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On the adaptive stabilization of MIMO systems with uncertain control direction

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Abstract—This paper addresses the problem of adaptive stabilization of MIMO systems whose constant high-frequency gain matrix is nonsingular but has completely unknown sign structure. Unlike most existing approaches, no positivity, symmetry, or diagonal dominance assumptions on the gain matrix are required. Building upon the unit-vector framework and the concept of unmixing sets, a modified discontinuous adaptive control strategy is proposed. The key novelty consists in decoupling the mechanism that cycles through the elements of the unmixing set from the adaptation of the control input magnitude. In particular, the adaptive gain is used exclusively to govern the switching among candidate gain matrices, while the control input magnitude is fixed *a priori*. A Lyapunov-based analysis proves global asymptotic stabilization for driftless systems. The results are further extended to systems affected by matched bounded disturbances under mild additional assumptions. Simulation studies illustrate the effectiveness of the proposed approach and highlight its advantages in terms of reduced control authority and alleviated transient peaking.

I. INTRODUCTION

The problem of controlling single-input single-output (SISO) systems with unknown sign of the high-frequency gain (HFG) has attracted considerable attention in the literature [13], [5], [2], [1]. Extending such results from the single-input case to multi-input systems, however, is far from trivial. Most of the available approaches rely on the fulfillment of restrictive structural assumptions on the HFG matrix G , such as positivity ($G > 0$), symmetric positive definiteness ($G + G^T > 0$), or diagonal dominance. In many contributions, it is further assumed that a constant matrix K is known beforehand, such that the right-multiplied matrix GK belongs to one of the aforementioned classes. This assumption significantly reduces the admissible set of uncertain system dynamics.

A significant advancement in this research area has been achieved through the development of methods that only require the knowledge of an “unmixing set” for the class $\mathcal{G}$ to which the HFG matrix belongs. A set of matrices $\mathcal{U}$ is said to be unmixing for $\mathcal{G}$ if, for every $M \in \mathcal{G}$, there exists at least one matrix $K \in \mathcal{U}$ such that the spectrum satisfies $\lambda(MK) \subset C_0$.

In this work, the class $\mathcal{G} = GL(m, \mathbb{R})$ of real-valued, nonsingular square matrices of dimension m is considered. The existence of unmixing sets with finite cardinality for $GL(m, \mathbb{R})$ has been established in [11] for all $m \in \mathbb{N}$.

In [15], it is assumed that an unmixing set is available *a priori* to the control designer. The proposed control law

takes the form of a high-gain adaptive discontinuous feedback based on the unit-vector approach. The gain matrix is selected from the unmixing set, and a specific rule is introduced to cycle through its elements. A key component of this scheme is a scalar adaptive gain $k(t)$, which is nonnegative and nondecreasing. This gain both premultiplies the control input and governs the switching mechanism among the elements of the unmixing set.

The primary objective of the present paper is to decouple the mechanism that cycles the control gain matrix through the unmixing set from the online adaptation of the control input magnitude. To this end, a modified control strategy is proposed in which the adaptive gain $k(t)$ solely governs the switching among the elements of the unmixing set, while it no longer appears as a multiplicative factor in the control input. As a consequence, the cycling of the gain matrix is separated from the adjustment of the control magnitude, which is instead selected as a constant value assigned *a priori*.

It is also worth mentioning the related work [14], where the availability of an unmixing set is similarly assumed and a discontinuous unit-vector control law is employed without the use of adaptive high-gain terms. In that approach, the switching mechanism among the elements of the unmixing set differs substantially from that in [15], as it is based on so-called “monitoring functions”. The construction of these functions, however, requires prior knowledge of lower and upper bounds on the norm of the HFG matrix G .

The remainder of the paper is organized as follows. Section (I-A) introduces the notation and recalls preliminary results and formulas used throughout the paper. Section II reviews the main concepts and some recent developments related to unmixing sets. In Section III, the control problem is formulated and solved for unperturbed system dynamics using Lyapunov-based arguments. Section IV extends the analysis to uncertain systems affected by matched drift disturbances. Simulation results are presented and discussed in Section V, while concluding remarks and directions for future research are provided in Section VI.

A. Notation and useful formulas

The sets of natural, real and complex numbers are denoted by $\mathbb{N}$, $\mathbb{R}$ and $\mathbb{C}$. $\mathbb{N}_0$ and $\mathbb{R}_+$ denote respectively the set of natural numbers including zero and the set of real positive numbers. The real and imaginary parts of $z \in \mathbb{C}$ are denoted by $\text{Re}(z)$ and $\text{Im}(z)$, respectively.

For $\alpha \in \mathbb{R}$, define $C_\alpha = \{z \in \mathbb{C} : \text{Re}(z) > \alpha\}$. Note that C_0 denotes the open right half complex plane.

Let $v = (v_1, \dots, v_n) \in \mathbb{R}^n$. Its Euclidean norm is defined as $\|v\|_2 = \sqrt{\sum_{i=1}^n |v_i|^2}$.

For $n \in \mathbb{N}$, $I_n \in \mathbb{R}^{n \times n}$ denotes the n -dimensional identity matrix. Let $W \in \mathbb{R}^{n \times n}$ be a real-valued square matrix. The set of eigenvalues of W is referred to as the spectrum of W and is denoted by $\lambda(W) = [\lambda_1(W), \lambda_2(W), \dots, \lambda_n(W)]$.

Let $v, w \in \mathbb{R}^n$ be real-valued column vectors, and let $M \in \mathbb{R}^{n \times n}$ be a symmetric positive definite matrix. Then $\lambda(M) = [\lambda_1(M), \lambda_2(M), \dots, \lambda_n(M)] \in \mathbb{R}_+^n$. Define

$$\lambda_{\min}(M) \equiv \min_{i=1,2,\dots,n} \lambda_i(M), \quad \lambda_{\max}(M) \equiv \max_{i=1,2,\dots,n} \lambda_i(M) \quad (1)$$

The so-called “axis–angle” representation of a special orthogonal matrix $R \in SO(3)$ (also referred to as the “Rodrigues formula”) is given as follows:

$$R = R(\mathbf{r}, \theta) = \cos(\theta) I_3 + (1 - \cos\theta) \mathbf{r} \mathbf{r}^T + \sin\theta S(\mathbf{r}), \quad (2)$$

where $\mathbf{r} = [r_x \ r_y \ r_z] \in \mathbb{R}^3$, $\|\mathbf{r}\| = 1$, is the rotation axis, $\theta \in [-\pi, \pi]$ is the rotation angle and

$$S(\mathbf{r}) = \begin{bmatrix} 0 & -r_z & r_y \\ r_z & 0 & -r_x \\ -r_y & r_x & 0 \end{bmatrix}. \quad (3)$$

II. UNMIXING SETS

A matrix set $\mathcal{U} = \{K_i^N\} = \{K_1, K_2, \dots, K_N\}$ is unmixing $\mathcal{G}$ if for any $M \in \mathcal{G}$ there is at least one element $K_i \in \mathcal{U}$ such that the spectrum of MK_i satisfies $\lambda(MK_i) \subset \mathcal{C}_0$.

The existence of unmixing sets with finite cardinality for the set $GL(m, \mathbb{R})$ has been proven in [11] for any $m \in \mathbb{N}$. Interestingly, it was shown in [11] that it is always possible to identify an unmixing set composed of orthogonal matrices.

Unfortunately, the cardinality of the unmixing sets given by the general construction in [11] is far too large than would be reasonable for applications, and hardly anything is known on the minimum cardinality of the unmixing sets for arbitrary values of m . For the case $m = 2$ an unmixing set of cardinality 6 is given in [11]. Zhu [17] has constructed an unmixing set having cardinality 32 for the case $m = 3$. In [3] an unmixing set for $GL(3, \mathbb{R})$ having cardinality 25, i.e. reduced cardinality compared to that proposed in [17], was derived, and it is recalled in the next Subsection.

A. An unmixing set for $GL(3, \mathbb{R})$

The construction method proposed in [3] of an unmixing set for $GL(3, \mathbb{R})$ is strictly related to the structure and properties of the regular dodecahedron inscribed in the unitary sphere centered at the origin. Let B_i ($i = 1, \dots, 12$) be the baricenters of the twelve pentagons which form the dodecahedron (the corresponding coordinates are explicitly given in [3]). Define

$$\mathbf{b}_i = \frac{(B_i - 0)}{\|B_i - 0\|}, \quad i = 1, \dots, 12 \quad (4)$$

and

$$T = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}. \quad (5)$$

Lemma 1: see [3, Lemma 1] The set of matrices $\{K_i^{25}\} = \{K_1, K_2, \dots, K_{25}\}$ defined by

$$K_i = \begin{cases} R(\mathbf{b}_i, \beta^*) & i = 1, 2, \dots, 12 \\ TR(\mathbf{b}_{i-12}, \beta^*) & i = 13, 14, \dots, 24 \end{cases} \quad (6)$$

where $\beta^* = -3\pi/4$ and $K_{25} = I$ is an unmixing set for $GL(3, \mathbb{R})$.

III. ADAPTIVE UNIT-VECTOR CONTROL

Consider a driftless square system

$$\dot{\mathbf{y}} = G\mathbf{u}, \quad \mathbf{y}, \mathbf{u} \in \mathbb{R}^n, \quad G \in GL(n, \mathbb{R}), \quad (7)$$

where $\mathbf{y} = [y_1, y_2, \dots, y_n]^T$ is the state vector, $\mathbf{u}$ is the manipulable input vector, and G is the uncertain nonsingular high-frequency gain (HFG) matrix. No additional information on G is assumed.

By exploiting the availability of an unmixing set $\{K_i^N\}$ for $GL(n, \mathbb{R})$, in [15] the adaptive unit-vector control strategy

$$\mathbf{u} = -k(t) K_{\Sigma(k(t))} \frac{\mathbf{y}}{\|\mathbf{y}\|_2}, \quad (8)$$

$$\dot{k}(t) = \gamma \|\mathbf{y}\|_2, \quad k(0) = 0, \quad \gamma > 0, \quad (9)$$

was proposed to achieve the asymptotic stabilization of (7), where function $\Sigma: \mathbb{R} \rightarrow \{1, 2, \dots, N\}$, is given by

$$\Sigma(\kappa) = i. \quad \kappa \in \mathcal{T}_i \equiv \bigcup_{j \in \mathbb{N}} [\kappa_{(i-1)+(j-1)N}, \kappa_{i+(j-1)N}), \quad (10)$$

and the sequence $\{\kappa_h\}_{h=0,1,2,\dots}$ fulfills the properties

$$\kappa_i < \kappa_{i-1} \quad \forall i \in \mathbb{N}, \quad \lim_{h \rightarrow \infty} \frac{\kappa_{h-1}}{\kappa_h} = 0. \quad (11)$$

A sequence meeting such properties (11) is referred to as a “switching sequence”. Note that (see [6]) a switching sequence necessarily satisfies

$$\lim_{h \rightarrow \infty} \kappa_h = \infty. \quad (12)$$

An example of a switching sequence is given as follows

$$\kappa_0 = 0, \quad \kappa_1 > 1, \quad \kappa_{h+1} = \kappa_h^q, \quad q > 1, \quad h = 1, 2, \dots \quad (13)$$

A major drawback of the control law (8)-(9) is that the same adaptive gain $k(t)$ (which is nonnegative and monotonically increasing) drives the cycling mechanism through the elements of the unmixing set (generated by the adaptive matrix gain $K_{\Sigma(k(t))}$ in (8)) and pre-multiplies the control vector (8). Indeed, the adaptive gain $k(t)$ can reach large values during the convergence transient which causes unnecessarily large control authority and risk of actuator saturation.

The main goal of this paper is to devise a de-coupling between the cycling mechanism through the elements of the unmixing set and the adaptive magnitude of the control input. More precisely, we are going to investigate the performance of the modified control law where (8) is replaced by

$$\mathbf{u} = -\rho K_{\Sigma(k(t))} \frac{\mathbf{y}}{\|\mathbf{y}\|_2}, \quad (14)$$

where ρ is a positive constant. We are going to show that ρ can actually be chosen arbitrarily small, thus allowing one to impose by construction the constraint

$$\|\mathbf{u}\|_2 \leq \rho \quad (15)$$

The stability proof will be enabled by a technical result to be demonstrated in the following Lemma.

Lemma 2: Let $\{\kappa_h\}_{h=0,1,2,\dots}$ be a switching sequence with $\kappa_0 = 0$, and let $\Sigma(x)$ be defined as in (10). Let $\beta \in \mathbb{R}^+$. Define

$$\nu_i^\alpha(x) = \begin{cases} 1 & \text{if } \Sigma(x) = i \\ -\alpha & \text{if } \Sigma(x) \neq i \end{cases} \quad (16)$$

where i is an arbitrary integer such that $1 \leq i \leq N$. Then it follows that

$$\lim_{x \rightarrow \infty} \int_0^x \nu_i^\alpha(x) dx = +\infty \quad (17)$$

Proof of Lemma 2. Without restriction of generality we only consider the function $\nu_N^\alpha(\cdot)$, i.e. $i = N$. It follows from the definition of $\Sigma(\cdot)$ that for any $j = 0, 1, 2, \dots$ one has that

$$\begin{aligned} \int_{\kappa_{jN}}^{\kappa_{(j+1)N}} \nu_N^\alpha(x) dx &= -\alpha \int_{\kappa_{jN}}^{\kappa_{(j+1)N-1}} dx + \int_{\kappa_{(j+1)N-1}}^{\kappa_{(j+1)N}} dx \\ &= \alpha \kappa_{jN} - (1 + \alpha) \kappa_{(j+1)N-1} + \kappa_{(j+1)N} \end{aligned} \quad (18)$$

Considering that $\kappa_0 = 0$, the next relation holds

$$\begin{aligned} \int_0^{\kappa_{(\ell+1)N}} \nu_N^\alpha(x) dx &= \sum_{j=0}^{\ell} \int_{\kappa_{jN}}^{\kappa_{(j+1)N}} \nu_N^\alpha(x) dx \\ &= \sum_{j=0}^{\ell-1} \int_{\kappa_{jN}}^{\kappa_{(j+1)N}} \nu_N^\alpha(x) dx \\ &\quad + \alpha \kappa_{\ell N} - (1 + \alpha) \kappa_{(\ell+1)N-1} + \kappa_{(\ell+1)N} \end{aligned} \quad (19)$$

Clearly, the integral limit in (17) is obtained by getting the limit for $\ell \rightarrow +\infty$ of (19). The last three terms in the right-hand side of (19) are going to be estimated and manipulated as follows

$$\begin{aligned} &\alpha \kappa_{\ell N} - (1 + \alpha) \kappa_{(\ell+1)N-1} + \kappa_{(\ell+1)N} \\ &\geq -(1 + \alpha) \kappa_{(\ell+1)N-1} + \kappa_{(\ell+1)N} \\ &= \kappa_{(\ell+1)N} \left[1 - (1 + \alpha) \frac{\kappa_{(\ell+1)N-1}}{\kappa_{(\ell+1)N}} \right] \end{aligned} \quad (20)$$

According to the properties of the switching sequences, the ratio $\frac{\kappa_{(\ell+1)N-1}}{\kappa_{(\ell+1)N}}$ tends to zero as ℓ approaches infinity, whereas $\kappa_{(\ell+1)N}$ tends clearly to infinity. Thus,

$$\lim_{\ell \rightarrow \infty} [\alpha \kappa_{\ell N} - (1 + \alpha) \kappa_{(\ell+1)N-1} + \kappa_{(\ell+1)N}] = +\infty \quad (21)$$

It remains to prove that the term $\sum_{j=0}^{\ell-1} \int_{\kappa_{jN}}^{\kappa_{(j+1)N}} \nu_N^\alpha(x) dx$ in (19) is bounded from below by a constant $\bar{L} \in \mathbb{R}$

independent of ℓ . It can be derived that

$$\begin{aligned} &\sum_{j=0}^{\ell-1} \int_{\kappa_{jN}}^{\kappa_{(j+1)N}} \nu_N^\alpha(x) dx \\ &= \sum_{j=0}^{\ell-1} [\alpha \kappa_{jN} - (1 + \alpha) \kappa_{(j+1)N-1} + \kappa_{(j+1)N}] \\ &\geq \sum_{j=0}^{\ell-1} [-(1 + \alpha) \kappa_{(j+1)N-1} + \kappa_{(j+1)N}] \\ &= \sum_{j=0}^{\ell-1} \kappa_{(j+1)N} \left[1 - (1 + \alpha) \frac{\kappa_{(j+1)N-1}}{\kappa_{(j+1)N}} \right] \end{aligned} \quad (22)$$

As previously noticed, the properties of the switching sequences ensure that the ratio $\frac{\kappa_{(\ell+1)N-1}}{\kappa_{(\ell+1)N}}$ tends to zero as $\ell \rightarrow \infty$. Thus, there exists ℓ^* such that

$$(1 + \alpha) \frac{\kappa_{(\ell+1)N-1}}{\kappa_{(\ell+1)N}} < 1, \quad \forall \ell \geq \ell^* \quad (23)$$

We now decompose the right-hand side of (22) as follows

$$\begin{aligned} &\sum_{j=0}^{\ell-1} \kappa_{(\ell+1)N} \left[1 - (1 + \alpha) \frac{\kappa_{(\ell+1)N-1}}{\kappa_{(\ell+1)N}} \right] \\ &= \sum_{j=0}^{\ell^*} \kappa_{(\ell+1)N} \left[1 - (1 + \alpha) \frac{\kappa_{(\ell+1)N-1}}{\kappa_{(\ell+1)N}} \right] \\ &\quad + \sum_{j=\ell^*+1}^{\ell-1} \kappa_{(\ell+1)N} \left[1 - (1 + \alpha) \frac{\kappa_{(\ell+1)N-1}}{\kappa_{(\ell+1)N}} \right] \\ &\geq \sum_{j=0}^{\ell^*} \kappa_{(\ell+1)N} \left[1 - (1 + \alpha) \frac{\kappa_{(\ell+1)N-1}}{\kappa_{(\ell+1)N}} \right] = \bar{L}. \end{aligned} \quad (24)$$

The statement (17) derives by combining together (19), (21), (22) and (24). Lemma 2 is proven. $\square$

The next Theorem is in force.

Theorem 1: Consider system (7) and let $\{K_i^N\}$ be an unmixing set of cardinality N for $GL(n, \mathbb{R})$, where $K_i \in O(n) \forall i = 1, 2, \dots, N$, i.e. all elements of the unmixing set are orthogonal matrices. The application of the discontinuous feedback (14), (9)-(10), with the switching sequence $\{\kappa_h\}_{h=0,1,2,\dots}$ defined in (13) and with arbitrary $\rho > 0$ guarantees that the closed-loop system approaches the origin asymptotically whereas $k(t)$ settles to a constant finite value k^* .

Proof of Theorem 1. Let K_{j^*} be an element of the unmixing set such that $\sigma(GK_{j^*}) \subset C^+$. Then, there exists a unique solution $Q = Q^T > 0$ to the Lyapunov equation

$$Q GK_{j^*} + (GK_{j^*})^T Q = I_n \quad (25)$$

Consider the following Lyapunov candidate function

$$V = \frac{1}{2} \mathbf{y}^T Q \mathbf{y}. \quad (26)$$

Computing the time derivative of (26) along the solutions of (7), (14) yields

$$\dot{V} = \mathbf{y}^T Q \dot{\mathbf{y}} = -\rho \frac{1}{\|\mathbf{y}\|_2} \mathbf{y}^T Q G K_{\Sigma(k(t))} \mathbf{y}. \quad (27)$$

When $\Sigma(k(t)) = j^*$, i.e. when the control gain matrix $K_{\Sigma(k(t))}$ coincides with a specific matrix K_{j^*} of the unmixing set ensuring that matrix $-GK_{j^*}$ is Hurwitz, the right-hand side of (27) is estimated as follows by virtue of (25):

$$\dot{V} = -\rho \frac{1}{\|\mathbf{y}\|_2} \mathbf{y}^T Q G K_{j^*} \mathbf{y} \quad (28)$$

$$= -\frac{\rho}{2} \frac{1}{\|\mathbf{y}\|_2} \mathbf{y}^T [Q G K_{j^*} + (G K_{j^*})^T Q] \mathbf{y} \quad (29)$$

$$= -\frac{\rho}{2} \|\mathbf{y}\|_2 \quad (30)$$

When $\Sigma(k(t)) \neq j^*$, exploiting the fact that all elements of the unmixing set are orthogonal matrices (that is, $\|K_i\| = 1 \forall i = 1, 2, \dots, N$) one straightforwardly derives from

$$\dot{V} \leq \rho \|Q G\| \|\mathbf{y}\|_2 \quad (31)$$

Combining (30) (which is valid when $\Sigma(k(t)) = j^*$) and (31) (which holds when $\Sigma(k(t)) \neq j^*$) it yields that

$$\dot{V} \leq -\frac{\rho}{2} \nu_{j^*}^\alpha(k(t)) \|\mathbf{y}\|_2 \quad (32)$$

with $\alpha = 2\|Q G\|$. By virtue of (9) one rewrites (32) as follows

$$\dot{V} \leq -\frac{\rho}{2\gamma} \nu_{j^*}^\alpha(k(t)) \dot{k}(t) \quad (33)$$

Integrating the left and right-hand sides of (33) yields

$$0 \leq V(t) \leq V(0) - \frac{\rho}{2\gamma} \int_0^t \nu_{j^*}^\alpha(k(t)) \dot{k}(t) dt \quad (34)$$

The integral term in the right-hand side of (34) can be manipulated as follows

$$\int_0^t \nu_{j^*}^\alpha(k(t)) \dot{k}(t) dt = \int_{k(0)}^{k(t)} \nu_{j^*}^\alpha(k(t)) dk(t) \quad (35)$$

Then, by replacing (35) into (34) and observing that $k(0) = 0$ we have that

$$0 \leq V(t) \leq V(0) - \frac{\rho}{2\gamma} \int_0^{k(t)} \nu_{j^*}^\alpha(\theta) d\theta \quad (36)$$

Seeking for a contradiction, suppose that the monotone function $k(t)$ is unbounded, i.e. $\lim_{t \rightarrow \infty} k(t) = +\infty$. This implies, according to Lemma 2, that

$$\lim_{t \rightarrow \infty} V(t) \leq V(0) - \frac{\rho}{2\gamma} \lim_{x \rightarrow \infty} \int_0^x \nu_{j^*}^\alpha(x) dx = -\infty \quad (37)$$

which contradicts (36) since $V(t)$ is nonnegative by construction, and so $k(t)$ is bounded. Being $\dot{k}(t)$ strictly nonnegative by (9), this implies that $k(t)$ eventually converges to a positive constant.

The fact that $k(t)$ admits a finite limit for $t \rightarrow \infty$, along with the uniform continuity of $\dot{k}(t)$, ensures that $\dot{k}(t)$ is asymptotically vanishing according to the Barbalat Lemma. A sufficient condition ensuring the uniform continuity of $\dot{k}(t)$ is the uniform boundedness of $\ddot{k}(t)$. Differentiating (9) yields

$$\ddot{k}(t) = \gamma \frac{\mathbf{y}^T \dot{\mathbf{y}}}{\|\mathbf{y}\|_2}. \quad (38)$$

By the Cauchy-Schwarz inequality one estimates (38) as follows

$$|\ddot{k}(t)| \leq \gamma \frac{\|\mathbf{y}\|_2 \|\dot{\mathbf{y}}\|_2}{\|\mathbf{y}\|_2} = \gamma \|\dot{\mathbf{y}}\|_2. \quad (39)$$

By (7) and (14), $\dot{\mathbf{y}}$ takes the explicit form

$$\dot{\mathbf{y}} = -\rho G K_{\Sigma(k(t))} \frac{\mathbf{y}}{\|\mathbf{y}\|_2}. \quad (40)$$

Exploiting once more the fact that all elements of the unmixing set are orthogonal matrices, one derives the next estimation

$$\|\dot{\mathbf{y}}\|_2 \leq \rho \|G\| \quad (41)$$

Combining (39) and (41) it straightforwardly follows that $\ddot{k}(t)$ is uniformly bounded. Therefore, $\dot{k}(t)$ is asymptotically vanishing according to the Barbalat Lemma. This implies, according to (9), that $\|\mathbf{y}\|_2$ is also asymptotically vanishing. Theorem 1 is proven. $\square$

IV. PERTURBED SYSTEMS

The results presented in the previous section can be generalized to controlled systems affected by perturbations. This generalization, however, requires more informations on the HFG matrix G than the simple requirement of being nonsingular, as it was previously the case. Then, we now consider the perturbed process

$$\dot{\mathbf{y}} = f(\mathbf{y}, t) + G\mathbf{u}, \quad \mathbf{y}, \mathbf{u} \in \mathbb{R}^n, \quad G \in GL(n, \mathbb{R}), \quad (42)$$

The drift perturbation is assumed to be uniformly bounded according to the next

Assumption 1: There exist a known nonnegative constant F such that $\|f(\mathbf{y}, t)\|_2 \leq F \forall t \geq 0$.

The additional informations on the the HFG matrix G previously mentioned are formalized by the next

Assumption 2: There exist a known nonnegative constant ϵ_m and a matrix K_{j^*} belonging to the unmixing set such that

$$\lambda_{\min}(G K_{j^*} + (G K_{j^*})^T) \geq \epsilon_m. \quad (43)$$

Parameter ϵ_m was evaluated in [4] as $\epsilon_m = 1$ with reference to the unmixing set for $GL(2, \mathbb{R})$ therein presented. The same value $\epsilon_m = 1$ is also in force with reference to the unmixing set for $GL(3, \mathbb{R})$ proposed in [3] (this derivation was not explicitly made in [3], however it can be proven rather easily). Notice that, rather than being an additional information requirement about the HFG matrix G only, the assumption 2 is a requirement involving both the HFG matrix and the specific unmixing set employed in the control law.

For asymptotically stabilizing system (42) subject to Assumptions 1 and 2, a control law of the same form (14), (9)-(10), (13) is taken. The main difference from the unperturbed case dealt with the previous section is that the control gain ρ is now to be taken large enough, according to

$$\rho \geq \frac{2F}{\epsilon_m} + \chi^2, \quad \chi \neq 0 \quad (44)$$

The next Theorem establishes the underlying convergence properties.

Theorem 2: Consider system (42) subject to Assumption 1. Let $\{K_i^N\}$ be an unmixing set of cardinality N for $GL(n, \mathbb{R})$, where $K_i \in O(n) \forall i = 1, 2, \dots, N$, i.e. all elements of the unmixing set are orthogonal matrices, and let Assumption 2 be in force. Then, the application of the discontinuous feedback (14), (9)-(10), (13) and with ρ chosen according to (44) guarantees that the closed-loop system approaches the origin asymptotically whereas $k(t)$ settles to a constant finite value k^* .

Proof of Theorem 2. The proof of Theorem 2 shares much in common with the proof of Theorem 1. Only the main differences are highlighted hereinafter. The same Lyapunov function (26) is employed. In light of Assumption 1, and exploiting trivial matrix manipulations, equation (30) (which is valid when $\Sigma(k(t)) = j^*$) is now estimated as

$$\dot{V} = \mathbf{y}^T Q f(\mathbf{y}, t) - \frac{\rho}{2} \|\mathbf{y}\|_2 \leq \left[-\frac{\rho}{2} - F \|Q\|_2 \right] \|\mathbf{y}\|_2 \quad (45)$$

Exploiting (by virtue of Assumption 2) the known bound

$$\|Q\|_2 \leq \frac{1}{\lambda_{\min}(GK_{j^*} + (GK_{j^*})^T)} \leq \frac{1}{\epsilon_m} \quad (46)$$

involving the solution Q to the Lyapunov equation (25), the inequality (45) is further elaborated as follows in light of (44)

$$\dot{V} \leq \left[-\frac{\rho}{2} - \frac{F}{\epsilon_m} \right] \|\mathbf{y}\|_2 \leq -\frac{1}{2} \chi^2 \|\mathbf{y}\|_2. \quad (47)$$

Equation (31), in force when $\Sigma(k(t)) \neq j^*$, is now changed as follows

$$\dot{V} \leq \frac{\rho}{\epsilon_m} (F + \|G\|_2) \|\mathbf{y}\|_2 \quad (48)$$

Combining (47) and (48) one thus derives the modified version of (32) as follows

$$\dot{V} \leq -\frac{\chi^2}{2\gamma} \nu_{j^*}^\alpha(k(t)) \dot{k}(t) \quad (49)$$

with $\alpha = \frac{2\gamma\rho}{\chi^2\epsilon_m} (F + \|G\|_2)$. From now on, analogous considerations as those made in the Proof of Theorem 1 after the derivation of (32), allow us to establish that $k(t)$ is bounded and eventually converges to a positive constant. To establish that $\dot{k}(t)$ (and so also $\|\mathbf{y}\|_2$) are asymptotically vanishing, one needs to show that $\ddot{k}(t)$ is uniformly bounded in order to exploit the Barbalat Lemma. Straightforward manipulations yield

$$|\ddot{k}(t)| \leq \gamma \|\dot{\mathbf{y}}\|_2 \leq \gamma (F + \rho \|G\|_2) \quad (50)$$

Theorem 2 is proven. $\square$

V. SIMULATION RESULTS

Consider system (7) with the nonsingular HFG matrix

$$G = \begin{bmatrix} 1 & -2 & 0.3 \\ -1 & -1 & 0.5 \\ 0.5 & 0 & -2 \end{bmatrix}. \quad (51)$$

The initial condition is $y(0) = [1, 2, 3]^T$. The unmixing set $\{K_1, K_2, \dots, K_{25}\}$ is constructed according to the procedure

outlined in the Lemma 1. The coordinates of points B_i ($i=1,2,\dots,12$) are given in the Appendix Section. The “good” matrices of the unmixing set, i.e. those matrices K_i such that $\sigma(GK_i) \in C^+$, correspond to the indexes $i \in \{6, 11, 12\}$.

In the first simulation (TEST 1) the control law (8)-(10), (13) proposed in [15] was applied with parameters $\gamma = 0.002$ and $q = 1.2$. In a second simulation run (TEST 2), the novel control law described in Theorem 1 is implemented, with the same γ and q values used in TEST 1 and with the gain parameter $\rho = 1$.

Figure 1 shows the time evolution of the vector $\mathbf{y}$ components in TEST 1 (left plot) and in TEST 2 (right plot). It is apparent that in TEST 1 the transient peaking of the state variables is much higher than that obtained in TEST 2. This is due to the fact that unlike TEST 1 (where the control law 8) is employed) in TEST 2 the gain $k(t)$ does not multiply anymore the manipulable input vector, hence less control authority is employed which causes less transient peaking. As expected, for the same reason, the transient time needed to achieve a vicinity of the origin is longer in TEST 2 compared to TEST 1.

Figures 2 and 3 depict, respectively, the cycling function $\Sigma(k(t))$ and the adaptive gain $k(t)$ in TEST 1 and TEST 2. The corresponding time evolutions are comparable. In both the simulation runs the cycling function $\Sigma(k(t))$ settles to the value 12 which corresponds to one of the “good” matrices of the unmixing set and the adaptive gain $k(t)$ grows monotonically until it settles to a constant value.

Figures 4 and 5 display the entries $u_1(t), u_2(t), u_3(t)$ of the manipulable input vector $u(t)$. The benefits in terms of control authority alleviation obtained in TEST 2 thanks to the proposed modification to the control law are apparent. In the steady state, where all control inputs exhibits fast oscillations (induced by the discontinuous nature of the unit vector control at the origin of the state space) those arising in TEST 2 are of reduced amplitude, with positive implications in terms of chattering alleviation.

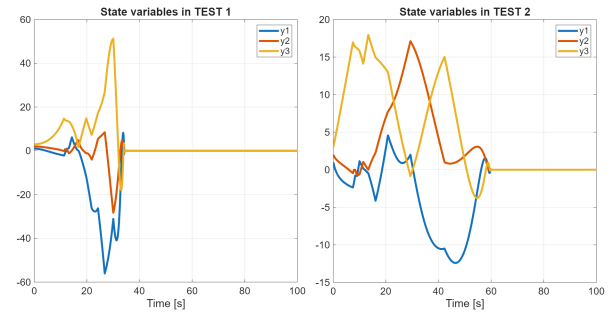


Fig. 1. State vector components in TEST 1 (left) and TEST 2 (right).

VI. CONCLUSIONS

The adaptive stabilization of MIMO systems has been investigated under the sole assumption that the uncertain high-frequency gain (HFG) matrix G is nonsingular. The adaptive unit-vector control strategy introduced in [15] has been revisited by removing the adaptive gain $k(t)$ as a

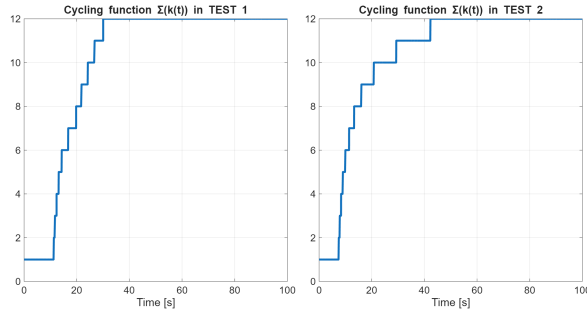


Fig. 2. Cycling function $\Sigma(k(t))$ in TEST 1 (left) and TEST 2 (right).

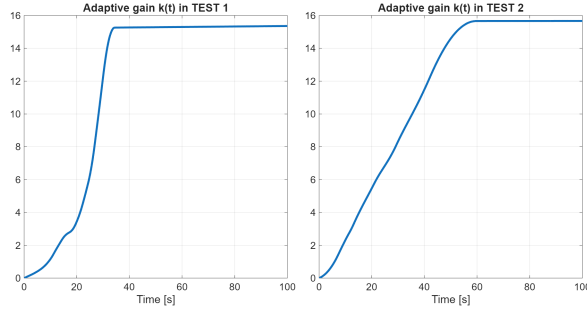


Fig. 3. Adaptive gain $k(t)$ in TEST 1 (left) and TEST 2 (right).

premultiplicative factor in the control law, with the aim of reducing excessive control authority. A novel Lyapunov-based analysis has been developed to characterize the convergence properties of the resulting controller. The approach is subsequently extended to perturbed MIMO systems, showing that the same convergence guarantees are retained under a mild increase in prior knowledge of the system's HFG matrix, provided that the constant scalar ρ acting as a multiplicative gain on the control input is chosen sufficiently large. Future work will focus on broadening the class of admissible systems by incorporating internal dynamics and allowing for state- and time-dependent HFG matrices.

REFERENCES

- [1] Bartolini G, Pisano A., Usai E. "On the second-order sliding mode control of nonlinear systems with uncertain control direction", *IEEE Trans. Automatic Control*, **45**, 2982–2985 (2009)
- [2] Bartolini, G., Ferrara, A., Giacomini, L. A switching controller for systems with hard uncertainties, *IEEE Trans. Circ. and Syst. - I Fund. Th. and Applic.*, **50**, 8, 984–990 (2003)
- [3] Bartolini G, Pilloni A, Pisano A., Usai E. "Adaptive stabilization of 3rd-order MIMO systems with uncertain control direction", Proc. 17-th International Workshop on Variable Structure Systems (VSS 2024), pp. 81–86.
- [4] Bartolini G, Pilloni A, Pisano A., Usai E. "On the adaptive unit-vector stabilization of second-order MIMO systems with uncertain control direction", in "Sliding-mode, Variable-Structure and Discontinuous Control Systems", eds: I. Boiko and A. Ferrara, Springer, in press.
- [5] T. Guo, Y. Liu and Y. Man. "Adaptive controller of nonlinear systems with unknown control directions and unknown input powers", *Int. J. Robust Nonlinear Control*, **30**, 17, 7670–7689 (2021)
- [6] A. Ilchmann and H. Logemann "High-gain adaptive stabilization of multivariable linear systems - revisited", *System and Control Letters*, **18**, 355–364 (1992)
- [7] A. Ilchmann "Non-identifier based adaptive control of dynamical systems: a survey", *IMA J. Math. Contr. & Informat.*, **8**, 321–366 (1991)
- [8] H. Khalil, *Nonlinear Systems. Third edition*. Prentice Hall, 2002.
- [9] K. Li, K. Zhao, X. Jia and R. Gao. (2023) Adaptive global prescribed performance tracking control of uncertain robotic systems with unknown control directions. *Int. J. Robust Nonlinear Control*, 335956–5974.
- [10] H. Logemann and A. Ilchmann "An adaptive servomechanism for a class of infinite-dimensional systems", *SIAM J. Contr. & Optimiz.*, **32**, 917–936 (1994)
- [11] Martensson B., "The unmixing problem", *IMA J. Math. Contr. & Informat.*, **8**, 367–377 (1991)
- [12] Martensson B., "Adaptive stabilization of multivariable nonlinear systems", *Contemporary Mathematics*, **68**, 191–225 (1987)
- [13] T. R. Oliveira, L. Hsu and J. Peixoto. "Output-feedback global tracking for unknown control direction plants with application to extremum-seeking control", *Automatica*, **47**, 9, 2029–2038 (2011)
- [14] T. R. Oliveira, J. Peixoto, and L. Hsu. "Sliding Mode Control of Uncertain Multivariable Nonlinear Systems With Unknown Control Direction via Switching and Monitoring Function", *IEEE Trans. Automatic Control*, **55**, 4, 1028–1034 (2010)
- [15] E.P. Ryan. "Adaptive stabilization of multi-input nonlinear systems", *Int. J. Robust Nonlinear Control*, **3**, 169–181 (1993)
- [16] E.P. Ryan. "A nonlinear universal servomechanism", *IEEE Trans. Automatic Control*, **39**, 753–761 (1994)
- [17] Zhu X.-J. "A finite spectrum unmixing set for $GL(3, \mathbb{R})$ ", in "Computation and Control III", Bower and Lund eds., Birkhäuser, Boston, p. 403–410, 1989.
- [18] Mac Lean K.J.M. A Geometric Analysis of the Platonic Solids and other Semi-Regular Polyhedra, Second Edition, Marv. Spirit Press; , 2019.

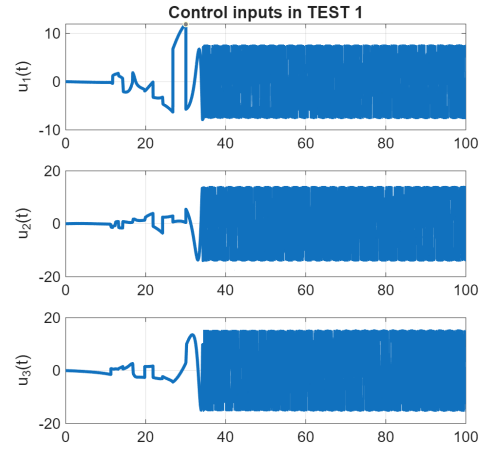


Fig. 4. Manipulable control components in TEST 1.

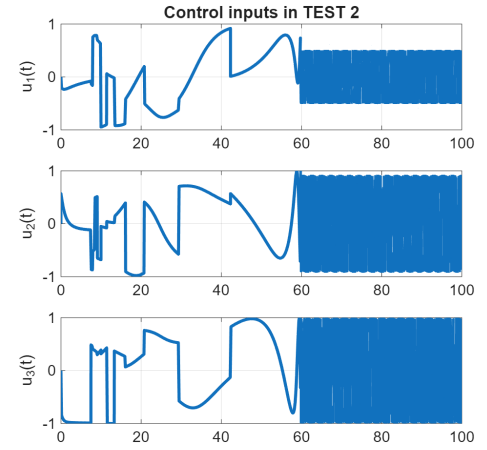


Fig. 5. Manipulable control components in TEST 2.