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TUMOR-IMMUNE CELL INTERACTIONS BY A FULLY PARABOLIC CHEMOTAXIS MODEL WITH LOGISTIC SOURCE

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Abstract. This work studies the existence of classical solutions to a class of chemotaxis systems by

$$\begin{cases} u_t = \Delta u - \chi\nabla\cdot(u\nabla v) + \mu_1 u^k - \mu_2 u^{k+1}, & \text{in } \Omega\times(0,T_{\max}),\\ v_t = \Delta v + \alpha w - \beta v - \gamma uv, & \text{in } \Omega\times(0,T_{\max}),\\ w_t = \Delta w - \delta uw + \mu_3 w(1-w), & \text{in } \Omega\times(0,T_{\max}),\\ \dfrac{\partial u}{\partial\nu} = \dfrac{\partial v}{\partial\nu} = \dfrac{\partial w}{\partial\nu} = 0, & \text{on } \partial\Omega\times(0,T_{\max}),\\ u(x,0)=u_0(x), \quad v(x,0)=v_0(x), \quad w(x,0)=w_0(x), & x\in\overline{\Omega}, \end{cases}$$

that model interactions between tumor (i.e., w) and immune cells (i.e., u) with a logistic-type source term $\mu_1 u^k - \mu_2 u^{k+1}$, $k \geq 1$, also in presence of a chemical signal (i.e., v). The model parameters $\chi, \mu_1, \mu_2, \mu_3, \alpha, \beta, \gamma$, and δ are all positive. The vector ν is the outward unit normal on $\partial\Omega$. The value $T_{\max}$ indicates the maximum instant of time up to which solutions are defined. Our focus is on examining the global existence in a bounded domain $\Omega \subset \mathbb{R}^n, n \geq 3$, under Neumann boundary conditions. We distinguish between two scenarios: $k > 1$ and $k = 1$. The first case allows to prove boundedness under smaller assumptions relying only on the model parameters instead of on the initial data, while the second case requires an extra condition relating the parameters χ, μ_2, n, and the initial data $\|v_0\|_{L^\infty(\Omega)}$. This model can be seen as an extension of those previously examined in [11] and [4], the former being a system with only two equations and the latter the same model without logistic.

1. **Introduction.** This work studies the dynamics between tumor cells and immune cells influenced by the presence of a chemical signal released by the latter. The interplay between these units has been broadly studied as a chemotaxis model [2, 16]. Chemotaxis, as a model where units (microorganisms as bacteria or organisms as animals [22]) move in response to chemical gradients, was first modeled by Keller and Segel considering mainly two dynamics. The first is the so-called *production model* [9], where a population of cells $u = u(x,t)$ (e.g., amoebae in the

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first study), $x \in \Omega$ a bounded region of $\mathbb{R}^2$, $t \in \mathbb{R}$, releases a chemical substance $v = v(x,t)$ (in that case the acrasin), related to the simplified mathematical model

$$\begin{cases} u_t = \Delta u - \chi \nabla \cdot (u \nabla v), & \text{in } \Omega \times (0, T_{\max}) \\ v_t = \Delta v - v + u, & \text{in } \Omega \times (0, T_{\max}) \\ \dfrac{\partial u}{\partial \nu} = \dfrac{\partial v}{\partial \nu} = 0, & \text{on } \partial\Omega \times (0, T_{\max}), \\ u(x,0) = u_0(x), \quad v(x,0) = v_0(x), & x \in \overline{\Omega}, \end{cases} \tag{1}$$

with χ the *chemotactic coefficient* measuring the strength of chemotaxis, i.e., the term $-\chi \nabla \cdot (u \nabla v)$ represents the movement of the cells towards the chemical signal. The vector ν represents the outward unit normal on the boundary $\partial\Omega$. The value $T_{\max} \in (0, \infty]$ indicates the maximum instant of time up to which solutions are defined. All the models discussed within this work are assumed to have homogeneous Neumann boundary conditions. Henceforth, for the sake of brevity, this homogeneous Neumann boundary condition will be implicitly assumed for all models and omitted from their explicit statements.

The second is the *consumption model* [10] which, in a simplified form, reads:

$$\begin{cases} u_t = \Delta u - \chi \nabla \cdot (u \nabla v), & \text{in } \Omega \times (0, T_{\max}) \\ v_t = \Delta v - uv, & \text{in } \Omega \times (0, T_{\max}) \\ u(x,0) = u_0(x), \quad v(x,0) = v_0(x), & x \in \overline{\Omega}, \end{cases} \tag{2}$$

where the population of bacteria $u(x,t)$ moves toward higher concentration of oxygen $v(x,t)$, thereby depleting it. For more details on these models and the main theoretical techniques to tackle their studies, the reader may address the surveys and papers [2, 5, 6, 7, 17, 18, 19, 1, 16, 12, 13]. Those works deal with analyses concerning local and/or global solutions for models defined on bounded and smooth domains Ω of $\mathbb{R}^n$, for $n \geq 1$; specifically, some aspects regarding the possibility of blow-up in finite time are considered.

To summarize some of these results, regarding model (1), as mentioned in [18] and the references cited there, for $n = 1$, all the solutions are global in time and bounded; meanwhile, for $n = 2$, it is required that $\|u_0\|_{L^1(\Omega)} < 4\pi$. Otherwise, for $n \geq 3$, global and bounded solutions exist provided that $\|u_0\|_{L^{\frac{n}{2}+\delta}(\Omega)} < \varepsilon$ and $\|\nabla v_0\|_{L^{n+\delta}(\Omega)} < \varepsilon$ for some $\varepsilon > 0$ and any $\delta > 0$ [25]. However, blow up either in finite or infinite time may occur for $n = 2$ if $\|u_0\|_{L^1(\Omega)} > 4\pi$. Also, for $n \geq 3$, the solution may be unbounded either in finite or infinite time when Ω is a ball.

On the other hand, for the consumption model, in [17] it was proved that system (2) possesses a unique classical global uniformly-in-time bounded solution if $0 < \chi \leq \frac{1}{6(n+1)\|v_0\|_{L^\infty(\Omega)}}$ for a smooth bounded domain $\Omega \subset \mathbb{R}^n$ and $n \geq 2$. Note the required relationship between the parameter χ, the initial condition $v_0(x)$, and the dimension n. That restriction was sharpened by [1], always linking those three values. In case of arbitrary large initial data, global weak solutions are obtained in smooth bounded convex domains $\Omega \subset \mathbb{R}^3$ [19].

The aggregation processes mentioned above are essentially connected to an uncontrolled increasing of the cells' distribution; this is typical for production models (1), where the intensity of attraction actions are higher. In this sense, the presence of an external source of the form $u - u^k$ for $k > 1$ (i.e., of logistic-type), in the equation for u provides a dampening effect possibly limiting the presence of blow

up; despite this, for k close to 1, even this presence is not sufficient to prevent gathering formation [21, 26].

Conversely, for consumption models (2) (which are of our interest in this paper), the situation is quite different. Indeed, even though this is a challenging problem, it is conjectured that such mechanisms do not present blow up phenomena. As indicated above, globality and boundedness are nonetheless achieved, provided a smallness assumption on the product $\chi\|v_0\|_{L^\infty(\Omega)}$. As a consequence, it appears natural to introduce logistic sources not to prevent blow up, but rather to ensure boundedness even for large values of $\chi\|v_0\|_{L^\infty(\Omega)}$. This choice can be traced in the model studied by Lankeit and Wang [11]:

$$\begin{cases} u_t = \Delta u - \chi\nabla\cdot(u\nabla v) + \mu_1 u - \mu_2 u^2, & \text{in } \Omega\times(0,T_{\max}),\\ v_t = \Delta v - uv, & \text{in } \Omega\times(0,T_{\max}),\\ u(x,0)=u_0(x), \quad v(x,0)=v_0(x), & x\in\overline{\Omega}, \end{cases} \tag{3}$$

with $\Omega\subset\mathbb{R}^n$ a smooth bounded domain, $n\in\mathbb{N}$. The presence of such an external source of logistic-type benefits the boundedness of the global solution provided a restriction on the problem parameter μ_2 reading

$$\mu_2 > K_1(n)\|\chi v_0\|^{\frac{1}{n}}_{L^\infty(\Omega)} + K_2(n)\|\chi v_0\|^{2n}_{L^\infty(\Omega)}, \tag{4}$$

for some positive constants $K_1 = K_1(n)$ and $K_2 = K_2(n)$.

Several extensions to those models have been proposed, including as well a third equation in the model relating prey and predator [22]. In this context, the predator moves toward the gradient of concentration of some chemical released by prey instead of moving directly toward the higher density of prey [23]. The dynamics of prey and predator consider both the production and consumption models, in the sense that the prey (i.e., w) produces the chemical (i.e., v), while it is consumed by the predator (i.e., u). Let us consider as a reference point the model

$$\begin{cases} u_t = \Delta u - \chi\nabla\cdot(u\nabla v) + g(u) & \text{in } \Omega\times(0,T_{\max}),\\ v_t = \Delta v + h_1(u,v,w) & \text{in } \Omega\times(0,T_{\max}),\\ w_t = \Delta w + h_2(u,w) & \text{in } \Omega\times(0,T_{\max}), \end{cases} \tag{5}$$

and the respective initial conditions for each function.

In [8], Hu and Tao identified a critical mass of lymphocytes necessary for the coexistence of tumor and immune cells, highlighting the importance of immune response in tumor dynamics. The proposed model describes the interaction of two cells with one chemical secreted by the tumor cells, where $u = u(x,t)$ represents the density of lymphocytes, $v = v(x,t)$ describes the chemical signal secreted by tumor cells, and $w = w(x,t)$ denotes the density of tumor cells. In this case, $x\in\Omega\subset\mathbb{R}^2$, a bounded convex domain with smooth boundary. The term $-\chi\nabla\cdot(u\nabla v)$ in the first equation in (5) represents the movement of the lymphocytes toward the chemical signal, and the external source is ignored, i.e., $g(u) = 0$. The sources for the second and third equations are chosen as $h_1(u,v,w) = w - v - uv$, which combines the production of chemical signal and the consumption by the immune cells, and $h_2(u,w) = -uw + w(1-w)$, combining the consumption of the tumor cells by the immune cells with a natural logistic growth term of the tumor cells. The system is proven to admit a unique global uniformly-in-time bounded classical solution for all nonnegative initial conditions $(u_0,v_0,w_0)\in C^0(\overline{\Omega})\times W^{1,\infty}(\Omega)\times W^{1,\infty}(\Omega)$, with $u_0\not\equiv 0$, $v_0 > 0$, and $w_0\not\equiv 0$.

The lack of parameters scaling each term in the equations for their choices of $h_1(u,v,w)$ and $h_2(u,v)$ loses the sense of dimensionality of the model, as observed in [22]. That was a motivation in [4] to propose a model with $h_1(u,v,w) = \alpha w - \beta v - \gamma uv$ and $h_2(u,w) = -\delta uw + \mu w(1-w)$. The parameters χ, α, β, γ, δ, and μ are assumed to be positive constants. The constant α is the production rate of the chemical, by the tumor cells, β is the decay rate of the chemical and γ is the consumption rate of the chemical by the lymphocytes. The parameter δ models the destruction rate of the tumor cells by the lymphocytes. Lastly, μ is the growth coefficient for the cancer cells.

In that work, $\Omega \subset \mathbb{R}^n$ is considered to be a bounded domain with $n \geq 3$. The studied fully parabolic system admits a unique nonnegative global classical solution uniformly bounded-in-time, provided

$$\max\left\{\frac{\alpha}{\beta}, \frac{\alpha}{\beta}\|w_0\|_{L^\infty(\Omega)}, \|v_0\|_{L^\infty(\Omega)}\right\} < \frac{\pi}{\chi}\sqrt{\frac{2}{n}}. \tag{6}$$

A natural following up is to consider an external source for the first equation $g(u) = \mu_1 u^k - \mu_2 u^{k+1}$, for $k \geq 1$ and positive μ_1 and μ_2. Let us consider the model, extending the proposals in [11, 4], describing the tumor-immune cell interactions by

$$\begin{cases} u_t = \Delta u - \chi\nabla\cdot(u\nabla v) + \mu_1 u^k - \mu_2 u^{k+1}, & \text{in } \Omega\times(0,T_{\max}),\\ v_t = \Delta v + \alpha w - \beta v - \gamma uv, & \text{in } \Omega\times(0,T_{\max}),\\ w_t = \Delta w - \delta uw + \mu_3 w(1-w), & \text{in } \Omega\times(0,T_{\max}),\\ u(x,0) = u_0(x), \quad v(x,0) = v_0(x), \quad w(x,0) = w_0(x), & x\in\overline{\Omega}, \end{cases} \tag{7}$$

where $\chi, \alpha, \beta, \gamma, \delta$, and μ_3 are positive constants. The considered domain $\Omega \subset \mathbb{R}^n$, with $n \geq 3$, is bounded and with smooth boundary $\partial\Omega$.

The logistic source term for lymphocytes $g(u) = \mu_1 u^k - \mu_2 u^{k+1} = \mu_1 u^k\left(1 - \frac{\mu_2}{\mu_1}u\right)$ captures the dynamics of lymphocyte population growth and proliferation, reflecting the self-regulating nature of the immune response in the presence of a tumor [24], [15]. As the number of lymphocytes increases, the rate of growth slows down, eventually leading to a plateau in the population size. To the best of our knowledge, there are no studies carried out for system (7) with logistic source of that form. With $k = 1$, we recover the well-known logistic source term studied in [11].

For $w = 0$, that is, when no indirect effects are taken into account, we refer to [20] for analyses concerning models closely related to (7) as a particular case of (5).

1.1. **Main result and structure of the paper.** In order to formally present the aforementioned analysis, let the initial data (u_0, v_0, w_0) satisfy

$$\begin{cases} u_0 \in C^0(\overline{\Omega}), \quad \text{with} \quad u_0 \geq 0 \quad \text{in } \overline{\Omega},\\ v_0, w_0 \in C^1(\overline{\Omega}), \quad \text{with} \quad v_0, w_0 \geq 0 \quad \text{in } \overline{\Omega}. \end{cases} \tag{8}$$

With these initial conditions, we are able to state what follows.

Theorem 1.1. *Let $\Omega \subset \mathbb{R}^n$, with $n \geq 3$, be a bounded domain with smooth boundary, and let $\chi, \alpha, \beta, \delta, \gamma, \mu_1, \mu_2$, and μ_3 be positive constants. Let u_0, v_0, and w_0 be initial conditions to (7) fulfilling the hypotheses in (8). For $M_2 := \max\left\{1, \|w_0\|_{L^\infty(\Omega)}\right\}$ and $M_3 := \max\left\{\frac{\alpha}{\beta}M_2, \|v_0\|_{L^\infty(\Omega)}\right\}$, let us assume that those parameters satisfy for $k > 1$,*

$$\delta > \mu_3 \quad \text{and} \quad \alpha < 2\min\{\beta+\gamma, \delta-\mu_3\} \tag{9}$$

and, for $k = 1$, satisfy both (9) and

$$\mu_2 > \; A_1(n)\chi^{2+\frac{4}{n}} M_3^{\frac{4}{n}} + A_2(n)\gamma^{\frac{n+2}{2}} M_3^n + A_3(n)\delta^{\frac{n+2}{2}} M_2^n, \tag{10}$$

for certain computable constants $A_1(n), A_2(n)$, and $A_3(n)$.

Then, problem (7) admits a unique classical solution, globally and uniformly bounded in time.

Remark 1.2. The same result, under the same hypotheses, applies to the model obtained by replacing the first equation with $u_t = \Delta u - \chi\nabla\cdot(u\nabla v) + \mu_1 u^l - \mu_2 u^{k+1}$ for any $l \in [1, k+1)$. This remark will be further clarified in Section 4.1 following the proof of the preceding theorem.

Remark 1.3. It is worth noting that for $k > 1$, large values of $\chi\|v_0\|_{L^\infty(\Omega)}$ are allowed, and the restrictions are only on the problem parameters $\alpha, \beta, \delta, \gamma$, and μ_3.

Remark 1.4. If we compare the condition in (4) with (10), considering model (7) as a generalization for $k = 1$ of model (3), it becomes evident that adding that equation for w does not change the type of constraint on the data, rather it only adds an additional term associated with w. However, for $k > 1$, the benefit of the constraint that does not depend on $\|v_0\|_{L^\infty(\Omega)}$ and $\|w_0\|_{L^\infty(\Omega)}$ is noticeable.

On the other hand, comparing (6) with (10), we can see how the extension of the model proposed in [4] to model (7) has the advantage of eliminating the constraint for the value $\|w_0\|_{L^\infty(\Omega)}$.

Remark 1.5. The work of Hu and Tao [8] investigated a dimensionless variant of our model (7) in $\mathbb{R}^2$, notably lacking a logistic-type source term. Their finding of global existence for uniformly-in-time bounded classical solutions, achieved without imposing specific conditions, significantly reduces the necessity of incorporating a logistic source term solely to prevent potential solution blow-up in this two-dimensional setting. Consequently, our subsequent analysis specifically concentrates on scenarios where $n \geq 3$.

Theorem 1.1 is proved as a consequence of some lemmas and quoted inequalities. To clearly present the proof, the remainder of the paper is organized as follows. Section 2 presents some known results from the literature, which are used during the proofs in this paper. The next Section 3 proves the local existence of solutions (u, v, w) for the considered model. These solutions are positive, and their boundedness in some Sobolev spaces is proved. Moreover, an extensibility criterion is given which supports the globality of the existing local solution. Finally, Section 4 proves the main result already stated in this section (i.e., Theorem 1.1) as a consequence of some *a priori* inequalities and the results in Section 3. To conclude this section, some thoughts are devoted to further refining the arguments in *Remark 1.4*.

Throughout the proofs, many constants, denoted by $c_i, i \in \mathbb{N}$, are explicitly computed, supporting the statement in Theorem 1.1 to provide computable constants. However, their explicit forms can be cumbersome. At the same time, showing them makes it easier for the reader to follow the passages along the inequalities, hence the decision to include Appendix A where the explicit values of some constants are shown.

2. Some preliminary results.

We dedicate this section to present some lemmas supporting the main results within this work.

With the Gagliardo-Nirenberg inequality, we refer to the version proposed in [14]. In particular, we claim that for all $\phi \in W^{1,2}(\Omega) \cap L^{\frac{2}{p}}(\Omega)$, where $p \geq 1$, it holds that

$$\|\phi\|_{L^2(\Omega)} \leq C_{\mathrm{GN}} \left(\|\nabla \phi\|^{\theta}_{L^2(\Omega)} \|\phi\|^{(1-\theta)}_{L^{\frac{2}{p}}(\Omega)} + \|\phi\|_{L^{\frac{2}{p}}(\Omega)} \right), \tag{11}$$

with $\theta = \frac{p-1}{p-1+\frac{2}{n}} \in (0,1)$ and for a positive constant $C_{\mathrm{GN}} := C_{\mathrm{GN}}(p,n,\Omega)$.

Let us present some inequalities from [11, Lemma 2.1, Lemma 2.2] in the following lemma. The term $D^2\psi$ denotes the Hessian of ψ and $|D^2\psi|^2 = \sum_{i,j=1}^{n} \left(\frac{\partial^2 \psi}{\partial x_i x_j} \right)^2$.

Lemma 2.1. *1. For any function $\psi \in C^2(\Omega)$,*

$$|\Delta \psi|^2 \leq n |D^2 \psi|^2 \tag{12}$$

throughout Ω.

2. Let $p \in [1,\infty)$. For every $\eta > 0$, there is $C_\eta > 0$ such that every function $\psi \in C^2(\bar{\Omega})$ with $\frac{\partial \psi}{\partial \nu} = 0$ on $\partial\Omega$ satisfies

$$\int_{\partial\Omega} |\nabla \psi|^{2p-2} \frac{\partial |\nabla \psi|^2}{\partial \nu} \leq \eta \int_\Omega |\nabla \psi|^{2p-4} |\nabla |\nabla \psi|^2|^2 + C_\eta \left(\int_\Omega |\nabla \psi|^2 \right)^p. \tag{13}$$

3. Let $q \in [1,\infty)$. Then, for any $\psi \in C^2(\overline{\Omega})$ satisfying $\psi \frac{\partial \psi}{\partial \nu} = 0$ on $\partial\Omega$, the inequality

$$\|\nabla \psi\|^{2(q+1)}_{L^{2(q+1)}(\Omega)} \leq 2(4q^2+n) \|\psi\|^2_{L^\infty(\Omega)} \||\nabla \psi|^{q-1} D^2 \psi\|^2_{L^2(\Omega)} \tag{14}$$

holds.

The following two known results will turn out to be useful as well, and are presented here without proof.

Lemma 2.2. *For any function $\psi \in C^2(\Omega)$ the following identity holds:*

$$\Delta |\nabla \psi|^2 = 2 \nabla \psi \cdot \nabla \Delta \psi + 2 |D^2 \psi|^2. \tag{15}$$

Lemma 2.3. *For any function $\psi \in C^2(\Omega)$ and $p > 1$, the following inequality holds:*

$$|\nabla |\nabla \psi|^p|^2 \leq p^2 |\nabla \psi|^{2p-2} |D^2 \psi|^2. \tag{16}$$

The following lemma presents several useful inequalities. For clarity, these inequalities are expressed using generic functions ψ and φ, which represent any of the state variables u, v, or w from our system (7) that satisfy the conditions outlined in *Lemma 3.1*. Specific instantiations of these inequalities, substituting u, v, or w for ψ and φ, will be used in subsequent proofs.

Lemma 2.4. *For any nonnegative functions $\psi, \varphi \in C^2(\overline{\Omega})$, any positive ε_1 and ε_2, and any $p > 1$, the following inequalities are valid.*

1. For any $k > 1$, there exists a computable positive constant $c_1(\varepsilon_1, \varepsilon_2)$ such that

$$\int_\Omega \psi^p |\nabla \varphi|^2 \leq \varepsilon_1 \int_\Omega \psi^{p+k} + \varepsilon_2 \int_\Omega |\nabla \varphi|^{2(p+1)} + c_1(\varepsilon_1, \varepsilon_2). \tag{17}$$

2. There exists a computable positive constant $c_2(\varepsilon_1)$ such that

$$\int_\Omega \psi^p |\nabla \varphi|^2 \leq \varepsilon_1 \int_\Omega |\nabla \varphi|^{2(p+1)} + c_2(\varepsilon_1) \int_\Omega \psi^{p+1}. \tag{18}$$

3. For any $k > 1$, there exists a computable positive constant $c_3(\varepsilon_1, \varepsilon_2)$ such that

$$\int_\Omega \psi^2 |\nabla \varphi|^{2p-2} \leq \varepsilon_1 \int_\Omega \psi^{p+k} + \varepsilon_2 \int_\Omega |\nabla \varphi|^{2(p+1)} + c_3(\varepsilon_1, \varepsilon_2). \tag{19}$$

4. *There exists a computable positive constant* $c_4(\varepsilon_1)$ *such that*

$$\int_\Omega \psi^2 |\nabla\varphi|^{2p-2} \le \varepsilon_1 \int_\Omega |\nabla\varphi|^{2(p+1)} + c_4(\varepsilon_1) \int_\Omega \psi^{p+1}. \tag{20}$$

5. *There exists a computable positive constant* $c_5(\varepsilon_1, \varepsilon_2)$ *such that*

$$\int_\Omega |\nabla\psi|^{2p-1} |\nabla\varphi| \le \varepsilon_1 \int_\Omega |\nabla\psi|^{2(p+1)} + \varepsilon_2 \int_\Omega |\nabla\varphi|^{2(p+1)} + c_5(\varepsilon_1, \varepsilon_2). \tag{21}$$

6. *There exists a computable positive constant* $c_6(\varepsilon_1)$ *such that*

$$\int_\Omega |\nabla\psi|^{2p} \le \varepsilon_1 \int_\Omega |\nabla|\nabla\psi|^p|^2 + c_6(\varepsilon_1). \tag{22}$$

Proof. Since the proof of some of the inequalities follows a similar line of reasoning, let us then show the proof for just a few of them. For instance, to prove the inequalities in (17), (19), and (21), we can consider the first one of them. By the Young inequality,

$$\begin{aligned}\int_\Omega \psi^p |\nabla\varphi|^2 &\le \varepsilon_1 \int_\Omega \psi^{p+k} + \tfrac{k}{p+k} \left(\tfrac{p}{\varepsilon_1(p+k)}\right)^{\frac{p}{k}} \int_\Omega |\nabla\varphi|^{\frac{2(p+k)}{k}} \\ &\le \varepsilon_1 \int_\Omega \psi^{p+k} + \varepsilon_2 \int_\Omega |\nabla\varphi|^{2(p+1)} + c_1(\varepsilon_1, \varepsilon_2),\end{aligned}$$

with $c_1(\varepsilon_1, \varepsilon_2)$ as in (59).

On the other hand, inequalities (18) and (20) are derived similarly. In particular, for (18) we have

$$\int_\Omega \psi^p |\nabla\varphi|^2 \le \varepsilon_1 \int_\Omega |\nabla\varphi|^{2(p+1)} + c_2(\varepsilon_1) \int_\Omega \psi^{p+1}$$

with $c_2(\varepsilon_1)$ defined in (60).

Lastly, regarding (22), from the *arithmetic mean - quadratic mean* inequality (i.e., $(a+b)/2 \le \sqrt{(a^2+b^2)/2}$) and the Gagliardo-Nirenberg inequality in (11) for $\phi = |\nabla\psi|^p$, we deduce

$$\begin{aligned}\int_\Omega |\nabla\psi|^{2p} \le\ & 2C_{\mathrm{GN}} \|\nabla|\nabla\psi|^p\|_{L^2(\Omega)}^{2\theta} \||\nabla\psi|^p\|_{L^{\frac{2}{p}}(\Omega)}^{2(1-\theta)} + 2C_{\mathrm{GN}} \||\nabla\psi|^p\|_{L^{\frac{2}{p}}(\Omega)}^2 \\ \le\ & 2C_{\mathrm{GN}} \left(\int_\Omega |\nabla|\nabla\psi|^p|^2\right)^\theta \left(\int_\Omega |\nabla\psi|^2\right)^{p(1-\theta)} + 2C_{\mathrm{GN}} \left(\int_\Omega |\nabla\psi|^2\right)^p,\end{aligned} \tag{23}$$

with $\theta = \frac{p-1}{p-1+\frac{2}{n}}$ as already defined.

Using the bound $\int_\Omega |\nabla\psi|^2 < C$ for some constant $C > 0$ (which will be established for the solutions later), the Young inequality applied to (23) leads to

$$\begin{aligned}\int_\Omega |\nabla\psi|^{2p} \le\ & 2C_{\mathrm{GN}} C^{p(1-\theta)} \left(\int_\Omega |\nabla|\nabla\psi|^p|^2\right)^\theta + 2C_{\mathrm{GN}} C^p \\ \le\ & \varepsilon_1 \int_\Omega |\nabla|\nabla\psi|^p|^2 + c_6(\varepsilon_1)\end{aligned}$$

with $c_6(\varepsilon_1)$ defined in (64). This completes the proof of (22). □

Remark 2.5. It is important to recognize that the constants c_i for $i = 1, \dots, 5$ appearing in inequalities (17)-(22) depend on the parameter p. Their explicit forms, reflecting this dependence, are detailed in (59)-(64). We further emphasize that this

p-dependence is a general characteristic for many constants derived throughout this work, as can be systematically traced via the detailed computations and expressions in Appendix A.

3. **Local existence and extensibility criterion.** First, we state the local solvability of the system of equations (7), and we prove some properties of the local solution (u, v, w), for instance the boundedness of u in $L^1(\Omega)$ and of v, w in $L^\infty(\Omega)$. An extensibility criterion is presented as support to derive the globality of the local solution obtained.

Lemma 3.1 (Local existence and positivity)**.** *Suppose that $\Omega \subset \mathbb{R}^n$, $n \geq 2$, is a bounded domain with smooth boundary and $q > n$. Then, for nonnegative initial data satisfying (8) and for positive constants $k \geq 1$, $\chi, \alpha, \beta, \gamma, \delta, \mu_1, \mu_2$, and μ_3, there exists $T_{max} \in (0, \infty]$ such that system (7) admits a unique nonnegative solution (u, v, w) belonging to*

$$u \in C^0\left(\overline{\Omega} \times [0, T_{max})\right) \cap C^{2,1}\left(\overline{\Omega} \times (0, T_{max})\right),$$
$$v, w \in C^0\left(\overline{\Omega} \times [0, T_{max})\right) \cap C^{2,1}\left(\overline{\Omega} \times (0, T_{max})\right) \cap L^\infty_{loc}\left([0, T_{max}); W^{1,q}(\Omega)\right).$$

Furthermore, if $T_{max} < \infty$, then

$$\limsup_{t \to T_{max}} \left(\|u(\cdot, t)\|_{L^\infty(\Omega)} + \|v(\cdot, t)\|_{W^{1,q}(\Omega)} + \|w(\cdot, t)\|_{W^{1,q}(\Omega)}\right) = \infty.$$

Proof. The proof follows the same arguments as in [4, Lemma 2.1]. □

Then, we proceed by stating the mass limitation for u and boundedness for v and w as a foundation for further analysis.

Lemma 3.2 (Mass limitation and boundedness)**.** *Let the hypotheses of* Lemma 3.1 *be satisfied. Then, the solution (u, v, w) satisfies*

$$\|u(\cdot, t)\|_{L^1(\Omega)} \leq M_1 := \max\left\{\|u_0\|_{L^1(\Omega)}, \frac{2\mu_1\ |\Omega| k^{\frac{k}{k+1}}}{\mu_2(k+1)}\right\}, \quad \forall\ t \in (0, T_{max}) \tag{24}$$

and

$$\|w(\cdot, t)\|_{L^\infty(\Omega)} \leq M_2 := \max\left\{1, \|w_0\|_{L^\infty(\Omega)}\right\}, \qquad \forall\ t \in (0, T_{max}), \tag{25}$$

$$\|v(\cdot, t)\|_{L^\infty(\Omega)} \leq M_3 := \max\left\{\frac{\alpha}{\beta} M_2, \|v_0\|_{L^\infty(\Omega)}\right\}, \qquad \forall\ t \in (0, T_{max}). \tag{26}$$

Proof. By differentiating the functional $\int_\Omega u$ and using the Neumann boundary conditions, we have

$$\frac{d}{dt}\int_\Omega u \leq \mu_1 \int_\Omega u^k - \mu_2 \int_\Omega u^{k+1} \quad \text{for all } t \in (0, T_{\max}). \tag{27}$$

The Young inequality gives

$$\frac{d}{dt}\int_\Omega u \leq -\frac{\mu_2}{2}\int_\Omega u^{k+1} + c_7, \qquad \forall\ t \in (0, T_{\max}), \tag{28}$$

where c_7 as in (65). Moreover, the Hölder inequality leads to

$$\frac{d}{dt}\int_\Omega u \leq -\frac{\mu_2}{2|\Omega|^k}\left(\int_\Omega u\right)^{k+1} + c_7 \quad \text{for all } t \in (0, T_{\max}).$$

The last differential inequality, by denoting $y(t) = \int_\Omega u$ with initial condition $y(0) = \int_\Omega u_0$, provides the bound in (24).

Further, (26) and (25) are consequences of the parabolic maximum principle (see, e.g., [3, §6.4 and §7.1.4]). Indeed, the first equation entails that $u \equiv 0$ is a subsolution, therefore we can assure that any solution u is a nonnegative function. As a consequence, the third equation implies that $\overline{w} \equiv 1$ is a supersolution such that $0 \leq w \leq 1$ for all $(x,t) \in \bar{\Omega} \times (0, T_{\max})$. We exploit this information to observe that $v \equiv \frac{\alpha}{\beta} M_2$ is a supersolution for the second equation. The claim follows by also considering the initial conditions for v and w. □

From now on, (u, v, w) indicates the local solution to problem (7) provided by *Lemma* 3.1 and fulfilling the properties in *Lemma 3.2*.

In model (7), we not only obtain the mass limitation for u, but further properties, such as the inclusion of ∇v and ∇w in $L^\infty((0, T_{\max}); L^2(\Omega))$.

Lemma 3.3 (Gradient limitation)**.** *Let $k \geq 1$, $\delta > \mu_3$, and $\alpha < 2 \min\{\beta + \gamma, \delta - \mu_3\}$. Then, there exists a constant $C > 0$ such that*

$$\int_\Omega |\nabla v(\cdot, t)|^2 \leq C \quad \textit{and} \quad \int_\Omega |\nabla w(\cdot, t)|^2 \leq C, \quad \forall\, t \in (0, T_{max}). \tag{29}$$

Proof. The thesis is derived from a differential inequality by studying the evolution of the functional

$$\varphi(t) := \tfrac{\gamma^2 M_3^2 + \delta^2 M_2^2}{\mu_2} \int_\Omega u + \int_\Omega |\nabla v|^2 + \int_\Omega |\nabla w|^2, \qquad \forall\, t \in (0, T_{\max}). \tag{30}$$

For the second term in the sum, integrating by parts and by the Neumann boundary conditions, we obtain for all $t \in (0, T_{\max})$

$$\begin{aligned} \frac{d}{dt} \int_\Omega |\nabla v|^2 &= 2 \int_\Omega \nabla v \cdot \nabla v_t = 2 \int_\Omega \nabla v \cdot \nabla (\Delta v + \alpha w - \beta v - \gamma u v) \\ &= -2 \int_\Omega |\Delta v|^2 + 2\alpha \int_\Omega \nabla v \cdot \nabla w - 2\beta \int_\Omega |\nabla v|^2 + 2\gamma \int_\Omega u v \Delta v \\ &= -2 \int_\Omega |\Delta v|^2 + 2\alpha \int_\Omega \nabla v \cdot \nabla w - 2\beta \int_\Omega |\nabla v|^2 \\ &\quad + 2\gamma \int_\Omega (u-1) v \Delta v - 2\gamma \int_\Omega |\nabla v|^2 \end{aligned} \tag{31}$$

After neglecting some non-positive terms, by considering the Cauchy-Schwarz inequality and the upper bound in (26), it follows for all $t \in (0, T_{\max})$ that

$$\begin{aligned} \frac{d}{dt} \int_\Omega |\nabla v|^2 &\leq -2 \int_\Omega |\Delta v|^2 + 2\alpha \int_\Omega \nabla v \cdot \nabla w - 2(\beta + \gamma) \int_\Omega |\nabla v|^2 \\ &\quad + \gamma^2 \int_\Omega (u-1)^2 v^2 + \int_\Omega |\Delta v|^2 \\ &\leq 2\alpha \int_\Omega \nabla v \cdot \nabla w - 2(\beta + \gamma) \int_\Omega |\nabla v|^2 + \gamma^2 \int_\Omega (u-1)^2 v^2 \\ &\leq (\alpha - 2\beta - 2\gamma) \int_\Omega |\nabla v|^2 + \alpha \int_\Omega |\nabla w|^2 + \gamma^2 M_3^2 \int_\Omega (u-1)^2. \end{aligned} \tag{32}$$

On the other hand, with analogous reasoning as for $\int_\Omega uv\Delta v$ in (31), we have for all $t \in (0, T_{\max})$ that

$$\begin{aligned}\frac{d}{dt}\int_\Omega |\nabla w|^2 &= 2\int_\Omega \nabla w\cdot\nabla w_t = 2\int_\Omega \nabla w\cdot\nabla\left(\Delta w - \delta uw + \mu_3 w(1-w)\right)\\ &= -2\int_\Omega |\Delta w|^2 + 2\delta\int_\Omega uw\Delta w + 2\mu_3\int_\Omega |\nabla w|^2 - 4\mu_3\int_\Omega w|\nabla w|^2\\ &\le \delta^2\int_\Omega w^2(u-1)^2 - 2(\delta-\mu_3)\int_\Omega |\nabla w|^2 - 4\mu_3\int_\Omega w|\nabla w|^2\\ &\le \delta^2 M_2^2\int_\Omega (u-1)^2 - 2(\delta-\mu_3)\int_\Omega |\nabla w|^2,\end{aligned} \tag{33}$$

after neglecting some non-positive terms in this occasion as well.

Let us differentiate $\varphi(t)$ in (30), considering (27), (32), and (33) we obtain

$$\begin{aligned}\frac{d\varphi(t)}{dt} &\le \frac{\mu_1 c_8}{\mu_2}\int_\Omega u^k - c_8\int_\Omega u^{k+1} + (\alpha-2\beta-2\gamma)\int_\Omega |\nabla v|^2\\ &\quad + (\alpha-2\delta+2\mu_3)\int_\Omega |\nabla w|^2 + c_8\int_\Omega (u-1)^2\\ &\le \frac{\mu_1 c_8}{\mu_2}\int_\Omega u^k - c_8\int_\Omega u^{k+1} + (\alpha-2\beta-2\gamma)\int_\Omega |\nabla v|^2\\ &\quad + (\alpha-2\delta+2\mu_3)\int_\Omega |\nabla w|^2 + c_8\int_\Omega (u+1)^2\end{aligned} \tag{34}$$

with $c_8 := \gamma^2 M_3^2 + \delta^2 M_2^2$ and $t \in (0, T_{\max})$.

From here, we study (34) under the two cases: $k > 1$ and $k = 1$.

Case $k > 1$). Let us denote $c_9 := \max\{2\beta + 2\gamma - \alpha, 2\delta - 2\mu_3 - \alpha\} > 0$, since from hypotheses $2\beta + 2\gamma - \alpha > 0$ and $2\delta - 2\mu_3 - \alpha > 0$. From the Young inequality for the terms $\int_\Omega u^k$ and $\int_\Omega u^2$, after adding and subtracting $\frac{c_8 c_9}{\mu_2}\int_\Omega u$, we conclude that

$$\begin{aligned}\frac{d\varphi(t)}{dt} &\le c_{10} - \tfrac{c_8 c_9}{\mu_2}\int_\Omega u - (2\beta+2\gamma-\alpha)\int_\Omega |\nabla v|^2 - (2\delta-2\mu_3-\alpha)\int_\Omega |\nabla w|^2\\ &\le c_{10} - c_9\ \varphi(t), \qquad \forall\ t \in (0, T_{\max}),\end{aligned}$$

with c_{10} as in (66).

Thus, for $C_1 := \max\left\{\varphi(0), \frac{c_{10}}{c_9}\right\}$, it holds that

$$\tfrac{\gamma^2 M_3^2 + \delta^2 M_2^2}{\mu_2}\int_\Omega u + \int_\Omega |\nabla v|^2 + \int_\Omega |\nabla w|^2 \le C_1, \qquad \text{on } (0, T_{\max}).$$

Case $k = 1$). In this case, we recall the bound (24), and then (34) becomes

$$\begin{aligned}\frac{d\varphi(t)}{dt} &\le \frac{\mu_1 c_8}{\mu_2}\int_\Omega u - c_8\int_\Omega u^2 + (\alpha-2\beta-2\gamma)\int_\Omega |\nabla v|^2\\ &\quad + (\alpha-2\delta+2\mu_3)\int_\Omega |\nabla w|^2 + c_8\int_\Omega (u+1)^2\\ &\le -\tfrac{c_8 c_9}{\mu_2}\int_\Omega u - (2\beta+2\gamma-\alpha)\int_\Omega |\nabla v|^2 - (2\delta-2\mu_3-\alpha)\int_\Omega |\nabla w|^2 + c_{11}\\ &\le c_{11} - c_9\ \varphi(t),\end{aligned}$$

with c_{11} as in (67), obtaining in this way a differential inequality valid on $(0, T_{\max})$.

As in the case for $k > 1$, if we take $C_2 := \max\left\{\varphi(0), \frac{c_{11}}{c_9}\right\}$, it holds that

$$\frac{\gamma^2 M_3^2+\delta^2 M_2^2}{\mu_2}\int_\Omega u + \int_\Omega |\nabla v|^2 + \int_\Omega |\nabla w|^2 \le C_2, \qquad \forall\, t \in (0, T_{\max}).$$

The proof is given for $C := \max\{C_1, C_2\}$. □

The following result allows us to extend the local-in-time existence of classical solution to globally bounded solutions. Extensibility criteria are known in the context of chemotaxis [2]; herein, for consistency with the nomenclature, for details we refer to [4].

Lemma 3.4 (Extensibility criterion)**.** *Suppose there exists $p > \frac{n}{2} \ge 1$ such that*

$$\sup_{t\in(0,T_{max})} \|u(\cdot,t)\|_{L^p(\Omega)} < \infty.$$

Then, for all $q \ge 1$, we have

$$\sup_{t\in(0,T_{max})} \left(\|u(\cdot,t)\|_{L^\infty(\Omega)} + \|v(\cdot,t)\|_{W^{1,q}(\Omega)} + \|w(\cdot,t)\|_{W^{1,q}(\Omega)}\right) < \infty.$$

In particular, $T_{max} = \infty$ and $u \in L^\infty((0,\infty), L^\infty(\Omega))$.

Proof. The proof follows the same reasoning as in [4, Lemma 2.3], with the main difference lying in the variation-of-constants formula for $u(\cdot,t)$ in (7). For each $t \in (0, T_{\max})$ and $t_0 = \max\{0, t-1\}$, we instead have

$$\begin{aligned} u(\cdot,t) = e^{(t-t_0)\Delta}u(\cdot,t_0) - \chi\int_{t_0}^t e^{(t-s)\Delta}\nabla\cdot(u(\cdot,s)\nabla v(\cdot,s))\,\mathrm{d}s \\ + \mu_1\int_{t_0}^t e^{(t-s)\Delta}u^k\left(1-\tfrac{\mu_2}{\mu_1}u\right)\mathrm{d}s. \end{aligned} \tag{35}$$

The function $z^k\left(1-\frac{\mu_2}{\mu_1}z\right)$, for $z \in \mathbb{R}^+$, attains the maximum value $c_{12} := \frac{k^k\mu_1^k}{(k+1)^{k+1}\mu_2^k}$ at $z = \frac{k\mu_1}{(k+1)\mu_2} > 0$. Therefore, $u^k\left(1-\frac{\mu_2}{\mu_1}u\right) \le c_{12}$, and by the positivity of the heat semigroup, it follows that

$$\int_{t_0}^t e^{(t-s)\Delta}u^k\left(1-\tfrac{\mu_2}{\mu_1}u\right)\mathrm{d}s \le c_{12}.$$

The $L^\infty(\Omega)$ norm of each side in (35) leads to

$$\begin{aligned} \|u(\cdot,t)\|_{L^\infty(\Omega)} \le \left\|e^{(t-t_0)\Delta}u(\cdot,t_0)\right\|_{L^\infty(\Omega)} \\ + \chi\int_{t_0}^t \left\|e^{(t-s)\Delta}\nabla\cdot(u(\cdot,s)\nabla v(\cdot,s))\right\|_{L^\infty(\Omega)}\mathrm{d}s + \mu_1 c_{12}. \end{aligned} \tag{36}$$

The bounds in [4] for the two first integrals on the right-hand side in (36) prove the boundedness for the left-hand side.

The proof follows as in [4]. □

4. **A priori inequalities.** With the aim to prove the boundedness of the solution (u, v, w) of (7) in the space $L^p(\Omega)$, for some $p > 1$, let us consider the energy functional

$$\Phi(t) = \int_\Omega u^p + \int_\Omega |\nabla v|^{2p} + \int_\Omega |\nabla w|^{2p}, \qquad \text{for all } \ t \in (0, T_{\max}), \tag{37}$$

with $p \in (1, \infty)$. The following lemmas pursue that goal, treating each term in (37) separately before obtaining an ordinary differential equation for $\Phi(t)$.

Lemma 4.1. *Let the hypotheses in* Lemma 3.3 *be satisfied. Then, it holds, for all* $p > 1$ *and* $t \in (0, T_{max})$*, it holds that*

$$\frac{d}{dt}\int_\Omega u^p + \frac{2(p-1)}{p}\int_\Omega |\nabla u^{\frac{p}{2}}|^2 \leq \frac{\chi^2 p(p-1)}{2}\int_\Omega u^p |\nabla v|^2 - \frac{\mu_1 p}{2}\int_\Omega u^{p+k} + c_{13}, \tag{38}$$

with c_{13} *a positive constant.*

Proof. Derivating the functional $\int_\Omega u^p$ and considering the first equation in (7), we obtain on $(0, T_{\max})$ that

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega u^p =& p\int_\Omega u^{p-1}u_t = p\int_\Omega u^{p-1}\left(\Delta u - \chi\nabla\cdot(u\nabla v) + \mu_1 u^k - \mu_2 u^{k+1}\right)\\
=& -p(p-1)\int_\Omega u^{p-2}|\nabla u|^2 + \chi p(p-1)\int_\Omega u^{p-1}\nabla u\cdot\nabla v + \mu_1 p\int_\Omega u^{p+k-1}\\
&- \mu_2 p\int_\Omega u^{p+k}\\
\leq& -\frac{p(p-1)}{2}\int_\Omega u^{p-2}|\nabla u|^2 + \frac{\chi^2 p(p-1)}{2}\int_\Omega u^p|\nabla v|^2 + \mu_1 p\int_\Omega u^{p+k-1}\\
&- \mu_2 p\int_\Omega u^{p+k}\\
\leq& -\frac{2(p-1)}{p}\int_\Omega |\nabla u^{\frac{p}{2}}|^2 + \frac{\chi^2 p(p-1)}{2}\int_\Omega u^p|\nabla v|^2 - \frac{\mu_2 p}{2}\int_\Omega u^{p+k} + c_{13},
\end{aligned}$$

after applying the Young inequality with c_{13} defined in (68). □

Lemma 4.2. *Let the hypotheses in* Lemma 3.3 *be satisfied. Then, for any* $p > 1$*,* $\eta \in (0, p-1)$*, and* $t \in (0, T_{max})$*, it holds that*

$$\begin{aligned}
\frac{d}{dt}\int_\Omega |\nabla v|^{2p} \leq& -p\int_\Omega |\nabla v|^{2p-2}|D^2 v|^2 + c_{14}\int_\Omega |\nabla v|^{2(p+1)} + c_{15}\int_\Omega |\nabla w|^{2(p+1)}\\
&+ c_{16}(n,\eta)\gamma^2 M_3^2\int_\Omega u^2|\nabla v|^{2p-2} + c_{17}(\eta),
\end{aligned} \tag{39}$$

where c_{14}, c_{15}*,* $c_{16}(n,\eta)$*, and* $c_{17}(\eta)$ *are positive constants.*

Proof. Similarly to what was done before, after substituting v_t from (7) and integrating by parts some integral terms on $(0, T_{\max})$, we have

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |\nabla v|^{2p} &= p\int_\Omega |\nabla v|^{2p-2} 2\nabla v\cdot\nabla v_t \\
&= p\int_\Omega |\nabla v|^{2p-2} 2\nabla v\cdot\nabla\left(\Delta v + \alpha w - \beta v - \gamma uv\right) \\
&= p\int_\Omega |\nabla v|^{2p-2}\left(\Delta|\nabla v|^2 - 2|D^2 v|^2 + 2\alpha\nabla v\cdot\nabla w - 2\beta|\nabla v|^2\right. \\
&\qquad \left. -2\gamma\nabla v\cdot\nabla(uv)\right) \\
&= p\int_{\partial\Omega} |\nabla v|^{2p-2}\,\frac{\partial|\nabla v|^2}{\partial\nu} - p(p-1)\int_\Omega |\nabla v|^{2p-4}|\nabla|\nabla v|^2|^2 \\
&\quad - 2p\int_\Omega |\nabla v|^{2p-2}|D^2 v|^2 + 2\alpha p\int_\Omega |\nabla v|^{2p-2}\nabla v\cdot\nabla w - 2\beta p\int_\Omega |\nabla v|^{2p} \\
&\quad - 2\gamma p\int_\Omega |\nabla v|^{2p-2}\nabla v\cdot\nabla(uv) \\
&= p\int_{\partial\Omega} |\nabla v|^{2p-2}\,\frac{\partial|\nabla v|^2}{\partial\nu} - p(p-1)\int_\Omega |\nabla v|^{2p-4}|\nabla|\nabla v|^2|^2 \\
&\quad - 2p\int_\Omega |\nabla v|^{2p-2}|D^2 v|^2 + 2\alpha p\int_\Omega |\nabla v|^{2p-2}\nabla v\cdot\nabla w - 2\beta p\int_\Omega |\nabla v|^{2p} \\
&\quad + 2\gamma p\int_\Omega uv|\nabla v|^{2p-2}\Delta v + 2\gamma p(p-1)\int_\Omega uv|\nabla v|^{2p-4}\nabla v\cdot\nabla|\nabla v|^2,
\end{aligned}$$

with the second line justified by the identity (15).

After ignoring the non-positive fifth term, we use the inequality in (13) for some $\eta\in(0,p-1)$ to treat the boundary integral. Successively, applying the bounds in (26) and (29) along with the Cauchy-Schwarz inequality leads to

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |\nabla v|^{2p} \le\ & -p\left(p-1-\eta\right)\int_\Omega |\nabla v|^{2p-4}|\nabla|\nabla v|^2|^2 + pC_\eta\left(\int_\Omega |\nabla v|^2\right)^p \\
& - 2p\int_\Omega |\nabla v|^{2p-2}|D^2 v|^2 + 2\alpha p\int_\Omega |\nabla v|^{2p-2}\nabla v\cdot\nabla w \\
& + 2\gamma p M_3\int_\Omega u|\nabla v|^{2p-2}\Delta v + 2\gamma p(p-1)M_3\int_\Omega u|\nabla v|^{2p-4}\nabla v\cdot\nabla|\nabla v|^2 \\
\le\ & -p\left(p-1-\eta\right)\int_\Omega |\nabla v|^{2p-4}|\nabla|\nabla v|^2|^2 - 2p\int_\Omega |\nabla v|^{2p-2}|D^2 v|^2 \\
& + 2\alpha p\int_\Omega |\nabla v|^{2p-1}|\nabla w| + 2\gamma p M_3\int_\Omega u|\nabla v|^{2p-2}|\Delta v| \\
& + 2\gamma p(p-1)M_3\int_\Omega u|\nabla v|^{2p-3}|\nabla|\nabla v|^2| + c_{18}(\eta)
\end{aligned}$$

for all $t \in (0, T_{\max})$, where $c_{18}(\eta) := p\, C_\eta C^p$ for the constants C obtained in (29) and C_η in (13).

After applying the Young inequality for $u|\nabla v|^{2p-2}|\Delta v|$ and $u|\nabla v|^{2p-3}|\nabla|\nabla v|^2|$, it follows from (12) and suitable choices for the parameters in (21) that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |\nabla v|^{2p} \leq & -p\int_\Omega |\nabla v|^{2p-2}|D^2 v|^2 + 2\alpha p\int_\Omega |\nabla v|^{2p-1}|\nabla w| \\ & + c_{16}(n,\eta)\gamma^2 M_3^2 \int_\Omega u^2|\nabla v|^{2p-2} + c_{18}(\eta) \\ \leq & -p\int_\Omega |\nabla v|^{2p-2}|D^2 v|^2 + c_{14}\int_\Omega |\nabla v|^{2(p+1)} + c_{15}\int_\Omega |\nabla w|^{2(p+1)} \\ & + c_{16}(n,\eta)\gamma^2 M_3^2 \int_\Omega u^2|\nabla v|^{2p-2} + c_{17}(\eta)\end{aligned}$$

with $c_{14} := \frac{p}{12(4p^2+n)M_3^2}$, $c_{15} := \frac{p}{16(4p^2+n)M_2^2}$, $c_{16}(n,\eta) := p\left(n + \frac{(p-1)^2}{p-1-\eta}\right)$, and $c_{17}(\eta)$ defined in (69). □

Lemma 4.3. *Let the hypotheses in* Lemma 3.3 *be satisfied. Then, for $p > 1$, by a positive constant $c_{19}(\eta)$, with $\eta \in (0, p-1)$ (and $c_{16}(n,\eta)$ defined as in (39)), it holds, for all $t \in (0, T_{max})$, that*

$$\begin{aligned}\frac{d}{dt}\int_\Omega |\nabla w|^{2p} \leq & -p\int_\Omega |\nabla w|^{2p-2}|D^2 w|^2 + \frac{1}{4p}\int_\Omega |\nabla|\nabla w|^p|^2 \\ & + c_{16}(n,\eta)\delta^2 M_2^2 \int_\Omega u^2|\nabla w|^{2p-2} + c_{19}(\eta).\end{aligned} \tag{40}$$

Proof. Following an analogous procedure as before, for the third integral term in (37), we have for all $t \in (0, T_{\max})$ that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |\nabla w|^{2p} &= p\int_\Omega |\nabla w|^{2p-2} 2\nabla w \cdot \nabla\left(\Delta w - \delta uw + \mu_3(1-w)\right) \\ &= p\int_\Omega |\nabla w|^{2p-2}\left(\Delta|\nabla w|^2 - 2|D^2 w|^2\right) - 2p\delta\int_\Omega |\nabla w|^{2p-2}\nabla w\cdot\nabla(uw) \\ &\quad + 2p\mu_3\int_\Omega |\nabla w|^{2p} - 4p\mu_3\int_\Omega w|\nabla w|^{2p} \\ &\leq -p\int_\Omega |\nabla w|^{2p-2}|D^2 w|^2 + 2p\mu_3\int_\Omega |\nabla w|^{2p} \\ &\quad + c_{16}(n,\eta)\delta^2 M_2^2\int_\Omega u^2|\nabla w|^{2p-2} + c_{18}(\eta) \\ &\leq -p\int_\Omega |\nabla w|^{2p-2}|D^2 w|^2 + \frac{1}{p^2}\int_\Omega |\nabla|\nabla w|^p|^2 \\ &\quad + c_{16}(n,\eta)\delta^2 M_2^2\int_\Omega u^2|\nabla w|^{2p-2} + c_{19}(\eta),\end{aligned}$$

after using (22) with $\varepsilon_1 = \frac{1}{4p}$ and $c_{19}(\eta)$ as defined in (70). □

Finally, let us use the estimates obtained from the previous three lemmas to prove the boundedness of the functional defined in (37).

Lemma 4.4. *Let the hypotheses of* Lemma 3.3 *be satisfied,* $p > 1$*, and either* $k > 1$ *or* $k = 1$ *with*

$$\mu_2 > \; A_1(p,n)\chi^{2+\frac{2}{p}} M_3^{\frac{2}{p}} + A_2(p,n)\gamma^{p+1} M_3^{2p} + A_3(p,n)\delta^{p+1} M_2^{2p} \tag{41}$$

hold, where

$$A_1(p,n) := \; \tfrac{(p-1)p^2}{2(p+1)} \left(\tfrac{6\ (p-1)\left(4p^2+n\right)}{p+1} \right)^{1/p} \tag{42}$$

$$A_2(p,n) := \; \tfrac{2\ c_{16}(n,0)}{p+1} \left(\tfrac{12\ c_{16}(n,0)\left(4p^2+n\right)}{p(p+1)} \right)^{\frac{p-1}{2}} \tag{43}$$

$$A_3(p,n) := \; \tfrac{2\ c_{16}(n,0)}{p+1} \left(\tfrac{16\ c_{16}(n,0)(p-1)\left(4p^2+n\right)}{p(p+1)} \right)^{\frac{p-1}{2}} . \tag{44}$$

(with $c_{16}(n,\eta)$ *defined as in* (39)*).*

Then, there exists a constant $\widetilde{C}$ *such that*

$$\int_\Omega u^p + \int_\Omega |\nabla v|^{2p} + \int_\Omega |\nabla w|^{2p} \leq \widetilde{C} \qquad \forall\ t \in (0, T_{max}).$$

Proof. We start by combining (38), (39), and (40) to obtain an estimate for $\frac{\mathrm{d}}{\mathrm{d}t}\Phi(t)$ on $(0, T_{\max})$ from (37):

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\Phi(t) &= \frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega u^p + \frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |\nabla v|^{2p} + \frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |\nabla w|^{2p} \\
&\leq -\tfrac{2(p-1)}{p}\int_\Omega |\nabla u^{\frac{p}{2}}|^2 - p\int_\Omega |\nabla v|^{2p-2}|D^2 v|^2 - p\int_\Omega |\nabla w|^{2p-2}|D^2 w|^2 \\
&\quad - \tfrac{\mu_2 p}{2}\int_\Omega u^{p+k} + \tfrac{\chi^2 p(p-1)}{2}\int_\Omega u^p |\nabla v|^2 + c_{16}(n,\eta)\gamma^2 M_3^2 \int_\Omega u^2 |\nabla v|^{2p-2} \\
&\quad + c_{16}(n,\eta)\delta^2 M_2^2 \int_\Omega u^2 |\nabla w|^{2p-2} + c_{14}\int_\Omega |\nabla v|^{2(p+1)} + c_{15}\int_\Omega |\nabla w|^{2(p+1)} \\
&\quad + \tfrac{1}{4p}\int_\Omega |\nabla |\nabla w|^p|^2 + c_{20}(c_{14}, c_{15}, \eta),
\end{aligned} \tag{45}$$

with $c_{20}(c_{14}, c_{15}, \eta) := c_{13} + c_{17}(c_{14}, c_{15}, \eta) + c_{19}(\eta)$.

From here on, we distinguish between two cases: $k > 1$ and $k = 1$.

Case $k > 1$*).* Let us choose the values $\varepsilon_1 = \frac{\mu_2 p}{6}$ and $\varepsilon_2 = \frac{p}{12(4p^2+n)M_3^2}$ in (17) and (19) for $\psi = u$ and $\varphi = v$. Meanwhile, we pick $\varepsilon_1 = \frac{\mu_2 p}{6}$ and $\varepsilon_2 = \frac{p}{16(4p^2+n)M_3^2}$ in (19) for $\psi = u$ and $\varphi = w$. Incorporating the obtained inequalities into (45), we

obtain for $t \in (0, T_{\max})$ that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\Phi(t) \leq & -\frac{2(p-1)}{p}\int_\Omega |\nabla u^{\frac{p}{2}}|^2 - p\int_\Omega |\nabla v|^{2p-2}|D^2 v|^2 - p\int_\Omega |\nabla w|^{2p-2}|D^2 w|^2 \\ & + \frac{p}{4(4p^2+n)M_3^2}\int_\Omega |\nabla v|^{2(p+1)} + \frac{p}{8(4p^2+n)M_2^2}\int_\Omega |\nabla w|^{2(p+1)} \\ & + \frac{1}{4p}\int_\Omega |\nabla|\nabla w|^p|^2 + c_{21}(\eta)\end{aligned} \tag{46}$$

where $c_{21}(\eta) := c_{20}(\eta) + c_{22} + c_{23}(\eta) + c_{24}(\eta)$ with the constants c_{22}, $c_{23}(\eta)$, and $c_{24}(\eta)$ obtained from the Young inequality.

Then, thanks to (14) for an arbitrary value of $\eta \in (0, p-1)$ and for every $t \in (0, T_{\max})$, the inequality in (46) becomes

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\Phi(t) \leq & -\tfrac{2(p-1)}{p}\int_\Omega |\nabla u^{\frac{p}{2}}|^2 - \tfrac{p}{2}\int_\Omega |\nabla v|^{2p-2}|D^2 v|^2 - \tfrac{3p}{4}\int_\Omega |\nabla w|^{2p-2}|D^2 w|^2 \\ & + \tfrac{1}{4p}\int_\Omega |\nabla|\nabla w|^p|^2 + c_{21} \\ \leq & -\tfrac{2(p-1)}{p}\int_\Omega |\nabla u^{\frac{p}{2}}|^2 - \tfrac{1}{2p}\int_\Omega |\nabla|\nabla v|^p|^2 - \tfrac{1}{2p}\int_\Omega |\nabla|\nabla w|^p|^2 + c_{21},\end{aligned} \tag{47}$$

due to inequality (16).

By applying the Gagliardo-Nirenberg inequality (11) with specific choices for the function ϕ, and utilizing the bounds in (29), we derive the following inequalities on $(0, T_{\max})$. For $\phi = u^{\frac{p}{2}}$, we obtain:

$$\int_\Omega u^p \leq 2C_{GN}^2 M_1^{1-\theta}\left(\int_\Omega |\nabla u^{\frac{p}{2}}|^2\right)^\theta + 2C_{GN}^2 M_1^2. \tag{48}$$

Similarly, for $\phi = |\nabla v|^p$, the inequality yields:

$$\int_\Omega |\nabla v|^{2p} \leq 2C_{GN}^2 C^{1-\theta}\left(\int_\Omega |\nabla|\nabla v|^p|^2\right)^\theta + 2C_{GN}^2 C^2. \tag{49}$$

An analogous inequality holds for $\int_\Omega |\nabla w|^{2p}$ when setting $\phi = |\nabla w|^p$.

Here, C_{GN} is the constant from (11), C is the constant specified in *Lemma 3.3*, and $\theta = \frac{p-1}{p-1+\frac{2}{n}} \in (0,1)$. These constants are consistent across all applications.

Therefore, recalling $(a+b)^{\frac{1}{\theta}} \leq 2^{\frac{1-\theta}{\theta}}\left(a^{\frac{1}{\theta}} + b^{\frac{1}{\theta}}\right)$ provided $\theta < 1$ and $a, b \geq 0$, and rearranging terms for inequalities (48) and (49), we have on $(0, T_{\max})$ that

$$-\int_\Omega |\nabla u^{\frac{p}{2}}|^2 \leq -c_{25}\left(\int_\Omega u^p\right)^{\frac{1}{\theta}} + c_{26} \tag{50}$$

and

$$-\int_\Omega |\nabla|\nabla v|^p|^2 \leq -c_{27}\left(\int_\Omega |\nabla v|^{2p}\right)^{\frac{1}{\theta}} + c_{28}, \tag{51}$$

with a similar expression to the latter for the function w,

$$c_{25} := \frac{1}{\left(2C_{GN}^2 M_1^{1-\theta}\right)^{\frac{1}{\theta}}}, \quad c_{26} := \left(2C_{GN}^2 M_1^2\right)^{\frac{1}{\theta}} \ c_{25},$$

$$c_{27} := \frac{1}{\left(2C_{GN}^2 C^{1-\theta}\right)^{\frac{1}{\theta}}}, \quad c_{28} := \left(2C_{GN}^2 C^{1-\theta}\right)^{\frac{1}{\theta}} \ c_{27}.$$

Inserting inequalities (50) and (51) into (47), we conclude

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\Phi(t) \leq & -\frac{2(p-1)}{p} c_{25} \left(\int_\Omega u^p\right)^{\frac{1}{\theta}} - \frac{c_{27}}{2p}\left(\int_\Omega |\nabla v|^{2p}\right)^{\frac{1}{\theta}} - \frac{c_{27}}{2p}\left(\int_\Omega |\nabla w|^{2p}\right)^{\frac{1}{\theta}} \\
& + \frac{2(p-1)}{p} c_{26} + \frac{c_{28}}{p} + c_{21} \\
\leq & - c_{29}\left(\left(\int_\Omega u^p\right)^{\frac{1}{\theta}} + \left(\int_\Omega |\nabla v|^{2p}\right)^{\frac{1}{\theta}} + \left(\int_\Omega |\nabla w|^{2p}\right)^{\frac{1}{\theta}}\right) + c_{30} \\
\leq & - \frac{c_{29}}{3^{\frac{1-\theta}{\theta}}}\Phi(t)^{\frac{1}{\theta}} + c_{30}, \qquad \forall\ t \in (0, T_{\max}),
\end{aligned} \tag{52}$$

where $c_{29} := \min\left\{\frac{2(p-1)c_{25}}{p}, \frac{c_{27}}{2p}\right\}$, $c_{30} := \frac{2(p-1)c_{26}}{p} + \frac{c_{28}}{p} + c_{21}$, and the last inequality is justified by the generalized mean inequality $(a+b+c)^{\frac{1}{\theta}} \leq 3^{\frac{1-\theta}{\theta}}\left(a^{\frac{1}{\theta}} + b^{\frac{1}{\theta}} + c^{\frac{1}{\theta}}\right)$, with $\theta < 1$ and $a, b, c \geq 0$. Thus, we obtain the differential inequality

$$\begin{cases}
\Phi'(t) \leq -\frac{c_{29}}{3^{\frac{1-\theta}{\theta}}}\Phi(t)^{\frac{1}{\theta}} + c_{30}, & \forall\ t \in (0, T_{\max}) \\
\Phi(0) = \int_\Omega u_0^{2p} + \int_\Omega |\nabla v_0|^{2p} + \int_\Omega |\nabla w_0|^{2p}.
\end{cases} \tag{53}$$

From the comparison principle, for every $t \in (0, T_{\max})$, it follows the upper bound

$$\int_\Omega u^p + \int_\Omega |\nabla v|^{2p} + \int_\Omega |\nabla w|^{2p} \leq C_3 := \max\left\{\Phi(0), 3^{\frac{1}{1+n(p-1)}} \ \left(\frac{c_{30}}{c_{29}}\right)^{\frac{p-1}{p-1+\frac{1}{n}}}\right\}. \tag{54}$$

Case $k = 1$). Analogous to the previous case, we choose the values $\varepsilon_1 = \frac{p}{12(4p^2+n)M_3^2}$ in (18) and (20) for $\psi = u$ and $\varphi = v$. Meanwhile, we pick $\varepsilon_1 = \frac{p}{16(4p^2+n)M_3^2}$ in (20) for $\psi = u$ and $\varphi = w$. Incorporating the obtained inequalities into (45) leads to

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\Phi(t) \leq & -\tfrac{2(p-1)}{p}\int_\Omega |\nabla u^{\frac{p}{2}}|^2 - p\int_\Omega |\nabla v|^{2p-2}|D^2 v|^2 - p\int_\Omega |\nabla w|^{2p-2}|D^2 w|^2 \\
& + \tfrac{p}{4(4p^2+n)M_3^2}\int_\Omega |\nabla v|^{2(p+1)} + \tfrac{p}{8(4p^2+n)M_2^2}\int_\Omega |\nabla w|^{2(p+1)} \\
& + \tfrac{1}{4p^2}\int_\Omega |\nabla |\nabla w|^p|^2 + \left(c_{31} + c_{32}(n,\eta) + c_{33}(n,\eta) - \tfrac{\mu_2 p}{2}\right)\int_\Omega u^{p+1} \\
& + c_{20}(\eta), \quad \forall\ t \in (0, T_{\max}),
\end{aligned} \tag{55}$$

where the constants $c_{31}, c_{32}(n,\eta)$, and $c_{33}(n,\eta)$ are obtained from the Young inequality as follows:

$$c_{31} := \tfrac{p^2\chi^2(p-1)}{2(p+1)} \left(\tfrac{6\chi^2(p-1)(4p^2+n)M_3^2}{p+1} \right)^{\frac{1}{p}},$$

$$c_{32}(n,\eta) := \left(c_{16}(n,\eta)\gamma^2 M_3^2\right)^{\frac{p+1}{2}} \ \tfrac{2}{p+1} \left(\tfrac{p(p+1)}{12(p-1)(4p^2+n)M_3^2} \right)^{\frac{1-p}{2}},$$

and

$$c_{33}(n,\eta) := \left(c_{16}(n,\eta)\delta^2 M_2^2\right)^{\frac{p+1}{2}} \ \tfrac{2}{p+1} \left(\tfrac{p(p+1)}{16(p-1)(4p^2+n)M_2^2} \right)^{\frac{1-p}{2}}.$$

Let us impose that $c_{31} + c_{32}(n,\eta) + c_{33}(n,\eta) \leq \frac{\mu_2 p}{2}$. This inequality is valid whenever the parameter μ_2 is chosen large enough.

We then write the restriction on μ_2 as $\mu_2 \geq f(p,\eta)$, with

$$\begin{aligned} f(p,\eta) := {} & \tfrac{(p-1)p^2}{2(p+1)} \left(\tfrac{6\ (p-1)\left(4p^2+n\right)}{p+1} \right)^{1/p} \chi^{2+\frac{2}{p}} M_3^{\frac{2}{p}} \\ & + \tfrac{2\ c_{16}(n,\eta)}{p+1} \left(\tfrac{12\ c_{16}(n,\eta)\left(4p^2+n\right)}{p(p+1)} \right)^{\frac{p-1}{2}} \gamma^{p+1} M_3^{2p} \\ & + \tfrac{2\ c_{16}(n,\eta)}{p+1} \left(\tfrac{16\ c_{16}(n,\eta)(p-1)\left(4p^2+n\right)}{p(p+1)} \right)^{\frac{p-1}{2}} \delta^{p+1} M_2^{2p}. \end{aligned}$$

From the definition of $f(p,\eta)$, we can rewrite the hypothesis (41) as

$$\begin{aligned} & \mu_2 - f(p,0) \\ & = \mu_2 - A_1(p,n)\chi^{2+\frac{2}{p}} M_3^{\frac{2}{p}} - A_2(p,n)\gamma^{p+1} M_3^{2p} - A_3(p,n)\delta^{p+1} M_2^{2p} > 0, \end{aligned} \tag{56}$$

with $A_1(p,n), A_2(p,n)$, and $A_3(p,n)$ already defined in (42), (43), and (44), respectively. With $\mu_2 - f(p,\eta)$ a continuous function on both variables p and η, there exists an $\eta \in (0, p-1)$ such that $\mu_2 - f(p,\eta) \geq 0$. This implies that the coefficient of the term $\int_\Omega u^{p+1}$ in (55) is non-positive. Therefore, after neglecting that term and with analogous reasoning as from (47) to (52), the inequalities in (50) and (51) imply that

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\Phi(t) \leq & -\frac{2(p-1)}{p} \int_\Omega |\nabla u^{\frac{p}{2}}|^2 - \frac{1}{2p} \int_\Omega |\nabla |\nabla v|^p|^2 - \frac{1}{2p} \int_\Omega |\nabla |\nabla w|^p|^2 + c_{20}(\eta) \\ \leq & -c_{29} \left(\left(\int_\Omega u^p \right)^{\frac{1}{\theta}} + \left(\int_\Omega |\nabla v|^{2p} \right)^{\frac{1}{\theta}} + \left(\int_\Omega |\nabla w|^{2p} \right)^{\frac{1}{\theta}} \right) + c_{34} \\ \leq & -\frac{c_{29}}{3^{\frac{1-\theta}{\theta}}} \Phi(t)^{\frac{1}{\theta}} + c_{34}, \qquad \forall\ t \in (0, T_{\max}), \end{aligned}$$

with $c_{34} := \frac{2(p-1)c_{26}}{p} + \frac{c_{28}}{p} + c_{20}(\eta)$.

With the same argument as in (53), we obtain the upper bound

$$\int_\Omega u^p + \int_\Omega |\nabla v|^{2p} + \int_\Omega |\nabla w|^{2p} \leq C_4 := \max\left\{ \Phi(0), 3^{\frac{1}{1+n(p-1)}} \left(\tfrac{c_{34}}{c_{29}} \right)^{\frac{p-1}{p-1+\frac{1}{n}}} \right\}, \tag{57}$$

for all $t \in (0, T_{\max})$.

It is enough to consider as the upper bound in the thesis $\widetilde{C} = \max\{C_3, C_4\}$, with C_3 defined in (54) and C_4 in (57). □

With the previous result, we are able to provide a proof to *Theorem 1.1*.

Proof of Theorem 1.1. We divide the proof into two cases, $k > 1$ and $k = 1$, which rely on different hypotheses.

For $k > 1$, the hypotheses in (9) agree with the requirements of *Lemma 3.3*. Thus, *Lemma 4.4* is valid, and, as a consequence of *Lemma 3.4*, it the thesis follows.

On the other hand, for $k = 1$, let us consider the function $\mu_2 - f(p, 0)$ already defined in (56), which is continuous for the variable p by the definition of $A_1(p, n), A_2(p, n)$, and $A_3(p, n)$. With abuse of notation, let us define those constants as: $A_1(n) = A_1(\frac{n}{2}, n), A_2(n) = A_2(\frac{n}{2}, n)$, and $A_3(n) = A_3(\frac{n}{2}, n)$. The hypothesis in (10) states that $\mu_2 - f(\frac{n}{2}, 0) > 0$ Then, there exists $p > \frac{n}{2}$ such that the hypothesis in (41) of *Lemma 4.4* holds. Then, *Lemma 3.3* follows from the hypotheses in (9), and consequently *Lemma 4.4* applies again. □

4.1. **Further reflections.** We now return to *Remark 1.2*, where we replaced the first equation in (7) with $u_t = \Delta u - \chi\nabla\cdot(u\nabla v) + \mu_1 u^l - \mu_2 u^{k+1}$ for any $l \in [1, k+1)$. Applying Young's inequality to the new form of the mass inequality (27), which now reads

$$\frac{d}{dt}\int_\Omega u \leq \mu_1 \int_\Omega u^l - \mu_2 \int_\Omega u^{k+1}$$

leads to the same inequality (28) with the appropriate adjustment to c_7. Consequently, *Lemma 3.2* still holds in this case. Analogously, for the proof of *Lemma 4.1*, the constant c_{13} must be adjusted accordingly. Lastly, regarding the proof of *Lemma 3.4*, the necessary modifications involve demonstrating the boundedness of the term in (35)

$$\int_\Omega e^{(t-s)\Delta} u^l \left(1 - \tfrac{\mu_2}{\mu_1} u^{k+1-l}\right) \mathrm{d}s.$$

Indeed, since for any such l the function $u^l\left(1 - \frac{\mu_2}{\mu_1}u^{k+1-l}\right)$ is upper bounded, the validity of the mentioned result is assured. The remaining lemmas hold with the same proofs, and, as a consequence, the remark holds.

On the other hand, let us consider the case $\alpha = \beta = 0$. Then, the studied model becomes

$$\begin{cases} u_t = \Delta u - \chi\nabla\cdot(u\nabla v) + \mu_1 u^k - \mu_2 u^{k+1}, & \text{in } \Omega\times(0, T_{\max}), \\ v_t = \Delta v - \gamma uv, & \text{in } \Omega\times(0, T_{\max}), \\ w_t = \Delta w - \delta uw + \mu_3 w(1-w), & \text{in } \Omega\times(0, T_{\max}), \\ u(x,0) = u_0(x), \quad v(x,0) = v_0(x), \quad w(x,0) = w_0(x), & x \in \overline{\Omega}, \end{cases} \tag{58}$$

where the second and third equations are decoupled. The first two equations give the consumption model studied in [11] if we set $\gamma = k = 1$.

Lemma 3.2 still holds by the parabolic maximum principle, although equation (26) becomes

$$\|v(\cdot,t)\|_{L^\infty(\Omega)} \leq M_3 := \|v_0\|_{L^\infty(\Omega)}, \qquad \forall\, t \in (0, T_{\max}).$$

Regarding the boundedness of the gradients in Lemma 3.3, after setting $\alpha = \beta = 0$, the only hypothesis required is then $\delta > \mu_3$, meaning that the rate of consumption

of tumor cells by the lymphocytes must be greater than the growth coefficient of the former. Such a requirement is consistent with biological intuition.

The proof of *Theorem 1.1* relies on the ODI obtained for the functional $\Phi(t)$ in (37), i.e., the proof of *Lemma 4.4*. Without the problem parameters α and β, it only changes the value of constant $c_{17}(\eta)$ in *Lemma 4.2*. The proof of the theorem remains valid, giving the conclusion.

From a biological perspective, it may be unappealing to decouple the second and third equations in (7) as done in (58). The lack of connection between w and v in (58) makes no sense, the latter being the chemical signal released by the former. However, as an extension of the model in [11] for $k > 1$, it is worthwhile to realize that large values of $\chi\|v_0\|_{L^\infty(\Omega)}$ are allowed without the need to impose further restrictions on the parameters as in (4) and (10).

The model in [4] does not admit global classical solutions for large values of $\|w_0\|_{L^\infty(\Omega)}$. We can notice, as an advantage of the model proposed in this work, that the presence of a source term for the first equation in (7) makes the existence of such solutions independent on $\|w_0\|_{L^\infty(\Omega)}$. It would also be interesting to study whether the global solution stabilizes for large times as in [11] and [8]. After rewriting the first equation in (7) as $u_t = \Delta u - \chi\nabla\cdot(u\nabla v) + \mu_1 u^k\left(1 - \frac{\mu_2}{\mu_1}u\right)$, we expect from the logistic term that the constant $\frac{\mu_1}{\mu_2}$ represents the carrying capacity of the lymphocytes (i.e., u). Moreover, aligned with the observations in [8], with the amount of lymphocytes bounded in time, they eventually eliminate both cancer cells (i.e., w) and the chemical signal (i.e., v). Therefore, since $\left(\frac{\mu_1}{\mu_2}, 0, 0\right)$ is a constant stationary solution of (7), we conjecture that

$$(u(\cdot,t), v(\cdot,t), w(\cdot,t)) \to \left(\tfrac{\mu_1}{\mu_2}, 0, 0\right) \quad \textbf{in } L^\infty(\Omega), \qquad \textbf{as } t \to \infty.$$

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Appendix A. **Computation of constants.** This section is devoted to showing the constants computed in the text. It supports the statement that we have computable constants in our results, except the constant related to the Gagliardo-Nirenberg inequality.

We define

$$C_{\mathrm{Y}}(\varepsilon, r) := \tfrac{r-1}{r}(\varepsilon r)^{\frac{1}{1-r}},$$

representing the constant dependent on ε in the Young inequality

$$ab \leq \varepsilon a^r + C_{\mathrm{Y}}(\varepsilon, r)\, b^{\frac{r}{r-1}},$$

for real $a, b, \varepsilon > 0$ and $r > 1$. This approach facilitates the use of the inequality when choosing parameters r and ε in each case.

In the proof of Lemma 2.4, some parameters are left as degrees of freedom for the a priori inequalities later on. Along with them, some constants dependent on them are computed, for instance, in (17) and (18). They are

$$c_1(\varepsilon_1, \varepsilon_2) := \left[C_{\mathrm{Y}}\left(\varepsilon_1, \tfrac{p+k}{p}\right)\right]^{\frac{(p+1)k}{p(k-1)}} C_{\mathrm{Y}}\left(\varepsilon_2, \tfrac{(p+1)k}{p+k}\right)|\Omega|, \tag{59}$$

$$c_2(\varepsilon_1) := C_{\mathrm{Y}}(\varepsilon_1, p+1), \tag{60}$$

$$c_3(\varepsilon_1, \varepsilon_2) := \left[C_{\mathrm{Y}}\left(\varepsilon_1, \tfrac{p+k}{2}\right)\right]^{\frac{(p+1)(p+k-2)}{2(k-1)}} C_{\mathrm{Y}}\left(\varepsilon_2, \tfrac{(p+1)(p+k-2)}{(p-1)(p+k)}\right)|\Omega|, \tag{61}$$

$$c_4(\varepsilon_1) := C_{\mathrm{Y}}\left(\varepsilon_1, \tfrac{p+1}{p-1}\right), \tag{62}$$

$$c_5(\varepsilon_1, \varepsilon_2) := \left[C_{\mathrm{Y}}\left(\varepsilon_1, \tfrac{2(p+1)}{p-1}\right)\right]^{\frac{3}{2}} C_{\mathrm{Y}}(\varepsilon_2, 3)\,|\Omega|, \tag{63}$$

and

$$c_6(\varepsilon_1) := 2C_{\mathrm{GN}}C^{p(1-\theta)} C_{\mathrm{Y}}\left(\frac{\varepsilon_1}{2C_{\mathrm{GN}}C^{p(1-\theta)}}, \frac{1}{\theta}\right) + 2C_{\mathrm{GN}}C^p. \tag{64}$$

In Section 3, we define

$$c_7 := \tfrac{\mu_1}{k+1}\left(\tfrac{2\mu_1 k}{\mu_2(k+1)}\right)^p |\Omega|. \tag{65}$$

In Section 4, during the proof of Lemma 3.3, we compute

$$\begin{aligned} c_{10} :=& c_8|\Omega|\left(1 + 2M_1 + \frac{c_9 M_1}{\mu_2}\right) + c_8 C_{\mathrm{Y}}\left(\frac{1}{2}, \frac{k+1}{2}\right)|\Omega|^{\frac{k+1}{k-1}} \\ &+ \frac{\mu_1 c_8}{\mu_2} C_{\mathrm{Y}}\left(\frac{\mu_2}{2\mu_1}, \frac{k+1}{k}\right)|\Omega|^{k+1} \end{aligned} \tag{66}$$

and

$$c_{11} := c_8|\Omega| + \left(\frac{\mu_1 c_8}{\mu_2} + \tfrac{c_8 c_9}{\mu_2} + 2c_8\right) M_1. \tag{67}$$

Next, for the proof of Lemma 4.1, we define

$$c_{13} := \mu_1 p\; C_{\mathrm{Y}}\left(\frac{\mu_2}{2\mu_1}, \frac{p+k}{p+k-1}\right)|\Omega|. \tag{68}$$

Lemma 4.2 relies on the constant

$$c_{17}(c_{14}, c_{15}, \eta) := c_{18}(\eta) + (2\alpha p)^{\frac{3}{2}} c_5\left(\tfrac{c_{14}}{2\alpha p}, c_{15}\right). \tag{69}$$

Finally, Lemma 4.3 obtains the constant

$$c_{19}(\eta) := c_{18}(\eta) + 4p\mu_3 C_{\mathrm{GN}}C^p\left(C_{\mathrm{Y}}\left(\frac{1}{16p^2\mu_3 C_{\mathrm{GN}}C^{p(1-\theta)}}, \frac{1}{\theta}\right) C^{-p\theta} + 1\right), \tag{70}$$

with the constants $\theta = \frac{p-1}{p-1+\frac{2}{n}}$ and C from (29), and C_{GN} from the Gagliardo-Nirenberg inequality as cited in (22).