**Research Article**

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# Risk Aversion and Uniqueness of Equilibrium in Economies with Two Goods and Arbitrary Endowments

https://doi.org/10.1515/bejte-2021-0150
Received November 13, 2021; accepted September 20, 2022

**Abstract:** We study the connection between risk aversion, the number of consumers, and the uniqueness of equilibrium. We consider an economy with two goods and $I$ impatience types, where each type has additive separable preferences with HARA Bernoulli utility function, $u_{\mathrm{H}}(x) := \frac{\gamma}{1-\gamma}\left(b + \frac{a}{\gamma}x\right)^{1-\gamma}$. We show that if $\gamma \in \left(1, \frac{I}{I-1}\right]$, the economy has a unique regular equilibrium. Moreover, the methods used, including Newton's symmetric polynomials and Descartes' rule of signs, enable us to offer new sufficient conditions for uniqueness in a closed-form expression that highlight the role played by endowments, patience, and specific HARA parameters. Finally, we derive new necessary and sufficient conditions that ensure uniqueness for the particular case of CRRA Bernoulli utility functions with $\gamma = 3$.

**Keywords:** uniqueness of equilibrium, excess demand function, risk aversion, polynomial approximation, Descartes' rule of signs, Newton's symmetric polynomials

**JEL Classification:** C62, D51, D58

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# 1 Introduction and the Statements of Main Results

Uniqueness of equilibrium plays a crucial role in comparative statics and stability. However, as highlighted by (Kehoe 1998), it is rarely possible to provide easy analytical conditions that guarantee uniqueness in applied models. In a recent paper, Geanakoplos and Walsh (2018) presented new sufficient conditions to ensure uniqueness and stability of equilibrium in an economy with two goods, where $I$ agents, $I \geq 2$, ordered according to a parameter $\beta = (\beta_1, \ldots, \beta_I)$, $\beta_1 < \cdots < \beta_I$, representing patience, have identical endowments and the same Bernoulli utility function displaying decreasing (non-increasing) absolute risk aversion (DARA[1]). The role played in that paper by the DARA assumption is twofold: it ensures that income effects can be ordered across types and determines a positive covariance between consumption and income effect, hence bounding the market income effect.

For our purposes, their main results (see Proposition 2 and Proposition 5 in Geanakoplos and Walsh 2018 for further details) can be summarized as follows.

**Theorem GW.** *Let u be a Bernoulli utility function and let impatience type i's preferences be represented by*

$$u_i(x, y) = u(x) + \beta_i u(y), \quad i = 1, \ldots, I,$$

*where $(e_i, f_i)$ denote consumer i's endowments of goods x and y, respectively.*

*The equilibrium price is unique if:*

(1) *u satisfies DARA and agents have identical endowments; that is, $(e_i, f_i) = (e_j, f_j)$ for all i and j;*

(2) *u satisfies CRRA and the following restrictions on patience and endowments hold: $\beta_i \leq \beta_j$, $e_i \leq e_j$, and $f_i \geq f_j$, for any $i < j$.*

The authors point out that although the assumption of identical endowments is used in several papers, it is rather restrictive. They highlight that, under DARA preferences, there is no evident condition that ensures uniqueness if the assumption of identical endowments is dropped. They conjecture that there should not be "too much heterogeneity" across agents to ensure uniqueness, arguing instead that heterogeneity should involve a condition on the patience

1 We say that a Bernoulli utility function $u$ exhibits decreasing absolute risk aversion (DARA) if the absolute risk aversion coefficient $-u''(x)/u'(x)$ is decreasing.

parameter $\beta$, endowments and the particular Bernoulli utility function that is used.

In this paper, we are mainly interested in exploring sufficient conditions on the parameter $\gamma$ that guarantee uniqueness of equilibrium without imposing any restriction on endowments, unlike Theorem GW. Moreover, the methods that we have used enable us to address the issue of heterogeneity that was raised by Geanakoplos and Walsh (2018). They also enable us to obtain necessary and sufficient conditions for CRRA preferences.

More precisely, we consider an economy with two goods and $I$ impatience types, where type $i$ has preferences represented by the utility function

$$u_i(x, y) = u_{\mathrm{H}}(x) + \beta_i u_{\mathrm{H}}(y), \tag{1}$$

where $u_{\mathrm{H}}$ is HARA;[2] that is,

$$u_{\mathrm{H}}(x) := \frac{\gamma}{1-\gamma}\left(b + \frac{a}{\gamma}x\right)^{1-\gamma}, \quad \gamma > 0,\ \gamma \neq 1,\ a > 0,\ b \geq 0. \tag{2}$$

Note that HARA is an important subclass of DARA preferences that is extensively used in the literature, which also encompasses the CRRA case by setting $b = 0$.

Our main result is the following theorem, which establishes a connection between uniqueness, $\gamma$, and the number $I$ of impatience types in the economy:

**Theorem 1.** *Let $u_{\mathrm{H}}$ be the Bernoulli utility function* (2) *and let utility of impatience type i be given by* (1). *If*

$$\gamma \in \left(1, \frac{I}{I-1}\right], \tag{3}$$

*the economy then has a unique regular equilibrium.*[3]

This result is also made interesting by the fact that "utility functions with concavity parameters in the range of 1–2 are widely considered plausible in the literature", as Kaplow observes (see Kaplow 2008, Chapter 3 and references therein).

For CRRA preferences, Theorem 1 links the coefficient of relative risk aversion $\gamma$ to the number of consumers and also answers the issue raised by Toda and Walsh (2017); that is, whether or not a value of $\gamma$ in the interval $(1, 2]$ is compatible with multiple equilibria for CRRA preferences with 2 consumers. Moreover, it

2 We will not consider the well-known case $\gamma = 1$, where the Bernoulli utility is logarithmic.

3 Observe that economies are "generically" regular (e.g. see Mas-Colell et al. 1995, Section 17.D).

provides a generalization for any number of consumers and arbitrary endowments allocations.

It is known (see Mas-Colell 1991 and references therein; see also Hens and Loeffler 1995 and Mas-Colell, Whinston, and Green 1995) that for $C^2$ separable preferences $\sum_l u_l(x)$, where each $u_l$ is monotonic and concave, relative risk aversion less or equal to 1 implies uniqueness. In particular, under CRRA preferences (i.e. when $b = 0$), if $\gamma$ belongs to $(0, 1)$, then we have uniqueness. Toda and Walsh (2017) show, for CRRA preferences, the possibility of multiple equilibria in an economy with two goods and two consumers when $\gamma > 2$. However, the question of whether or not multiplicity is possibile for $1 < \gamma \leq 2$ was left as an open question (Toda and Walsh 2017, Remark 1) (see Subsection 2.1 below for a deeper analysis of the Toda–Walsh example). Corollary 10 rules out this possibility.

As far as the heterogeneity issue raised by Geanakoplos and Walsh (2018) is concerned, our contribution is Theorem 2, where we provide a closed-form expression that embodies what Geanakoplos and Walsh (2018) argue for arbitrary DARA preferences, highlighting the role played by endowments, patience, and specific Bernoulli utility parameters in ensuring uniqueness.

Given that we believe that the approach that we have used is as interesting as the results, we provide an intuition here, leaving the details in Section 3. As usual, from the maximization problem, one obtains the excess demand function $Z$, which is expressed as an implicit function of the price raised to a positive function depending on the parameter $\gamma$. The strategy is to turn $Z$, by algebraic manipulations, into a polynomial $P$. Because $\gamma$ can be irrational, we use the denseness of $\mathbb{Q}$ in $\mathbb{R}$ to approximate $\gamma$. This strategy "generically" works for regular values by the implicit function theorem. Finally, we write $P$ in terms of Newton's symmetric polynomials and we apply Descartes' rule of signs, which states that the number of positive roots of a real polynomial, arranged in ascending or descending powers, cannot exceed the number of sign variations in consecutive coefficients (e.g. see Anderson, Jackson, and Sitharam 1998).

It is remarkable how useful and powerful this simple method can be. In fact, it allows us to study the number of equilibria without ad hoc restrictions on the set of parameters. Moreover, it can provide necessary and sufficient conditions for uniqueness, as we show in Theorem 5, which establishes a connection between the number of equilibria, their type (regular and critical), endowments, and utility weights, thus complementing Toda and Walsh (2017)'s analysis of a CRRA economy with relative risk aversion $\gamma = 3$.

The literature on uniqueness is vast. For a survey, the reader is primarily referred to (Kehoe 1998; Mas-Colell et al. 1991), and the references therein. As we have already pointed out, a feature of Theorem 1 is that we do not impose restrictions on endowments. From this point of view, it is related to the strand of the literature that provides sufficient conditions to guarantee uniqueness globally (i.e. for every possible allocation of resources among consumers). Two recent papers that belong to this line of research are (Loi and Matta 2018, 2021), where uniqueness is globally characterized through the geometric properties of the equilibrium manifold. Another recent result (Toda and Walsh 2020, Theorem 1) shows, under certain assumptions, the existence of uniqueness of equilibrium in an economy with heterogeneous and homothetic preferences, extending Chipman and Moore (1979)'s result to an incomplete market setup. For a different approach that provides sufficient conditions on offer curves in a two-commodity, two-agent exchange economy, see (Gimémez 2019). We finally refer the reader to Kubler, Renner, and Schmedders (2014), and the references therein, for a survey of how to solve economic equilibria that are described as solutions to systems of polynomial equations.

We believe that uniqueness of equilibrium and the methods that we have used in this paper deserve further attention and research. In particular, consider extending Theorem 2 to an arbitrary number of consumers without imposing restrictions on $\gamma$. Moreover, it could be interesting to apply these methods to different Bernoulli utility functions to achieve new sufficient conditions on uniqueness.

The rest of this paper is organized as follows. In Section 2, we apply this polynomial approach to the case of $I = 2$ consumer types and $\gamma = 3$, and we prove Theorems 2 and 5, that provide, respectively, new sufficient and necessary and sufficient for uniqueness under HARA and CRRA preferences. In Section 3 we prove Theorem 1 and derive Corollary 10). The mathematical appendix contains all of the proofs of the intermediate results.

# 2 The Case $I = 2$ and $\gamma = 3$

In this section, we show how our polynomial approach enables us to study the equilibria of the aggregate excess demand function and to establish connections between endowments, patience, and parameters of the Bernoulli utility function. In particular, we study the number of equilibria in the HARA case with $I = 2$ consumers and $\gamma = 3$.

By maximizing (1) under the budget constraint

$$px + y \leq pe_i + f_i, \quad i = 1, 2$$

we obtain (e.g. see Appendix A.1) the aggregate excess demand function for good $x$:

$$\sum_{i=1}^{2} \frac{b - bp^{\frac{1}{3}}\sigma_i + \frac{a}{3}(pe_i + f_i)}{\frac{a}{3}\left(p + \sigma_i p^{\frac{1}{3}}\right)} - (e_1 + e_2),$$

where $\sigma_i = \beta_i^{\frac{1}{3}}, i = 1, 2$.

After combining terms over a common denominator and taking the numerator, we have

$$3\left(p + \sigma_2 p^{\frac{1}{3}}\right)\left(b - bp^{\frac{1}{3}}\sigma_1 + \frac{a}{3}(pe_1 + f_1)\right) + 3\left(p + \sigma_1 p^{\frac{1}{3}}\right)$$
$$\times \left(b - bp^{\frac{1}{3}}\sigma_2 + \frac{a}{3}(pe_2 + f_2)\right) - a(e_1 + e_2)\left(p + \sigma_1 p^{\frac{1}{3}}\right)\left(p + \sigma_2 p^{\frac{1}{3}}\right).$$

By expanding the previous expression and collecting the terms in $p$, we obtain

$$p^{4/3}(-\sigma_1(ae_1 + 3b) - \sigma_2(ae_2 + 3b)) + p(af_1 + af_2 + 6b)$$
$$- p^{2/3}\sigma_1\sigma_2(ae_1 + ae_2 + 6b) + p^{\frac{1}{3}}(\sigma_1(af_2 + 3b) + \sigma_2(af_1 + 3b)).$$

Dividing by $p^{\frac{1}{3}}$, a simple algebraic manipulation yields that $p$ is an equilibrium price if and only if $q := p^{\frac{1}{3}}$ is a zero of the following expression

$$P(q) = A(e, \sigma, a, b)q^3 + B(f, \sigma, a, b)q^2 + C(e, \sigma, a, b)q + D(f, \sigma, a, b), \tag{4}$$

where

$$\begin{aligned}
A(e, \sigma, a, b) &:= -(e_1\sigma_1 + e_2\sigma_2) - \frac{3b}{a}(\sigma_1 + \sigma_2) < 0,\\
B(f, \sigma, a, b) &:= f_1 + f_2 + \frac{6b}{a} > 0,\\
C(e, \sigma, a, b) &:= -(e_1 + e_2)\sigma_1\sigma_2 - \frac{6b}{a}\sigma_1\sigma_2 < 0,\\
D(f, \sigma, a, b) &:= f_1\sigma_2 + f_2\sigma_1 + \frac{3b}{a}(\sigma_1 + \sigma_2) > 0.
\end{aligned} \tag{5}$$

Our main result is the following theorem:

**Theorem 2.** *In the HARA case with two impatience types and for $\gamma = 3$, we have at most three equilibria. Moreover, if we assume that*

$$\beta_1 < \beta_2, \quad e_1 \le e_2, \quad f_1 \ge f_2, \tag{6}$$

*and*

$$b \ge \frac{a}{3}\left(\frac{\beta_2}{\beta_1}\right)^{\frac{2}{3}}(e_2 + f_1) \tag{7}$$

*are satisfied, then the equilibrium is unique.*

To prove Theorem 2, we need a simple algebraic lemma.

**Lemma 3.** *Assume that the polynomial* $P(x) = Ax^3 + Bx^2 + Cx + D$ *has three sign changes and* $ABCD \neq 0$. *If*

$$AD - BC < 0, \tag{8}$$

*then* $P(x)$ *has a unique positive root.*

*Proof.* The discriminant of $P(x)$ is given by:

$$\Delta = B^2C^2 - 4AC^3 - 4B^3D - 27A^2D^2 + 18ABCD. \tag{9}$$

The assumption (8) gives $A^2D^2 > B^2C^2$ and $A^2D^2 > ABCD$. Then $\Delta < 0$. Hence, the polynomial has a unique root that is positive by Descartes' rule of sign Anderson, Jackson, and Sitharam (1998). □

*Proof of Theorem 2.* The boundedness of the number of equilibria is immediate, because the polynomial (4) has degree three. To prove the second part of the theorem, by Lemma 3 we need to verify that (6) and (7) imply

$$A(e, \sigma, a, b)D(f, \sigma, a, b) - B(f, \sigma, a, b)C(e, \sigma, a, b) < 0$$

where $A(e, \sigma, a, b), B(f, \sigma, a, b), C(e, \sigma, a, b), D(f, \sigma, a, b)$ are given by (5) above. A long but straightforward computation yields

$$\begin{aligned} &A(e, \sigma, a, b)D(f, \sigma, a, b) - B(f, \sigma, a, b)C(e, \sigma, a, b) \\ &\quad = (\sigma_2 - \sigma_1)(e_1 f_2 \sigma_1 - e_2 f_1 \sigma_2) + E(e, f, \sigma, a, b), \end{aligned} \tag{10}$$

where

$$\begin{aligned} E(e, f, \sigma, a, b) := & -\frac{9b^2}{a^2}(\sigma_1 - \sigma_2)^2 + \frac{3b}{a}\big[(e_1 + e_2 + f_1 + f_2)\sigma_1\sigma_2 \\ & - (e_1 + f_2)\sigma_1^2 - (e_2 + f_1)\sigma_2^2\big]. \end{aligned}$$

Notice that by (6), the first summand of (10), namely $(\sigma_2 - \sigma_1)(e_1 f_2 \sigma_1 - e_2 f_1 \sigma_2)$, is strictly less than zero:

$$\begin{aligned} (\sigma_2 - \sigma_1)(e_1 f_2 \sigma_1 - e_2 f_1 \sigma_2) &\leq (\sigma_2 - \sigma_1) f_1 (e_1 \sigma_1 - e_2 \sigma_2) \\ &< (\sigma_2 - \sigma_1) f_1 \sigma_2 (e_1 - e_2) \leq 0. \end{aligned}$$

Moreover, $E(e, f, \sigma, a, b) \leq 0$ if and only if

$$b \geq \frac{a}{3} \frac{\left[(e_1 + e_2 + f_1 + f_2)\sigma_1\sigma_2 - (e_1 + f_2)\sigma_1^2 - (e_2 + f_1)\sigma_2^2\right]}{(\sigma_1 - \sigma_2)^2}.$$

Again, by (6) one can find an upper bound of the right-hand side of this inequality, namely:

$$\begin{aligned}
&\frac{a}{3} \frac{\left[(e_1 + e_2 + f_1 + f_2)\sigma_1\sigma_2 - (e_1 + f_2)\sigma_1^2 - (e_2 + f_1)\sigma_2^2\right]}{(\sigma_1 - \sigma_2)^2} \\
&\quad < \frac{a}{3} \frac{\left[(e_1 + e_2 + f_1 + f_2)\sigma_1\sigma_2\right]}{(\sigma_1 - \sigma_2)^2} \\
&\quad < \frac{a}{3} \left(\frac{\sigma_2}{\sigma_1}\right)^2 (e_2 + f_1).
\end{aligned}$$

Thus, if (6) and $b \geq \frac{a}{3}\left(\frac{\sigma_2}{\sigma_1}\right)^2 (e_2 + f_1)$ hold true, then we get that (10) is strictly less than zero and by Lemma 3 there is uniqueness of equilibria. Hence, the proof of the theorem follows, because $\sigma_i = \beta_i^{\frac{1}{3}}$, $i = 1, 2$. □

Condition (7) confirms what has been claimed by Geanakoplos and Walsh (2018). In particular, the parameters involved $a, b, \gamma$ affect risk tolerance (i.e. the inverse of absolute risk aversion). Following Geanakoplos and Walsh (2018)'s insight, this closed-form expression ensures the effect of positive covariance across types. Moreover, it allows for more heterogeneity in allocations than might be expected.

**Remark 4.** Conditions (6) are exactly those of (Geanakoplos and Walsh 2018, Proposition 5), which holds for CRRA (homotethic) preferences. In fact, by assuming only conditions (6) and CRRA preferences, one can follow the same argument of the Proof of Theorem 2 and achieve Proposition 5 in a different way. Hence, Theorem 2 encompasses Proposition 5 as a special case, when $\gamma = 3$ (see Remark 6).[4] Condition (7) enables us to extend the result to HARA preferences and appraise the connection in ensuring uniqueness between endowments, patience, and Bernoulli utility parameters affecting the concavity of the Bernoulli utility function.

4 We thank again Alexis Akira Toda for suggesting the analysis of the general case, arbitrary $\gamma$ with two consumers, which will be the subject of our future research.

## 2.1 The Toda–Walsh Example with CRRA Preferences

In this subsection, we illustrate how this polynomial approach is a simple and powerful tool for studying the number of equilibria with arbitrary endowments. For this purpose, we consider the example of multiple equilibria, which was provided by Toda and Walsh (2017), of an economy with two goods and two consumers under the assumption of CRRA preferences and symmetric endowments. Observe that this CRRA example can be obtained from HARA (see (2)) by setting $b = 0$ and

$$\beta_1 = \left(\frac{1-\alpha}{\alpha}\right)^{\gamma}, \quad \beta_2 = \left(\frac{\alpha}{1-\alpha}\right)^{\gamma}, \quad 0 < \alpha < 1. \tag{11}$$

More precisely, preferences are given by

$$u_1(x_1, x_2) = \frac{1}{1-\gamma}\left(\alpha^{\gamma} x_1^{1-\gamma} + (1-\alpha)^{\gamma} x_2^{1-\gamma}\right)$$
$$u_2(x_1, x_2) = \frac{1}{1-\gamma}\left((1-\alpha)^{\gamma} x_1^{1-\gamma} + \alpha^{\gamma} x_2^{1-\gamma}\right)$$

and the consumers' endowments are symmetric:

$$(e_1, f_1) = (e, 1-e), \quad (e_2, f_2) = (1-e, e), \quad 0 < e < 1. \tag{12}$$

To study the equilibria prices, we have to analyze the roots of (4) in this particular case. By substituting (11) and (12) into (5), we are reduced to analyze the polynomial

$$P(q) = -\delta(\alpha, e)q^3 + q^2 - q + \delta(\alpha, e),$$

where

$$\delta(\alpha, e) := \frac{\alpha^2 - (2\alpha - 1)e}{\alpha - \alpha^2}, \quad 0 < \alpha, e < 1. \tag{13}$$

The discriminant of this polynomial (cfr. (9)) is

$$\Delta = -(3\delta(\alpha, e) - 1)^3(\delta(\alpha, e) + 1).$$

Because $\Delta \geq 0 (< 0)$ iff and only if $\delta(\alpha, e) \leq \frac{1}{3} (> \frac{1}{3})$ – that is, $(\alpha - 3e)(2\alpha - 1) \leq 0 (> 0)$ –, we get the following result.

**Theorem 5.** *In the CRRA symmetric case with $\gamma = 3$, the following facts hold true:*
(1) *There is uniqueness of equilibria if and only if $(\alpha - 3e)(2\alpha - 1) > 0$;*
(2) *There are critical equilibria if and only if $\alpha = 3e$ or $\alpha = \frac{1}{2}$;*
(3) *There are three equilibria if and only if $(\alpha - 3e)(2\alpha - 1) < 0$.*

This theorem complements (Toda and Walsh 2017, Proposition 1) by providing necessary and sufficient conditions for uniqueness in the CRRA economy with relative risk aversion $\gamma = 3$. It establishes a connection between the number of equilibria, their type (regular and critical), endowments, and utility weights. We finally note that for $\alpha = \frac{1}{7}$ and $e = \frac{1}{49}$, one has $\delta(\alpha, e) = \frac{2}{7}$ and the polynomial $P(q)$ becomes $-\frac{2}{7}q^3 + q^2 - q + \frac{2}{7}$ which has three solutions, $\{\frac{1}{2}, 1, 2\}$, which correspond, because $p = q^3$, to $\{\frac{1}{8}, 1, 8\}$ in accordance with (Toda and Walsh 2017)'s example.

**Remark 6.** Theorem 5 has shown how this polynomial approach makes it easier to deal with the particular case $\gamma = 3$. We believe that this method deserves future investigation given that it can be used to tackle the general case of $I$ consumers under CRRA and HARA preferences and arbitrary $\gamma$.

# 3 Proof of the Main Result

In this section, we lead the reader through all the intermediate steps that are necessary to prove our main result, Theorem 1. For ease of exposition, we have relegated the proofs of these steps to Appendix A.2.

We consider a pure exchange economy with two goods and an arbitrary (finite) number ($I$) of impatience types, where type $i$ has preferences represented by (1).

Consumer $i$'s endowments is denoted by $(e_i, f_i)$. We have $\sum_{i=1}^{I} e_i = r_x(I)$ and $\sum_{i=1}^{I} f_i = r_y(I)$, where $(r_x(I), r_y(I))$ is the total resources vector.

Under the budget constraint $px + y \leq pe_i + f_i$, consumer $i$'s maximization problem, for $i = 1, \ldots, I$, leads to (see the proof in the Appendix A) the aggregate excess demand function for good $x$:

$$Z(e, f, p, \epsilon, \sigma, I) := \sum_{i=1}^{I} \frac{b - bp^{\epsilon}\sigma_i + a\epsilon(pe_i + f_i)}{a\epsilon(p + \sigma_i p^{\epsilon})} - r_x(I), \tag{14}$$

where we set

$$\epsilon := \frac{1}{\gamma}, \quad \sigma_i := \beta_i^{\epsilon}, \quad i = 1, \ldots, I$$

and we denote

$$e = (e_1, \ldots, e_I), \quad f = (f_1, \ldots, f_I), \quad \sigma = (\sigma_1, \ldots, \sigma_I).$$

To make algebraic manipulations of expression (14) easier (see Proposition 7), we will exploit the presence of symmetric polynomials within (14) via Newton's identities. We refer the interested reader to Edwards (1984), although this present paper is self-contained.

For each integer $t$ such that $1 \leq t \leq I$, set $S_0(X) = 1$ and

$$S_t(X) := \sum_{i_1<\ldots<i_t}^{I} X_{i_1} \ldots X_{i_t}, \quad i_l = 1, \ldots, I, \quad l = 1, \ldots, t, \tag{15}$$

where $X = (X_1, \ldots, X_I)$ is a generic vector of variables. For example, if $I = 3$, $S_1(X) = X_1 + X_2 + X_3$, $S_2(X) = X_1X_2 + X_1X_3 + X_2X_3$, $S_3(X) = X_1X_2X_3$, and so on. Moreover for $i = 1, \ldots, I$ we denote by $X_{-i}$ the vector of $I - 1$ variables obtained by deleting from $X$ the $i$th component, namely

$$X_{-i} = (X_1, \ldots, X_{i-1}, X_{i+1}, \ldots X_I).$$

Note that, as a straightforward consequence of these definitions, for all $I \geq 2$, $1 \leq t \leq I$ and $i = 1, \ldots, I$,

$$S_t(X) = S_t(X_{-I}) + X_I S_{t-1}(X_{-I}), \tag{16}$$

$$S_t(X_{-i}) = S_t(X_{-(I,i)}) + X_I S_{t-1}(X_{-(I,i)}), \tag{17}$$

where $X_{-(I,i)} := (X_{-i})_{-I}$.

The Proof of Theorem 1 uses the following Proposition and Lemmas (see the Appendix A for their proofs):

**Proposition 7.** *The zero set of the aggregate excess demand function* (14) *equals that of the following function:*

$$\begin{aligned} z(e, f, p, \epsilon, \sigma, I) := & -\sigma_1 \ldots \sigma_I \left( r_x(I) + \frac{Ib}{a\epsilon} \right) p^{I(\epsilon-1)+1} + \\ & - \sum_{t=1}^{I-1} \left[ \xi(e, \sigma, I, t) + u(\sigma, I, t) \right] p^{t(\epsilon-1)+1} \\ & + \sum_{t=1}^{I-1} v(f, \sigma, I, t) p^{t(\epsilon-1)} + r_y(I) + \frac{Ib}{a\epsilon}, \end{aligned} \tag{18}$$

*where*

$$\xi(e, \sigma, I, t) := \left[ r_x(I) S_t(\sigma) - \sum_{i=1}^{I} e_i S_t(\sigma_{-i}) \right], \tag{19}$$

$$u(\sigma, I, t) := \frac{b}{a\epsilon} \sum_{i=1}^{I} \sigma_i S_{t-1}(\sigma_{-i}), \tag{20}$$

$$v(f, \sigma, I, t) := \sum_{i=1}^{I} \left( f_i + \frac{b}{a\epsilon} \right) S_t(\sigma_{-i}) \tag{21}$$

**Lemma 8.** *For* $\sigma = (\sigma_1, \dots, \sigma_I)$ *set*

$$F(t, I) := r_x(I) S_t(\sigma) - \sum_{i=1}^{I} e_i S_t(\sigma_{-i}). \tag{22}$$

*Then,* $F(t, I) > 0$ *for each integer* $t$ *such that* $1 \leq t \leq I - 1$.

We recall that an equilibrium is regular if the slope of excess demand is nonzero. The next lemma can only be applied to regular equilibria, being robust to sufficiently small perturbations:

**Lemma 9.** *Let us denote by* $N(e, f, \epsilon, \sigma, I)$ *the (finite) cardinality of the set of regular equilibria of the aggregate excess demand* (14)*, or equivalently* (18)*, in the HARA case. For every* $\epsilon_0 \in (0, 1)$*, and for all* $e, f, \sigma, I$*, there exist two natural numbers* $m, n$*,* $0 < m < n$*, where* $\frac{m}{n}$ *is sufficiently close to* $\epsilon_0$*, such that*

$$N(e, f, \epsilon_0, \sigma, I) \leq N\left(e, f, \frac{m}{n}, \sigma, I\right).$$

This lemma points out that the number of regular equilibria cannot decrease after the perturbation. The following figure provides an insight into the previous lemma. The black curve represents the aggregate excess demand curve for a given $\epsilon$. The perturbation induced in the original curve by replacing $\epsilon$ with $\frac{m}{n}$ is represented by the red curve. In this case, the number of regular equilibria is such that

$$3 = N(e, f, \epsilon, \sigma, I) \leq N\left(e, f, \frac{m}{n}, \sigma, I\right) = 5.$$

Observe that if the perturbation had “opposite direction”, then the number of regular equilibria would remain unchanged (Figure 1).

The reader who is interested in the proofs of these intermediate results is referred to Appendix A.2. Otherwise, the reader can skip it and follow the line of reasoning below to see how these steps are connected in the strategy of the Proof of Theorem 1.

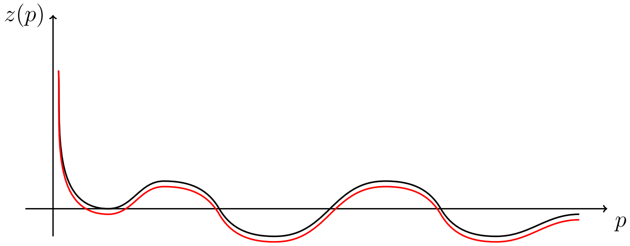


**Figure 1:** Aggregate excess demand after replacing $\epsilon$ with $\frac{m}{n}$.

*Proof of Theorem 1.* By inserting $\epsilon = \frac{m}{n}$, a rational number, as in Lemma 9 into (18) and denoting $q = p^{\frac{1}{n}}$, we deduce that

$$z\left(e, f, q, \frac{m}{n}, \sigma, I\right) := -\sigma_1 \dots \sigma_I \left(r_x(I) + \frac{Ib}{a\epsilon}\right) q^{I(m-n)+n} - \sum_{t=1}^{I-1} \left[\xi(e, \sigma, I, t) + u(\sigma, I, t)\right] q^{t(m-n)+n} + \sum_{t=1}^{I-1} v(f, \sigma, I, t) q^{t(m-n)} + r_y + \frac{Ib}{a\epsilon} \tag{23}$$

We claim that for $\frac{m}{n} \in \left[\frac{I-1}{I}, 1\right)$ and for $e, f, \sigma, I$ arbitrarily chosen, there exists a unique $q_0 > 0$ that is a zero of the function

$$z\left(e, f, q, \frac{m}{n}, \sigma, I\right), \tag{24}$$

that is, $z\left(e, f, q_0, \frac{m}{n}, \sigma, I\right) = 0$. Then, by applying Lemma 9, we deduce that $p_0 = q_0^n$ is the only positive solution of (18) for $\epsilon \in \left[\frac{I-1}{I}, 1\right)$ (i.e. we have uniqueness of price equilibrium in the HARA economy under the assumption (3)), thus arriving at a conclusion of the proof of the theorem.

To prove this claim, we fix $e, f, \sigma, a, b, I$ and set

$$\mu_I := \sigma_1 \dots \sigma_I \left(r_x(I) + \frac{Ib}{a\epsilon}\right), \quad \mu_t := \xi_t + u_t, \quad v_t, \; v_0 = r_y + \frac{Ib}{a\epsilon},$$

where for $t = 1, \dots, I-1$ we define

$$\xi_t := \xi(e, \sigma, I, t), \quad u_t := u(\sigma, I, t), \quad v_t := v(f, \sigma, I, t).$$

By multiplying Eq. (24) by the monomial $q^{(I-1)(n-m)}$ $(n > m)$ and using (23), one sees that there exists a positive zero of (24) if and only if the following polynomial in the variable $q$ has a unique positive root:

$$P(q) := v_{I-1} + v_{I-2} q^{n-m} + \cdots + v_1 q^{(I-2)(n-m)} + v_0 q^{(I-1)(n-m)} - \mu_I q^m - \mu_{I-1} q^n - \cdots - \mu_1 q^{(I-2)(n-m)+n} \tag{25}$$

Notice now that by the definition of the symbols involved, $\mu_I$, $v_0$, $u_t$ and $v_t$ are strictly positive real numbers for all $t = 1, \dots, I-1$. By applying Lemma 8, we also deduce that $\xi_t$ is strictly positive and hence $\mu_t > 0$ for all $t = 1, \dots, I-1$. Because the assumption $\frac{m}{n} \in \left[\frac{I-1}{I}, 1\right)$ is equivalent to $(I-1)(n-m) \le m$, it follows that the monomials appearing in $P(q)$ are written in increasing order. Thus, by Descartes' rule of sign (Anderson, Jackson, and Sitharam 1998), the polynomial $P(q)$ has a unique positive root and the theorem follows. □

Finally, the following corollary, which follows immediately by Theorem 1, answers the question left open by Toda and Walsh of whether multiplicity is possible or not for $\gamma \leq 2$.

**Corollary 10.** *In the HARA case with two agents, if $\gamma$ belongs to the interval* $(1, 2]$, *then the equilibrium price is unique.*

**Acknowledgement:** We would like to acknowledge Alexis Akira Toda and an anonymous referee for their useful comments and critical remarks.
**Research funding:** The first author was supported by INdAM. GNSAGA – Gruppo Nazionale per le Strutture Algebriche, Geometriche e le loro Applicazioni. Both authors were supported by STAGE – Funded by Fondazione di Sardegna.
**Author contribution:** All the authors have accepted responsibility for the entire content of this submitted manuscript and approved submission.
**Conflict of interest statement:** The authors declare no conflicts of interest regarding this article.

# Appendix A

## A.1 Derivation of Eq. (14)

Consumer $i$ maximizes (see (1))

$$u_i(x, y) = u_H(x) + \beta_i u_H(y),$$

under the constraint

$$pe_i + f_i \leq px + y,$$

where

$$u_H(x) := \frac{\gamma}{1-\gamma}\left(b + \frac{a}{\gamma}x\right)^{1-\gamma}, \quad \gamma > 0,\ \gamma \neq 1,\ a > 0,\ b \geq 0.$$

By monotonicity of preferences, the budget constraint is fulfilled as an equality. By substituting $y = pe_i + f_i - px$ into the objective function, we turn the constrained maximization problem into the unconstrained problem of maximizing the following function

$$\frac{\gamma}{1-\gamma}\left(b+\frac{a}{\gamma}x\right)^{1-\gamma}+\beta_i\frac{\gamma}{1-\gamma}\left(b+\frac{a}{\gamma}(pe_i+f_i-px)\right)^{1-\gamma}.$$

The necessary (and sufficient) condition is

$$a\left(b+\frac{ax}{\gamma}\right)^{-\gamma}-ap\beta\left(b+\frac{a(f+ep-px)}{\gamma}\right)^{-\gamma}=0.$$

By setting $\epsilon := \frac{1}{\gamma}$ and $\sigma_i := \beta_i^{\epsilon}$ we obtain $i$'s demand for good $x$:

$$\frac{b-bp^{\epsilon}\sigma_i+a\epsilon(pe_i+f_i)}{a\epsilon(p+\sigma_i p^{\epsilon})}.$$

By summing over consumers $i=1,\dots,I$ and denoting $\sum_{i=1}^{I}e_i=r_x(I)$, we obtain (14), the aggregate excess demand function for good $x$.

## A.2 Proofs of the Intermediate Results

*Proof of Proposition 7.* At an equilibrium price $p$, the function (14)

$$-pr_x(I)+\sum_{i=1}^{I}\frac{pe_i+f_i+\frac{b}{a\epsilon}-\frac{b}{a\epsilon}\sigma_i p^{\epsilon}}{1+\sigma_i p^{\epsilon-1}},$$

vanishes, or equivalently

$$-pr_x(I)\prod_{i=1}^{I}\left(1+\sigma_i p^{\epsilon-1}\right)+\sum_{i=1}^{I}\left[\left(pe_i+f_i+\frac{b}{a\epsilon}-\frac{b}{a\epsilon}\sigma_i p^{\epsilon}\right)\prod_{j=1}^{I}\frac{1+\sigma_j p^{\epsilon-1}}{1+\sigma_i p^{\epsilon-1}}\right]=0.$$

By Eq. (15), we can write these products as

$$\prod_{i=1}^{I}\left(1+\sigma_i p^{\epsilon-1}\right)=1+\sum_{t=1}^{I-1}S_t(\sigma)p^{t(\epsilon-1)}+\sigma_1\dots\sigma_I p^{I(\epsilon-1)}$$

$$\prod_{j=1}^{I}\frac{1+\sigma_j p^{\epsilon-1}}{1+\sigma_i p^{\epsilon-1}}=1+\sum_{t=1}^{I-1}S_t(\sigma_{-i})p^{t(\epsilon-1)}$$

and rewrite the expression accordingly:

$$-pr_x(I)\left[1+\sum_{t=1}^{I-1}S_t(\sigma)p^{t(\epsilon-1)}+\sigma_1\dots\sigma_I p^{I(\epsilon-1)}\right]$$
$$+\sum_{i=1}^{I}\left(pe_i+f_i+\frac{b}{a\epsilon}-\frac{b}{a\epsilon}\sigma_i p^{\epsilon}\right)\left[1+\sum_{t=1}^{I-1}S_t(\sigma_{-i})p^{t(\epsilon-1)}\right].$$

By expanding and rearranging, we immediately get

$$-r_x(I)\sigma_1\dots\sigma_I p^{I(\epsilon-1)+1}-\sum_{t=1}^{I-1}\left[r_x(I)S_t(\sigma)-\sum_{i=1}^{I}e_iS_t(\sigma_{-i})\right]p^{t(\epsilon-1)+1}$$
$$+\sum_{t=1}^{I-1}\sum_{i=1}^{I}\left(f_i+\frac{b}{a\epsilon}\right)S_t(\sigma_{-i})p^{t(\epsilon-1)}$$
$$-\frac{b}{a\epsilon}\left(\sum_{i=1}^{I}\sigma_i\right)p^{\epsilon}-\frac{b}{a\epsilon}\sum_{t=1}^{I-1}\sum_{i=1}^{I}\sigma_iS_t(\sigma_{-i})p^{t(\epsilon-1)+\epsilon}+r_y(I)+\frac{Ib}{a\epsilon}.$$

Note now that by the change of index $u:=t+1$, one gets

$$-\frac{b}{a\epsilon}\sum_{t=1}^{I-1}\sum_{i=1}^{I}\sigma_iS_t(\sigma_{-i})p^{t(\epsilon-1)+\epsilon}=-\frac{b}{a\epsilon}\sum_{t=1}^{I-1}\sum_{i=1}^{I}\sigma_iS_t(\sigma_{-i})p^{(t+1)(\epsilon-1)+1}$$
$$=-\frac{b}{a\epsilon}\sum_{u=2}^{I}\sum_{i=1}^{I}\sigma_iS_{u-1}(\sigma_{-i})p^{u(\epsilon-1)+1}=\frac{b}{a\epsilon}\sum_{i=1}^{I}\sigma_iS_t(\sigma_{-i})p^{(\epsilon-1)+1}$$
$$-\frac{b}{a\epsilon}\sum_{t=1}^{I-1}\sum_{i=1}^{I}\sigma_iS_{t-1}(\sigma_{-i})p^{t(\epsilon-1)+1}-\frac{b}{a\epsilon}\sum_{i=1}^{I}\sigma_iS_{I-1}(\sigma_{-i})p^{I(\epsilon-1)+1}$$
$$=\frac{b}{a\epsilon}\left(\sum_{i=1}^{I}\sigma_i\right)p^{\epsilon}-\frac{b}{a\epsilon}\sum_{t=1}^{I-1}\sum_{i=1}^{I}\sigma_iS_{t-1}(\sigma_{-i})p^{t(\epsilon-1)+1}$$
$$-\frac{Ib}{a\epsilon}\sigma_1\dots\sigma_I p^{I(\epsilon-1)+1},$$

where in the last equality we use $\sum_{i=1}^{I}\sigma_iS_{I-1}(\sigma_{-i})=I\sigma_1\cdots\sigma_I$.

By inserting this last equality into the previous expression, one gets

$$-\sigma_1\dots\sigma_I\left(r_x(I)+\frac{Ib}{a\epsilon}\right)p^{I(\epsilon-1)+1}-\sum_{t=1}^{I-1}\left[r_x(I)S_t(\sigma)\right.$$
$$\left.-\sum_{i=1}^{I}e_iS_t(\sigma_{-i})+\frac{b}{a\epsilon}\sum_{i=1}^{I}\sigma_iS_{t-1}(\sigma_{-i})\right]p^{t(\epsilon-1)+1}$$
$$+\sum_{t=1}^{I-1}\sum_{i=1}^{I}\left(f_i+\frac{b}{a\epsilon}\right)S_t(\sigma_{-i})p^{t(\epsilon-1)}+r_y(I)+\frac{Ib}{a\epsilon},$$

and the proposition follows. □

*Proof of Lemma 8.* We work by induction on $I \geq 2$ for all $t$ such that $1 \leq t \leq I-1$. The base on the induction is immediate ($\sigma = (\sigma_1, \sigma_2)$):

$$\begin{aligned} F(1,2) &= (e_1 + e_2)S_1(\sigma) = -e_1 S_1(\sigma_{-1}) - e_2 S_1(\sigma_{-2}) \\ &= (e_1 + e_2)(\sigma_1 + \sigma_2) - e_1\sigma_2 - e_2\sigma_1 \\ &= e_1\sigma_1 + e_2\sigma_2 > 0 \\ F(2,2) &= (e_1 + e_2)S_2(\sigma) - e_1 S_2(\sigma_{-1}) - e_2 S_2(\sigma_{-2}) = (e_1 + e_2)\sigma_1\sigma_2 > 0. \end{aligned}$$

Assume now, by the induction hypothesis, that

$$F(t, I-1) > 0$$

for each integer $1 \leq t \leq I-2$. By (16) and (17), Eq. (22) reads as

$$\begin{aligned} F(t,I) = {} & r_x(I-1)S_t(\sigma_{-I}) + \sigma_I r_x(I) S_{t-1}(\sigma_{-I}) \\ & - \sum_{j=1}^{I-1} e_j \left[ S_t(\sigma_{-(I,j)}) + \sigma_I S_t(\sigma_{-(I,j)}) \right], \end{aligned}$$

that we can rewrite as

$$F(t,I) = F(t, I-1) + \sigma_I F(t-1, I-1) + \sigma_I e_I S_{t-1}(\sigma_{-I}),$$

which is strictly positive by the induction assumption. □

*Proof of Lemma 9.* For fixed $(e, f, \sigma, I)$, the aggregate excess demand function (18) depends on the price $p$ and the parameter $\epsilon$. Let $p_0$ be a regular equilibrium of the function of one variable $z(e, f, p, \epsilon_0, \sigma, I)$. By the implicit function theorem, the regularity property holds true after a small perturbation of $\epsilon_0$. Hence, the result follows by the denseness of $\mathbb{Q}$ in $\mathbb{R}$. □

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