FIVE SOLUTIONS FOR THE FRACTIONAL p-LAPLACIAN WITH NONCOERCIVE ENERGY

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Abstract. We deal with a Dirichlet problem driven by the degenerate fractional p-Laplacian and involving a nonlinear reaction which satisfies, among other hypotheses, a $(p-1)$-linear growth at infinity with non-resonance above the first eigenvalue. The energy functional governing the problem is thus noncoercive. We focus on the behavior of the reaction near the origin, assuming that it has a $(p-1)$-sublinear growth at zero, vanishes at three points, and satisfies a reverse Ambrosetti-Rabinowitz condition. Under such assumptions, by means of critical point theory and Morse theory, and using suitably truncated reactions, we show the existence of five nontrivial solutions: two positive, two negative, and one nodal.

1. Introduction

Nonlinear boundary value problems driven by nonlocal operators of fractional order are the subject of a vast and rapidly developing literature, both for their intrinsic mathematical interest and for their numerous applications (see for instance [7, 10]). Most authors have focused on the semilinear case, i.e, on problems driven by the fractional Laplacian or variants of it with several kernels (see [32] and the monograph [25]).

In the quasilinear case, which has the fractional p-Laplacian as its prototype operator, the fundamental theory is still developing (for an introduction to the subject see [26, 28]). The simplest problem to be studied in such framework is the following Dirichlet problem:

$$\begin{cases} (-\Delta)_p^s\, u = f(x,u) & \text{in } \Omega \\ u = 0 & \text{in } \Omega^c. \end{cases} \tag{1.1}$$

Here $\Omega \subset \mathbb{R}^N$ ($N \geqslant 2$) is a bounded domain with $C^{1,1}$ boundary, $p \geqslant 2$, $s \in (0,1)$ s.t. $N > ps$, and the leading operator is the degenerate fractional p-Laplacian, defined for all $u : \mathbb{R}^N \to \mathbb{R}$ smooth enough and all $x \in \mathbb{R}^N$ by

$$(-\Delta)_p^s\, u(x) = 2 \lim_{\varepsilon \to 0^+} \int_{B_\varepsilon^c(x)} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))}{|x-y|^{N+ps}}\, dy \tag{1.2}$$

(which in the linear case $p = 2$ reduces to the fractional Laplacian, up to a dimensional constant). The reaction $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory mapping, subject to a subcritical growth condition on the real variable. The current literature on problem (1.1) stems from the study of the corresponding nonlinear eigenvalue problem in [6, 24], and it includes existence results based on Morse theory [16], as well as regularity results [5, 19, 20], a version of the 'Sobolev vs. Hölder minima' result [21], maximum and comparison principles [9, 22], sub-supersolutions [12, 23], and existence/multiplicity results for several types of reactions based on both topological and variational methods [3, 4, 8, 13, 17, 18, 31].

In dealing with nonlinear elliptic boundary value problems, the most delicate case is when the reaction is asymptotically $(p-1)$-linear at infinity, namely, when uniformly for a.e. $x \in \Omega$

$$-\infty < \liminf_{|t| \to \infty} \frac{f(x,t)}{|t|^{p-2}t} \leqslant \limsup_{|t| \to \infty} \frac{f(x,t)}{|t|^{p-2}t} < \infty, \tag{1.3}$$

in particular when such limits lie above the principal eigenvalue of the leading operator. Indeed, such case prevents both coercivity of the energy functional (as in the sublinear case), and the Ambrosetti-Rabinowitz condition at infinity (which is typically required in the superlinear case and allows for the use of the mountain pass theorem, see [2]). Usually one requires some type of non-resonance condition, i.e., that the limits in (1.3)

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do not coincide with any eigenvalue (see [1]). Under such additional assumption, the existence of nontrivial solutions can be proved either via topological or variational methods (see [13, 16], respectively, for problem (1.1)). Several results have been proved for (local) p-Laplacian problems with noncoercive energy, see for instance [11, 29].

In this paper, we apply the state-of-the-art theory on the fractional p-Laplacian to a quite specific situation, taking inspiration from an interesting work of Papageorgiou-Smyrlis [30] dealing with the p-Laplacian. Our hypotheses are the following: $f(x,\cdot)$ is asymptotically $(p-1)$-linear at $\pm\infty$ with the limiting slopes in (1.3) above the principal eigenvalue; also, $f(x,\cdot)$ is $(p-1)$-sublinear at 0, crosses the 0-line at $a_- < 0 < a_+$, and satisfies a quasi-monotonicity condition. Under such hypotheses we prove that (1.1) has at least four nontrivial constant sign solutions, two positive and two negative (Theorem 3.5). Adding a reverse Ambrosetti-Rabinowitz condition near the origin, (1.1) gains a fifth, nodal solution (Theorem 4.3).

Our approach is variational, based on critical point theory and Morse theory, and also makes a wide use of truncations, sub-supersolutions, and comparison principle (we will use the recent results of [18]). Note that, while assuming that the limits in (1.3) are above the principal eigenvalue, we do not assume non-resonance, thus turning our attention on the behavior of $f(x,\cdot)$ near the origin (in such sense we generalize the result of [30]). We confine ourselves to the degenerate case $p \geqslant 2$ essentially due to regularity reasons (see [20]), while the variational machinery also works, with minor adjustments, for the singular case $1 < p < 2$. Also, we remark that our result is new, even in the semilinear case $p = 2$ (fractional Laplacian).

The paper has the following structure: in Section 2 we recall the functional-analytic framework and some well-known results about fractional p-Laplacian problems; in Section 3 we show the existence of constant sign solutions (two positive and two negative); and in Section 4 we prove the existence of a nodal solution.

Notation: Throughout the paper, for any $A \subset \mathbb{R}^N$ we shall set $A^c = \mathbb{R}^N \setminus A$. For any two measurable functions $f, g : \Omega \to \mathbb{R}$, $f \leqslant g$ in Ω will mean that $f(x) \leqslant g(x)$ for a.e. $x \in \Omega$ (and similar expressions). The positive (resp., negative) part of f is denoted f^+ (resp., f^-). If X is an ordered Banach space, then X_+ will denote its non-negative order cone. For all $r \in [1,\infty]$, $\|\cdot\|_r$ denotes the standard norm of $L^r(\Omega)$ (or $L^r(\mathbb{R}^N)$, which will be clear from the context). Every function u defined in Ω will be identified with its 0-extension to $\mathbb{R}^N$. The constant functions of the type $u(x) = a$ are simply denoted by a. Moreover, C will denote a positive constant (whose value may change case by case).

2. Preliminaries

In this section we introduce the functional-analytic framework for problem (1.1) and recall some useful results about the fractional p-Laplacian.

First, for any Banach space $(X, \|\cdot\|)$ with topological dual $(X^*, \|\cdot\|_*)$ and all $u_0 \in X$, $\rho > 0$ we set

$$\overline{B}_\rho(u_0) = \big\{u \in X : \|u - u_0\| \leqslant \rho\big\}.$$

Let $\Phi \in C^1(X)$ be a functional. For all $c \in \mathbb{R}$ we set

$$\Phi^c = \big\{u \in X : \Phi(u) \leqslant c\big\}.$$

We say that Φ satisfies the Palais-Smale condition at level $c \in \mathbb{R}$, shortly $(PS)_c$, if every sequence (u_n) in X, s.t. $\Phi(u_n) \to c$ and $\Phi'(u_n) \to 0$ in X^*, admits a (strongly) convergent subsequence. We say that Φ satisfies (PS), if Φ satisfies $(PS)_c$ for any $c \in \mathbb{R}$.

We denote by $K(\Phi)$ the set of all critical points of Φ, and by $K_c(\Phi)$ the set of all $u \in K(\Phi)$ s.t. $\Phi(u) = c$. We say that $u \in K_c(\Phi)$ is an isolated critical point, if there exists a neighborhood $U \subset X$ of u s.t. $K(\Phi) \cap U = \{u\}$, and in such case, for all $k \in \mathbb{N}$ we define the k-th critical group of Φ at u as

$$C_k(\Phi, u) = H_k(\Phi^c \cap U, \Phi^c \cap U \setminus \{u\}),$$

where $H_k(\cdot,\cdot)$ denotes the k-th singular homology group for a topological pair (see [27, Chapter 6]).

Following [10], for all measurable $u : \mathbb{R}^N \to \mathbb{R}$ we set

$$[u]_{s,p}^p = \iint_{\mathbb{R}^N \times \mathbb{R}^N} \frac{|u(x) - u(y)|^p}{|x - y|^{N+ps}}\, dx\, dy,$$

and define the fractional Sobolev spaces

$$W^{s,p}(\mathbb{R}^N) = \big\{u \in L^p(\mathbb{R}^N) : [u]_{s,p} < \infty\big\},$$

$$W_0^{s,p}(\Omega)=\big\{u\in W^{s,p}(\mathbb{R}^N):\,u(x)=0\text{ in }\Omega^c\big\},$$

the latter being a uniformly convex, separable Banach space with norm $\|u\|=[u]_{s,p}$. The dual space of $W_0^{s,p}(\Omega)$ is denoted $W^{-s,p'}(\Omega)$, with norm $\|\cdot\|_{-s,p'}$. The embedding $W_0^{s,p}(\Omega)\hookrightarrow L^q(\Omega)$ is continuous for all $q\in[1,p_s^*]$ and compact for all $q\in[1,p_s^*)$, with $p_s^*=Np/(N-ps)$.

Also we recall from [19, Definition 2.1] the following space

$$\widetilde{W}^{s,p}(\Omega)=\Big\{u\in L^p_{\rm loc}(\mathbb{R}^N):\,\exists\,\Omega'\Supset\Omega\text{ s.t. }u\in W^{s,p}(\Omega'),\int_{\mathbb{R}^N}\frac{|u(x)|^{p-1}}{(1+|x|)^{N+ps}}\,dx<\infty\Big\}.$$

Clearly, any constant function $u(x)=a$ in $\mathbb{R}^N$ lies in $\widetilde{W}^{s,p}(\Omega)$. Further, $W_0^{s,p}(\Omega)\subseteq\widetilde{W}^{s,p}(\Omega)$, and vice versa we have:

Lemma 2.1. *If $u\in\widetilde{W}^{s,p}(\Omega)$ and $u=0$ in Ω^c, then $u\in W_0^{s,p}(\Omega)$.*

Proof. By assumption, $u\in L^p(\mathbb{R}^N)$ and $u=0$ in Ω^c, so there remains to show that $[u]_{s,p}<\infty$. Indeed, by symmetry we have

$$\begin{aligned}\iint_{\mathbb{R}^N\times\mathbb{R}^N}\frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}}\,dx\,dy&=\iint_{\Omega'\times\Omega'}\frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}}\,dx\,dy+2\iint_{\Omega'\times\Omega'^c}\frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}}\,dx\,dy\\&\quad+\iint_{\Omega'^c\times\Omega'^c}\frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}}\,dx\,dy.\end{aligned}$$

We note that the first integral on the right-hand side is finite since $u\in W^{s,p}(\Omega')$ and the third integral is zero by the condition on Ω^c. So we focus on the second integral:

$$\begin{aligned}\iint_{\Omega'\times\Omega'^c}\frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}}\,dx\,dy&=\iint_{\Omega\times\Omega'^c}\frac{|u(x)|^p}{|x-y|^{N+ps}}\,dx\,dy\\&\leqslant\iint_{\Omega\times\Omega'^c}\frac{|u(x)|^p}{C(1+|y|^{N+ps})}\,dx\,dy\\&=\frac{1}{C}\int_\Omega|u(x)|^p\,dx\int_{\Omega'^c}\frac{1}{(1+|y|^{N+ps})}\,dy<\infty.\end{aligned}$$

Hence $[u]_{s,p}<\infty$, so $u\in W_0^{s,p}(\Omega)$. □

By [19, Lemma 2.3], we can rephrase the fractional p-Laplacian as an operator $(-\Delta)_p^s:\widetilde{W}^{s,p}(\Omega)\to W^{-s,p'}(\Omega)$ defined for all $u\in\widetilde{W}^{s,p}(\Omega)$, $v\in W_0^{s,p}(\Omega)$ by

$$\langle(-\Delta)_p^s\,u,v\rangle=\iint_{\mathbb{R}^N\times\mathbb{R}^N}\frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))(v(x)-v(y))}{|x-y|^{N+ps}}\,dx\,dy.$$

This definition is equivalent to (1.2), provided u is sufficiently smooth. The restricted operator $(-\Delta)_p^s:W_0^{s,p}(\Omega)\to W^{-s,p'}(\Omega)$ is continuous, and satisfies the $(S)_+$-property, i.e., whenever (u_n) is a sequence in $W_0^{s,p}(\Omega)$ s.t. $u_n\rightharpoonup u$ in $W_0^{s,p}(\Omega)$ and

$$\limsup_n\,\langle(-\Delta)_p^s\,u_n,u_n-u\rangle\leqslant 0,$$

then $u_n\to u$ in $W_0^{s,p}(\Omega)$ (see [12, Lemma 2.1], [14, Lemma 3.2]). By [17, Lemma 2.1], for all $u\in W_0^{s,p}(\Omega)$ we have

$$\|u^\pm\|^p\leqslant\langle(-\Delta)_p^s\,u,\pm u^\pm\rangle. \tag{2.1}$$

Moreover, the operator $(-\Delta)_p^s$ is strictly (T)-monotone, namely:

Proposition 2.2. [24, proof of Lemma 9] *Let $u,v\in\widetilde{W}^{s,p}(\Omega)$ s.t. $(u-v)^+\in W_0^{s,p}(\Omega)$ satisfy*

$$\langle(-\Delta)_p^s\,u-(-\Delta)_p^s\,v,(u-v)^+\rangle\leqslant 0,$$

then $u\leqslant v$ in Ω.

We consider problem (1.1) under the following hypothesis:

- $\mathbf{H}_0$ $f:\Omega\times\mathbb{R}\to\mathbb{R}$ is a Carathéodory function, and there exist $c_0>0$, $r\in(1,p_s^*)$ s.t. for a.e. $x\in\Omega$ and all $t\in\mathbb{R}$

$$|f(x,t)|\leqslant c_0(1+|t|^{r-1}).$$

We say that $u \in \widetilde{W}^{s,p}(\Omega)$ is a *(weak) supersolution* of (1.1), if $u \geqslant 0$ in Ω^c and for all $v \in W_0^{s,p}(\Omega)_+$

$$\langle (-\Delta)_p^s\, u, v\rangle \geqslant \int_\Omega f(x,u)v\,dx.$$

Similarly, we define a *(weak) subsolution* of (1.1). We say that $(\underline{u},\overline{u}) \in \widetilde{W}^{s,p}(\Omega)\times\widetilde{W}^{s,p}(\Omega)$ is a *sub-supersolution pair* of (1.1), if $\underline{u}$ is a subsolution, $\overline{u}$ is a supersolution, and $\underline{u} \leqslant \overline{u}$ in Ω. Finally, $u \in W_0^{s,p}(\Omega)$ is a *(weak) solution* of (1.1) if u is both a super- and a subsolution, i.e., if for all $v \in W_0^{s,p}(\Omega)$

$$\langle (-\Delta)_p^s\, u, v\rangle = \int_\Omega f(x,u)v\,dx.$$

On spaces $W_0^{s,p}(\Omega), \widetilde{W}^{s,p}(\Omega)$ we consider the pointwise partial ordering, inducing a lattice structure. Set

$$u \wedge v = \min\{u,v\},\ \ u \vee v = \max\{u,v\}.$$

We recall that the pointwise minimum of supersolutions is a supersolution, as well as the maximum of subsolutions is a subsolution (see also [23]):

Proposition 2.3. [12, Lemma 3.1] *Let* $\mathbf{H}_0$ *hold and* $u_1, u_2 \in \widetilde{W}^{s,p}(\Omega)$*:*

(i) if u_1, u_2 *are supersolutions of* (1.1)*, then so is* $u_1 \wedge u_2$*;*
(ii) if u_1, u_2 *are subsolutions of* (1.1) *then so is* $u_1 \vee u_2$*.*

For any sub-supersolution pair $(\underline{u},\overline{u})$ of (1.1) we define the constrained solution set

$$\mathcal{S}(\underline{u},\overline{u}) = \big\{u \in W_0^{s,p}(\Omega) : u \text{ solves (1.1)},\ \underline{u} \leqslant u \leqslant \overline{u} \text{ in } \Omega\big\}.$$

The properties of $\mathcal{S}(\underline{u},\overline{u})$ were studied in [12], and amount at the following:

Proposition 2.4. [12, Theorem 3.5] *Let* $\mathbf{H}_0$ *hold,* $(\underline{u},\overline{u})$ *be a sub-supersolution pair of* (1.1)*. Then* $\mathcal{S}(\underline{u},\overline{u})$ *is both upward and downward directed, compact in* $W_0^{s,p}(\Omega)$*, and it contains a smallest and a biggest element.*

Some words about regularity of solutions are now in order. First, we have a uniform *a priori* bound:

Proposition 2.5. [8, Theorem 3.3] *Let* $\mathbf{H}_0$ *hold,* $u \in W_0^{s,p}(\Omega)$ *be a solution of* (1.1)*. Then,* $u \in L^\infty(\Omega)$ *with* $\|u\|_\infty \leqslant C$*, for some* $C = C(\|u\|) > 0$*.*

Solutions of fractional order equations have a good interior regularity, but they may fail to be smooth up to the boundary. Thus, an important role in regularity theory for nonlocal operators is played by the following weighted Hölder spaces, with weight

$$\mathrm{d}_\Omega^s(x) = \mathrm{dist}(x,\Omega^c)^s.$$

Set

$$C_s^0(\overline{\Omega}) = \Big\{u \in C^0(\overline{\Omega}) : \frac{u}{\mathrm{d}_\Omega^s} \text{ has a continuous extension to } \overline{\Omega}\Big\},$$

and for all $\alpha \in (0,1)$

$$C_s^\alpha(\overline{\Omega}) = \Big\{u \in C^0(\overline{\Omega}) : \frac{u}{\mathrm{d}_\Omega^s} \text{ has a } \alpha\text{-Hölder continuous extension to } \overline{\Omega}\Big\},$$

with norms, respectively,

$$\|u\|_{0,s} = \Big\|\frac{u}{\mathrm{d}_\Omega^s}\Big\|_\infty,\ \ \|u\|_{\alpha,s} = \|u\|_{0,s} + \sup_{x\neq y}\frac{|u(x)/\mathrm{d}_\Omega^s(x) - u(y)/\mathrm{d}_\Omega^s(y)|}{|x-y|^\alpha}.$$

The embedding $C_s^\alpha(\overline{\Omega}) \hookrightarrow C_s^0(\overline{\Omega})$ is compact for all $\alpha \in (0,1)$. By [16, Lemma 5.1], the positive cone $C_s^0(\overline{\Omega})_+$ of $C_s^0(\overline{\Omega})$ has a nonempty interior given by

$$\mathrm{int}(C_s^0(\overline{\Omega})_+) = \Big\{u \in C_s^0(\overline{\Omega}) : \inf_\Omega \frac{u}{\mathrm{d}_\Omega^s} > 0\Big\}.$$

Combining Proposition 2.5 and [20, Theorem 1.1], we have the following global regularity result for the degenerate case $p \geqslant 2$:

Proposition 2.6. *Let* $\mathbf{H}_0$ *hold,* $u \in W_0^{s,p}(\Omega)$ *be a solution of* (1.1)*. Then,* $u \in C_s^\alpha(\overline{\Omega})$ *for some* $\alpha \in (0,s]$*.*

We recall from [18] two general maximum/comparison principles:

Proposition 2.7. [18, Theorem 2.6] *Let* $g \in C^0(\mathbb{R}) \cap BV_{\rm loc}(\mathbb{R})$, $u \in \widetilde{W}^{s,p}(\Omega) \cap C^0(\bar{\Omega}) \setminus \{0\}$ *s.t.*

$$\begin{cases} (-\Delta)_p^s\, u + g(u) \geqslant g(0) & \text{in } \Omega \\ u \geqslant 0 & \text{in } \mathbb{R}^N. \end{cases}$$

Then,

$$\inf_{\Omega} \frac{u}{{\rm d}_\Omega^s} > 0.$$

In particular, if $u \in C_s^0(\overline{\Omega})$, *then* $u \in {\rm int}(C_s^0(\overline{\Omega})_+)$.

Proposition 2.8. [18, Theorem 2.7] *Let* $g \in C^0(\mathbb{R}) \cap BV_{\rm loc}(\mathbb{R})$, $u \in W_0^{s,p}(\Omega) \cap C^0(\overline{\Omega})$, $v \in \widetilde{W}^{s,p}(\Omega) \cap C^0(\bar{\Omega})$, $u \neq v$ *and* $C > 0$ *s.t.*

$$\begin{cases} (-\Delta)_p^s\, u + g(u) \leqslant (-\Delta)_p^s\, v + g(v) \leqslant C & \text{in } \Omega \\ 0 < u \leqslant v & \text{in } \Omega \\ v \geqslant 0 & \text{in } \Omega^c. \end{cases}$$

Then,

$$\inf_{\Omega} \frac{v-u}{{\rm d}_\Omega^s} > 0.$$

In particular, if $u, v \in {\rm int}(C_s^0(\overline{\Omega})_+)$, *then* $v - u \in {\rm int}(C_s^0(\overline{\Omega})_+)$.

Now we define an energy functional for problem (1.1) by setting for all $(x,t) \in \Omega \times \mathbb{R}$

$$F(x,t) = \int_0^t f(x,\tau)\, d\tau,$$

and for all $u \in W_0^{s,p}(\Omega)$

$$\Phi(u) = \frac{\|u\|^p}{p} - \int_\Omega F(x,u)\, dx.$$

By $\mathbf{H}_0$, it is easily seen that $\Phi \in C^1(W_0^{s,p}(\Omega))$ with Gâteaux derivative given for all $u, v \in W_0^{s,p}(\Omega)$ by

$$\langle \Phi'(u), v\rangle = \langle (-\Delta)_p^s\, u, v\rangle - \int_\Omega f(x,u) v\, dx.$$

Clearly, then, critical points of Φ coincide with the solutions of (1.1). Also, Φ is sequentially weakly lower semicontinuous, and every bounded (PS)-sequence for Φ has a convergent subsequence (see [16, Proposition 2.1]). Another useful property is that local minima of Φ in the topologies of $W_0^{s,p}(\Omega)$ and $C_s^0(\overline{\Omega})$, respectively, coincide:

Proposition 2.9. [20, Theorem 1.1] *Let* $\mathbf{H}_0$ *hold,* $u \in W_0^{s,p}(\Omega)$. *Then, the following are equivalent:*

- (i) *there exists* $\rho > 0$ *s.t.* $\Phi(u+v) \geqslant \Phi(u)$ *for all* $v \in W_0^{s,p}(\Omega) \cap C_s^0(\overline{\Omega})$, $\|v\|_{0,s} \leqslant \rho$;
- (ii) *there exists* $\sigma > 0$ *s.t.* $\Phi(u+v) \geqslant \Phi(u)$ *for all* $v \in W_0^{s,p}(\Omega)$, $\|v\| \leqslant \sigma$.

Finally, we recall some properties of the following weighted eigenvalue problem, with weight $m \in L^\infty(\Omega)_+ \setminus \{0\}$:

$$\begin{cases} (-\Delta)_p^s\, u = \lambda m(x)|u|^{p-2}u & \text{in } \Omega \\ u = 0 & \text{in } \Omega^c. \end{cases} \tag{2.2}$$

Problem (2.2) admits an unbounded, nondecreasing sequence of positive variational (Lusternik-Schnirelmann) eigenvalues $(\lambda_k(m))$. In particular, as a special case of [15, Theorem 1.1] (see also [14, Proposition 3.4]) we have:

Proposition 2.10. *Let* $m \in L^\infty(\Omega)_+ \setminus \{0\}$. *Then, the smallest eigenvalue of* (2.2) *is*

$$\lambda_1(m) = \inf_{u \in W_0^{s,p}(\Omega)\setminus\{0\}} \frac{\|u\|^p}{\int_\Omega m(x)|u|^p\, dx} > 0.$$

Besides,

- (i) $\lambda_1(m)$ *is simple, isolated, with constant sign eigenfunctions, while for any eigenvalue* $\lambda > \lambda_1(m)$ *of* (2.2) λ*-eigenfunctions are nodal;*
- (ii) *if* $\tilde{m} \in L^\infty(\Omega)_+ \setminus \{0\}$ *is s.t.* $m \leqslant \tilde{m}$ *in* Ω *and* $m \not\equiv \tilde{m}$, *we have* $\lambda_1(m) > \lambda_1(\tilde{m})$.

When $m = 1$, we set $\lambda_1(1) = \lambda_1$ and we denote by $e_1 \in \mathrm{int}(C^0_s(\overline{\Omega})_+)$ the unique, positive λ_1-eigenfunction s.t. $\|e_1\|_p = 1$.

3. Constant sign solutions

In this section we study problem (1.1) under the following hypotheses:

$\mathbf{H}_1$ $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory mapping satisfying

(*i*) there exists $c_0 > 0$ s.t. for a.e. $x \in \Omega$ and all $t \in \mathbb{R}$
$$|f(x,t)| \leqslant c_0(1 + |t|^{p-1});$$

(*ii*) there exist $\eta_1, \eta_2 \in L^\infty(\Omega)$ s.t. $\eta_1 \geqslant \lambda_1$ in Ω, $\eta_1 \not\equiv \lambda_1$, and uniformly for a.e. $x \in \Omega$
$$\eta_1(x) \leqslant \liminf_{|t|\to\infty} \frac{f(x,t)}{|t|^{p-2}t} \leqslant \limsup_{|t|\to\infty} \frac{f(x,t)}{|t|^{p-2}t} \leqslant \eta_2(x);$$

(*iii*) there exist $\delta_0, c_1 > 0$, $q \in (1,p)$ s.t. for a.e. $x \in \Omega$
$$f(x,t) \begin{cases} \geqslant c_1 t^{q-1} & \text{for all } t \in [0,\delta_0], \\ \leqslant c_1|t|^{q-2}t & \text{for all } t \in [-\delta_0, 0]; \end{cases}$$

(*iv*) there exist a_-, a_+ with $\min\{a_+, -a_-\} > \delta_0$ s.t. for a.e. $x \in \Omega$
$$f(x,a_-) = f(x,0) = f(x,a_+) = 0;$$

(*v*) there exists $c_2 > 0$ s.t. for a.e. $x \in \Omega$ the map
$$t \mapsto f(x,t) + c_2|t|^{p-2}t$$
is nondecreasing in $[a_-, a_+]$.

Hypotheses $\mathbf{H}_1$ include a subcritical growth condition (*i*), $(p-1)$-linear asymptotic growth with non-resonance on the first eigenvalue (*ii*), a $(p-1)$-sublinear behavior near the origin (*iii*), vanishing at $a_- < 0 < a_+$ (not necessarily with a sign change) (*iv*), and a quasi-monotonicity condition (*v*). We note that, contrary to the assumptions of [30], we do not require that $\eta_2 < \lambda_2$ in Ω, that is, we allow resonance on higher eigenvalues.

Clearly, $\mathbf{H}_1$ imply $\mathbf{H}_0$ (with $r = p$), so all results of Section 2 apply here, with F, Φ defined as above. Also, by $\mathbf{H}_1$ (*iv*) problem (1.1) admits the trivial solution $u = 0$. Without loss of generality, we may assume that (1.1) has only *finitely many* solutions.

Example 3.1. We present here an autonomous reaction $f \in C^0(\mathbb{R})$ satisfying $\mathbf{H}_1$. Set for simplicity $p = 2$, and fix real numbers $\eta > \lambda_1$, $\gamma > \eta + 1$, and set for all $t \geqslant 0$
$$f(t) = \frac{\eta t^2 - \gamma t + \sqrt{t}}{t+1}.$$
We focus on the positive semiaxis. Clearly, f satisfies (*i*). Since
$$\lim_{t\to\infty} \frac{f(t)}{t} = \eta,$$
then we have (*ii*). Also, we obtain
$$\lim_{t\to 0^+} \frac{f(t)}{\sqrt{t}} = 1,$$
which implies (*iii*) with $q = 3/2$. Since $f(t) > 0$ for all $t > 0$ small enough and
$$f(1) = \frac{\eta - \gamma + 1}{2} < 0,$$
then there exists $a_+ \in (0,1)$ s.t. $f(a_+) = 0$, whence (*iv*). Finally, since $f'(t)$ is bounded below in $(0,\infty)$, we can find $c_2 > 0$ s.t.
$$t \mapsto f(t) + c_2 t$$
is increasing (*v*). Taking $f(t) = -f(-t)$ for all $t < 0$, we complete the definition.

We begin our study by proving the existence of a first positive solution:

Lemma 3.2. *If $\mathbf{H}_1$ holds, then (1.1) has a solution $u_0 \in \mathrm{int}(C^0_s(\overline{\Omega})_+)$, which is a local minimizer of Φ.*

Proof. Set for all $(x,t) \in \Omega \times \mathbb{R}$

$$f_0(x,t) = \begin{cases} 0 & \text{if } t < 0, \\ f(x,t) & \text{if } 0 \leqslant t \leqslant a_+, \\ 0 & \text{if } t > a_+, \end{cases}$$

with $a_+ > 0$ as in $\mathbf{H}_1$ *(iv)*. By $\mathbf{H}_1$ *(i)* *(iv)* we see that $f_0 : \Omega \times \mathbb{R} \to \mathbb{R}$ satisfies $\mathbf{H}_0$. Also, it is bounded. Accordingly, set for all $(x,t) \in \Omega \times \mathbb{R}$

$$F_0(x,t) = \int_0^t f_0(x,\tau)\,d\tau,$$

and for all $u \in W_0^{s,p}(\Omega)$

$$\Phi_0(u) = \frac{\|u\|^p}{p} - \int_\Omega F_0(x,u)\,dx.$$

As seen in Section 2, $\Phi_0 \in C^1(W_0^{s,p}(\Omega))$ is sequentially weakly lower semicontinuous. Besides, Φ_0 is coercive. Indeed, for all $u \in W_0^{s,p}(\Omega)$

$$\Phi_0(u) \geqslant \frac{\|u\|^p}{p} - \int_\Omega C|u|\,dx \geqslant \frac{\|u\|^p}{p} - C\|u\|,$$

and the latter tends to ∞ as $\|u\| \to \infty$. So there exists $u_0 \in W_0^{s,p}(\Omega)$ s.t.

$$\Phi_0(u_0) = \inf_{u \in W_0^{s,p}(\Omega)} \Phi_0(u) =: m_0. \tag{3.1}$$

Let $e_1 \in \mathrm{int}(C_s^0(\overline{\Omega})_+)$ be defined as in Proposition 2.10. In particular, $e_1 \in L^\infty(\Omega)$, so for all $\tau > 0$ small enough (recalling $\mathbf{H}_1$ *(iv)*) we have in Ω

$$0 < \tau e_1 \leqslant \delta_0 < a_+.$$

Hence, by $\mathbf{H}_1$ *(iii)* we have

$$\Phi_0(\tau e_1) \leqslant \frac{\tau^p}{p}\|e_1\|^p - \int_\Omega \frac{c_1}{q}(\tau e_1)^q\,dx = \frac{\lambda_1}{p}\tau^p - \frac{c_1\|e_1\|_q^q}{q}\tau^q,$$

and the latter is negative for all $\tau > 0$ small enough (as $q < p$). So in (3.1) we have

$$m_0 = \Phi_0(u_0) < 0 = \Phi_0(0),$$

hence $u_0 \neq 0$. By (3.1) we have $\Phi_0'(u_0) = 0$, i.e. weakly in Ω

$$(-\Delta)_p^s\, u_0 = f_0(x,u_0). \tag{3.2}$$

Testing (3.2) with $-u_0^- \in W_0^{s,p}(\Omega)$ and using (2.1), we get

$$\|u_0^-\|^p \leqslant \langle(-\Delta)_p^s\, u_0, -u_0^-\rangle = \int_\Omega f_0(x,u_0)(-u_0^-)\,dx = 0,$$

so $u_0 \geqslant 0$ in Ω. Besides, note that $a_+ \in \widetilde{W}^{s,p}(\Omega)$ satisfies weakly in Ω

$$(-\Delta)_p^s\, a_+ = 0.$$

By Lemma 2.1 we have $(u_0 - a_+)^+ \in W_0^{s,p}(\Omega)$. By using such function as test in (3.2) and the above equation, we get

$$\langle(-\Delta)_p^s\, u_0 - (-\Delta)_p^s\, a_+, (u_0 - a_+)^+\rangle = \int_\Omega f_0(x,u_0)(u_0 - a_+)^+\,dx = 0.$$

By Proposition 2.2 we have $0 \leqslant u_0 \leqslant a_+$ in Ω. Thus, in (3.2) we can replace f_0 with f and see that $u_0 \in W_0^{s,p}(\Omega) \setminus \{0\}$ is a solution of (1.1). By Proposition 2.6 we have $u_0 \in C_s^\alpha(\overline{\Omega}) \setminus \{0\}$. Further, by $\mathbf{H}_1$ *(v)* we have weakly in Ω

$$\begin{cases} (-\Delta)_p^s\, u_0 + c_2 u_0^{p-1} = f(x,u_0) + c_2 u_0^{p-1} \geqslant f(x,0) = 0 \\ u_0 \geqslant 0. \end{cases}$$

By Proposition 2.7 (with $g(t) = c_2|t|^{p-2}t$), we deduce $u_0 \in \mathrm{int}(C_s^0(\overline{\Omega})_+)$.

Finally, we prove that u_0 is a local minimizer of Φ. By $\mathbf{H}_1$ *(iv) (v)* we have weakly in Ω

$$\begin{cases} (-\Delta)_p^s\, u_0 + c_2 u_0^{p-1} \leqslant f(x,a_+) + c_2 a_+^{p-1} = (-\Delta)_p^s\, a_+ + c_2 a_+^{p-1} = c_2 a_+^{p-1} \\ 0 < u_0 \leqslant a_+, \end{cases}$$

while $u_0 \in W^{s,p}_0(\Omega)$ clearly implies $u_0 \neq a_+$. So, by Proposition 2.8 (with the same choice of g), we infer that

$$\inf_{\Omega} \frac{a_+ - u_0}{\mathrm{d}_\Omega^s} > 0,$$

in particular $0 < u_0 < a_+$ in Ω. Set

$$\mathcal{U} = \big\{u \in W^{s,p}_0(\Omega) \cap C^0_s(\overline{\Omega}) : 0 < u < a_+ \text{ in } \Omega\big\},$$

an open set in $C^0_s(\overline{\Omega})$ s.t. $u_0 \in \mathcal{U}$. By (3.1) we have for all $u \in \mathcal{U}$

$$\Phi(u) = \Phi_0(u) \geqslant \Phi_0(u_0) = \Phi(u_0).$$

So, u_0 is a local minimizer of Φ in $C^0_s(\overline{\Omega})$. By Proposition 2.9, u_0 is a local minimizer of Φ in $W^{s,p}_0(\Omega)$ as well. □

Exploiting the asymptotic behavior of $f(x,\cdot)$ at ∞, we find a second positive solution:

Lemma 3.3. *If $\mathbf{H}_1$ holds, then (1.1) has a solution $u_1 \in \mathrm{int}(C^0_s(\overline{\Omega})_+)$ s.t. $u_1 \geqslant u_0$ in Ω, $u_1 \neq u_0$.*

Proof. Set for all $(x,t) \in \Omega \times \mathbb{R}$

$$f_1(x,t) = \begin{cases} f(x,u_0) & \text{if } t < u_0(x), \\ f(x,t) & \text{if } t \geqslant u_0(x), \end{cases}$$

with $u_0 \in \mathrm{int}(C^0_s(\overline{\Omega})_+)$ given by Lemma 3.2. By $\mathbf{H}_1$ *(i)*, $f_1 : \Omega \times \mathbb{R} \to \mathbb{R}$ satisfies $\mathbf{H}_0$. So, for all $(x,t) \in \Omega \times \mathbb{R}$ we set

$$F_1(x,t) = \int_0^t f_1(x,\tau)\,d\tau.$$

We define $\Phi_1 \in C^1(W^{s,p}_0(\Omega))$ by setting for all $u \in W^{s,p}_0(\Omega)$

$$\Phi_1(u) = \frac{\|u\|^p}{p} - \int_\Omega F_1(x,u)\,dx.$$

We prove first that Φ_1 satisfies (PS). Indeed, let (w_n) be a sequence in $W^{s,p}_0(\Omega)$ s.t. $|\Phi_1(w_n)| \leqslant C$ and $\Phi_1'(w_n) \to 0$ in $W^{-s,p'}(\Omega)$, i.e., there exists a sequence (ε_n) in $\mathbb{R}^+_0$ s.t. $\varepsilon_n \to 0^+$ and for all $n \in \mathbb{N}$, $v \in W^{s,p}_0(\Omega)$

$$\Big|\langle(-\Delta)_p^s\, w_n, v\rangle - \int_\Omega f_1(x,w_n)v\,dx\Big| \leqslant \varepsilon_n\|v\|. \tag{3.3}$$

We aim at proving that (w_n) is bounded in $W^{s,p}_0(\Omega)$. Arguing by contradiction, assume that up to a subsequence $\|w_n\| \to \infty$. Testing (3.3) with $-w_n^- \in W^{s,p}_0(\Omega)$ and applying (2.1), we have for all $n \in \mathbb{N}$

$$\begin{aligned} \|w_n^-\|^p &\leqslant \langle(-\Delta)_p^s\, w_n, -w_n^-\rangle \\ &\leqslant \int_\Omega f_1(x,w_n)(-w_n^-)\,dx + \varepsilon_n\|w_n^-\| \\ &= \int_\Omega f(x,u_0)(-w_n^-)\,dx + \varepsilon_n\|w_n^-\| \leqslant C\|w_n^-\|, \end{aligned}$$

where we have used the definition of f_1 and $\mathbf{H}_1$ *(i)*. Since $p > 1$, (w_n^-) is bounded in $W^{s,p}_0(\Omega)$. So

$$\|w_n^+\| = \|w_n + w_n^-\| \geqslant \|w_n\| - \|w_n^-\| \to \infty.$$

Set for all $n \in \mathbb{N}$

$$\hat{w}_n = \frac{w_n}{\|w_n\|},$$

so $\hat{w}_n \in W^{s,p}_0(\Omega)$ with $\|\hat{w}_n\| = 1$. Passing to a subsequence, we have $\hat{w}_n \rightharpoonup \hat{w}$ in $W^{s,p}_0(\Omega)$, $\hat{w}_n \to \hat{w}$ in $L^p(\Omega)$, and $\hat{w}_n(x) \to \hat{w}(x)$ for a.e. $x \in \Omega$. Since $\hat{w}_n^- \to 0$ in $W^{s,p}_0(\Omega)$, we deduce that $\hat{w} \geqslant 0$ in Ω. By (3.3) we have for all $n \in \mathbb{N}$, $v \in W^{s,p}_0(\Omega)$

$$\langle(-\Delta)_p^s\, \hat{w}_n, v\rangle \leqslant \int_\Omega \frac{f_1(x,w_n)}{\|w_n\|^{p-1}}v\,dx + \frac{\varepsilon_n}{\|w_n\|^{p-1}}\|v\|. \tag{3.4}$$

By $\mathbf{H}_1$ (i) and the definition of f_1, we have for all $n \in \mathbb{N}$

$$\int_\Omega \Big|\frac{f_1(x,w_n)}{\|w_n\|^{p-1}}\Big|^{p'}\,dx \leqslant \int_\Omega \frac{C(1+|w_n|^{p-1})^{p'}}{\|w_n\|^p}\,dx \leqslant C.$$

By reflexivity of $L^{p'}(\Omega)$, passing to a subsequence we have

$$\frac{f_1(\cdot,w_n)}{\|w_n\|^{p-1}} \rightharpoonup h_1 \text{ in } L^{p'}(\Omega). \tag{3.5}$$

Setting $v = \hat{w}_n - \hat{w} \in W^{s,p}_0(\Omega)$ in (3.4), we have for all $n \in \mathbb{N}$

$$\langle(-\Delta)^s_p\,\hat{w}_n, \hat{w}_n - \hat{w}\rangle \leqslant \int_\Omega \frac{f_1(x,w_n)}{\|w_n\|^{p-1}}(\hat{w}_n - \hat{w})\,dx + \frac{\varepsilon_n}{\|w_n\|^{p-1}}\|\hat{w}_n - \hat{w}\|,$$

and the latter tends to 0 as $n \to \infty$. By the $(S)_+$-property of $(-\Delta)^s_p$, we have $\hat{w}_n \to \hat{w}$ in $W^{s,p}_0(\Omega)$.
We claim that there exists $\beta_1 \in L^\infty(\Omega)_+$ s.t. $\eta_1 \leqslant \beta_1 \leqslant \eta_2$ in Ω and $h_1 = \beta_1\hat{w}^{p-1}$. Indeed, set

$$\Omega^+ = \big\{x \in \Omega : \hat{w}(x) > 0\big\}, \quad \Omega^0 = \big\{x \in \Omega : \hat{w}(x) = 0\big\},$$

and for all $\varepsilon > 0$, $n \in \mathbb{N}$

$$\Omega^+_{\varepsilon,n} = \Big\{x \in \Omega : w_n(x) > 0,\ \eta_1(x) - \varepsilon \leqslant \frac{f_1(x,w_n(x))}{w_n^{p-1}(x)} \leqslant \eta_2(x) + \varepsilon\Big\}.$$

By $\mathbf{H}_1$ (ii) we have $\chi_{\Omega^+_{\varepsilon,n}} \to 1$ a.e. in Ω^+ with dominated convergence, hence by (3.5)

$$\chi_{\Omega^+_{\varepsilon,n}}\frac{f_1(\cdot,w_n)}{\|w_n\|^{p-1}} \rightharpoonup h_1 \text{ in } L^{p'}(\Omega^+).$$

So for all $\varepsilon > 0$, $n \in \mathbb{N}$ big enough, in Ω^+ we have both $\hat{w}_n > 0$ and

$$(\eta_1 - \varepsilon)\chi_{\Omega^+_{\varepsilon,n}}\hat{w}_n^{p-1} \leqslant \chi_{\Omega^+_{\varepsilon,n}}\frac{f_1(\cdot,w_n)}{\|w_n\|^{p-1}} \leqslant (\eta_2 + \varepsilon)\chi_{\Omega^+_{\varepsilon,n}}\hat{w}_n^{p-1}.$$

Passing to the limit as $n \to \infty$, we get in Ω^+

$$(\eta_1 - \varepsilon)\hat{w}^{p-1} \leqslant h_1 \leqslant (\eta_2 + \varepsilon)\hat{w}^{p-1}.$$

Further, letting $\varepsilon \to 0^+$, we obtain in Ω^+

$$\eta_1\hat{w}^{p-1} \leqslant h_1 \leqslant \eta_2\hat{w}^{p-1}.$$

Similarly we see that $h_1 = 0$ in Ω^0, and thus prove our claim.
Passing to the limit in (3.4) as $n \to \infty$ we have weakly in Ω

$$(-\Delta)^s_p\,\hat{w} = \beta_1(x)\hat{w}^{p-1}. \tag{3.6}$$

By $\mathbf{H}_1$ (ii) we have $\beta_1 \geqslant \lambda_1$ in Ω, $\beta_1 \not\equiv \lambda_1$. Hence, by Proposition 2.10 (ii) we have

$$\lambda_1(\beta_1) < \lambda_1(\lambda_1) = 1.$$

So, by (3.6) $\hat{w} \in W^{s,p}_0(\Omega)_+$ is a non-principal eigenfunction of $(-\Delta)^s_p$ with weight $\beta_1 \in L^\infty(\Omega)_+$, against Proposition 2.10 (i). Such contradiction implies that (w_n) is bounded in $W^{s,p}_0(\Omega)$. By the bounded (PS)-property of Φ_1, (w_n) admits a convergent subsequence.
The next step consists in proving that

$$\lim_{\tau\to\infty}\Phi_1(\tau e_1) = -\infty, \tag{3.7}$$

with $e_1 \in \mathrm{int}(C^0_s(\overline{\Omega})_+)$ given by Proposition 2.10. Indeed, by $\mathbf{H}_1$ (ii) we have

$$\int_\Omega \eta_1(x)e_1^p\,dx > \lambda_1\int_\Omega e_1^p\,dx = \lambda_1.$$

So we can fix $\varepsilon > 0$ s.t.

$$\int_\Omega \eta_1(x)e_1^p\,dx > \lambda_1 + \varepsilon.$$

By $\mathbf{H}_1$ (ii) again, there exists $T_\varepsilon > \|u_0\|_\infty$ s.t. for a.e. $x \in \Omega$ and all $t > T_\varepsilon$

$$f(x,t) \geqslant (\eta_1(x) - \varepsilon)t^{p-1},$$

which, along with $\mathbf{H}_1$ (i), implies

$$\begin{aligned}F_1(x,t) &= \int_0^{u_0} f(x,u_0)\,d\tau + \int_{u_0}^{T_\varepsilon} f(x,\tau)\,d\tau + \int_{T_\varepsilon}^t f(x,\tau)\,d\tau \\ &\geqslant f(x,u_0)u_0 - \int_{u_0}^{T_\varepsilon} c_0(1+\tau^{p-1})\,d\tau + \int_{T_\varepsilon}^t (\eta_1(x)-\varepsilon)\tau^{p-1}\,d\tau \\ &\geqslant \frac{\eta_1(x)-\varepsilon}{p}t^p - C_\varepsilon,\end{aligned}$$

for some $C_\varepsilon > 0$. Applying $\mathbf{H}_1$ (i), we easily extend such estimate to all $t \geqslant 0$, possibly with a bigger $C_\varepsilon > 0$. Therefore we have for all $\tau > 0$

$$\begin{aligned}\Phi_1(\tau e_1) &\leqslant \frac{\tau^p\|e_1\|^p}{p} - \int_\Omega \Big(\frac{\eta_1(x)-\varepsilon}{p}\tau^p e_1^p - C_\varepsilon\Big)\,dx \\ &\leqslant \frac{\lambda_1\tau^p}{p} - \frac{\tau^p}{p}\int_\Omega \eta_1(x)e_1^p\,dx + \frac{\varepsilon\tau^p}{p} + C \\ &= \Big(\lambda_1 - \int_\Omega \eta_1(x)e_1^p\,dx + \varepsilon\Big)\frac{\tau^p}{p} + C,\end{aligned}$$

and the latter tends to $-\infty$ as $\tau \to \infty$, proving (3.7).

We define now a new truncation. Recalling that $0 < u_0 < a_+$ in Ω, set for all $(x,t) \in \Omega \times \mathbb{R}$

$$\hat f_1(x,t) = \begin{cases} f(x,u_0) & \text{if } t < u_0(x), \\ f(x,t) & \text{if } u_0(x) \leqslant t \leqslant a_+, \\ 0 & \text{if } t > a_+. \end{cases}$$

As usual, by $\mathbf{H}_1$ (i) (iv) we see that $\hat f_1 : \Omega \times \mathbb{R} \to \mathbb{R}$ satisfies $\mathbf{H}_0$. Setting

$$\hat F_1(x,t) = \int_0^t \hat f_1(x,\tau)\,d\tau$$

and for all $u \in W_0^{s,p}(\Omega)$

$$\hat\Phi_1(u) = \frac{\|u\|^p}{p} - \int_\Omega \hat F_1(x,u)\,dx,$$

we deduce that $\hat\Phi_1 \in C^1(W_0^{s,p}(\Omega))$ is sequentially weakly lower semicontinuous. As in the proof of Lemma 3.2 we see that $\hat\Phi_1$ is coercive in $W_0^{s,p}(\Omega)$, so there exists $\hat u_1 \in W_0^{s,p}(\Omega)$ s.t.

$$\hat\Phi_1(\hat u_1) = \inf_{u \in W_0^{s,p}(\Omega)} \hat\Phi_1(u). \tag{3.8}$$

In particular we have $\hat\Phi_1'(\hat u_1) = 0$ in $W^{-s,p'}(\Omega)$, i.e., weakly in Ω

$$(-\Delta)_p^s\, \hat u_1 = \hat f_1(x,\hat u_1). \tag{3.9}$$

Testing (3.9) with $(u_0 - \hat u_1)^+ \in W_0^{s,p}(\Omega)$ and recalling that u_0 solves (1.1),

$$\langle (-\Delta)_p^s\, u_0 - (-\Delta)_p^s\, \hat u_1, (u_0-\hat u_1)^+\rangle = \int_\Omega \big(f(x,u_0) - \hat f_1(x,\hat u_1)\big)(u_0-\hat u_1)^+\,dx = 0,$$

hence, by Proposition 2.2, we have $\hat u_1 \geqslant u_0$ in Ω. Similarly, testing (3.9) with $(\hat u_1 - a_+)^+ \in W_0^{s,p}(\Omega)$ we get $\hat u_1 \leqslant a_+$ in Ω. By construction, we can replace $\hat f_1$ with f in (3.9), so $\hat u_1$ is a solution of (1.1). By Proposition 2.6 we have $\hat u_1 \in C_s^\alpha(\overline\Omega)$. Also, since $\hat u_1 \geqslant u_0$, we easily get $\hat u_1 \in \mathrm{int}(C_s^0(\overline\Omega)_+)$.

To conclude, we distinguish two cases:

(a) If $\hat u_1 \neq u_0$, then by $\mathbf{H}_1$ (v) and Proposition 2.8 we have $\hat u_1 > u_0$ in Ω, and setting $u_1 = \hat u_1$ we conclude.

(b) If $\hat u_1 = u_0$, then set

$$\mathcal{U} = \big\{u \in W_0^{s,p}(\Omega) \cap C_s^0(\overline\Omega) : u < a_+ \text{ in } \Omega\big\},$$

an open set in $C_s^0(\overline\Omega)$ s.t. $u_0 \in \mathcal{U}$. By (3.8) we have for all $u \in \mathcal{U}$

$$\Phi_1(u) = \hat\Phi_1(u) \geqslant \hat\Phi_1(u_0) = \Phi_1(u_0).$$

So u_0 is a local minimizer of Φ_1 in $C^0_s(\overline{\Omega})$, hence by Proposition 2.9 it is such as well in $W^{s,p}_0(\Omega)$. Namely, there exists $\rho > 0$ s.t. $\Phi_1(u) \geqslant \Phi_1(u_0)$ for all $u \in \overline{B}_\rho(u_0)$. Recalling (3.7), we see that Φ_1 exhibits the mountain pass geometry, while we have seen that it satisfies (PS). By the mountain pass theorem (see for instance [27, Theorem 5.40], or [2]), there exists $u_1 \in W^{s,p}_0(\Omega) \setminus \{u_0\}$ s.t. $\Phi_1'(u_1) = 0$ in $W^{-s,p'}(\Omega)$ and $\Phi_1(u_1) \geqslant \Phi_1(u_0)$. So we have

$$\langle(-\Delta)^s_p\, u_0 - (-\Delta)^s_p\, u_1, (u_0 - u_1)^+\rangle = \int_\Omega (f(x, u_0) - f_1(x, u_1))(u_0 - u_1)^+ \, dx = 0,$$

hence, by Proposition 2.2, we obtain $u_1 \geqslant u_0$ in Ω. So $f_1(x, u_1) = f(x, u_1)$ for a.e. $x \in \Omega$, and we deduce that u_1 solves (1.1). Finally, by $u_1 \geqslant u_0$ and reasoning as above, we have $u_1 \in \mathrm{int}(C^0_s(\overline{\Omega})_+)$.

In both cases, the proof is concluded. □

Remark 3.4. One natural question about Lemma 3.3 is the following: under hyotheses $\mathbf{H}_1$, can we ensure that $u_1 > u_0$ in Ω? In general, the answer is negative, as in case (b) above $u_1 > a_+$ may occur in Ω, so we cannot use the quasi-monotonicity condition $\mathbf{H}_1$ (v) and Proposition 2.8. Nevertheless, under the stricter assumption that

$$t \mapsto f(x, t) + c_2|t|^{p-2}t$$

is nondecreasing in $\mathbb{R}$ for a.e. $x \in \Omega$, we can prove that $u_1 > u_0$ in Ω.

Hypotheses $\mathbf{H}_1$ are somewhat symmetric, so we can argue as in Lemmas 3.2, 3.3 on the negative semiaxis and produce two negative solutions. As a whole, we have four constant sign solutions:

Theorem 3.5. *If $\mathbf{H}_1$ holds, then (1.1) has at least four nontrivial solutions:*

(i) $u_0, u_1 \in \mathrm{int}(C^0_s(\overline{\Omega})_+)$, s.t. $0 < u_0 < a_+$, $u_0 \leqslant u_1$ in Ω,
(ii) $v_0, v_1 \in -\mathrm{int}(C^0_s(\overline{\Omega})_+)$, s.t. $a_- < v_0 < 0$, $v_1 \leqslant v_0$ in Ω.

Under the same hypotheses $\mathbf{H}_1$, we can reach more precise information about constant sign solutions (which will be useful in proving existence of a nodal solution in the next section):

Lemma 3.6. *If $\mathbf{H}_1$ holds, then (1.1) has a smallest positive solution $u_+ \in \mathrm{int}(C^0_s(\overline{\Omega})_+)$ and a biggest negative solution $v_- \in -\mathrm{int}(C^0_s(\overline{\Omega})_+)$.*

Proof. First we set $\overline{u} = a_+ \in \widetilde{W}^{s,p}(\Omega)$ ($a_+ > 0$ introduced in $\mathbf{H}_1$ (iv)), which is a supersolution of (1.1). Indeed, by $\mathbf{H}_1$ (iv), we have weakly in Ω

$$(-\Delta)^s_p\, \overline{u} = 0 = f(x, \overline{u}).$$

Now let $e_1 \in \mathrm{int}(C^0_s(\overline{\Omega})_+)$ be defined as in Proposition 2.10, and (τ_n) be a decreasing sequence in $\mathbb{R}$ s.t. $\tau_n \to 0^+$, with $\tau_1 > 0$ small enough s.t. in Ω

$$0 < \tau_1 e_1 \leqslant \delta_0 < a_+, \quad \lambda_1(\tau_1 e_1)^{p-q} < c_1,$$

with $\delta_0, c_1 > 0$, $q \in (1, p)$ as in $\mathbf{H}_1$ (iii) (and recalling from $\mathbf{H}_1$ (iv) that $\delta_0 < a_+$). For any $n \in \mathbb{N}$ set $\underline{u}_n = \tau_n e_1 \in W^{s,p}_0(\Omega)$. By $\mathbf{H}_1$ (iii) and (2.2) (with $m = 1$) we have weakly in Ω

$$\begin{aligned}(-\Delta)^s_p\, \underline{u}_n &= \lambda_1(\tau_n e_1)^{p-1} < c_1(\tau_n e_1)^{q-1} \\ &\leqslant f(x, \tau_n e_1) = f(x, \underline{u}_n).\end{aligned}$$

So, $\underline{u}_n$ is a (strict) subsolution of (1.1). Moreover, for all $n \in \mathbb{N}$ we have $\underline{u}_n \leqslant \overline{u}$ in Ω, so $(\underline{u}_n, \overline{u})$ is a sub-supersolution pair. By Proposition 2.4, for all $n \in \mathbb{N}$ we can find a function $\check{u}_n \in W^{s,p}_0(\Omega)_+$ s.t.

$$\check{u}_n = \min \mathcal{S}(\underline{u}_n, \overline{u}). \tag{3.10}$$

In particular, for all $n \in \mathbb{N}$ we have weakly in Ω

$$(-\Delta)^s_p\, \check{u}_n = f(x, \check{u}_n). \tag{3.11}$$

Testing (3.11) with $\check{u}_n \in W^{s,p}_0(\Omega)$ and using $\mathbf{H}_1$ (i), we get for all $n \in \mathbb{N}$

$$\begin{aligned}\|\check{u}_n\|^p = \langle(-\Delta)^s_p\, \check{u}_n, \check{u}_n\rangle &= \int_\Omega f(x, \check{u}_n)\check{u}_n \, dx \\ &\leqslant \int_\Omega c_0(\check{u}_n + \check{u}_n^p) \, dx \leqslant c_0(a_+ + a_+^p)|\Omega|.\end{aligned}$$

Hence $(\breve{u}_n)$ is bounded in $W^{s,p}_0(\Omega)$. Passing to a subsequence, $\breve{u}_n \rightharpoonup u_+$ in $W^{s,p}_0(\Omega)$, $\breve{u}_n \to u_+$ in both $L^p(\Omega)$ and $L^1(\Omega)$, for some $u_+ \in W^{s,p}_0(\Omega)_+$. Testing (3.11) with $(\breve{u}_n - u_+)$ leads to

$$\begin{aligned}\langle(-\Delta)^s_p\, \breve{u}_n, \breve{u}_n - u_+\rangle &= \int_\Omega f(x, \breve{u}_n)(\breve{u}_n - u_+)\, dx \\ &\leqslant \int_\Omega c_0(1 + \breve{u}_n^{p-1})(\breve{u}_n - u_+)\, dx \\ &\leqslant c_0\big(\|\breve{u}_n - u_+\|_1 + \|\breve{u}_n\|_p^{p-1}\|\breve{u}_n - u_+\|_p\big)\end{aligned}$$

(where we have used $\mathbf{H}_1$ *(i)* and Hölder's inequality), and the latter tends to 0 as $n \to \infty$. By the $(S)_+$-property of $(-\Delta)^s_p$, we get $\breve{u}_n \to u_+$ in $W^{s,p}_0(\Omega)$. So we may pass to the limit as $n \to \infty$ in (3.11), and we have weakly in Ω

$$(-\Delta)^s_p\, u_+ = f(x, u_+).$$

Therefore $u_+ \in C^\alpha_s(\overline{\Omega})_+$ is a solution of (1.1) (see Proposition 2.6). The next step consists in proving that

$$u_+ \neq 0. \tag{3.12}$$

We introduce the auxiliary problem (with c_1, q as in $\mathbf{H}_1$ *(iii)*):

$$\begin{cases} (-\Delta)^s_p\, w = c_1 w^{q-1} & \text{in } \Omega, \\ w \geqslant 0 & \text{in } \Omega, \\ w = 0 & \text{in } \Omega^c. \end{cases} \tag{3.13}$$

The energy functional $\Psi \in C^1(W^{s,p}_0(\Omega))$ of (3.13) is defined by setting for all $u \in W^{s,p}_0(\Omega)$

$$\Psi(u) = \frac{\|u\|^p}{p} - c_1\frac{\|u^+\|_q^q}{q},$$

and it is coercive and sequentially weakly lower semicontinuous. Besides, for all $\tau > 0$ we have

$$\Psi(\tau e_1) = \frac{\lambda_1}{p}\tau^p - c_1\frac{\|e_1\|_q^q}{q}\tau^q,$$

and the latter is negative for $\tau > 0$ small enough. Hence there exists $w \in W^{s,p}_0(\Omega)$ s.t.

$$\Psi(w) = \inf_{u \in W^{s,p}_0(\Omega)} \Psi(u) < 0.$$

In particular, we have $\Psi'(w) = 0$ in $W^{-s,p'}(\Omega)$. Testing such relation with $-w^- \in W^{s,p}_0(\Omega)$ and using (2.1), we get

$$\begin{aligned}\|w^-\|^p &\leqslant \langle(-\Delta)^s_p\, w, -w^-\rangle \\ &= \int_\Omega c_1(w^+)^{q-1}(-w^-)\, dx = 0,\end{aligned}$$

hence $w \in W^{s,p}_0(\Omega)_+ \setminus \{0\}$ and it solves (3.13). By Propositions 2.6 and 2.7 (with $g(t) = 0$) we easily get $w \in \mathrm{int}(C^0_s(\overline{\Omega})_+)$. Let $\theta \in (0, 1)$ be s.t. $0 < \theta w \leqslant \delta_0$ in Ω, then set $\breve{w} = \theta w \in \mathrm{int}(C^0_s(\overline{\Omega})_+)$. For all $n \in \mathbb{N}$ we have in Ω

$$\breve{w} \leqslant \breve{u}_n. \tag{3.14}$$

Indeed, since $\breve{w}, \breve{u}_n \in \mathrm{int}(C^0_s(\overline{\Omega})_+)$, for all $\sigma > 0$ small enough we have $\breve{u}_n - \sigma\breve{w} \in \mathrm{int}(C^0_s(\overline{\Omega})_+)$, hence in Ω

$$\sigma\breve{w} \leqslant \breve{u}_n.$$

Arguing by contradiction, let $\sigma \in (0, 1)$ be maximal in the inequality above. In particular we have $\sigma\breve{w} \neq \breve{u}_n$ and by (3.13) we have weakly in Ω

$$\begin{aligned}(-\Delta)^s_p\,(\sigma\breve{w}) + c_2(\sigma\breve{w})^{p-1} &= (\sigma\theta)^{p-1}c_1 w^{q-1} + c_2(\sigma\breve{w})^{p-1} < c_1(\sigma\breve{w})^{q-1} + c_2(\sigma\breve{w})^{p-1} \\ &\leqslant f(x, \sigma\breve{w}) + c_2(\sigma\breve{w})^{p-1} \leqslant f(x, \breve{u}_n) + c_2\breve{u}_n^{p-1} \\ &= (-\Delta)^s_p\, \breve{u}_n + c_2\breve{u}_n^{p-1},\end{aligned}$$

where we have used $\mathbf{H}_1$ *(iii)* *(v)* and (3.11). We apply Proposition 2.8 (with $g(t) = c_2|t|^{p-2}t$) and find that $\breve{u}_n - \sigma\breve{w} \in \mathrm{int}(C^0_s(\overline{\Omega})_+)$, hence as above we can find $\sigma' \in (\sigma, 1)$ s.t. in Ω

$$\sigma'\breve{w} < \breve{u}_n,$$

against maximality of σ. Thus we have (3.14). Passing to the limit in (3.14) as $n \to \infty$ and using (3.10) we have $\breve{w} \leqslant u_+$ in Ω, which proves (3.12).

Recall that $0 \leqslant u_+ \leqslant a_+$ in Ω, hence by $\mathbf{H}_1$ *(v)* we have

$$\begin{cases} (-\Delta)^s_p\, u_+ + c_2(u_+)^{p-1} = f(x, u_+) + c_2(u_+)^{p-1} \geqslant 0 & \text{weakly in } \Omega \\ u_+ \geqslant 0 & \text{in } \mathbb{R}^N. \end{cases}$$

By Proposition 2.7 (with $g(t) = c_2|t|^{p-2}t$) we have $u_+ \in \mathrm{int}(C^0_s(\overline{\Omega})_+)$.

Finally, we need to show that u_+ is the smallest solution of (1.1). Indeed, let $u \in \mathrm{int}(C^0_s(\overline{\Omega})_+)$ be any positive solution of (1.1). By Lemma 2.1 we have

$$\breve{u} = u \wedge \overline{u} \in W^{s,p}_0(\Omega),$$

and by Proposition 2.3 $\breve{u}$ is a supersolution of (1.1). Also, for all $n \in \mathbb{N}$ big enough we have $\underline{u}_n \leqslant \breve{u} \leqslant \overline{u}$ in Ω. By Proposition 2.4 there exists $\hat{u}_n \in \mathcal{S}(\underline{u}_n, \breve{u})$, i.e., $\hat{u}_n \in \mathrm{int}(C^0_s(\overline{\Omega})_+)$ solves (1.1) and in Ω we have

$$0 < \underline{u}_n \leqslant \hat{u}_n \leqslant \breve{u} \leqslant \overline{u}.$$

By (3.10) we have in Ω

$$\breve{u}_n \leqslant \hat{u}_n \leqslant u.$$

Passing to the limit as $n \to \infty$ we have $u_+ \leqslant u$ in Ω, which proves our claim.

Reasoning similarly on the negative semiaxis, we produce a biggest negative solution $v_- \in -\mathrm{int}(C^0_s(\overline{\Omega})_+)$. □

4. Nodal solution

In this section we seek a nodal (i.e., sign-changing) solution of (1.1), taking values between the extremal constant sign solutions seen in Lemma 3.6. We are going to exploit some Morse theory (computation of critical groups of the energy functional). For such a purpose, we need to stregthen slightly our hypotheses on the reaction f, adding a reverse Ambrosetti-Rabinowitz condition near the origin:

$\mathbf{H}_2$ $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory mapping satisfying *(i)*-*(v)* as in $\mathbf{H}_1$ and
(vi) there exists $\mu \in (1, p)$ s.t. for a.e. $x \in \Omega$ and all $|t| \leqslant \delta_0$

$$\mu F(x, t) \geqslant f(x, t)t.$$

Clearly, $\mathbf{H}_2$ implies $\mathbf{H}_1$ (and hence $\mathbf{H}_0$). Thus, all results of Sections 2, 3 apply.

Example 4.1. The function $f \in C^0(\mathbb{R})$ introduced in Example 3.1 satisfies $\mathbf{H}_2$ as well. Indeed, a straightforward calculation leads to

$$f(t) = t^{\frac{1}{2}} + \mathbf{o}(t^{\frac{1}{2}}),\ F(t) = \frac{2}{3}t^{\frac{3}{2}} + \mathbf{o}(t^{\frac{3}{2}}),$$

as $t \to 0^+$. Hence, taking $\mu \in (3/2, 2)$ (recall that $p = 2$), for all $t > 0$ small enough we have

$$\mu F(t) \geqslant f(t)t$$

(the case $t < 0$ is studied similarly). Thus f satisfies $\mathbf{H}_2$ *(vi)*.

By Lemma 3.6, under $\mathbf{H}_2$ problem (1.1) admits the extremal constant sign solutions $u_+ \in \mathrm{int}(C^0_s(\overline{\Omega})_+)$, $v_- \in -\mathrm{int}(C^0_s(\overline{\Omega})_+)$. Set for all $(x, t) \in \Omega \times \mathbb{R}$

$$\tilde{f}(x, t) = \begin{cases} f(x, v_-) & \text{if } t < v_-, \\ f(x, t) & \text{if } v_- \leqslant t \leqslant u_+, \\ f(x, u_+) & \text{if } t > u_+, \end{cases}$$

and accordingly set

$$\tilde{F}(x, t) = \int_0^t \tilde{f}(x, \tau)\, d\tau.$$

Further, set for all $u \in W^{s,p}_0(\Omega)$

$$\tilde{\Phi}(u) = \frac{\|u\|^p}{p} - \int_\Omega \tilde{F}(x,u)\,dx.$$

By $\mathbf{H}_2$ *(i)* we see that $\tilde{f} : \Omega \times \mathbb{R} \to \mathbb{R}$ satisfies $\mathbf{H}_0$, hence $\tilde{\Phi} \in C^1(W^{s,p}_0(\Omega))$ with derivative given for all $u, v \in W^{s,p}_0(\Omega)$ by

$$\langle \tilde{\Phi}'(u), v\rangle = \langle (-\Delta)^s_p\, u, v\rangle - \int_\Omega \tilde{f}(x,u)v\,dx.$$

By $\mathbf{H}_2$ *(iv)* we have $\tilde{\Phi}'(0) = 0$ in $W^{-s,p'}(\Omega)$. Consistently with the general assumptions that (1.1) has finitely many solutions, without loss of generality we may assume that 0 is an isolated critical point of $\tilde{\Phi}$ (see Section 2).

In the next lemma we compute the critical groups of $\tilde{\Phi}$ at 0:

Lemma 4.2. *If $\mathbf{H}_2$ holds, then $C_k(\tilde{\Phi}, 0) = 0$ for all $k \in \mathbb{N}$.*

Proof. Fix any $r \in (p, p^*_s)$. Combining $\mathbf{H}_2$ *(i) (iii)*, we can find $C_1, C_2 > 0$ s.t. for a.e. $x \in \Omega$ and all $t \in \mathbb{R}$

$$\tilde{F}(x,t) \geqslant C_1|t|^q - C_2|t|^r. \tag{4.1}$$

Now fix $u \in W^{s,p}_0(\Omega) \setminus \{0\}$. By (4.1) we have for all $\tau > 0$

$$\begin{aligned}\tilde{\Phi}(\tau u) &\leqslant \frac{\tau^p\|u\|^p}{p} - \int_\Omega \big(C_1\tau^q|u|^q - C_2\tau^r|u|^r\big)\,dx \\ &= \frac{\tau^p\|u\|^p}{p} - C_1\tau^q\|u\|_q^q + C_2\tau^r\|u\|_r^r,\end{aligned}$$

and the latter is negative for $\tau > 0$ small enough (since $q < p < r$). By Lemma 2.1 we have

$$w_0 = \min\{u_+, -v_-, \delta_0\} \in W^{s,p}_0(\Omega).$$

We begin studying the behavior of $\tilde{\Phi}$ near 0. First, we claim that there exists $\rho > 0$ s.t. for all $u \in \overline{B}_\rho(0) \setminus \{0\}$ with $\tilde{\Phi}(u) = 0$ we have

$$\frac{d}{d\tau}\tilde{\Phi}(\tau u)\Big|_{\tau=1} > 0. \tag{4.2}$$

Indeed, fix $u \in W^{s,p}_0(\Omega) \setminus \{0\}$ with $\tilde{\Phi}(u) = 0$. Using $\mathbf{H}_2$ *(vi)* and the definition of $\tilde{f}$, we compute for all $\tau > 0$

$$\begin{aligned}\frac{d}{d\tau}\tilde{\Phi}(\tau u)\Big|_{\tau=1} &= \langle \tilde{\Phi}'(u), u\rangle = \|u\|^p - \int_\Omega \tilde{f}(x,u)u\,dx \\ &= \Big(1 - \frac{\mu}{p}\Big)\|u\|^p + \int_\Omega \big(\mu\tilde{F}(x,u) - \tilde{f}(x,u)u\big)\,dx \\ &\geqslant \Big(1 - \frac{\mu}{p}\Big)\|u\|^p + \int_{\{|u|\geqslant w_0\}} \big(\mu\tilde{F}(x,u) - \tilde{f}(x,u)u\big)\,dx.\end{aligned} \tag{4.3}$$

There exists $C > 0$ (independent of u) s.t.

$$\int_{\{|u|\geqslant w_0\}} \big(\mu\tilde{F}(x,u) - \tilde{f}(x,u)u\big)\,dx \geqslant -C\|u\|_r^r. \tag{4.4}$$

Indeed, fix $x \in \Omega$ s.t. $u(x) \geqslant w_0(x)$. Three cases may occur:

(*a*) If $\delta_0 \leqslant u(x) \leqslant u_+(x)$, then by $\mathbf{H}_2$ *(i)* and (4.1) we have

$$\begin{aligned}\mu\tilde{F}(x,u) - \tilde{f}(x,u)u &= \mu F(x,u) - f(x,u)u \geqslant \mu C(u^q - u^r) - C(u + u^p) \\ &\geqslant -C(u + u^p + u^r) \geqslant -C\Big(\frac{u^r}{\delta_0^{r-1}} + \frac{u^r}{\delta_0^{r-p}} + u^r\Big) \geqslant -Cu^r.\end{aligned}$$

(*b*) If $u_+(x) \leqslant u(x) \leqslant \delta_0$, then, recalling the definition of $\tilde{f}$, that $\mu > 1$, and that $f(x, u_+) \geqslant 0$ (by $\mathbf{H}_2$ (*iii*) (*vi*)), we obtain

$$\begin{aligned}\mu\tilde{F}(x,u) - \tilde{f}(x,u)u &= \mu\int_0^{u_+} f(x,t)\,dt + \mu\int_{u_+}^{u} f(x,u_+)\,dt - f(x,u_+)u \\ &= \mu F(x,u_+) + \mu f(x,u_+)(u-u_+) - f(x,u_+)u \\ &\geqslant \mu F(x,u_+) - f(x,u_+)u_+ \geqslant 0.\end{aligned}$$

(*c*) If $-v_-(x) \leqslant u(x) \leqslant \min\{u_+(x), \delta_0\}$, simply use $\mathbf{H}_2$ (*vi*).

Similarly we deal with the case $u(x) < 0$. Then, integrating in $\{|u| \geqslant w_0\}$ we get (4.4). Therefore, plugging (4.4) into (4.3) we get

$$\frac{d}{d\tau}\tilde{\Phi}(\tau u)\Big|_{\tau=1} \geqslant \Big(1 - \frac{\mu}{p}\Big)\|u\|^p - C\|u\|^r,$$

and the latter is positive whenever $\|u\| > 0$ is small enough. So we have (4.2).

Our next claim is that for all $u \in \overline{B}_\rho(0) \setminus \{0\}$ s.t. $\tilde{\Phi}(u) = 0$ and all $\tau \in [0, 1]$

$$\tilde{\Phi}(\tau u) \leqslant 0. \tag{4.5}$$

Arguing by contradiction, let $\tau_1 \in (0, 1)$ s.t. $\tilde{\Phi}(\tau_1 u) > 0$. By the mean value theorem there exists $\tau_2 \in (\tau_1, 1]$ minimal s.t. $\tilde{\Phi}(\tau_2 u) = 0$. So $\tilde{\Phi}(\tau u) > 0$ for all $\tau \in [\tau_1, \tau_2)$. Set $w = \tau_2 u$, then $w \in \overline{B}_\rho(0) \setminus \{0\}$ and $\tilde{\Phi}(w) = 0$. By (4.2) we have

$$\frac{d}{d\tau}\tilde{\Phi}(\tau w)\Big|_{\tau=1} > 0.$$

Besides, since $\tilde{\Phi}(\tau w) > 0$ for all $\tau \in (\tau_1/\tau_2, 1)$ we have

$$\frac{d}{d\tau}\tilde{\Phi}(\tau w)\Big|_{\tau=1} \leqslant 0,$$

a contradiction. So (4.5) is proved.

Taking $\rho > 0$ even smaller if necessary, we have

$$K(\tilde{\Phi}) \cap \overline{B}_\rho(0) = \{0\}.$$

Set

$$A = \big\{u \in \overline{B}_\rho(0) : \tilde{\Phi}(u) \leqslant 0\big\}.$$

Clearly $0 \in A$. Plus, A is a star-shaped set centered at 0. Indeed, for all $u \in A \setminus \{0\}$, $\tau \in [0, 1]$ we have $\tilde{\Phi}(\tau u) \leqslant 0$. Otherwise, there would exist $0 < \tau_1 < \tau_2 < 1$ s.t.

$$\tilde{\Phi}(\tau_1 u) > 0 = \tilde{\Phi}(\tau_2 u),$$

against (4.5). By [27, Remark 6.23], the set A is contractible. Now consider $u \in \overline{B}_\rho(0) \setminus A$, i.e., satisfying $\tilde{\Phi}(u) > 0$. As seen above, $\tilde{\Phi}(\tau u) < 0$ for all $\tau \in (0, 1)$ small enough, so there exists $\tau \in (0, 1)$ s.t.

$$\tilde{\Phi}(\tau u) = 0.$$

We claim that such $\tau \in (0, 1)$ is unique. Arguing by contradiction, let $0 < \tau_1 < \tau_2 < 1$ be s.t.

$$\tilde{\Phi}(\tau_1 u) = \tilde{\Phi}(\tau_2 u) = 0.$$

By (4.5) we have for all $\sigma \in [0, 1]$

$$\tilde{\Phi}(\sigma\tau_2 u) \leqslant 0.$$

Now set for all $\sigma \in [0, 1]$

$$g(\sigma) = \tilde{\Phi}(\sigma\tau_2 u),$$

so that the map $g \in C^1([0, 1])$ attains its maximum at $\sigma = \tau_1/\tau_2 \in (0, 1)$, hence

$$g'\Big(\frac{\tau_1}{\tau_2}\Big) = 0.$$

So we have

$$\frac{d}{d\tau}\tilde{\Phi}(\tau u)\Big|_{\tau=\tau_1} = \frac{1}{\tau_2}g'\Big(\frac{\tau_1}{\tau_2}\Big) = 0,$$

against (4.2). By the implicit function theorem (see [27, Theorem 7.3]), we can construct a continuous map $\hat{\tau} : \overline{B}_\rho(0) \setminus A \to (0,1)$ s.t. for all $u \in \overline{B}_\rho(0) \setminus A$, $\tau \in (0,1)$

$$\tilde{\Phi}(\tau u) = \begin{cases} < 0 & \text{if } 0 < \tau < \hat{\tau}(u) \\ = 0 & \text{if } \tau = \hat{\tau}(u) \\ > 0 & \text{if } \hat{\tau}(u) < \tau < 1. \end{cases}$$

Set for all $u \in \overline{B}_\rho(0) \setminus \{0\}$

$$\hat{h}(u) = \begin{cases} u & \text{if } u \in A \setminus \{0\} \\ \hat{\tau}(u)u & \text{if } u \in \overline{B}_\rho(0) \setminus A. \end{cases}$$

Then $\hat{h} : \overline{B}_\rho(0) \setminus \{0\} \to A \setminus \{0\}$ is a continuous retraction. Since $\dim(W_0^{s,p}(\Omega)) = \infty$, the set $\overline{B}_\rho(0) \setminus \{0\}$ is contractible [27, Example 6.45 (b)]. Then, being a retract of $\overline{B}_\rho(0) \setminus \{0\}$, $A \setminus \{0\}$ is contractible as well. So, by [27, Propositions 6.24, 6.25] we have for all $k \geqslant 0$

$$C_k(\tilde{\Phi}, 0) = H_k(A, A \setminus \{0\}) = 0,$$

which concludes the proof. □

Finally, we can prove our complete multiplicity result:

Theorem 4.3. *If $\mathbf{H}_2$ holds, then (1.1) has at least five nontrivial solutions:*

(i) $u_0, u_1 \in \mathrm{int}(C_s^0(\overline{\Omega})_+)$, *s.t.* $0 < u_0 < a_+$, $u_0 \leqslant u_1$ *in* Ω,

(ii) $v_0, v_1 \in -\mathrm{int}(C_s^0(\overline{\Omega})_+)$, *s.t.* $a_- < v_0 < 0$, $v_1 \leqslant v_0$ *in* Ω,

(iii) $\tilde{u} \in C_s^0(\overline{\Omega}) \setminus \{0\}$ *nodal, s.t.* $v_0 \leqslant \tilde{u} \leqslant u_0$ *in* Ω.

Proof. From Theorem 3.5 we have (i), (ii). So, there remains to prove (iii).

By Lemma 3.6 we know that (1.1) admits extremal constant sign solutions $u_+ \in \mathrm{int}(C_s^0(\overline{\Omega})_+)$, $v_- \in \mathrm{int}(C_s^0(\overline{\Omega})_+)$. Without loss of generality and consistently with (i), (ii), we may assume that

$$u_+ = u_0, \quad v_- = v_0.$$

In particular, that implies that no nontrivial, constant sign solution may exist in the set $\mathcal{S}(v_0, u_0)$. Now set for all $(x,t) \in \Omega \times \mathbb{R}$

$$\tilde{f}_\pm(x,t) = \tilde{f}(x, \pm t^\pm),$$

and

$$\tilde{F}_\pm(x,t) = \int_0^t \tilde{f}_\pm(x,\tau)\, d\tau.$$

Further, set for all $u \in W_0^{s,p}(\Omega)$

$$\tilde{\Phi}_\pm(u) = \frac{\|u\|^p}{p} - \int_\Omega \tilde{F}_\pm(x,u)\, dx.$$

Clearly $\tilde{f}_\pm : \Omega \times \mathbb{R} \to \mathbb{R}$ satisfies $\mathbf{H}_0$, so $\tilde{\Phi}_\pm \in C^1(W_0^{s,p}(\Omega))$ with derivative given for all $u, v \in W_0^{s,p}(\Omega)$ by

$$\langle \tilde{\Phi}_\pm(u), v \rangle = \langle (-\Delta)_p^s\, u, v \rangle - \int_\Omega \tilde{f}_\pm(x,u) v\, dx.$$

We now focus on $\tilde{\Phi}_+ \in C^1(W_0^{s,p}(\Omega))$, proving that for all $u \in K(\tilde{\Phi}_+)$ we have in Ω

$$0 \leqslant u \leqslant u_0. \tag{4.6}$$

Indeed, we have $\tilde{\Phi}_+'(u) = 0$ in $W^{-s,p'}(\Omega)$, which rephrases as

$$(-\Delta)_p^s\, u = \tilde{f}_+(x,u) \tag{4.7}$$

weakly in Ω. Testing (4.7) with $(u - u_0)^+ \in W_0^{s,p}(\Omega)$ and recalling that u_0 solves (1.1), we get

$$\langle (-\Delta)_p^s\, u - (-\Delta)_p^s\, u_0, (u-u_0)^+ \rangle = \int_\Omega \big(\tilde{f}_+(x,u) - f(x,u_0)\big)(u-u_0)^+\, dx = 0.$$

By Proposition 2.2, we have $u \leqslant u_0$ in Ω. In a similar way, testing (4.7) with $-u^- \in W_0^{s,p}(\Omega)$ and using (2.1) we have

$$\|u^-\|^p \leqslant \langle (-\Delta)_p^s\, u, -u^- \rangle = \int_\Omega \tilde{f}_+(x,u)(-u^-)\, dx = 0,$$

hence $u \geqslant 0$ in Ω. So (4.6) is proved. More precisely, we have

$$K(\tilde{\Phi}_+) = \{0, u_0\}. \tag{4.8}$$

Indeed, by construction of $\tilde{f}_+$, clearly we have in $W^{-s,p'}(\Omega)$

$$\tilde{\Phi}'_+(0) = \tilde{\Phi}'_+(u_0) = 0.$$

Vice versa, let $u \in K(\tilde{\Phi}_+)$, i.e., u satisfies (4.7). By (4.6) we can replace $\tilde{f}_+$ with f in (4.7), hence $u \in C_s^\alpha(\overline{\Omega})$ solves (1.1) (see Proposition 2.6). Assume $u \neq 0$. Then by $\mathbf{H}_2$ (v) we have

$$\begin{cases} (-\Delta)_p^s\, u + c_2 u^{p-1} = f(x,u) + c_2 u^{p-1} \geqslant 0 & \text{in } \Omega \\ u \geqslant 0 & \text{in } \Omega. \end{cases}$$

By Proposition 2.7 (with $g(t) = c_2|t|^{p-2}t$), we get $u \in \mathrm{int}(C_s^0(\overline{\Omega})_+)$. Therefore u is a positive solution of (1.1) s.t. $u \leqslant u_0$. By minimality we deduce $u = u_0$, which proves (4.8).

Next, we claim that u_0 is a local minimizer of $\tilde{\Phi}$. Indeed, since $\tilde{f}_+ : \Omega \times \mathbb{R} \to \mathbb{R}$ is bounded, then $\tilde{\Phi}_+$ is coercive and sequentially weakly lower semicontinuous. So there exists $\tilde{u}_+ \in W_0^{s,p}(\Omega)$ s.t.

$$\tilde{\Phi}_+(\tilde{u}_+) = \inf_{u \in W_0^{s,p}(\Omega)} \tilde{\Phi}_+(u) = \tilde{m}_+. \tag{4.9}$$

Let once again $e_1 \in \mathrm{int}(C_s^0(\overline{\Omega})_+)$ be as in Proposition 2.10. For all $\tau > 0$ small enough we have in Ω

$$0 < \tau e_1 \leqslant \min\{u_0, \delta_0\}.$$

Therefore, by $\mathbf{H}_2$ (iii) we have in Ω

$$\tilde{f}_+(x, \tau e_1) = f(x, \tau e_1) \geqslant c_1 (\tau e_1)^{q-1}.$$

Hence, for all $\tau > 0$ small enough

$$\tilde{\Phi}_+(\tau e_1) \leqslant \frac{\tau^p \|e_1\|^p}{p} - \int_\Omega \frac{c_1}{q} (\tau e_1)^q \, dx = \frac{\lambda_1}{p} \tau^p - \frac{c_1}{q} \tau^q \|e_1\|_q^q,$$

and the latter is negative for $\tau > 0$ even smaller if necessary (since $q < p$). So, in (4.9) we have $\tilde{m}_+ < 0$, hence $\tilde{u}_+ \neq 0$. Therefore we have $\tilde{u}_+ \in K(\tilde{\Phi}_+) \setminus \{0\}$, which by (4.8) implies $\tilde{u}_+ = u_0$. Further, since $u_0 \in \mathrm{int}(C_s^0(\overline{\Omega})_+)$, we can find $\rho > 0$ s.t. for all $u \in W_0^{s,p}(\Omega) \cap C_s^0(\overline{\Omega})$ with $\|u - u_0\|_{0,s} \leqslant \rho$ we have $u > 0$ in Ω. So for any such u we get

$$\tilde{\Phi}(u) = \tilde{\Phi}_+(u) \geqslant \tilde{\Phi}_+(u_0) = \tilde{\Phi}(u_0),$$

i.e., u_0 is a local minimizer of $\tilde{\Phi}$ in $C_s^0(\overline{\Omega})$. By Proposition 2.9, u_0 is as well a local minimizer of $\tilde{\Phi}$ in $W_0^{s,p}(\Omega)$. Reasoning in a similar way, we see that

$$K(\tilde{\Phi}_-) = \{0, v_0\},$$

and that v_0 is a local minimizer of $\tilde{\Phi}$.

Now let us turn to the functional $\tilde{\Phi} \in C^1(W_0^{s,p}(\Omega))$. This is coercive, hence it satisfies (PS) (see Section 2) and admits two distinct local minimizers $u_0, v_0 \in W_0^{s,p}(\Omega)$. Without loss of generality, we may assume that $K(\tilde{\Phi})$ is a finite set, hence in particular that u_0 and v_0 are strict local minimizers. As in the proof of (4.6), we see that for all $u \in K(\tilde{\Phi})$ we have in Ω

$$v_0 \leqslant u \leqslant u_0. \tag{4.10}$$

Now set

$$\Gamma = \big\{\gamma \in C([0,1], W_0^{s,p}(\Omega)) : \gamma(0) = u_0,\ \gamma(1) = v_0\big\}.$$

Then we have

$$c = \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} \tilde{\Phi}(\gamma(t)) > \max\{\tilde{\Phi}(u_0),\, \tilde{\Phi}(v_0)\}.$$

By Hofer's version of the mountain pass theorem and the characterization of critical groups at mountain pass type critical points (see [27, Theorem 6.99, Proposition 6.100]), there exists $\tilde{u} \in K_c(\tilde{\Phi})$ s.t.

$$C_1(\tilde{\Phi}, \tilde{u}) \neq 0. \tag{4.11}$$

Then we have weakly in Ω

$$(-\Delta)_p^s\, \tilde{u} = \tilde{f}(x, \tilde{u}).$$

By (4.10) we have in Ω

$$v_0 \leqslant \tilde{u} \leqslant u_0.$$

Therefore, by construction of $\tilde{f}$, we see that $\tilde{u}$ solves (1.1). Hence, by Proposition 2.6, we have $\tilde{u} \in C_s^\alpha(\overline{\Omega})$. There remains to prove that $\tilde{u}$ is nodal. Recall that $u_0, v_0 \in K(\tilde{\Phi})$ are strict local minimizers and isolated critical points. So, by [27, Example 6.45 (a)], we obtain

$$C_1(\tilde{\Phi}, u_0) = C_1(\tilde{\Phi}, v_0) = 0.$$

Besides, by Lemma 4.2 we have

$$C_1(\tilde{\Phi}, 0) = 0.$$

So, (4.11) implies

$$\tilde{u} \in K(\tilde{\Phi}) \setminus \{0, u_0, v_0\}.$$

Assume now that $\tilde{u} \geqslant 0$ in Ω. Then, by Proposition 2.7 we would have $\tilde{u} \in \mathrm{int}(C_s^0(\overline{\Omega})_+)$ and $\tilde{u} \leqslant u_0$, against minimality of u_0. Similarly, assuming that $\tilde{u} \leqslant 0$ in Ω leads to a contradiction. Thus, $\tilde{u}$ changes sign in Ω, which completes the argument for *(iii)*. $\square$

Remark 4.4. As seen in Remark 3.4, in general we cannot be sure that all solutions lie in the order interval $[a_-, a_+]$, so we cannot use Proposition 2.8 and get strict inequalities. These can be retrieved by strengthening the quasi-monotonicity hypothesis $\mathbf{H}_2$ *(v)*.

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