NUMERICAL METHODS FOR CAUCHY BISINGULAR INTEGRAL EQUATIONS OF THE FIRST KIND ON THE SQUARE

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Abstract. In this paper we investigate the numerical solution of Cauchy bisingular integral equations of the first kind on the square. We propose two different methods based on a global polynomial approximation of the unknown solution. The first one is a discrete collocation method applied to the original equation and then is a "direct"method. The second one is an "indirect"procedure of discrete collocation-type since we act on the so-called regularized Fredholm equation. In both cases, the convergence and the stability of the method is proved in suitable weighted spaces of functions, and the well conditioning of the linear system is showed. In order to illustrate the efficiency of the proposed procedures, some numerical tests are given.

Keywords: Cauchy bisingular integral equations and cubature method and collocation method and Lagrange interpolation.

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1. Introduction

Singular integral equations with Cauchy kernels arise in the mathematical modelling of several problems of the Applied Sciences like aerodynamics, elasticity, fluid flow problems and crack theory [1, 11, 30].

For the univariate case, a general theory on such type of equations is well developed and described in the monographs [9, 27, 28, 31] and several numerical methods have been extensively investigated [3, 5, 7, 12, 14, 15, 17, 18, 19, 23] in terms of stability, convergence, well-conditioning and accuracy of the results.

Concerning the multivariate case, the theoretical analysis of these equations is well studied in the books [20, 26] and several authors focus their research on bisingular integral equations arising from the 3D Helmholtz equations. An example is the following bivariate singular integral equation of the first kind which is strictly related to the stationary problem of a flow past a rectangular airfoil of large span [8]

$$\frac{1}{\pi^2} \oint_{-1}^{1} \oint_{-1}^{1} \frac{F(x,y)}{(x-t)(y-s)}\, dx\, dy = g(t,s),$$

where here and in the sequel the symbol $\oint$ means that the integral has to be interpreted in the Cauchy Principal Value sense.

However, even if these equations are of applicative nature, according to our knowledge, very few numerical methods are disposable in the literature [13, 16].

The principal aim of this paper is to investigate on the numerical treatment of the more general bisingular integral equation of the first kind defined on the square

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$S = [-1, 1] \times [-1, 1]$

$$\frac{1}{\pi^2} \oint_S \frac{F(x,y)}{(x-t)(y-s)}\, dx\, dy + \int_S k(x,y,t,s) F(x,y)\, dx\, dy = g(t,s), \tag{1.1}$$

where F is the bivariate unknown function and k and g are given functions defined on S^2 and S, respectively.

According to [8, 12], the solution of the above equation can be singular along two o more edges of the square S and the behavior of the singularities is known.

In this paper we consider the case when the solution turns to be unbounded at $x = y = -1$ and thus [8, 12] it can be expressed as

$$F(x,y) = f(x,y) \sqrt{\frac{1-x}{1+x}} \sqrt{\frac{1-y}{1+y}},$$

where f has to be determined. In a nutshell, the function F has a behaviour similar to that of the solution of the airfoil equation in the univariate case [31].

Hence, equation (1.1) can be rewritten as

$$(D + K)f = g, \tag{1.2}$$

where D is the dominant operator

$$Df(t,s) = \frac{1}{\pi^2} \oint_S \frac{f(x,y)}{(x-t)(y-s)} \sqrt{\frac{1-x}{1+x}} \sqrt{\frac{1-y}{1+y}}\, dx\, dy \tag{1.3}$$

and K is the perturbation operator

$$Kf(t,s) = \int_S k(x,y,t,s) f(x,y) \sqrt{\frac{1-x}{1+x}} \sqrt{\frac{1-y}{1+y}}\, dx\, dy. \tag{1.4}$$

In this paper, for the numerical treatment of (1.2), we propose two different approaches, both based on a global polynomial approximation of the unknown bivariate function f. The first one is a “direct”method since we act directly on the equation, while the second one is an “indirect”procedure, since we go to solve an equivalent regularized Fredholm equation.

In both cases, by using a suitable Lagrange interpolating operator, we project the considered equation into the subspace of polynomials and we discretize the integrals by using a suitable Gaussian cubature formula and by applying the fundamental invariance property of D on the orthogonal polynomials. Then, by collocation on suitable nodes, we end up with a linear system whose unknowns are the coefficients of the polynomial approximating the exact solution.

For both methods, we give a complete analysis in suitable weighted L^2 spaces. In details, we examine the stability, show the related convergence results and error estimates, and discuss the condition numbers of the systems we get.

Comparing the presented two procedures, they are equivalent in terms of convergence order and computational costs, at least when in the indirect approach we can compute exactly the involved integrals. Otherwise the indirect procedure is more expensive. Nevertheless the strategy of using the Fredholm equation equivalent to the Cauchy singular one, can be much easier extended to other functional spaces.

We underline that in order to achieve such theoretical analysis, we needed to prove some auxiliary results concerning the mapping properties of the involved integral operators and the bivariate Lagrange and Fourier operators. These auxiliary results are new and can also be used elsewhere.

The paper is structured into six sections. In Section 2, once the function spaces in which we address our investigation have been introduced, we give some preliminary results concerning the bivariate Fourier and Lagrange operators as well as the Gaussian cubature rule. In Section 3 we state the mapping properties of the integral operators D and K. Sections 4 and 5 are devoted to the two different methods we propose and whose numerical tests are showed in Section 6. At the end, in Section 7 we give the proofs of our results.

2. Preliminaries

2.1. **Function spaces.** Let $w(x,y)=w_1(x)w_2(y)$ the product of two Jacobi weights, $w_i(z)=(1-z)^{\alpha_i}(1+z)^{\beta_i}$, $\alpha_i,\beta_i>-1$, $i=1,2$, $z\in(-1,\,1)$.

We define the weighted Hilbert space $L^2_w(S)$ as the set of all weighted square integrable functions $f:S\to\mathbb{R}$ equipped with the inner product

$$\langle f,g\rangle_w=\int_S f(x,y)g(x,y)w(x,y)\,dx\,dy$$

and the norm

$$\|f\|_{L^2_w(S)}=\|f\sqrt{w}\|_2=\sqrt{\langle f,f\rangle_w}.$$

For brevity, from now on we set $\|f\|_{L^2_w}:=\|f\|_{L^2_w(S)}$.

For more regular functions and for a positive integer $r\geq 1$, let us also consider the following Sobolev-type subspace

$$W^r_w=\{f\in L^2_w(S):f^{(r-1)}\in AC((-1,1)^2),\ \|f\|_{W^r_w}=\|f\|_{L^2_w}+\mathcal{M}_r(f,w)<\infty\}$$

where the superscript $(r-1)$ denotes the $(r-1)$-th derivative with respect to each variable, $AC((-1,1)^2)$ stands for the set of all functions f which are absolutely continuous on every closed sub-domain of $(-1,\,1)^2$, and

$$\mathcal{M}_r(f,w)=\sup\left\{\left(\int_S\left|\frac{\partial^r f(x,y)}{\partial x^r}\varphi^r(x)\right|^2 w(x,y)dx\,dy\right)^{1/2},\right.$$
$$\left.\left(\int_S\left|\frac{\partial^r f(x,y)}{\partial y^r}\varphi^r(y)\right|^2 w(x,y)\,dx\,dy\right)^{1/2}\right\}$$

with $\varphi(z)=\sqrt{1-z^2}$.

2.2. **Bivariate Fourier and Lagrange operators.** Let $\{p_m(w_i)\}_{m=0}^\infty$ be the sequence of the orthonormal polynomials with positive leading coefficients, w.r.t. the weight w_i, $i=1,2$, i.e.

$$p_m(w_i,z)=\gamma_m(w_i)z^m+\ \text{terms of lower degree},\quad \gamma_m(w_i)>0.$$

For a function $f\in L^2_w(S)$, we define the bivariate Fourier sum as

$$S_{m,m}(f,w,x,y)=\sum_{i=0}^{m-1}\sum_{j=0}^{m-1}c_{ij}(f,w)p_i(w_1,x)p_j(w_2,y) \tag{2.1}$$

where

$$c_{ij}(f,w)=\int_S f(x,y)p_i(w_1,x)p_j(w_2,y)w(x,y)dx\,dy \tag{2.2}$$

are the Fourier coefficients.

The next two propositions show the behaviour of $S_{m,m}$ in the case when $f\in L^2_w(S)$ or $f\in W^r_w$.

To this end, let us define the error of best polynomial approximation in $L^2_w(S)$ as

$$E_{m,m}(f)_w=\inf_{P\in\mathbb{P}_{m,m}}\|f-P\|_{L^2_w}$$

where $\mathbb{P}_{m,m}$ denotes the set of all algebraic polynomials of two variables of degree at most m in each variable.

Moreover, in the sequel $\mathcal{C}$ denotes a positive constant which may have different values in different formulas. We will write $\mathcal{C}(a,b,\dots)$ to say that $\mathcal{C}$ depends only on the parameters $a,b,\dots$ and $\mathcal{C}\neq\mathcal{C}(a,b,\dots)$ to say that $\mathcal{C}$ is independent of the parameters $a,b,\dots$.

Proposition 2.1. *Let $f\in L^2_w(S)$. Then*

$$E^2_{m,m}(f)_w=\|f-\mathcal{S}_{m,m}(f,w)\|^2_{L^2_w}=\|f\|^2_{L^2_w}-\sum_{i=0}^{m-1}\sum_{j=0}^{m-1}c^2_{ij}(f,w). \tag{2.3}$$

Thus, according to the previous result, as in the univariate case (see [25] and the reference therein), $\mathcal{S}_{m,m}$ is the best polynomial approximation of $f\in L^2_w(S)$ and, since the Weierstrass Theorem holds true, by (2.3) we get the Parseval identity

$$\|f\|_{L^2_w}=\sqrt{\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}c^2_{ij}(f,w)} \tag{2.4}$$

and

$$E_{m,m}(f)_w=\sqrt{\sum_{i\geq m}\sum_{j\geq m}c^2_{ij}(f,w)}.$$

Proposition 2.2. *Let $f\in W^r_w$ and r_1 and r be two positive integers such that $r_1\leq r$. Then there exists a positive constant $\mathcal{C}\neq\mathcal{C}(m,f)$ such that the following estimate holds true*

$$\|f-\mathcal{S}_{m,m}(f,w)\|_{W^{r_1}_w}\leq\frac{\mathcal{C}}{m^{r-r_1}}\|f\|_{W^r_w}.$$

Now, for a function $f\in C((-1,1)^2)$ (i.e. f is continuous on every closed subset of the open square $(-1,1)^2$), let us consider the bivariate Lagrange polynomial

$$\mathcal{L}_{m,m}(f,w,x,y)=\sum_{i=1}^{m}\sum_{j=1}^{m}\ell_i(w_1,x)\ell_j(w_2,y)f(z_i,y_j), \tag{2.5}$$

where $\{z_i\}_{i=1}^m$ and $\{y_j\}_{j=1}^m$ are the zeros of $p_m(w_1)$ and $p_m(w_2)$, respectively and $\ell_i(w_1,x)$ and $\ell_j(w_2,y)$ denote the i-th and j-th fundamental Lagrange polynomial, respectively defined as

$$\ell_i(w_1,x)=\frac{p_m(w_1,x)}{p'_m(w_1,z_i)(x-z_i)},\quad \ell_i(w_2,y)=\frac{p_m(w_2,y)}{p'_m(w_2,y_j)(y-y_j)}. \tag{2.6}$$

Hence $\mathcal{L}_{m,m}$ is a polynomial of degree $m-1$ in each variable and by its definition it follows that $\mathcal{L}_{m,m}(P,w,x,y)=P(x,y)$ if $P\in\mathbb{P}_{m-1,m-1}$.

Next proposition shows the weighted-L^2 convergence of the Lagrange interpolating polynomial for every $f\in W^r_w$.

Proposition 2.3. *Let $f\in W^r_w$. Then there exists a positive constant $\mathcal{C}\neq\mathcal{C}(m,f)$ such that the following estimate holds true*

$$\|f-\mathcal{L}_{m,m}(f,w)\|_{L^2_w}\leq\frac{\mathcal{C}}{m^r}\|f\|_{W^r_w}. \tag{2.7}$$

2.3. **Gaussian cubature rule.** Let us now introduce the tensor-product Gauss rule which will be essential for our aims. The Gaussian cubature rule reads as [29]

$$\int_S f(x,y)w(x,y)dx\,dy=\sum_{i=1}^{m}\sum_{j=1}^{m}\lambda_i(w_1)\lambda_j(w_2)f(z_i,y_j)+\mathcal{R}_{m,m}(f) \tag{2.8}$$

where $\{z_i\}_{i=1}^m$ and $\{y_j\}_{j=1}^m$ are the zeros of $p_m(w_1)$ and $p_m(w_2)$, respectively, $\lambda_i(w_1)$, $\lambda_j(w_2)$ are the corresponding Christoffel numbers and $\mathcal{R}_{m,m}(f)$ denotes the remainder term. We point out that $\mathcal{R}_{m,m}(f)=0$ if $f\equiv P\in\mathbb{P}_{2m-1,2m-1}$.

3. Mapping properties of the operator D and K

In this section we investigate on the mapping properties of the operators D and K involved in equation (1.2). To this end, let v be the product of two fourth kind Chebyshev weight functions, i.e.

$$v(x,y) = u(x)u(y), \quad \text{with} \quad u(z) = \sqrt{\frac{1-z}{1+z}}. \tag{3.1}$$

According to the above notation, we rewrite the operator D introduced in (1.3) as

$$Df(t,s) = \frac{1}{\pi^2} \oint_S \frac{f(x,y)}{(x-t)(y-s)}\, v(x,y)dx\, dy. \tag{3.2}$$

By using standard arguments, it is not hard to prove that the adjoint operator of D has the following form

$$\widehat{D}f(t,s) = \frac{1}{\pi^2} \oint_S \frac{f(x,y)}{(x-t)(y-s)} v^{-1}(x,y)\, dx\, dy. \tag{3.3}$$

Now we recall the explicit expression for $p_m(u)$ and $p_m(u^{-1})$ (the fourth and third kind Chebyshev orthonormal polynomials with respect to the weights u and u^{-1}, respectively), namely [10, 21]

$$p_m(u,z) = \frac{\sin\left(\left(m+\frac{1}{2}\right)\theta\right)}{\sin\left(\frac{1}{2}\theta\right)}, \qquad z = \cos\theta, \quad 0 \le \theta \le \pi, \tag{3.4}$$

and

$$p_m(u^{-1},z) = \frac{\cos\left(\left(m+\frac{1}{2}\right)\theta\right)}{\cos\left(\frac{1}{2}\theta\right)}, \qquad z = \cos\theta, \quad 0 \le \theta \le \pi. \tag{3.5}$$

Next results state useful properties of the operators D, $\widehat{D}$ and K which are basic for our methods.

Lemma 3.1. *Let u be the weight introduced in* (3.1)*, $q_m(t,s) = p_m(u,t)p_m(u,s)$ and $r_m(t,s) = p_m(u^{-1},t)p_m(u^{-1},s)$. Then,*

$$Dq_m(t,s) = r_m(t,s) \tag{3.6}$$

and

$$\widehat{D}r_m(t,s) = q_m(t,s). \tag{3.7}$$

Proposition 3.2. *Let D and $\widehat{D}$ be the operators defined in* (3.2) *and* (3.3)*, respectively. Then*

$$D : W^r_v \to W^r_{v^{-1}} \tag{3.8}$$

is continuous and invertible and its two-sided inverse is the continuous operator

$$\widehat{D} : W^r_{v^{-1}} \to W^r_v. \tag{3.9}$$

From now on we denote by $k_{(x,y)}$ and $k_{(t,s)}$ the kernel function $k(x,y,t,s)$ in (1.4) as a function of the only variables (t,s) and (x,y), respectively.

Proposition 3.3. *Let K be defined in* (1.4) *and let us assume that the kernel function k satisfies the following conditions*

$$\sup_{(t,s)\in S} \|k_{(t,s)}\|_{W^r_v} < \infty, \qquad \sup_{(x,y)\in S} \|k_{(x,y)}\|_{W^{r_1}_{v^{-1}}} < \infty, \tag{3.10}$$

for some positive integers numbers r and r_1. Then the perturbation operator $K : L^2_v(S) \to W^{r_1}_{v^{-1}}$ is linear and bounded if $r_1 \le r$. Moreover, K is a compact operator for all $r_1 < r$.

Let us remark that, in virtue of Proposition 3.2 and 3.3, we can claim that under the assumptions (3.10) and if the null space Ker$\{D+K\}$ is trivial in $L^2_v(S)$, then the operator

$$D+K : W^{r_1}_v \to W^{r_1}_{v^{-1}}$$

is an invertible linear bounded operator for all $0 \le r_1 < r$. Hence, equation (1.2) has a unique solution $f \in W^{r_1}_v$, for each given right-hand side $g \in W^{r_1}_{v^{-1}}$.

4. A direct numerical method

The aim of this section is to present a "direct" numerical approach for the solution of equation (1.2). Inspired by the discrete collocation method proposed for the univariate case [18, 23], we first approximate operator K by means of

$$K_m f(t,s) = \int_S \mathcal{L}_{m,m}\left(k_{(t,s)}, v, x, y\right) f(x,y)\, v(x,y)\, dx\, dy. \tag{4.1}$$

Hence we project equation (1.2) with K_m instead of K by means of the interpolating operator $\mathcal{L}_{m,m}(v^{-1})$ and we search for a polynomial solution $f_m \in \mathbb{P}_{m-1,m-1}$, i.e. we solve the finite dimensional equation

$$\mathcal{L}_{m,m}\left((D+K_m)f_m, v^{-1}, t, s\right) = \mathcal{L}_{m,m}(g, v^{-1}, t, s),$$

namely

$$\mathcal{L}_{m,m}\left((D+K_m)f_m - g, v^{-1}, t, s\right) = 0. \tag{4.2}$$

Equation (4.2) is equivalent in the weighted space $L^2_{v^{-1}}$ to

$$\|\mathcal{L}_{m,m}\left((D+K_m)f_m - g, v^{-1}\right)\|_{L^2_{v^{-1}}} = 0$$

that is

$$\int_S \left|\mathcal{L}_{m,m}\left((D+K_m)f_m - g, v^{-1}, t, s\right)\right|^2 v^{-1}(t,s)\, dt\, ds = 0.$$

Thus, by approximating the integral by means of the Gaussian cubature rule (2.8) with $w_i = u^{-1}$, $i = 1, 2$, that in this case turns out to be exact, we have

$$\sum_{i=1}^{m}\sum_{j=1}^{m} \lambda_i(u^{-1})\lambda_j(u^{-1}) \left|\mathcal{L}_{m,m}\left((D+K_m)f_m - g, v^{-1}, t_i, t_j\right)\right|^2 = 0 \tag{4.3}$$

where [10, 21]

$$t_i = \cos\left(\frac{\left(m-i+\frac{1}{2}\right)\pi}{m+\frac{1}{2}}\right), \qquad i = 1, \ldots, m$$

are the nodes of the m-th third kind Chebyshev polynomial $p_m(u^{-1})$ defined in (3.5) and

$$\lambda_i(u^{-1}) = \frac{\pi}{m+\frac{1}{2}}(1+t_i), \qquad i = 1, \ldots, m$$

are the corresponding Christoffel numbers.

From (4.3) we deduce

$$\begin{aligned}\sqrt{\lambda_i(u^{-1})\lambda_j(u^{-1})}\,[Df_m(t_i,t_j) \quad &+ \quad K_m f_m(t_i,t_j)] \\ &= \quad \sqrt{\lambda_i(u^{-1})\lambda_j(u^{-1})}\, g(t_i,t_j), \quad i,j = 1,\ldots,m.\end{aligned} \tag{4.4}$$

Now we develop the terms $Df_m(t_i,t_j)$ and $K_m f_m(t_i,t_j)$ involved in the previous equations, in order to construct the approximated polynomial solution f_m in the form

$$f_m(t,s) = \mathcal{L}_{m,m}(f_m, v, t, s). \tag{4.5}$$

About the second term $K_m f_m(t_i, t_j)$, by using again the cubature formula (2.8) now with $w_i = u$, $i = 1, 2$, which is once again exact , we have

$$K_m f_m(t_i, t_j) = \sum_{h=1}^{m} \sum_{k=1}^{m} \lambda_h(u)\lambda_k(u) k(x_h, x_k, t_i, t_j) f_m(x_h, x_k), \quad i, j = 1, \ldots, m, \tag{4.6}$$

where [10, 21]

$$x_h = \cos\left(\frac{(m - h + 1)\pi}{m + \frac{1}{2}}\right), \qquad h = 1, \ldots, m \tag{4.7}$$

are the nodes of the m-th fourth kind Chebyshev polynomial $p_m(u)$ defined in (3.4) and

$$\lambda_h(u) = \frac{\pi}{m + \frac{1}{2}}(1 - x_h), \qquad h = 1, \ldots, m \tag{4.8}$$

are the corresponding Christoffel numbers.

Concerning to the first term $Df_m(t_i, t_j)$, we have the following proposition.

Proposition 4.1. *Let f_m be the polynomial defined in* (4.5) *and let $\{t_i\}_{i=1}^m$ and $\{x_h\}_{h=1}^m$ be the zeros of $p_m(u^{-1})$ and $p_m(u)$, respectively. Then,*

$$Df_m(t_i, t_j) = \frac{1}{\pi^2} \sum_{h=1}^{m} \sum_{k=1}^{m} \lambda_h(u)\lambda_k(u) \frac{f_m(x_h, x_k)}{(x_h - t_i)(x_k - t_j)} \tag{4.9}$$

for $i, j = 1, ..., m$.

Hence, by replacing (4.9) and (4.6) in (4.4), we get

$$\begin{aligned} \sqrt{\lambda_i(u^{-1})\lambda_j(u^{-1})} \sum_{h=1}^{m} \sum_{k=1}^{m} \sqrt{\lambda_h(u)\lambda_k(u)} \left[\frac{\pi^{-2}}{(x_h - t_i)(x_k - t_j)} + k(x_h, x_k, t_i, t_j)\right] a_{hk} \\ = \sqrt{\lambda_i(u^{-1})\lambda_j(u^{-1})}\, g(t_i, t_j), \qquad i, j = 1, \ldots, m, \end{aligned} \tag{4.10}$$

where we set $a_{hk} = \sqrt{\lambda_h(u)\lambda_k(u)}\, f_m(x_h, x_k)$.

This is a linear system of m^2 equations in the m^2 unknown a_{hk} that, once solved, allow us to approximate the solution we are looking for

$$f_m(t, s) = \sum_{h=1}^{m} \sum_{k=1}^{m} \frac{\ell_h(u, t)}{\sqrt{\lambda_h(u)}} \frac{\ell_k(u, s)}{\sqrt{\lambda_k(u)}} a_{hk}. \tag{4.11}$$

Let us remark that system (4.10) is well-defined, since $\min |x_h - t_i| = \mathcal{O}(1/m)$, $h, i = 1, \ldots, m$, [23] , and that it can be rewritten in a matrix form as

$$\mathbf{P}_m (\mathbf{D}_m + \mathbf{K}_m) \mathbf{P}_m \mathbf{a} = \mathbf{P}_m (\mathbf{g}\mathbf{P}_m)^T. \tag{4.12}$$

Here $\mathbf{P}_m$ is a m-blocks matrix in which each block is given by

$$\mathbf{P} = diag\left(\sqrt{\lambda_1(u^{-1})}, \cdots, \sqrt{\lambda_m(u^{-1})}\right),$$

the matrices $\mathbf{D}_m$ and $\mathbf{K}_m$ are the m-blocks matrix defined as

$$\mathbf{D}_m = \begin{pmatrix} \mathbf{D}^{(1,1)} & \mathbf{D}^{(1,2)} & \ldots & \mathbf{D}^{(1,m)} \\ \mathbf{D}^{(2,1)} & \mathbf{D}^{(2,2)} & \ldots & \mathbf{D}^{(2,m)} \\ \ldots & \ldots & \ldots & \\ \mathbf{D}^{(m,1)} & \mathbf{D}^{(m,2)} & \ldots & \mathbf{D}^{(m,m)} \end{pmatrix}, \quad \mathbf{K}_m = \begin{pmatrix} \mathbf{K}^{(1,1)} & \mathbf{K}^{(1,2)} & \ldots & \mathbf{K}^{(1,m)} \\ \mathbf{K}^{(2,1)} & \mathbf{K}^{(2,2)} & \ldots & \mathbf{K}^{(2,m)} \\ \ldots & \ldots & \ldots & \\ \mathbf{K}^{(m,1)} & \mathbf{K}^{(m,2)} & \ldots & \mathbf{K}^{(m,m)} \end{pmatrix}$$

with

$$\mathbf{D}^{(h,k)} = \left[\mathbf{D}^{(h,k)}\right]_{i,j=1}^{m} = \sqrt{\lambda_h(u)\lambda_k(u)}\frac{\pi^{-2}}{(x_h - t_i)(x_k - t_j)},$$

$$\mathbf{K}^{(h,k)} = \left[\mathbf{K}^{(h,k)}\right]_{i,j=1}^{m} = \sqrt{\lambda_h(u)\lambda_k(u)}k(x_h, x_k, t_i, t_j),$$

and $\mathbf{a} \in \mathbb{R}^{m^2}$ and $\mathbf{g} \in \mathbb{R}^{m^2}$ are the arrays of the unknown function and the right-hand side which have been obtained by reordering column by column the matrices $\mathbf{G}$ and $\mathbf{A}$, respectively defined as

$$\mathbf{G} = [g_{ij}]_{i,j=1}^{m} = g(t_i, t_j) \in \mathbb{R}^{m\times m}, \quad \mathbf{A} = [a_{hk}]_{h,k=1}^{m} = f_m(x_h, x_k) \in \mathbb{R}^{m\times m}$$

namely,

$$\mathbf{g}_{(j-1)m+i} = g_{ij}, \quad \mathbf{a}_{(k-1)m+h} = a_{hk}.$$

Next proposition, concerning with the operator introduced in (4.1), is essential for the analysis of the method.

Proposition 4.2. *Under the assumptions* (3.10)*, the estimate*

$$\|Kf - \mathcal{L}_{m,m}(K_m f, v^{-1})\|_{L^2_{v^{-1}}} \leq \frac{\mathcal{C}}{m^{r_1}}\|f\|_{L^2_v}$$

holds true with $\mathcal{C} \neq \mathcal{C}(m, f)$.

Next theorem assures that the proposed discrete collocation method is stable and convergent. It also states that, in the case when the right-hand side g belongs to a certain class of functions, namely the Sobolev-type space $W^{r_1}_{v^{-1}}$, then the solution f of (1.2) belongs to $W^{r_1}_v$. Moreover the theorem gives an estimate of the error of the approximate solution. Finally it shows that the condition number in the spectral norm of system (4.12)

$$cond(\mathbf{P}_m(\mathbf{D}_m + \mathbf{K}_m)\mathbf{P}_m) = \|\mathbf{P}_m(\mathbf{D}_m + \mathbf{K}_m)\mathbf{P}_m\| \, \|(\mathbf{P}_m(\mathbf{D}_m + \mathbf{K}_m)\mathbf{P}_m)^{-1}\|$$

is independent of the dimension of the matrix and uniformly bounded by the condition number of the operator $D + K$.

Theorem 4.3. *Assume that equation* (1.2) *has a unique solution* $f \in L^2_v$ *and the kernel function* k *satisfies* (3.10)*. Then, for sufficiently large* m*, say* $m \geq m_0$*, the system of equations* (4.12) *has a unique solution* f_m*. Moreover if the right-hand side* $g \in W^{r_1}_{v^{-1}}$ *then the solution* $f \in W^{r_1}_v$ *and the following estimate holds true*

$$\|f - f_m\|_{L^2_v} \leq \frac{\mathcal{C}}{m^{r_1}}\|f\|_{W^{r_1}_v} \tag{4.13}$$

with $\mathcal{C} \neq \mathcal{C}(m, f, g)$*. Furthermore,*

$$\limsup_m cond(\mathbf{P}_m(\mathbf{D}_m + \mathbf{K}_m)\mathbf{P}_m) \leq \mathcal{C}\, cond(D + K), \tag{4.14}$$

where here $\mathcal{C} \neq \mathcal{C}(m)$.

5. An indirect numerical method

In this section we propose an alternative numerical method still based on a polynomial approximation of the unknown solution written in the form

$$f_m(t, s) = \mathcal{L}_{m,m}(f_m, v, t, s), \qquad f_m \in \mathbb{P}_{m-1,m-1}.$$

The method takes advantages of the smoothness properties of the operators D and K stated in Section 3. In fact, thanks to the compactness of K and the invertibility of D, following [7], we can move from the equation (1.2) into the equivalent regularized Fredholm equation

$$(I + \widehat{D}K)f = \widehat{D}g, \tag{5.1}$$

where I is the identity operator in L^2_v.

Then, if we assume that the null space $\mathrm{Ker}\{I + \widehat{D}K\}$ is trivial, by applying the Fredholm Alternative Theorem, equation (5.1) has a unique solution for each given right hand side $\widehat{D}g \in L^2_v$.

For our convenience, let us rewrite (5.1) as

$$(I + \mathcal{K})f = \mathcal{G}, \tag{5.2}$$

where $\mathcal{G} = \widehat{D}g$ and $\mathcal{K} = \widehat{D}K$ i.e.

$$\mathcal{K}f(t,s) = \int_S \phi(\xi,\eta,t,s) f(\xi,\eta) v(\xi,\eta)\, d\xi\, d\eta,$$

with

$$\phi(\xi,\eta,t,s) = \widehat{D}k_{(\xi,\eta)}(t,s). \tag{5.3}$$

In order to approximate the solution of (5.2), let us project the equation on the finite dimensional space $\mathbb{P}_{m-1,m-1}$ by means of the interpolating operator $\mathcal{L}_{m,m}(v)$ and then let us consider the following finite dimensional equation

$$\mathcal{L}_{m,m}\left((I + \mathcal{K}_m) f_m, v, t, s\right) = \mathcal{L}_{m,m}\left(\mathcal{G}, v, t, s\right), \tag{5.4}$$

where

$$\mathcal{K}_m f(t,s) = \int_S \mathcal{L}_{m,m}\left(\phi(t,s), v, \xi, \eta\right) f(\xi,\eta) v(\xi,\eta)\, d\xi\, d\eta.$$

Equation (5.4), considered in L^2_v, means that

$$\int_S \left|\mathcal{L}_{m,m}\left((I + \mathcal{K}_m) f_m - \mathcal{G}, v, t, s\right)\right|^2 v(t,s)\, dt\, ds = 0$$

that is, for $i, j = 1, \dots, m$,

$$\sqrt{\lambda_i(u)\lambda_j(u)}\left[f_m(x_i,x_j) + \mathcal{K}_m f_m(x_i,x_j)\right] = \sqrt{\lambda_i(u)\lambda_j(u)}\mathcal{G}(x_i,x_j),$$

where x_i and $\lambda_i(u)$ were introduced in (4.7) and (4.8), respectively. Hence by approximating the operator $\mathcal{K}_m$ by means of the Gaussian cubature rule (2.8) we get the following linear system

$$\begin{aligned} \sqrt{\lambda_i(u)\lambda_j(u)} \sum_{h=1}^{m}\sum_{k=1}^{m} \left[\delta_{hk}^{ij} + \sqrt{\lambda_h(u)\,\lambda_k(u)}\,\phi(x_h,x_k,x_i,x_j)\right] a_{hk} \\ = \sqrt{\lambda_i(u)\lambda_j(u)}\mathcal{G}(x_i,x_j), \quad i,j = 1,\dots,m, \end{aligned} \tag{5.5}$$

where $a_{hk} = \sqrt{\lambda_h(u)\,\lambda_k(u)} f_m(x_h,x_k)$ and $\delta_{hk}^{ij} = \begin{cases} 1, & i = h \quad \text{and} \quad j = k \\ 0, & \textit{otherwise} \end{cases}$.

Once solved (5.5), the solution allows us to compute the approximate solution

$$f_m(t,s) = \sum_{hi=1}^{m}\sum_{k=1}^{m} \frac{\ell_h u, t)}{\sqrt{\lambda_k(u)}} \frac{\ell_h(u,s)}{\sqrt{\lambda_k(u)}} a_{hk}. \tag{5.6}$$

Note that the polynomial solution f_m just defined has the same expression of the solution f_m given in (4.11), obtained applying the method described in Section 4.

Let us also remark that in order to implement system (5.5) we need to evaluate the integrals

$$\phi(\xi,\eta,t,s) = \frac{1}{\pi^2} \oint_S \frac{k(x,y,\xi,\eta)}{(x-t)(y-s)} v^{-1}(x,y)\, dx\, dy$$

$$\mathcal{G}(t,s) = \frac{1}{\pi^2} \oint_S \frac{g(x,y)}{(x-t)(y-s)} v^{-1}(x,y)\, dx\, dy$$

whose analytical expressions are not always known. Then, in the case when we do not have such expressions, we propose to approximate the known involved functions k and g with

$$k(x,y,\xi,\eta) \simeq \mathcal{L}_{m,m}\left(k_{(\xi,\eta)},v,x,y\right), \qquad g(x,y) \simeq \mathcal{L}_{m,m}\left(g,v,x,y\right)$$

and then by proceeding as in the proof of Proposition 4.1, in virtue of Lemma 3.1, we end up to approximate $\phi(\xi,\eta,t,s)$ and $\mathcal{G}(t,s)$ with

$$\phi_m(x_h,x_k,x_i,x_j) = \frac{1}{\pi^2}\sum_{\iota=1}^{m}\sum_{\zeta=1}^{m}\lambda_\iota(u^{-1})\lambda_\zeta(u^{-1})\frac{k(t_\iota,t_\zeta,x_h,x_k)}{(t_\iota-x_i)(t_\zeta-x_j)},$$

and

$$\mathcal{G}_m(x_i,x_j) = \frac{1}{\pi^2}\sum_{\iota=1}^{m}\sum_{\zeta=1}^{m}\lambda_\iota(u^{-1})\lambda_\zeta(u^{-1})\frac{g(t_\iota,t_\zeta)}{(t_\iota-x_i)(t_\zeta-x_j)}.$$

Let us now rewrite (5.5) in a matrix form as

$$\boldsymbol{\mathcal{P}}_m\left(\mathbf{I}_m+\boldsymbol{\mathcal{K}}_m\right)\boldsymbol{\mathcal{P}}_m\mathbf{a} = \boldsymbol{\mathcal{P}}_m\ \left(\mathfrak{g}\,\boldsymbol{\mathcal{P}}_m\right)^T, \tag{5.7}$$

where $\boldsymbol{\mathcal{P}}_m$ is a m-blocks matrix in which each block is given by

$$\boldsymbol{\mathcal{P}} = diag\left(\sqrt{\lambda_1(u)},\ \cdots,\ \sqrt{\lambda_m(u)}\right),$$

the matrices $\mathbf{I}_m$ and $\mathbf{K}_m$ are the m-blocks matrix defined as

$$\mathbf{I}_m = \begin{pmatrix} \mathbf{I} & 0 & \ldots 0\\ 0 & \mathbf{I} & \ldots 0\\ \ldots & \ldots & \ldots\\ 0 & 0 & \ldots\mathbf{I}\end{pmatrix}, \quad \boldsymbol{\mathcal{K}}_m = \begin{pmatrix} \boldsymbol{\mathcal{K}}^{(1,1)} & \boldsymbol{\mathcal{K}}^{(1,2)} & \ldots\boldsymbol{\mathcal{K}}^{(1,m)}\\ \boldsymbol{\mathcal{K}}^{(2,1)} & \boldsymbol{\mathcal{K}}^{(2,2)} & \ldots\boldsymbol{\mathcal{K}}^{(2,m)}\\ \ldots & \ldots & \ldots\\ \boldsymbol{\mathcal{K}}^{(m,1)} & \boldsymbol{\mathcal{K}}^{(m,2)} & \ldots\boldsymbol{\mathcal{K}}^{(m,m)}\end{pmatrix},$$

where $\mathbf{I}$ denotes the identity matrix of order m,

$$\boldsymbol{\mathcal{K}}^{(h,k)} = \left[\boldsymbol{\mathcal{K}}^{(h,k)}\right]_{i,j=1}^{m} = \sqrt{\lambda_h(u)\lambda_k(u)}\phi(x_h,x_k,x_i,x_j)$$

and $\mathbf{a}\in\mathbb{R}^{m^2}$ and $\mathfrak{g}\in\mathbb{R}^{m^2}$ are the arrays of the unknown function and the right-hand side which have been obtained by reordering column by column the matrices $\boldsymbol{\mathcal{G}}$ and $\mathbf{A}$ respectively

$$\boldsymbol{\mathcal{G}} = [\mathcal{G}_{ij}]_{i,j=1}^{m} = \mathcal{G}(x_i,x_j)\in\mathbb{R}^{m\times m}, \quad \mathbf{A} = [a_{hk}]_{h,k=1}^{m} = f_m(x_h,x_k)\mathbb{R}^{m\times m}$$

namely,

$$\mathfrak{g}_{(j-1)m+i} = \mathcal{G}_{ij}, \quad \mathbf{a}_{(k-1)m+h} = a_{hk}.$$

Next proposition is essential for the stability and the convergence of the described method stated in Theorem 5.2.

Proposition 5.1. *Assume that kernel k satisfies the conditions* (3.10). *Then*

$$\|\mathcal{K}f - \mathcal{L}_{m,m}(\mathcal{K}_m f,v)\|_{L^2_v} \le \frac{\mathcal{C}}{m^r}\|f\|_{L^2_v}$$

where $\mathcal{C}\neq\mathcal{C}(m)$.

Theorem 5.2. *Assume that $Ker\{I+\widehat{D}K\}=\{0\}$, the assumptions of Proposition 5.1 are satisfied and the function g belongs to $W^{r_1}_{v^{-1}}$. Then, for sufficiently large m, say $m\ge m_0$, system* (5.7) *has a unique solution f_m and the following estimate holds true*

$$\|f-f_m\|_{L^2_v} \le \frac{\mathcal{C}}{m^{r_1}}\|f\|_{W^{r_1}_v} \tag{5.8}$$

with $\mathcal{C}\neq\mathcal{C}(m,f,g)$. Moreover

$$\limsup_m cond(\boldsymbol{\mathcal{P}}_m(\mathbf{I}_m+\boldsymbol{\mathcal{K}}_m)\boldsymbol{\mathcal{P}}_m) \le \mathcal{C}\,cond(I+\mathcal{K}),$$

Table 1. Numerical results of Example 1 via the direct method.

m	$\epsilon_{64,m}(0.5, 0.8)$	$\epsilon_{64,m}(0.1, -0.5)$	$\epsilon_{64,m}(-0.6, 0.7)$	$cond(\mathbf{P}_m(\mathbf{D}_m + \mathbf{K}_m)\mathbf{P}_m)$
4	2.89e-03	1.27e-03	6.73e-03	1.3931498886229416e+01
8	1.19e-08	1.24e-07	2.53e-08	1.3931550518318879e+01
16	6.24e-15	4.88e-15	4.08e-15	1.3931550518335689e+01
32	8.73e-16	5.75e-15	4.80e-16	1.3931550518335690e+01

Table 2. Numerical results of Example 1 via the indirect method.

m	$\epsilon_{64,m}(0.5, 0.8)$	$\epsilon_{64,m}(0.1, -0.5)$	$\epsilon_{64,m}(-0.6, 0.7)$	$cond(\boldsymbol{\mathcal{P}}_m(\mathbf{I}_m + \boldsymbol{\mathcal{K}}_m)\boldsymbol{\mathcal{P}}_m)$
4	2.89e-03	1.27e-03	6.73e-03	1.3931498886229420e+01
8	1.19e-08	1.24e-07	2.53e-08	1.3931550518318886e+01
16	1.25e-16	3.50e-15	3.12e-15	1.3931550518335696e+01
32	2.87e-15	2.38e-15	3.72e-15	1.3931550518335680e+01

where $\mathcal{C} \neq \mathcal{C}(m)$.

6. Numerical Tests

In this section, by means of some numerical tests, we show the performance of the methods described in the previous sections. In each example, for the direct method, we solve system (4.10) and compute the approximate solution f_m given in (4.11). For the indirect method through the unique solution of system (5.5) we compute f_m defined in (5.6).

Since the exact solutions of the equations we will consider are unknown, we assume as exact those obtained for a fixed value of $m = M$ that we will specify in each test and we compute the relative errors

$$\epsilon_{M,m}(t, s) = \frac{|f_M(t, s) - f_m(t, s)|}{|f_M(t, s)|}$$

in different points $(t, s) \in S$.

Example 1. Let us consider the equation

$$\frac{1}{\pi^2} \oint_S \frac{f(x, y)}{(x - t)(y - s)}\, v(x, y)dxdy + \int_S \log\,(4 + sx + ty)f(x, y)v(x, y)dxdy = e^{ts}.$$

In Tables 1 and 2 we report, for increasing value of m, the relative errors we get in three different points of the square and the condition number in the spectral norm of the systems we solve. As we can see the convergence is very fast in virtue of the smoothness properties of the kernel and right-hand side. Moreover, the sequence $\{cond(\mathbf{P}_m(\mathbf{D}_m + \mathbf{K}_m)\mathbf{P}_m)\}_m$ as well as $\{cond(\boldsymbol{\mathcal{P}}_m(\mathbf{I}_m + \boldsymbol{\mathcal{K}}_m)\boldsymbol{\mathcal{P}}_m)\}_m$ is convergent as m goes to infinity.

Example 2. Let us apply our methods to the following equation

$$\frac{1}{\pi^2} \oint_S \frac{f(x, y)}{(x - t)(y - s)}\, v(x, y)dx\, dy + \int_S \frac{xt}{5 + y^2 + s^2} f(x, y)v(x, y)dxdy = \log(10 - s - t).$$

Table 3 and 4 show the numerical results we get. As in the previous example, in virtue of the presence of a kernel and a right-hand side very smooth, by solving a system with $m = 32$, we get very accurate results.

Table 3. Numerical results of Example 2 via the direct method.

m	$\epsilon_{64,m}(0.7,0.2)$	$\epsilon_{64,m}(0.1,-0.5)$	$\epsilon_{64,m}(-0.6,0.7)$	$cond(\mathbf{P}_m(\mathbf{D}_m+\mathbf{K}_m)\mathbf{P}_m)$
4	2.14e-04	1.63e-06	4.76e-04	2.2455715459596859e+00
8	2.57e-06	1.62e-06	9.27e-07	2.2455977174082378e+00
16	6.13e-12	4.14e-16	3.58e-12	2.2455977175654063e+00
32	3.45e-16	0.00e+00	3.20e-16	2.2455977175654054e+00

Table 4. Numerical results of Example 2 via the indirect method.

m	$\epsilon_{64,m}(0.7,0.2)$	$\epsilon_{64,m}(0.1,-0.5)$	$\epsilon_{64,m}(-0.6,0.7)$	$cond(\boldsymbol{\mathcal{P}}_m(\mathbf{I}_m+\boldsymbol{\mathcal{K}}_m)\boldsymbol{\mathcal{P}}_m)$
4	2.14e-04	1.63e-06	4.76e-04	2.2455715459596877e+00
8	2.57e-06	1.62e-06	9.27e-07	2.2455977174082391e+00
16	6.13e-12	3.73e-15	3.58e-12	2.2455977175654072e+00
32	5.17e-15	1.24e-15	7.99e-16	2.2455977175654058e+00

Table 5. Numerical results for Example 3 via the direct method.

m	$\epsilon_{64,m}(0.1,-0.4)$	$\epsilon_{64,m}(0.3,-0.6)$	$\epsilon_{64,m}(-0.1,0.5)$	$cond(\mathbf{P}_m(\mathbf{D}_m+\mathbf{K}_m)\mathbf{P}_m)$
4	5.71e-04	1.33e-03	2.96e-04	9.4647134096191934e+01
8	1.02e-08	1.38e-08	1.25e-08	9.4646712492247204e+01
16	3.24e-15	3.21e-15	3.76e-16	9.4646712492247048e+01
32	1.80e-16	5.13e-16	1.25e-16	9.4646712492247090e+01

Table 6. Numerical results for Example 3 via the indirect method.

m	$\epsilon_{64,m}(0.1,-0.4)$	$\epsilon_{64,m}(0.3,-0.6)$	$\epsilon_{64,m}(-0.1,0.5)$	$cond(\boldsymbol{\mathcal{P}}_m(\mathbf{I}_m+\boldsymbol{\mathcal{K}}_m)\boldsymbol{\mathcal{P}}_m)$
4	3.95e-04	2.62e-04	1.55e-03	9.4647134096192175e+01
8	6.74e-09	1.23e-08	5.76e-09	9.4646712492247545e+01
16	3.77e-15	3.58e-15	7.43e-16	9.4646712492246621e+01
32	2.32e-15	1.73e-15	8.91e-16	9.4646712492247204e+01

Example 3. Let us consider again an equation which present a kernel and a right-hand side very smooth

$$\frac{1}{\pi^2}\oint_S \frac{f(x,y)}{(x-t)(y-s)}\, v(x,y)dx\, dy + \int_S e^{tsxy} f(x,y)v(x,y)dxdy = \sin(3+st).$$

In Tables 5 and 6 we give the relative errors and the condition number in the spectral norm. Once again, we get very accurate results.

Example 4. Let us test the performance of our methods to the equation which present a convolution kernel

$$\frac{1}{\pi^2}\oint_S \frac{f(x,y)}{(x-t)(y-s)}\, v(x,y)dx\, dy + \int_S |x-t|^3\,|y-s|^4 f(x,y)v(x,y)dxdy = \sqrt{\frac{e^{ts}}{9+ts}}.$$

As we can see through Tables 7 and 8, the numerical results confirm the theoretical estimates given in (4.13) and (5.8).

TABLE 7. Numerical results for Example 4 via the direct method.

m	$\epsilon_{175,m}(0.4,-0.4)$	$\epsilon_{175,m}(0.2,-0.6)$	$\epsilon_{175,m}(-0.1,0.8)$	$cond(\mathbf{P}_m(\mathbf{D}_m+\mathbf{K}_m)\mathbf{P}_m)$
4	2.84e-01	7.73e-02	3.49e-01	5.8341767720850817e+02
8	4.36e-04	1.75e-04	8.60e-04	5.4307032442099785e+02
16	1.77e-05	9.38e-06	1.89e-05	5.4309967621958026e+02
32	1.05e-06	6.28e-07	8.45e-07	5.4310149342166687e+02
64	6.44e-08	3.99e-08	4.73e-08	5.4310161017935911e+02
128	2.95e-09	1.84e-09	2.11e-09	5.4310161764433553e+02

TABLE 8. Numerical results for Example 4 via the indirect method.

m	$\epsilon_{175,m}(0.4,-0.4)$	$\epsilon_{175,m}(0.2,-0.6)$	$\epsilon_{175,m}(-0.1,0.8)$	$cond(\boldsymbol{\mathcal{P}}_m(\mathbf{I}_m+\boldsymbol{\mathcal{K}}_m)\boldsymbol{\mathcal{P}}_m)$
4	2.85e-01	7.76e-02	3.50e-01	5.8341767720850714e+02
8	4.37e-04	1.75e-04	8.62e-04	5.4307032442101217e+02
16	1.78e-05	9.38e-06	1.90e-05	5.4309967621958094e+02
32	1.05e-06	6.28e-07	8.51e-07	5.4310149342166756e+02
64	6.46e-08	3.98e-08	4.77e-08	5.4310161017936002e+02
128	2.96e-09	1.84e-09	2.13e-09	5.4310161764433508e+02

TABLE 9. Numerical results for Example 5 via the direct method.

m	$\epsilon_{175,m}(0.5,-0.7)$	$\epsilon_{175,m}(0.3,0.6)$	$\epsilon_{175,m}(0,0)$	$cond(\mathbf{P}_m(\mathbf{D}_m+\mathbf{K}_m)\mathbf{P}_m)$
4	1.33e-02	3.74e-03	2.06e-02	1.3576451986839258e+01
8	2.31e-04	6.08e-04	8.62e-04	1.3584012702947833e+01
16	5.45e-07	1.21e-06	4.92e-06	1.3584041062960397e+01
32	5.93e-09	1.46e-09	8.09e-08	1.3584041246139085e+01
64	2.18e-10	2.17e-12	1.69e-09	1.3584041247052387e+01
128	2.74e-12	8.31e-13	4.15e-11	1.3584041247056279e+01

Example 5. Let us test the performance of our method to the following equation in which the kernel $k(x,y,t,s)=|\sin(xs)|^{\frac{11}{2}}+yt$ belongs to the Sobolev-type space of index $r=5$,

$$\frac{1}{\pi^2}\oint_S \frac{f(x,y)}{(x-t)(y-s)}\,v(x,y)dx\,dy + \int_S \left(|\sin(xs)|^{\frac{11}{2}}+yt\right)\, f(x,y)v(x,y)dxdy = \cos(ts).$$

As shown in Tables 9 and 10, the two methods are equivalent in terms of order of convergence and the numerical evidence confirms our theoretical estimates.

7. PROOFS

Proof. of Proposition 2.1. We only give the main idea of the proof since the thesis can be proved, mutatis mutandis, in the same way of the univariate case (see [25] and the reference therein).

Let $Q_{m-1,m-1}$ be an arbitrary polynomial of degree $m-1$ in each variable:

$$Q_{m-1,m-1}(x,y)=\sum_{i=0}^{m-1}\sum_{j=0}^{m-1} b_{ij}\,p_i(w_1,x)p_j(w_2,y).$$

Table 10. Numerical results for Example 5 via the indirect method.

m	$\epsilon_{175,m}(0.5,-0.7)$	$\epsilon_{175,m}(0.3,0.6)$	$\epsilon_{175,m}(0,0)$	$cond(\boldsymbol{\mathcal{P}}_m(\mathbf{I}_m+\boldsymbol{\mathcal{K}}_m)\boldsymbol{\mathcal{P}}_m)$
4	1.33e-02	3.74e-03	2.06e-02	1.3576451986839267e+01
8	2.31e-04	6.08e-04	8.62e-04	1.3584012702947835e+01
16	5.45e-07	1.21e-06	4.92e-06	1.3584041062960395e+01
32	5.93e-09	1.46e-09	8.09e-08	1.3584041246139078e+01
64	2.18e-10	3.11e-12	1.69e-09	1.3584041247052406e+01
128	2.13e-12	4.18e-12	4.23e-11	1.3584041247056248e+01

Then, by standard arguments, we get

$$\|f-Q_{m-1,m-1}\|_{L^2_w}=\|f\|_{L^2_w}+\|Q_{m-1,m-1}\|_{L^2_w}-2\sum_{i=0}^{m-1}\sum_{j=0}^{m-1}b_{ij}c_{ij}(f,w)$$

where $c_{ij}(f,w)$ are the Fourier coefficients of the function f defined in (2.2). In virtue of the orthogonality of $\{p_m(w_1)\}_m$ and $\{p_m(w_2)\}_m$, we have

$$\|Q_{m-1,m-1}\|^2_{L^2_w}=\sum_{i=0}^{m-1}\sum_{j=0}^{m-1}b_{ij}^2.$$

We can claim that

$$\|f-Q_{m-1,m-1}\|_{L^2_w}=\|f\|_{L^2_w}+\sum_{i=0}^{m-1}\sum_{j=0}^{m-1}(b_{ij}-c_{ij}(f,w))^2-\sum_{i=0}^{m-1}\sum_{j=0}^{m-1}c_{ij}^2(f,w).$$

Hence, by replacing b_{ij} with $c_{ij}(f,w)$ we get the thesis. □

In order to prove Proposition 2.2 and 2.3, let us note that the bivariate Fourier and Lagrange operators defined in (2.1) and (2.5), respectively can be thought as a composition of two univariate Fourier and Lagrange operators, namely

$$\mathcal{S}_{m,m}(f,w,x,y)=\mathcal{S}_m(\mathcal{S}_m(f_y,w_1,x),w_2,y)=\mathcal{S}_m(\mathcal{S}_m(f_x,w_2,y),w_1,x)$$
$$\mathcal{L}_{m,m}(f,w,x,y)=\mathcal{L}_m(\mathcal{L}_m(f_y,w_1,x),w_2,y)=\mathcal{L}_m(\mathcal{L}_m(f_x,w_2,y),w_1,x)$$

where $\mathcal{S}_m$ identifies the univariate Fourier sum, $\mathcal{L}_m$ denotes the univariate Lagrange polynomial and f_x and f_y denote the function f as a univariate function of the variable y and x, respectively.

Let us also remind that if we consider a one dimensional function h belonging to the one dimensional Sobolev space, for $i=1,2$

$$\mathcal{W}^r_{w_i}=\{h\in L^2_{w_i}([-1,1]):\, h^{(r-1)}\in AC((-1,1))\,,\, \|h\|_{\mathcal{W}^r_{w_i}}=\|h\|_{L^2_{w_i}}+\|h^{(r)}\varphi^r\|_{L^2_{w_i}}<\infty\},$$

the following estimates hold true [4, 6, 22]

$$\|h-\mathcal{S}_m(h,w_i)\|_{\mathcal{W}^{r_1}_{w_i}}\leq\frac{\mathcal{C}}{m^{r-r_1}}\|h\|_{\mathcal{W}^r_{w_i}},\quad r_1\leq r, \tag{7.1}$$

$$\|S_m(h-S_m(h,w_i),w_i)\|_{\mathcal{W}^r_{w_i}}\leq\mathcal{C}\,\|h-S_m(h,w_i)\|_{\mathcal{W}^r_{w_i}}, \tag{7.2}$$

$$\|h-\mathcal{L}_m(h,w_i)\|_{L^2_{w_i}}\leq\frac{\mathcal{C}}{m^r}\|h\|_{\mathcal{W}^r_{w_i}}, \tag{7.3}$$

$$\|\mathcal{L}_m(h,w_i)\|_{L^2_{w_i}}\leq\|h\|_{L^2_{w_i}}+\frac{\mathcal{C}}{m^r}\|h\|_{\mathcal{W}^r_{w_i}}, \tag{7.4}$$

$$\|[h-\mathcal{L}_m(h,w_i)]^{(r)}\varphi^r\|_{L^2_{w_i}}\leq\mathcal{C}\left(\|h^{(r)}\varphi^r\|_{L^2_{w_i}}+m^r\|h-\mathcal{L}_m(h,w_i)\|_{L^2_{w_i}}\right), \tag{7.5}$$

where in all the inequalities $\mathcal{C}\neq\mathcal{C}(m,h)$.

Proof. of Proposition 2.2. We write

$$\begin{aligned}
&\|f-\mathcal{S}_{m,m}(f,w)\|_{W^{r_1}_w}\\
&\le \|f-\mathcal{S}_m(f,w_2)\|_{W^{r_1}_w}+\|\mathcal{S}_m(f,w_2)-\mathcal{S}_m(\mathcal{S}_m(f_y,w_1),w_2)\|_{W^{r_1}_w}\\
&=\left(\int_{-1}^{1}\|f_x-\mathcal{S}_m(f_x,w_2)\|^2_{\mathcal{W}^{r_1}_{w_2}}\,w_1(x)\,dx\right)^{1/2}\\
&+\quad\left(\int_{-1}^{1}\|\mathcal{S}_m(f_y-\mathcal{S}_m(f_y,w_1),w_2)\|^2_{\mathcal{W}^{r_1}_{w_2}}\,w_1(x)\,dx\right)^{1/2}.
\end{aligned}$$

Then, by applying (7.1) to the norm of the first term, (7.2) and again (7.1) to the norm of the second one, we get

$$\begin{aligned}
\|f-\mathcal{S}_{m,m}(f,w)\|_{W^{r_1}_w} &\le \frac{\mathcal{C}}{m^{r-r_1}}\left(\int_{-1}^{1}\|f_x\|^2_{\mathcal{W}^{r}_{w_2}}\,w_1(x)\,dx\right)^{1/2}\\
&+\frac{\mathcal{C}}{m^{r-r_1}}\left(\int_{-1}^{1}\|f_y\|^2_{\mathcal{W}^{r}_{w_1}}\,w_2(y)\,dy\right)^{1/2}\\
&\le \frac{\mathcal{C}}{m^{r-r_1}}\|f\|_{W^r_w}.
\end{aligned}$$

□

Proof. of Proposition 2.3. As in the previous proof, we begin by writing

$$\begin{aligned}
&\|f-\mathcal{L}_{m,m}(f,w)\|_{L^2_w}\\
&\le \|f-\mathcal{L}_m(f,w_2)\|_{L^2_w}+\|\mathcal{L}_m(f,w_2)-\mathcal{L}_m(\mathcal{L}_m(f_y,w_1),w_2)\|_{L^2_w}\\
&=\left(\int_{-1}^{1}\|f_x-\mathcal{L}_m(f_x,w_2)\|^2_{L^2_{w_2}}\,w_1(x)\,dx\right)^{1/2}\\
&+\quad\left(\int_{-1}^{1}\|\mathcal{L}_m(f_y-\mathcal{L}_m(f_y,w_1),w_2)\|^2_{L^2_{w_2}}\,w_1(x)\,dx\right)^{1/2}.
\end{aligned}$$

Hence by using (7.3) to the first term, (7.3), (7.4) and (7.5) to the second one, we get

$$\|f-\mathcal{L}_{m,m}(f,w)\|_{L^2_w}\le\frac{\mathcal{C}}{m^r}\left(\int_{-1}^{1}\|f_x-\mathcal{L}_m(f_x,w_2)\|^2_{\mathcal{W}^{r}_{w_2}}\,w_1(x)\,dx\right)^{1/2}$$

from which we deduce the thesis. □

Proof. of Lemma 3.1. Taking into account the definition of the dominant operator D, we write

$$\begin{aligned}
Dq_m(t,s)&=\frac{1}{\pi^2}\oint_S\frac{q_m(x,y)}{(x-t)(y-s)}\,u(x)\,u(y)\,dx\,dy\\
&=\left[\frac{1}{\pi}\oint_{-1}^{1}\frac{p_m(u,x)}{(x-t)}\,u(x)\,dx\right]\left[\frac{1}{\pi}\oint_{-1}^{1}\frac{p_m(u,y)}{(y-s)}\,u(y)\,dy\right]\\
&=p_m(u^{-1},t)\,p_m(u^{-1},s)\\
&=r_m(t,s)
\end{aligned}$$

being [27, 31]

$$\frac{1}{\pi}\oint_{-1}^{1}\frac{p_m(u,z)}{(z-\eta)}u(z)\,dz=p_m(u^{-1},\eta).$$

Analogously,

$$\begin{aligned}\widehat{D}r_m(t,s) &= \frac{1}{\pi^2}\oint_S \frac{r_m(x,y)}{(x-t)(y-s)}\,u^{-1}(x)\,u^{-1}(y)\,dx\,dy\\ &= \left[-\frac{1}{\pi}\oint_{-1}^{1}\frac{p_m(u^{-1},x)}{(x-t)}\,u^{-1}(x)\,dx\right]\left[-\frac{1}{\pi}\oint_{-1}^{1}\frac{p_m(u^{-1},y)}{(y-s)}\,u^{-1}(y)\,dy\right]\\ &= p_m(u,t)\,p_m(u,s)\\ &= q_m(t,s)\end{aligned}$$

since [27, 31]

$$-\frac{1}{\pi}\oint_{-1}^{1}\frac{p_m(u^{-1},z)}{(z-\eta)}\,u^{-1}(z)\,dz = p_m(u,\eta).$$

□

In order to prove Proposition 3.2, let us note that the dominant operator D can be rewritten in terms of the Hilbert transform of a 1D function h

$$H(h,t) = \frac{1}{\pi}\oint_{-1}^{1}\frac{h(x)}{(x-t)}\,u(x)\,dx$$

as follows

$$\begin{aligned}Df(t,s) &= \frac{1}{\pi^2}\oint_S \frac{f(x,y)}{(x-t)(y-s)}\,v(x,y)\,dx\,dy = \frac{1}{\pi}\oint_{-1}^{1}\frac{H(f_x,s)}{(x-t)}\,u(x)\,dx\\ &= \frac{1}{\pi}\oint_{-1}^{1}\frac{H(f_y,t)}{(y-s)}\,u(y)\,dy = H\left(H(f)\right)(t,s)\end{aligned}$$

where f_x and f_y denote the function f as a univariate function of the variable y and x, respectively. Let us also remind that for a univariate function h the following estimates hold true [24, 27]

$$\|(Hh)^{(r)}\varphi^r\|_{L^2_{u^{-1}}} \le \|h\|_{\mathcal{W}^r_u}, \qquad \text{and} \qquad \|Hh\|_{L^2_{u^{-1}}} \le \|h\|_{L^2_u}. \tag{7.6}$$

Proof. of Proposition 3.2. At first we note that, by definition, the operator D is a linear operator. Moreover, by (2.4) we have

$$\|Df\|^2_{L^2_{v^{-1}}} = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} c^2_{ij}(Df,v^{-1}) = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} c^2_{ij}(f,v) = \|f\|^2_{L^2_v} < \infty$$

being, in virtue of (3.7)

$$\begin{aligned}&c^2_{ij}(Df,v^{-1}) = \left(\int_S Df(x,y)\,p_i(u^{-1},x)\,p_j(u^{-1},y)\,v^{-1}(x,y)\,dx\,dy\right)^2\\ &= \left(\int_S \frac{1}{\pi^2}\oint_S\left[\frac{f(\eta,\xi)}{(\eta-x)(\xi-y)}v(\eta,\xi)d\eta\,d\xi\right]p_i(u^{-1},x)p_j(u^{-1},y)\,v^{-1}(x,y)\,dx\,dy\right)^2\\ &= \left(\int_S f(\eta,\xi)\left[\frac{1}{\pi^2}\oint_S\frac{p_i(u^{-1},x)\,p_j(u^{-1},y)}{(x-\eta)(y-\xi)}\,v^{-1}(x,y)\,dx\,dy\right]v(\eta,\xi)\,d\eta\,d\xi\right)^2\\ &= \left(\int_S f(\eta,\xi)\,p_i(u,\eta)\,p_j(u,\xi)\,v(\eta,\xi)\,d\eta\,d\xi\right)^2 = c^2_{ij}(f,v).\end{aligned}$$

Moreover, by applying (7.6) and taking into account that $(a+b)^2 \le 2(a^2+b^2)$ and $\sqrt{a+b} \le \sqrt{a} + \sqrt{b}$, we have

$$\begin{aligned}
&\left(\int_{-1}^{1}\int_{-1}^{1}\left|\frac{\partial^r}{\partial t^r}Df(t,s)\,\varphi^r(t)\right|^2 v^{-1}(t,s)\,dt\,ds\right)^{\frac{1}{2}}\\
&= \left(\int_{-1}^{1}\int_{-1}^{1}\left|H\left(\frac{\partial^r}{\partial t^r}H(f)\right)(t,s)\,\varphi^r(t)\right|^2 u^{-1}(t)\,u^{-1}(s)\,dt\,ds\right)^{\frac{1}{2}}\\
&\le \mathcal{C}\left(\int_{-1}^{1}u(s)\int_{-1}^{1}\left|\frac{\partial^r}{\partial t^r}H(f)(t,s)\,\varphi^r(t)\right|^2 u^{-1}(t)\,dt\,ds\right)^{\frac{1}{2}}\\
&\le \mathcal{C}\left(\int_{-1}^{1}u(s)\left[\int_{-1}^{1}\left|\frac{\partial^r}{\partial t^r}f(t,s)\,\varphi^r(t)\right|^2 u(t)\,dt + \int_{-1}^{1}|f(t,s)|^2\,u(t)\,dt\right]^2 ds\right)^{\frac{1}{2}}\\
&\le \mathcal{C}\left\{\left[\int_{-1}^{1}u(s)\left(\int_{-1}^{1}\left|\frac{\partial^r}{\partial t^r}f(t,s)\,\varphi^r(t)\right|^2 u(t)\,dt\right)^2 ds\right]^{\frac{1}{2}} + \right.\\
&\left. + \left[\int_{-1}^{1}u(s)\left(\int_{-1}^{1}|f(t,s)|^2\,u(t)\,dt\right)^2 ds\right]^{\frac{1}{2}}\right\} < \infty
\end{aligned}$$

which prove the boundedness of $D : W_v^r \to W_{v^{-1}}^r$ and consequently its continuity. Now we show that $\widehat{D}(Df) = f$ and $D(\widehat{D}f) = f$. Let $f \in L_v^2(S)$. By using the Fourier sum, taking into account the linearity of the operators D and $\widehat{D}$, and applying firstly (3.6) and then (3.7), we have

$$\widehat{D}(Df) = \widehat{D}\left(\sum_{i=0}^{m-1}\sum_{j=0}^{m-1} c_{ij}(f,v)p_i(u^{-1})p_j(u^{-1})\right) = \sum_{i=0}^{m-1}\sum_{j=0}^{m-1} c_{ij}(f,u)p_i(u)p_j(u) = f.$$

Proceeding in the same way, we can show also that $D(\widehat{D}f) = f$ and hence $\widehat{D} \equiv D^{-1}$. As regards to the mapping property (3.9) of $\widehat{D}$, this can be proved as done for the property (3.8). □

Proof. of Proposition 3.3. The linearity of the operator K is a trivial consequence of its definition (1.4) while the boundedness follows by

$$\|Kf\|_{W^{r_1}_{v^{-1}}} = \|Kf\|_{L^2_{v^{-1}}} + \mathcal{M}_{r_1}(Kf, v^{-1}) \le \mathcal{C}\,\|f\|_{L^2_v}. \tag{7.7}$$

In fact, by applying Schwarz's inequality and taking into account the first hypotheses on the kernel function k, we have

$$\begin{aligned}
\|Kf\|^2_{L^2_{v^{-1}}} &= \int_S |Kf(t,s)|^2 v^{-1}(t,s)\,dt\,ds\\
&= \int_S \left|\int_S k(x,y,t,s)f(x,y)v(x,y)dx\,dy\right|^2 v^{-1}(t,s)\,dt\,ds\\
&\le \|f\|^2_{L^2_v} \sup_{(t,s)\in S}\|k_{(t,s)}\|^2_{W^r_v}\int_S v^{-1}(t,s)\,dt\,ds\\
&\le \mathcal{C}\|f\|^2_{L^2_v}.
\end{aligned}$$

Moreover, by using again the Schwarz inequality we can write

$$\begin{aligned}\left|\frac{\partial^{r_1}(Kf)(t,s)}{\partial t^{r_1}}\right|^2 &= \left|\frac{\partial^{r_1}}{\partial t^{r_1}}\int_S k(x,y,t,s)f(x,y)\,v(x,y)\,dx\,dy\right|^2\\ &= \|f\|_{L^2_v}\left(\int_S\left|\frac{\partial^{r_1}k(x,y,t,s)}{\partial t^{r_1}}\right|^2 v(x,y)\,dx\,dy\right)\end{aligned}$$

from which we can deduce

$$\begin{aligned}&\int_S\left|\frac{\partial^{r_1}(Kf)(t,s)}{\partial t^{r_1}}\varphi^{r_1}(t)\right|^2 v^{-1}(t,s)\,dt\,ds\\ &\leq \|f\|_{L^2_v}\int_S\left(\int_S\left|\frac{\partial^{r_1}k(x,y,t,s)}{\partial t^{r_1}}\right|^2 v(x,y)dxdy\right)\varphi^{2r_1}(t)v^{-1}(t,s)\,dt\,ds\\ &= \|f\|_{L^2_v}\int_S\left(\int_S\left|\frac{\partial^{r_1}k(x,y,t,s)}{\partial t^{r_1}}\varphi^{r_1}(t)\right|^2 v^{-1}(t,s)dtds\right)v(x,y)\,dx\,dy\\ &\leq \mathcal{C}\|f\|_{L^2_v}\sup_{(x,y)\in S}\|k_{(x,y)}\|_{W^{r_1}_{v^{-1}}}.\end{aligned}$$

Analogously

$$\int_S\left|\frac{\partial^{r_1}(Kf)(t,s)}{\partial s^{r_1}}\varphi^{r_1}(s)\right|^2 v^{-1}(t,s)\,dt\,ds \leq \mathcal{C}\|f\|_{L^2_v}\sup_{(x,y)\in S}\|k_{(x,y)}\|_{W^{r_1}_{v^{-1}}}.$$

The only point remaining concerns the compactness. To this end let us note that we have

$$\begin{aligned}E_{m,m}(Kf)_{W^{r_1}_{v^{-1}}} \leq \|Kf-\mathcal{S}_{m,m}(Kf,v)\|_{W^{r_1}_{v^{-1}}} &\leq \frac{\mathcal{C}}{m^{r-r1}}\|Kf\|_{W^{r_1}_{v^{-1}}}\\ &\leq \frac{\mathcal{C}}{m^{r-r1}}\|f\|_{L^2_v}.\end{aligned}$$

Therefore, setting $T=\{f\in L^2_v:\ \|f\sqrt{v}\|_2\leq 1\}$, we have

$$\lim_m \sup_{f\in T} E_{m,m}(f)_v = 0$$

from which we deduce [32] that $K: L^2_v \to W^{r_1}_{v^{-1}}$ is a compact operator for all $r_1 < r$. □

Proof. of Proposition 4.1. By the definitions of the operator D and the function f_m, we get

$$\begin{aligned}Df_m(t_i,s_j) &= \frac{1}{\pi^2}\oint_S \frac{f_m(x,y)}{(x-t_i)(y-s_j)}v(x,y)\,dx\,dy\\ &= \frac{1}{\pi^2}\sum_{h=1}^m\sum_{k=1}^m f_m(x_h,y_k)\oint_S\frac{\ell(u,x)\ell(u,y)}{(x-t_i)(y-s_j)}u(x)u(y)\,dx\,dy.\end{aligned}$$

Moreover, by (2.6) we have

$$\begin{aligned}\frac{\ell(u,x)}{(x-t_i)}u(x) &= \frac{p_m(u,x)u(x)}{p'_m(u,x_h)(x-x_h)(x-t_i)}\\ &= \frac{p_m(u,x)u(x)}{p'_m(u,x_h)(x_h-t_i)}\left[\frac{1}{x-x_h}-\frac{1}{x-t_i}\right],\end{aligned}$$

and similarly

$$\frac{\ell(u,y)}{(y-s_j)}u(y) = \frac{p_m(u,y)u(y)}{p'_m(u,y_k)(y-y_k)(y-s_j)}$$
$$= \frac{p_m(u,y)u(y)}{p'_m(u,y_k)(y_k-s_j)}\left[\frac{1}{y-y_k} - \frac{1}{y-s_j}\right].$$

Then, setting $q_m(t,s) = p_m(u,t)p_m(u,s)$, $r_m(t,s) = p_m(u^{-1},t)p_m(u^{-1},s)$ and taking into account Lemma 3.1, we can write

$$\begin{aligned} &Df_m(t_i,s_j) \\ &= \sum_{h,k=1}^{m}\sum_{k=1}^{m}\left[\frac{f_m\left(x_h,y_k\right)\left\{Dq_m(x_h,y_k) - Dq_m(x_h,s_j) - Dq_m(t_i,y_k)\right\}}{q'_m(x_h,y_k)(x_h-t_i)(y_k-s_j)}\right. \\ &\left.+\frac{f_m\left(x_h,y_k\right)Dq_m(t_i,s_j)}{q'_m(x_h,y_k)(x_h-t_i)(y_k-s_j)}\right] \\ &= \sum_{h=1}^{m}\sum_{k=1}^{m}\frac{f_m\left(x_h,y_k\right)\left\{r_m(x_h,y_k) - r_m(x_h,s_j) - r_m(t_i,y_k) + r_m(t_i,s_j)\right\}}{q'_m(x_h,y_k)(x_h-t_i)(y_k-s_j)} \end{aligned}$$

and consequently,

$$Df_m(t_i,s_j) = \sum_{h=1}^{m}\sum_{k=1}^{m}\frac{f_m\left(x_h,y_k\right)r_m(x_h,y_k)}{q'_m(x_h,y_k)(x_h-t_i)(y_k-s_j)}.$$

Thus, the thesis can be deduced by observing that by using property (3.6), we have

$$\begin{aligned} r_m(x_h,y_k) &= Dq_m(x_h,y_k) \\ &= \frac{1}{\pi^2}q'_m(x_h,y_k)\oint_{-1}^{1}\ell_h(u,x)u(x)dx\oint_{-1}^{1}\ell_k(u,y)u(y)dy \\ &= \frac{1}{\pi^2}q'_m(x_h,y_k)\lambda_h(u)\lambda_k(u) \end{aligned}$$

where $\lambda_h(u)$ denotes the h-th Christoffel number w.r.t. the weight u. □

Proof. of *Proposition 4.2.* We start by writing

$$\begin{aligned} \|Kf - \mathcal{L}_{m,m}(K_mf,v^{-1})\|_{L^2_{v^{-1}}} &\le \left\|Kf - \mathcal{L}_{m,m}(Kf,v^{-1})\right\|_{L^2_{v^{-1}}} \\ &+ \left\|\mathcal{L}_{m,m}((K-K_m)f,v^{-1})\right\|_{L^2_{v^{-1}}} \\ &:= A + B. \end{aligned}$$

By using Proposition 2.3 and (7.7) we can deduce that

$$A \le \frac{\mathcal{C}}{m^{r_1}}\|Kf\|_{W^{r_1}_{v^{-1}}} \le \frac{\mathcal{C}}{m^{r_1}}\|f\|_{L^2_v}.$$

Moreover, by using the Gaussian cubature rule (2.8) with $w_i = u$, $i = 1, 2$, we have

$$\begin{aligned} B &= \left(\int_S\left|\mathcal{L}_{m,m}\left((K-K_m)f,v^{-1},t,s\right)\right|^2 v^{-1}(t,s)\,dt\,ds\right)^{\frac{1}{2}} \\ &= \left(\sum_{i=1}^{m}\sum_{j=1}^{m}\lambda_i(u^{-1})\lambda_j(u^{-1})\left|(K-K_m)\,f(t_i,t_j)\right|^2\right)^{\frac{1}{2}}. \end{aligned}$$

Since one has

$$\begin{aligned}|(K-K_m)\,f(t,s)|^2 &\le \|f\|^2_{L^2_v}\int_S \left|k(x,y,t,s)-\mathcal{L}_{m,m}\left(k_{(t,s)},v,x,y\right)\right|^2 v(x,y)\,dx\,dy\\ &= \|f\|^2_{L^2_v}\left\|k_{(t,s)}-\mathcal{L}_{m,m}\left(k_{(t,s)}\right)\right\|^2_{L^2_v}\\ &\le \frac{\mathcal{C}}{m^{2r}}\|f\|^2_{L^2_v}\left\|k_{(t,s)}\right\|^2_{W^r_v},\end{aligned}$$

from the first assumption in (3.10), it follows

$$\begin{aligned}B &\le \frac{\mathcal{C}}{m^r}\|f\|_{L^2_v}\left(\sum_{i=1}^m\sum_{j=1}^m \lambda_i(u^{-1})\lambda_j(u^{-1})\left\|k_{(t_i,t_j)}\right\|^2_{W^r_v}\right)^{\frac12}\\ &\le \frac{\mathcal{C}}{m^r}\|f\|_{L^2_v}\sup_{(t_i,t_j)\in S}\left\|k_{(t_i,t_j)}\right\|^2_{W^r_v}\left(\sum_{i=1}^m\sum_{j=1}^m \lambda_i(u^{-1})\lambda_j(u^{-1})\right)^{\frac12}\\ &\le \frac{\mathcal{C}}{m^r}\|f\|_{L^2_v}\left(\int_S v^{-1}(x,y)\,dx\,dy\right)^{\frac12}\\ &\le \frac{\mathcal{C}}{m^r}\|f\|_{L^2_v}.\end{aligned}$$

□

Proof. of Theorem 4.3. Taking into account Proposition 4.2, by standard arguments (see, for instance, Theorem 3.3.1 in [2]), it follows that for sufficiently large m, say $m \ge m_0$, the operators $D+\mathcal{L}_{m,m}K_m : L^2_v \to L^2_{v^{-1}}$ exist and are uniformly bounded being

$$\|(D+\mathcal{L}_{m,m}K_m)^{-1}\| \le \frac{\|(D+K)^{-1}\|}{1-\|(D+K)^{-1}\|\sup_{m\ge m_0}\|K-\mathcal{L}_{m,m}K_m\|} < \infty$$

(where the notation $\|\cdot\|$ denotes the norm of the operators), i.e. the method is stable. In order to prove the convergence estimate (4.13), we note that

$$f-f_m = (D+\mathcal{L}_{m,m}K_m)^{-1}\left[\left(g-\mathcal{L}_{m,m}\left(g,v^{-1}\right)\right)-\left(Kf-\mathcal{L}_{m,m}\left(K_mf,v^{-1}\right)\right)\right]$$

from which we deduce

$$\|f-f_m\|_{L^2_v} \le \mathcal{C}\,\|g-\mathcal{L}_{m,m}\left(g,v^{-1}\right)\|_{L^2_{v^{-1}}} + \|Kf-\mathcal{L}_{m,m}\left(K_mf,v^{-1}\right)\|_{L^2_{v^{-1}}}.$$

Then, by applying Proposition 2.3 to the first term and Proposition 4.2 to the second one we get (4.13). Let us now prove (4.14). To this end let us introduce an arbitrary array $\mathbf{c} = [c_{11},\,\dots,\,c_{1m},\,\dots,\,c_{m1},\,\dots c_{mm}]^T$ of length m^2, and let us denote by $\|\mathbf{c}\|_2 = \left(\sum_{i=1}^m\sum_{j=1}^m c_{ij}^2\right)^{1/2}$ its Euclidean norm. Then, the vector $\mathbf{b} = [b_{11},\,\dots,\,b_{1m},\,\dots,\,b_{m1},\,\dots b_{mm}]^T$ satisfies the system $\mathbf{P}_m(\mathbf{D}_m+\mathbf{K}_m)\mathbf{P}_m\mathbf{c} = \mathbf{b}$ if and only if $(D+\mathcal{L}_{m,m}K_m)f_m = g_m$ where f_m and g_m are the bivariate polynomials defined as

$$f_m(t,s) = \sum_{i=1}^m\sum_{j=1}^m \frac{\ell_i(u,t)}{\sqrt{\lambda_j(u)}}\frac{\ell_j(u,s)}{\sqrt{\lambda_j(u)}}c_{ij}$$

and

$$g_m(t,s) = \sum_{i=1}^m\sum_{j=1}^m \frac{\ell_i(u^{-1},t)}{\sqrt{\lambda_i(u^{-1})}}\frac{\ell_j(u^{-1},s)}{\sqrt{\lambda_j(u^{-1})}}b_{ij}.$$

Being

$$\|g_m\|^2_{L^2_{v^{-1}}} = \int_S |g_m(t,s)|^2 v^{-1}(t,s)\,dt\,ds = \sum_{i=1}^m \sum_{j=1}^m \lambda_i(u^{-1})\lambda_j(u^{-1})|g_m(t_i,t_j)|^2$$
$$= \sum_{i=1}^m \sum_{j=1}^m b_{ij}^2 = \|\mathbf{b}\|_2$$

and analogously $\|f_m\|_{L^2_v} = \|\mathbf{c}\|_2$, we have

$$\|\mathbf{P}_m(\mathbf{D}_m+\mathbf{K}_m)\mathbf{P}_m\| = \sup_{\substack{\mathbf{c}\in\mathbb{R}^{m^2}\\ \mathbf{c}\neq 0}} \frac{\|\mathbf{P}_m(\mathbf{D}_m+\mathbf{K}_m)\mathbf{P}_m\mathbf{c}\|_2}{\|\mathbf{c}\|_2}$$
$$= \sup_{\substack{\mathbf{f_m}\in\mathbb{P}_{m-1,m-1}\\ f_m\neq 0}} \frac{\|(D+\mathcal{L}_{m,m}K_m)f_m\|_{L^2_{v^{-1}}}}{\|f_m\|_{L^2_v}}$$
$$= \|D+\mathcal{L}_{m,m}K_m\|_{L^2_v\to L^2_{v^{-1}}}.$$

Then, in virtue of Proposition 4.2, for m sufficiently large,

$$\|\mathbf{P}_m(\mathbf{D}_m+\mathbf{K}_m)\mathbf{P}_m\| \le \mathcal{C}\,\|D+K\|_{L^2_v\to L^2_{v^{-1}}}. \tag{7.8}$$

In the same way we can prove that

$$\|(\mathbf{P}_m(\mathbf{D}_m+\mathbf{K}_m)\mathbf{P}_m)^{-1}\| = \|(D+\mathcal{L}_{m,m}K_m)^{-1}\|_{L^2_{v^{-1}}\to L^2_v}$$

from which, by applying again Proposition 4.2, we deduce that, for m sufficiently large,

$$\|(\mathbf{P}_m(\mathbf{D}_m+\mathbf{K}_m)\mathbf{P}_m)^{-1}\| \le \mathcal{C}\,\|(D+K)^{-1}\|_{L^2_{v^{-1}}\to L^2_v}. \tag{7.9}$$

Hence, the thesis (4.14) follows from (7.8) and (7.9). □

Proof. of Proposition 5.1. We can proceed analogously to the proof of Proposition 4.2. Therefore we only give the main sketch. We have

$$\|\mathcal{K}f - \mathcal{L}_{m,m}(\mathcal{K}_m f, v)\|_{L^2_v} \le \|\mathcal{K}f - \mathcal{L}_{m,m}(\mathcal{K}f, v)\|_{L^2_v}$$
$$+ \|\mathcal{L}_{m,m}((\mathcal{K}-\mathcal{K}_m)f, v)\|_{L^2_v}.$$

By noting that in virtue of Proposition 3.2 one has $\mathcal{K}f = (\widehat{D}K)(f) \in W^{r_1}_v$ and taking into account (2.7) and (3.10), we get

$$\|\mathcal{K}f - \mathcal{L}_{m,m}(\mathcal{K}f, v)\|_{L^2_v} \le \frac{\mathcal{C}}{m^{r_1}}\|\mathcal{K}f\|_{W^{r_1}_{v^{-1}}} \le \frac{\mathcal{C}}{m^{r_1}}\|f\|_{L^2_v}.$$

Moreover,

$$|(\mathcal{K}-\mathcal{K}_m)\,f(t,s)|^2 \le \frac{\mathcal{C}}{m^{2r}}\,\|f\|^2_{L^2_v}\,\left\|\phi_{(t,s)}\right\|^2_{W^r_v},$$

and by (5.3) and Proposition 3.2, we can write

$$\left\|\phi_{(t,s)}\right\|^2_{W^r_v} = \left\|\widehat{D}k_{(\xi,\eta)}\right\|^2_{W^r_v} \le \|k_{(\xi,\eta)}\|^2_{W^r_{v^{-1}}}.$$

Consequently, from the assumption (3.10), we can deduce

$$\|\mathcal{L}_{m,m}((\mathcal{K}-\mathcal{K}_m)f, v)\|_{L^2_v} \le \frac{\mathcal{C}}{m^r}\,\|f\|_{L^2_v}\left(\sum_{i=1}^m\sum_{j=1}^m \lambda_i(u)\lambda_j(u)\left\|k_{(x_i,x_j)}\right\|^2_{W^r_{v^{-1}}}\right)^{\frac{1}{2}}$$
$$\le \frac{\mathcal{C}}{m^r}\,\|f\|_{L^2_v}$$

from which the thesis follows. □

Proof. of Theorem 5.2. In order to prove this theorem it is sufficient to proceed as in the proof of Theorem 4.3 with I, $\mathcal{K}$, $\mathcal{G}$ in place of D, K and g, respectively. Moreover the thesis on the condition number can be proved as done for (4.14). □

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References

[1] R. S. Anderssen, F. R. de Hoog, and M. A. Lukas. *The application and numerical solution of integral equations.* Sijthoff and Noordhoff, Groningen, 1980.
[2] K. E. Atkinson. *The Numerical Solution of Integral Equations of the Second Kind*, volume 552 of *Cambridge Monographs on Applied and Computational Mathematics.* Cambridge University Press, Cambridge, 1997.
[3] D. Berthold, W. Hoppe, and B. Silbermann. A fast algorithm for solving the generalized airfoil equation. *J. Comp. Appl. Math.*, 43:185–219, 1992.
[4] G. Criscuolo and G. Mastroianni. Fourier and Lagrange operators in some weighted Sobolev-type spaces. *Acta Sci. Mat.*, 60:131–148, 1995.
[5] J. A. Cuminato. On the uniform convergence of a collocation method for a class of singular integral equations. *BIT*, 27(2):190–202, 1987.
[6] M. C. De Bonis and C. Laurita. A quadrature method for systems of Cauchy singular integral equations. *J. Integral Equations Appl.*, 24:241–270, 2012.
[7] M. C. De Bonis and G. Mastroianni. Projection methods and condition numbers in uniform norm for Fredholm and Cauchy singular integral equations. *SIAM Journal of Numerical Analysis*, 44:1351–1374, 2006.
[8] D. Elliott, I. K. Lifanov, and G. S. Litvinchuk. The solution in a class of singular functions of Cauchy type bisingular integral equations. *Journal of Integral Equations and Applications*, 9:237–251, 1997.
[9] F. D. Gakhov. *Boundary value problems.* Pergamon Press, Oxford, 1966.
[10] W. Gautschi. *Orthogonal Polynomials: Computation and Approximation.* Numerical mathematics and scientific computation. Oxford University Press, 2004.
[11] M. A. Golberg. *Solution methods for integral equations : Theory and applications.* Plenum Press, New York, 1979.
[12] M. A. Golberg. The numerical solution of Cauchy singular integral equations with constant coefficients. *J. Integral Equations*, 9:127–151, 1985.
[13] R. Hagen and B. Silbermann. A finite element collocation method for bisingular integral equations. *Applicable Analysis*, 19(2-3):117–135, 1985.
[14] P. Junghanns and U. Luther. Cauchy singular integral equations in spaces of continuous functions and methods for their numerical solution. *J. Comp. Appl. Math.*, 77:201–237, 1997.
[15] P. Junghanns and U. Luther. Uniform convergence of a fast algorithm for a Cauchy singular integral equations. *Proceedings of the Sixth Conference of the International Linear Algebra Society (Chemnitz 1996), Linear Algebra Appl.*, 275/276:327–347, 1998.
[16] L. E. Khairullina. Convergence of approximate solutions to two-dimensional singular integral equation with cauchy kernel in the integral metrics. *International Journal of Pharmacy and Technology*, 8(3):15008–15016, 2016.
[17] C. Laurita. Condition numbers for singular integral equations in weighted l^2 spaces. *J. Comp. Appl. Math.*, 116:23–40, 2000.
[18] C. Laurita and G. Mastroianni. Revisiting a quadrature method for Cauchy singular integral equations with a weakly singular perturbation kernel. In J. Elschner, I. Gohberg, and B. Silbermann, editors, *Oper. Theory Adv. Appl.*, volume 121, pages 307–326. Birkhäuser Verlag, Basel, 2001.
[19] C. Laurita, G. Mastroianni, and M. G. Russo. *Revisiting CSIE in L^2: Condition Numbers and Inverse Theorems*, volume 2 of *Integral and Integrodifferential equations, Ser. Math. Anal. Appl.* Gordon and Breach, Amsterdam, 2000.
[20] I. K. Lifanov, L. N. Poltavskii, and G. M. Vainikko. *Hipersingular integral equations and their applications.* CRC Press, New York, 2004.

[21] J. C. Mason and D. C. Handscomb. *Chebyshev polynomials.* Chapman and Hall/CRC, Boca Raton, 2003.
[22] G. Mastroianni and G. V. Milovanovic. *Interpolation Processes Basic Theory and Applications.* Springer Monographs in Mathematics. Springer Verlag, Berlin, 2009.
[23] G. Mastroianni and S. Prössdorf. A quadrature method for Cauchy integral equations with weakly singular perturbation kernel. *J. Integral Equations Appl.*, 4:205–228, 1992.
[24] G. Mastroianni and M. G. Russo. Lagrange interpolation in weighted Besov spaces. *Constr. Approx.*, 15:257–289, 1999.
[25] G. Mastroianni and M. G. Russo. Fourier sums in weighted spaces of functions. a survey. *Jaen Journal on Approximation*, 1(2):257–291, 2009.
[26] S. G. Mikhlin. *Multidimentional singular integrals and integral equations.* Pergamon Press, New York, 1965.
[27] S. G. Mikhlin and S. Prössdorf. *Singular Integral Operators.* Akademie-Verlag, Berlin, 1986.
[28] N. I. Muskhelishvili. *Singular Integral Equations.* Noordhoff, Groningen, 1953.
[29] D. Occorsio and M. G. Russo. Numerical methods for Fredholm integral equations on the square. *Appl. Math. and Comput.*, 218:2318–2333, 2011.
[30] V. Z. Parton and P. I. Perlin. *Integral Equations in elasticity.* Mir Publisher, Moscow, 1982.
[31] S. Prössdorf and B. Silbermann. *Numerical Analysis for Integral and Related Operator Equations.* Akademie-Verlag and Birkhäuser Verlag, Berlin, Basel, 1991.
[32] A. F. Timan. *Theory of Approximation of Functions of a Real Variable.* Dover, New York, 1994.

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