu^2$ and $\mathsf{B}_2\le\frac{1}{8}\nu^2$ and this implies the statement of Lemma 4.2. □

The second alternative is concluded estimating the measure of the set where $u^m(x,t)>\mu_+-\frac{\omega}{2^{s^*}}$, $s^*>s_2$ within a sub cylinder of $Q_R(\frac{\hat a}{2})$. This can be done via the following two lemmata which proofs can be deduced from Lemma 8.1 and Lemma 9.1, Chapter 3 in [5].

Lemma 4.3. *For every $\nu^* \in (0,1)$, there exists $s^* > s_2$, independent on R, ω such that*

$$\Big|\Big\{x \in Q_R(\frac{\hat{a}}{2}) : u^m(x,t) > \mu_+ - \frac{\omega}{2^{s^*}}\Big\}\Big| \le \nu^* \ |Q_R(\frac{\hat{a}}{2})|$$

with $A = 2^{s^*}$, $a_0 = \frac{\omega}{A}$.

Lemma 4.4. *The number ν^* (and s^*) can be chosen such that*

$$u^m(x,t) \le \mu_+ - \frac{\omega}{2^{s^*+1}}, \quad a.e. \ in \ Q\Big(\frac{R}{2}, \frac{1}{2}\Big(\frac{\omega}{2^{s*}}\Big)^{-\alpha}\Big(\frac{R}{2}\Big)^2\Big).$$

The reduction of the oscillation concludes the 2^{st} alternative.
Following the approach by Di Benedetto (see [5], Chapter III), the two alternatives imply the Hölder continuity of u^m, hence Theorem 1.1 is proved in the parabolic-parabolic case.

5. Hölder continuity to the parabolic-elliptic chemotaxis system

The aim of this section is to extend the results obtained in the previous sections for the system (1.1) with $\tilde{\tau} = 1$, to the following parabolic-elliptic degenerate system ($\tilde{\tau} = 0$) in $\mathbb{R}^N \times (t > 0)$

$$\begin{cases} u_t = \mathrm{div}(\nabla u^m) - \chi \mathrm{div}(u^{\mathfrak{q}-1}\nabla v) + B(x,t,u,\nabla u), \\ 0 = \Delta v - av + \ u, \end{cases}$$

with nonnegative initial data satisfying (1.2). Our approach is unitary and does not see the difference between the parabolic-parabolic and parabolic-elliptic cases. For this reason the proof of Hölder continuity of u^m follows almost entirely the steps of the Sections 3 and 4 for the parabolic-parabolic case. In this section we will focus our attention only on the main differences.
First, let us state a-priori elliptic L^p estimates.
Consider the elliptic equation

$$-\Delta v + av = w, \quad x \in \mathbb{R}^N.$$

By classical L^p regularity results ([2], [3])

$$||v(x)||_{W^{2,p}(\mathbb{R}^N)} \le c||w(x)||_{L^p(\mathbb{R}^N)} \tag{5.1}$$

where c is a constant depending upon p, N and a.
Let us start now the study of the 1^{st} Alternative.
In order to extend the result in Lemma 3.1 we must consider the terms containing $|\nabla v|$, and there, instead of (2.6), we have to use the estimate (5.1). The same must be done in the analogous of Lemma 3.2.
Let us explain some details. To construct the sequences X_i and Y_i the Lemma 2.5 must be applied. More precisely, we must change the estimate of the term with $|\nabla v|$ (see (2.11)) present in

$$\begin{aligned}&\int_{t_1}^{t_2}\int_{K_R\bigcap\{(k-u^m)>0\}} u^{2(\mathfrak{q}-1)}\eta^2|\nabla v|^2dxdt\\&\leq \mu_+^{\frac{2(\mathfrak{q}-1)}{m}}\Big(\int_{t_1}^{t_2}\Big(\int_{K_R}|\nabla v|^{2q}dx\Big)^{\frac{r}{q}}dt\Big)^{\frac{1}{r}}\Big(\int_{t_1}^{t_2}\Big|A_{k,R}(t)\Big|^{\frac{r(q-1)}{q(r-1)}}dt\Big)^{\frac{r-1}{r}}.\end{aligned}\tag{5.2}$$

By using (5.1) with $w=u$ and $p=2q$, we obtain

$$\Big(\int_{t_1}^{t_2}\Big(\int_{K_R}|\nabla v|^{2q}dx\Big)^{\frac{r}{q}}dt\Big)^{\frac{1}{r}}\leq c\Big(\int_{t_1}^{t_2}||u||_{2q}^{2r}dt\Big)^{\frac{1}{r}}\leq E_u,\tag{5.3}$$

with E_u a positive constant depending on $\sup\limits_{t_1<t<t_2}||u||_{2q}$. Replacing (5.3) in (5.2) and using (2.7) we get

$$\int_{t_1}^{t_2}\int_{K_R\bigcap\{(k-u^m)>0\}} u^{2(\mathfrak{q}-1)}\eta^2|\nabla v|^2dxdt\leq E_u\mu_+^{\frac{2(\mathfrak{q}-1)}{m}}\Big(\int_{t_1}^{t_2}\Big|A_{k,R}(t)\Big|^{\frac{\hat{r}}{\hat{q}}}dt\Big)^{\frac{2}{\hat{r}}(1+\kappa)}.$$

Inserting the last inequality in the computations of Lemma 3.1 and checking the validity of Lemma 3.2, we derive also for the parabolic-elliptic case that the oscillation of u^m is reduced by a fixed factor.
For the 2^{nd} Alternative, we observe that Lemmata 4.2 and 4.1 hold by replacing in the estimate of $|\nabla v|$ the constant I_u with the constant E_u defined in (5.3). So also in this case the oscillation is reduced. Exactly as in the parabolic-parabolic case, this implies the Hölder continuity of u^m.

Acknowledgments

The authors gratefully thank the anonymous reviewer for his helpful remarks and suggestions, which greatly improved the clarity of this paper.

The authors are members of the Gruppo Nazionale per l'Analisi Matematica, la Probabilità e le loro Applicazioni (GNAMPA) of the Istituto Nazionale di Alta Matematica (INdAM). M.M. is partially supported by the research project *Evolutive and Stationary Partial Differential Equations with a focus on biomathematics* (Fondazione di Sardegna 2019) and by the grant PRIN n.PRIN-2017AYM8XW: *Non-linear Differential Problems via Variational, Topological and Set-valued Methods*.

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