LEFT VARIABLE INCLUSION LOGICS ASSOCIATED WITH CLASSICAL LOGIC

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Abstract. Logics of significance have been proposed in an attempt to overcome the shortcomings of classical logic as a model of reasoning in the presence of non-significant (e.g. meaningless, ill-formed, unverifiable) sentences. Many-valued logicians have addressed this problem by introducing logics with *infectious* truth values. Cases in point are the *Weak Kleene* logics B_3 (paracomplete Weak Kleene logic) and PWK (paraconsistent Weak Kleene logic). Over time, it has become clear that the valid entailments of these significance logics obey variable inclusion patterns that link them to other, usually better known, logics – such patterns, however, allow for disturbing exceptions. Logics of pure (left or right) variable inclusion have been introduced with an eye to removing these exceptions. In this paper, we consider the pure left variable inclusion companion of classical logic and give a complete description of its subclassical extensions. We also provide relative axiomatisations and characteristic (sets of) matrices for each one of these extensions, as well as syntactic descriptions (in terms of variable inclusion criteria) for the valid entailments of some of them, and determine in each case the algebra reducts of the Suszko reduced matrix models.

1. Introduction

Recently, the relationships between logics of significance and logics of variable inclusion have stirred considerable attention. *Logics of significance* [4, 16, 14] have been proposed in an attempt to overcome the shortcomings of classical logic (CL) as a model of reasoning in the presence of non-significant (e.g. meaningless, ill-formed, unverifiable) sentences. It seems plausible to stipulate that sentential compounds containing a non-significant component are themselves non-significant. Hence, many-valued logicians have initially addressed this problem by introducing logics whose semantics provides for *infectious* truth values: namely, values that "spread" from

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propositional variables to any formula containing them. Cases in point are the *Weak Kleene* logics B_3 (paracomplete Weak Kleene logic) [4, 19, 8, 24] and PWK (paraconsistent Weak Kleene logic) [16, 19, 11, 5, 23, 8]; see below for more details.

Over time, it has become clear that the valid entailments of these significance logics obey some recurring variable inclusion patterns that link them to other, usually better known, logics [34, 11, 14, 12, 13]. For example, according to the Ciuni-Carrara theorem, if $\Gamma \cup \{\varphi\}$ is a set of formulas in the language of CL, we have that $\Gamma \vdash_{\mathrm{PWK}} \varphi$ iff there exists $\Delta \subseteq \Gamma$ such that $\Delta \vdash_{\mathrm{CL}} \varphi$ and such that the variables occurring in members of Δ are included in the variables occurring in φ. This situation is not specific to weak Kleene logics, but extends way further. In [5, 7, 9, 8, 24] a general theory has been developed to the effect that we can associate to any arbitrary logic L a *left variable inclusion companion* L^l and a *right variable inclusion companion* L^r, according as we have a demand that the variables in the premisses be included in the conclusion, or vice versa. For example, $\mathrm{CL}^l = \mathrm{PWK}$ and $\mathrm{CL}^r = \mathrm{B}_3$. For logics of variable inclusion, a general semantic framework has been introduced that encompasses as a very special case many-valued logics determined by matrices containing infectious truth values. This framework heavily borrows tools and techniques from the algebraic theories of regular varieties and Płonka sums of algebras [27, 32], by extending the Płonka sum construction from algebras to logical matrices.

As logics of variable inclusion, however, logics like B_3 or PWK have a clear drawback. For example, for distinct propositional variables x, y we have that $x \vdash_{\mathrm{PWK}} y \vee \neg y$. Indeed, there exists $\Delta \subseteq \{x\}$ – that is to say, the empty set – such that $\Delta \vdash_{\mathrm{CL}} y \vee \neg y$ and the variable inclusion constraint is (vacuously) respected. When it comes to entailments with a classical theorem for a conclusion, the variable inclusion strainer gets ineffective. A dual phenomenon, true to form, happens with B_3 and entailments with a classical antitheorem for a set of premisses. To avoid these exceptions, in [25], to any arbitrary logic L were associated a *pure left variable inclusion companion* L^{pl} and a *pure right variable inclusion companion* L^{pr}: see below for the precise definitions. Several results concerning CL^{pl} and CL^{pr} were proved, including the fact that the former has a 5-element characteristic matrix, while the latter has no single characteristic matrix. The rudiments of a general semantics for all logics of pure variable inclusion in terms of Płonka sums of matrices was also proposed.

In this paper we focus on the *left* variable inclusion companion of CL, the logic CL^{pl}. We provide a complete description of the lattice of all its extensions that are contained in CL (observe that CL^{pl}, being a theoremless logic, has relatively uninteresting extensions, like the almost inconsistent logic in its language; such logics will be barely mentioned in this paper). Interestingly, there are finitely many such extensions – 12, to be precise. All these logics are determined by finitely many (at most two) finite matrices, hence are finitary and decidable. For each one of them we provide axiomatic bases (relative to CL^{pl}), we identify their characteristic matrix or matrices, and we determine the class of the algebra reducts of their Suszko reduced models. In some cases, we also identify a form of variable inclusion constraint that links them more directly to CL, showing that these logics are, in a broad sense, variable inclusion logics in their own right.

In greater detail, here is the outline of our paper. In Section 2, we recapitulate some basic notions concerning universal algebra and abstract algebraic logic; Płonka sums of algebras and matrices; generalised involutive bisemilattices; logics of variable inclusion; and logics of pure variable inclusion. In Section 3 we describe the structure of the deductive filters, and hence of the matrix models, of CL^{pl}. Section 4 is the core of the paper, in that it contains an exhaustive description of all the extensions of CL^{pl} below CL. Finally, in Section 5, we associate to each logic in this lattice of extensions a quasivariety of generalised involutive bisemilattices, which acts as the class of the algebra reducts of its Suszko reduced models.

2. Preliminaries

2.1. **Universal Algebra and Abstract Algebraic Logic.** For unexplained terminology and notation on universal algebra, we refer the reader to [10]. We denote algebras by boldface capital letters and their universes by italicised capital letters. $Hom\,(\mathbf{A}, \mathbf{B})$ denotes the set of all homomorphisms from $\mathbf{A}$ to $\mathbf{B}$, with $End\,(\mathbf{A}) = Hom\,(\mathbf{A}, \mathbf{A})$. Given a class of algebras $\mathcal{K}$, we respectively denote by $I(\mathcal{K}), H(\mathcal{K}), S(\mathcal{K})$, $P(\mathcal{K})$, and $P_{SD}(\mathcal{K})$ the classes of isomorphic images, homomorphic images, subalgebras, products, and subdirect products of algebras in $\mathcal{K}$; $V(\mathcal{K})$ is the variety generated by $\mathcal{K}$. $\mathbf{B}_n$ will denote the finite Boolean algebra with n elements, for any positive integer n of the form 2^k. The elements of $\mathbf{B}_2$ will be denoted by 0 and 1.

In this paper, we only consider languages (similarity types) with at least an operation symbol of arity at least 2. The algebra of formulas of the language $\mathcal{L}$ ($\mathcal{L}$-formulas), over a countably infinite set $Var(\mathcal{L})$ of generators (variables), is denoted by $\mathbf{Fm}(\mathcal{L})$. Identities of language $\mathcal{L}$ ($\mathcal{L}$-identities) are ordered pairs of $\mathcal{L}$-formulas, noted $\varphi \approx \psi$. Given $\varphi \in Fm(\mathcal{L})$, we denote by $Var(\varphi)$ the set of variables occurring in φ. Similarly, given $\Gamma \subseteq Fm(\mathcal{L})$, we set $Var(\Gamma) = \bigcup\{Var(\gamma) \colon \gamma \in \Gamma\}$. An identity $\varphi \approx \psi$ of language $\mathcal{L}$ is *regular* provided that $Var(\varphi) = Var(\psi)$. For example, the associativity, the commutativity or the idempotency of a binary operation are all expressed by regular identities. Given an algebra $\mathbf{A}$, a function $p : A \to A$ is a *unary polynomial function* of $\mathbf{A}$ if there is a formula $\varphi(x, y_1, ..., y_m)$, and elements $b_1, \ldots, b_m \in A$ such that

$$p(a) = \varphi^{\mathbf{A}}(a, b_1, \ldots, b_m)$$

for every $a \in A$.

A distinguished language $\mathcal{L}_1$ will be assumed. It contains the connectives (operation symbols) $\neg$ (unary), $\wedge$ (binary) and $\vee$ (also binary). Clearly, the variety $\mathcal{BA}$ of Boolean algebras can be presented in the language $\mathcal{L}_1$, since the constants 0 and 1 are term-definable.

A *prevariety* of language $\mathcal{L}$ is a class $\mathcal{K}$ of $\mathcal{L}$-algebras that contains at least a trivial algebra and is closed under the formation of isomorphic images, subalgebras, and direct products: $\mathcal{K} = ISP(\mathcal{K})$. Whenever $\mathcal{K}$ is also closed under ultraproducts, it is said to be a *quasivariety*. It is well-known that $\mathcal{K}$ is a prevariety iff it can be axiomatised by infinitary formulas of equational logic of the form $\&_{i \in I} \varphi_i \approx \psi_i \Rightarrow \varphi \approx \psi$, where the φ_i's, the ψ_i's, φ and ψ are $\mathcal{L}$-formulas; $\mathcal{K}$ is a quasivariety in case the index set I can be taken to be finite. Of course, the prevariety generated by a finite class of finite algebras is the same as the quasivariety generated by it. The subprevarieties of a prevariety $\mathcal{K}$ form a poset under set inclusion.

A further notion will be needed later on. An *ordered structure* is a first-order structure where one of the relations is a partial order $\leq$. An ordered structure is said to be *semilattice-ordered* if $\leq$ is a semilattice order with induced meet $\wedge$.

For unexplained terminology and notation on abstract algebraic logic, the reader is referred to [15]. A *logic* of language $\mathcal{L}$ is an ordered pair $\mathrm{L} = \langle \mathbf{Fm}(\mathcal{L}), \vdash_{\mathrm{L}} \rangle$, where $\vdash_{\mathrm{L}} \subseteq \mathcal{P}(Fm(\mathcal{L})) \times Fm(\mathcal{L})$ is a consequence relation that is *substitution-invariant*, meaning that for every $\sigma \in End(\mathbf{Fm}(\mathcal{L}))$ and for every $\Gamma \cup \{\varphi\} \subseteq Fm(\mathcal{L})$, if $\Gamma \vdash_{\mathrm{L}} \varphi$, then $\sigma[\Gamma] \vdash_{\mathrm{L}} \sigma(\varphi)$. A logic L of language $\mathcal{L}$ is *finitary* when the following holds for all $\Gamma \cup \{\varphi\} \subseteq Fm(\mathcal{L})$: if $\Gamma \vdash_{\mathrm{L}} \varphi$, then there is a finite $\Delta \subseteq \Gamma$ s.t. $\Delta \vdash_{\mathrm{L}} \varphi$. Given $\Gamma, \Delta \subseteq Fm(\mathcal{L})$, we write $\Gamma \vdash_{\mathrm{L}} \Delta$ as shorthand for: $\Gamma \vdash_{\mathrm{L}} \varphi$, for all $\varphi \in \Delta$. We write $\vdash_{\mathrm{L}} \varphi$ in place of $\emptyset \vdash_{\mathrm{L}} \varphi$; in such case, we say that φ is a *theorem* of L. An *antitheorem* of L is a set $\Gamma \subset Fm(\mathcal{L})$ such that there is at least one variable y not occurring in Γ, and $\Gamma \vdash_{\mathrm{L}} x$ for every variable x not occurring in Γ. Equivalently, it is a set $\Gamma \subset Fm(\mathcal{L})$ such that for every $\sigma \in End(\mathbf{Fm}(\mathcal{L}))$ and for every $\varphi \in Fm(\mathcal{L})$, we have that $\sigma[\Gamma] \vdash_{\mathrm{L}} \varphi$.

An (inference) *rule* of language $\mathcal{L}$ is an ordered pair $\langle \Gamma, \varphi \rangle$, with $\Gamma = \{\varphi_1, ..., \varphi_n\}$ a finite subset of $Fm(\mathcal{L})$ and $\varphi \in Fm(\mathcal{L})$, written more conveniently in the form $\varphi_1, ..., \varphi_n \vdash \varphi$. The following rules of language $\mathcal{L}_1$ will be used later:

CS: $\varphi \wedge \psi \vdash \varphi, \psi$;
EFQ: $\varphi, \neg\varphi \vdash \psi$;
qEFQ: $(\varphi \wedge \neg\varphi) \vee \psi \vdash \psi$;
EFV: $\varphi, \neg\varphi \vdash \psi \vee \neg\psi$;
VEQ: $\varphi \vdash \psi \vee \neg\psi$;
MP: $\varphi, \neg\varphi \vee \psi \vdash \psi$;
qMP: $\varphi, \neg\varphi \vee \psi \vee \neg\psi \vdash \psi \vee \neg\psi$;
EM: $\vdash \varphi \vee \neg\varphi$;
eMP: $\psi \vee \neg\psi, \varphi, \neg\varphi \vee \psi \vdash \psi$;
EFVQ: $\varphi \vee \neg\varphi, \psi, \neg\psi \vdash \varphi$;
Res: $\varphi \vee \psi, \chi \vee \neg\psi \vdash \varphi \vee \chi$.

Let $\mathbf{A}$ be a semilattice-ordered structure of algebraic language $\mathcal{L}$ and relational language $\langle 2 \rangle$. The *semilattice-based logic* of $\mathbf{A}$ is the finitary logic $\mathrm{A}^{\leq} = \langle \mathbf{Fm}(\mathcal{L}), \vdash_{\mathbf{A}}^{\leq} \rangle$ of language $\mathcal{L}$, where, for every $\gamma_1, ..., \gamma_n, \varphi \in Fm(\mathcal{L})$:

- $\emptyset \vdash_{\mathbf{A}}^{\leq} \varphi \iff \forall a \in A, \forall v \in Hom(\mathbf{Fm}(\mathcal{L}), \mathbf{A}) : a \leq v(\varphi)$;
- $\gamma_1, ..., \gamma_n \vdash_{\mathbf{A}}^{\leq} \varphi \iff \forall v \in Hom(\mathbf{Fm}(\mathcal{L}), \mathbf{A}) : v(\gamma_1) \wedge ... \wedge v(\gamma_n) \leq v(\varphi)$.

An $\mathcal{L}$-*matrix* is an ordered pair $\langle \mathbf{A}, F \rangle$ where $\mathbf{A}$ is an algebra of language $\mathcal{L}$ and $F \subseteq A$. In this case, $\mathbf{A}$ is called the *algebra reduct* of $\langle \mathbf{A}, F \rangle$. The operations of taking substructures, direct and subdirect products, homomorphic and isomorphic images straightforwardly extend from algebras to matrices; with a slight abuse, we will employ the same symbols for such operations and the attendant class operators also in this case.

Every $\mathcal{L}$-matrix $\langle \mathbf{A}, F \rangle$ induces a logic of the same language whose consequence relation is determined as follows:

$$\Gamma \vdash_{\langle \mathbf{A}, F \rangle} \varphi \text{ iff for every } h \in Hom(\mathbf{Fm}(\mathcal{L}), \mathbf{A}), \text{ if } h[\Gamma] \subseteq F, \text{ then } h(\varphi) \in F.$$

For M a class of $\mathcal{L}$-matrices, we write $\Gamma \vdash_{\mathsf{M}} \varphi$ to mean $\Gamma \vdash_{\langle \mathbf{A}, F \rangle} \varphi$ for every $\langle \mathbf{A}, F \rangle \in \mathsf{M}$, and occasionally use $\mathsf{Log}(\mathsf{M})$ to denote the logic $\langle \mathbf{Fm}(\mathcal{L}), \vdash_{\mathsf{M}} \rangle$. A matrix $\langle \mathbf{A}, F \rangle$ is a *model* of the logic L, both of language $\mathcal{L}$, when $\vdash_{\mathrm{L}} \subseteq \vdash_{\langle \mathbf{A}, F \rangle}$; a logic L is *complete* w.r.t. a class of matrices M when $\vdash_{\mathrm{L}} = \vdash_{\mathsf{M}}$. A set $F \subseteq A$ is a *deductive filter* of L (or L-filter) on the algebra $\mathbf{A}$, when the matrix $\langle \mathbf{A}, F \rangle$ is a model of L. For $\mathbf{A}$ an algebra and $X \subseteq A$, we denote by $\mathrm{Fg}^{\mathbf{A}}_{\mathrm{L}}(X)$ the smallest L-filter on $\mathbf{A}$ containing X. When a logic is complete with respect to a single matrix, we call the matrix *characteristic* for the logic.

In the same way as we have done for Boolean algebras, we will consider *classical logic* CL as a logic of language $\mathcal{L}_1$:

$$\mathrm{CL} = \langle \mathbf{Fm}(\mathcal{L}_1), \vdash_{\mathrm{CL}} \rangle = \left\langle \mathbf{Fm}(\mathcal{L}_1), \vdash_{\langle \mathbf{B}_2, \{1\} \rangle} \right\rangle .$$

Let $\mathbf{A}$ be an algebra and $F \subseteq A$. A congruence θ of $\mathbf{A}$ is *compatible* with F when F is a union of θ-cosets. The largest congruence of $\mathbf{A}$ that is compatible with F always exists; this congruence is called the *Leibniz congruence* of F on $\mathbf{A}$, and is denoted by $\Omega^{\mathbf{A}} F$. The next lemma characterises Leibniz congruences:

Lemma 1. [15, Thm. 4.23] *Let* $\mathbf{A}$ *be an algebra,* $F \subseteq A$*, and* $a, b \in A$*.*

$$\langle a, b \rangle \in \Omega^{\mathbf{A}} F \iff \text{for every unary pol. function } p \text{ of } \mathbf{A}, \quad p(a) \in F \text{ if and only if } p(b) \in F.$$

The *Suszko congruence* of $F \subseteq A$ on $\mathbf{A}$ relative to the logic L, noted $\widetilde{\Omega}^{\mathbf{A}}_{\mathrm{L}} F$, on the other hand, is the intersection of all $\Omega^{\mathbf{A}} G$, where G is an L-filter on $\mathbf{A}$ that contains F. The next lemma characterises Suszko congruences:

Lemma 2. [15, Thm. 5.32] *Let* L *be a logic,* $\mathbf{A}$ *be an algebra,* $F \subseteq A$*, and* $a, b \in A$*.*

$$\langle a, b \rangle \in \widetilde{\Omega}^{\mathbf{A}}_{\mathrm{L}} F \iff \text{for every unary pol. function } p \text{ of } \mathbf{A}, \quad \mathrm{Fg}^{\mathbf{A}}_{\mathrm{L}}(F \cup \{p(a)\}) = \mathrm{Fg}^{\mathbf{A}}_{\mathrm{L}}(F \cup \{p(b)\}).$$

By means of the Leibniz and Suszko congruences, we can associate to logics three distinguished classes of models. More precisely, given a logic L of language $\mathcal{L}$, we set

$$\mathsf{Mod}(\mathrm{L}) = \{\langle \mathbf{A}, F \rangle : \langle \mathbf{A}, F \rangle \text{ is a model of L}\};$$
$$\mathsf{Mod}^*(\mathrm{L}) = \{\langle \mathbf{A}, F \rangle \in \mathsf{Mod}(\mathrm{L}) : \Omega^{\mathbf{A}} F = \Delta^{\mathbf{A}}\};$$
$$\mathsf{Mod}^{\mathrm{Su}}(\mathrm{L}) = \{\langle \mathbf{A}, F \rangle \in \mathsf{Mod}(\mathrm{L}) : \widetilde{\Omega}^{\mathbf{A}}_{\mathrm{L}} F = \Delta^{\mathbf{A}}\}.$$

The above classes of matrices are called, respectively, the classes of *models*, *Leibniz reduced models*, and *Suszko reduced models* of L.

Let $\mathrm{L} = \langle \mathbf{Fm}(\mathcal{L}), \vdash_{\mathrm{L}} \rangle$ be a logic of language $\mathcal{L}$. A *translation* of language $\mathcal{L}$ is a set $\tau = \{\gamma_i(x) \approx \delta_i(x)\}_{i \in I}$ of $\mathcal{L}$-identities in a single variable. We may also view a translation τ as a function which maps formulas in $Fm(\mathcal{L})$ to sets of identities of the same language. Thus, we let $\tau(\varphi)$ stand for the set

$$\{\gamma_i(x/\varphi) \approx \delta_i(x/\varphi)\}_{i \in I},$$

where $\gamma_i(x/\varphi)$ refers to the result of uniformly replacing any occurrences of x in γ_i by φ, and similarly for $\delta_i(x/\varphi)$. For $\Gamma \subseteq Fm(\mathcal{L})$, $\tau(\Gamma)$ is defined as $\bigcup \{\tau(\gamma) : \gamma \in \Gamma\}$. If $\mathbf{A}$ is an algebra of language $\mathcal{L}$ and τ is a translation of the same language, we denote by $Sol_{\tau}^{\mathbf{A}}$ the set of solutions $\{a \in A : \mathbf{A} \vDash \tau^{\mathbf{A}}(a)\}$.

A logic L of language $\mathcal{L}$ is:

- *protoalgebraic*, if there is a set of $\mathcal{L}$-formulas $\Delta(x, y)$ in two variables such that $\vdash_{\mathrm{L}} \Delta(x, x)$ and $x, \Delta(x, y) \vdash_{\mathrm{L}} y$;
- *equivalential*, if there is a set of $\mathcal{L}$-formulas $\Delta(x, y)$ in two variables such that for every $\langle \mathbf{A}, F \rangle \in \mathsf{Mod}(\mathrm{L})$,

$$\langle a, b \rangle \in \Omega^{\mathbf{A}} F \text{ iff } \Delta^{\mathbf{A}}(a, b) \subseteq F \text{ for all } a, b \in A.$$

- *truth-equational*, if there is a translation τ of language $\mathcal{L}$ such that for all $\langle \mathbf{A}, F \rangle \in \mathsf{Mod}^*(\mathrm{L})$, $a \in F$ iff $a \in Sol_{\tau}^{\mathbf{A}}$;
- *algebraisable*, if it is both equivalential and truth-equational.

Every equivalential logic is protoalgebraic. A finitary algebraisable logic L enjoys an especially tight and fruitful relationship with an attendant class of algebras $\mathcal{K}$, called the *equivalent algebraic semantics* for L. This relationship can be expressed in different ways — most notably, as an isomorphism of certain expanded lattices of theories of L and of theories of the equational consequence relation of $\mathcal{K}$, or as the presence of mutually inverse substitution-invariant mappings between entailments in L and in the equational consequence relation of $\mathcal{K}$: see [15] for more details.

A matrix is trivial if it is of the form $\langle \mathbf{A}, A \rangle$. Observe that the trivial matrix $\langle \mathbf{1}, \{1\} \rangle$ over the trivial algebra is a Leibniz reduced model of every logic. Moreover, if L is a logic and $\langle \mathbf{A}, A \rangle \in \mathsf{Mod}^*(\mathrm{L})$ is a trivial matrix, then $\langle \mathbf{A}, A \rangle = \langle \mathbf{1}, \{1\} \rangle$.

Given a logic L, we set

$$\mathsf{Alg}^*(\mathrm{L}) = \{\mathbf{A} : \text{ there is } F \subseteq A \text{ s.t. } \langle \mathbf{A}, F \rangle \in \mathsf{Mod}^*(\mathrm{L})\}.$$
$$\mathsf{Alg}(\mathrm{L}) = \{\mathbf{A} : \text{ there is } F \subseteq A \text{ s.t. } \langle \mathbf{A}, F \rangle \in \mathsf{Mod}^{\mathsf{Su}}(\mathrm{L})\}.$$

In other words, $\mathsf{Alg}^*(\mathrm{L})$ is the class of the algebra reducts of Leibniz reduced models of L, $\mathsf{Alg}(\mathrm{L})$ is the class of algebra reducts of

Suszko reduced models of L. This latter class is usually considered as the "algebraic counterpart" of L [15]. For a protoalgebraic logic L, $\mathsf{Alg}(\mathrm{L})$ coincides with $\mathsf{Alg}^*(\mathrm{L})$, but in general these two classes are different, although they generate the same prevariety.

Proposition 3. *Let* L *be a logic.*

(1) $\mathsf{Alg}(\mathrm{L}) = P_{SD}(\mathsf{Alg}^*(\mathrm{L}))$.
(2) *If* L *is protoalgebraic, then* $\mathsf{Mod}^*(\mathrm{L}) = \mathsf{Mod}^{\mathsf{Su}}(\mathrm{L})$ *and hence* $\mathsf{Alg}(\mathrm{L}) = \mathsf{Alg}^*(\mathrm{L})$.
(3) $\mathsf{Alg}(\mathrm{L})$ *is closed w.r.t. direct products.*

2.2. **Płonka sums of algebras and matrices.** Płonka sums (for which see e.g. [27, 28, 32]) are an extremely useful construction in universal algebra. They are especially designed for the investigation of varieties satisfying only regular identities. We recap hereafter the main definitions and concepts concerning them.

A *semilattice direct system (of $\mathcal{L}$-algebras)* is a triple

$$\mathbb{A} = \left\langle \{\mathbf{A}_i\}_{i\in I}, \mathbf{I}, \{p_{ij} : i \leq_{\mathbf{I}} j\} \right\rangle,$$

where:

- $\mathbf{I} = \langle I, \leq_{\mathbf{I}} \rangle$ is a join semilattice whose join is denoted by $\vee$;
- $\{\mathbf{A}_i\}_{i\in I}$ is a family of $\mathcal{L}$-algebras with pairwise disjoint universes;
- for every $i, j \in I$ such that $i \leq_{\mathbf{I}} j$, $p_{ij} \in Hom\left(\mathbf{A}_i, \mathbf{A}_j\right)$. Moreover, $p_{ii} = \Delta^{\mathbf{A}_i}$ for every $i \in I$, and if $i \leq_{\mathbf{I}} j \leq_{\mathbf{I}} k$, then $p_{ik} = p_{jk} \circ p_{ij}$.

If $\mathcal{L}$ does not contain constants and $\mathbb{A} = \left\langle \{\mathbf{A}_i\}_{i\in I}, \mathbf{I}, \{p_{ij} : i \leq_{\mathbf{I}} j\} \right\rangle$ is a semilattice direct system of $\mathcal{L}$-algebras, we define as follows a new $\mathcal{L}$-algebra $\mathcal{P}_{\text{ł}}(\mathbb{A})$, also noted $\mathcal{P}_{\text{ł}}(\mathbf{A}_i)$, called the *Płonka sum* over $\mathbb{A}$:

- its universe is the union $\bigcup_{i\in I} A_i$;
- for every n-ary basic operation f (with $n \geqslant 1$) in $\mathcal{L}$, and $a_1, \ldots, a_n \in \bigcup_{i\in I} A_i$,

$$f^{\mathcal{P}_{\text{ł}}(\mathbb{A})}(a_1, \ldots, a_n) = f^{\mathbf{A}_j}(p_{i_1 j}(a_1), \ldots, p_{i_n j}(a_n)),$$

  where $a_1 \in A_{i_1}, \ldots, a_n \in A_{i_n}$ and $j = i_1 \vee \cdots \vee i_n$.

We often refer to the algebras in $\{\mathbf{A}_i\}_{i\in I}$ as the *fibres* of $\mathcal{P}_{\text{ł}}(\mathbb{A})$. It will be expedient to avail ourselves of a special notation for Płonka sums with just two fibres $\mathbf{A}_i$ and $\mathbf{A}_j$, where $i <_{\mathbf{I}} j$ in the underlying semilattice on $\{i, j\}$. Such algebras will be denoted as $\mathbf{A}_i \oplus \mathbf{A}_j$; when

$\mathbf{A}_i$ is isomorphic to $\mathbf{A}_j$, the same symbol will be used for both summands with no danger of confusion. Whenever there is more than one algebra of this form, the homomorphisms p_{ij} will be specified to avoid ambiguities.

The problem as to whether all algebras in a variety are representable as Płonka sums over a certain semilattice direct system has been successfully addressed by Płonka [27, 28, 29]. It turns out that the Płonka construction preserves all regular identities – on the other hand, it fails to preserve any other identity that is satisfied in all the fibres. This is the content of the following:

Theorem 4. [27, Thm. I] *If $\mathbb{A}$ is a semilattice direct system of algebras containing at least two algebras, then all regular identities satisfied in all algebras of $\mathbb{A}$ are satisfied in $\mathcal{P}_{ł}(\mathbb{A})$, whereas any other identity is not satisfied in $\mathcal{P}_{ł}(\mathbb{A})$.*

We call *regular* a variety of algebras that satisfies only regular identities, and *irregular* a variety which is not regular. A variety $\mathcal{V}$ is *strongly irregular* if there is a formula $\varphi(x, y)$ such that $\mathcal{V} \models \varphi(x, y) \approx x$. In other words, a variety is strongly irregular if there is a formula realising the projection operation on the first component in all algebras in the variety. Obviously, any strongly irregular variety is also irregular. To each variety $\mathcal{V}$ is associated a variety $R(\mathcal{V})$ that satisfies all and only the regular identities holding in $\mathcal{V}$. $R(\mathcal{V})$ is called the *regularisation* of $\mathcal{V}$. Elements of the regularisation of a strongly irregular variety can always be represented as Płonka sums:

Theorem 5. [31, Thm. 7.1] *Let $\mathcal{V}$ be a strongly irregular variety of language $\mathcal{L}$, and let $\mathbf{A}$ be an $\mathcal{L}$-algebra. Then $\mathbf{A} \in R(\mathcal{V})$ iff $\mathbf{A}$ is decomposable as a Płonka sum over a semilattice direct system of algebras in $\mathcal{V}$.*

The construction of Płonka sums can be appropriately extended from algebras to matrices, as shown in [7] and in [9]. Roughly speaking, *l-direct systems* are the appropriate tools for the semantics of logics of left variable inclusion; *r-direct systems*, which are used to model logics of right variable inclusion, will not be addressed here.

An *l-direct system* of $\mathcal{L}$-matrices is an ordered pair $\mathbb{M} = \langle \mathbb{A}, \{F_i\}_{i \in I} \rangle$ such that:

(1) $\mathbb{A} = \left\langle \{\mathbf{A}_i\}_{i \in I}, \mathbf{I}, \{p_{ij} : i \leq_{\mathbf{I}} j\} \right\rangle$ is a semilattice direct system of $\mathcal{L}$-algebras;
(2) for every $i \in I$, $F_i \subseteq A_i$;
(3) for every $i, j \in I$ such that $i \leq_{\mathbf{I}} j$, $p_{ij}[F_i] \subseteq F_j$.

It may be expedient to view l-direct systems as the results of replacing, as a fibre of a semilattice direct system, each algebra $\mathbf{A}_i$ by the matrix $\langle \mathbf{A}_i, F_i \rangle$. The chosen name, therefore, is no misnomer.

If $\mathbb{M} = \langle \mathbb{A}, \{F_i\}_{i \in I} \rangle$ is an l-direct system of matrices, the *Płonka sum* over $\mathbb{M}$ is the matrix $\mathcal{P}_{\text{ł}}(\mathbb{M}) = \langle \mathcal{P}_{\text{ł}}(\mathbb{A}), \bigcup_{i \in I} F_i \rangle$. More generally, whenever K is a class of matrices, we denote by $\mathcal{P}^{l}_{\text{ł}}(\mathsf{K})$ the class of Płonka sums over the class of all l-direct systems of matrices in K.

2.3. **Generalised involutive bisemilattices.** Recall from Section 2.2 that $R(\mathcal{V})$ denotes the regularisation of the variety $\mathcal{V}$. The variety $\mathcal{GIB}$ is defined as $R(\mathcal{BA})$; in other words, it is the variety that satisfies exactly the regular identities satisfied in all Boolean algebras. Generalised involutive bisemilattices are the members of $\mathcal{GIB}$ [5, 23].

Generalised involutive bisemilattices are so called because they are expansions of *distributive bisemilattices*, i.e., regularisations of distributive lattices. This is the first regular variety that received substantial attention in the literature: see e.g. [26, 2, 18]. While there is a consolidated tradition of studies on distributive bisemilattices and their non-distributive generalisations, the literature on generalised involutive bisemilattices (and variants thereof in slightly different similarity types) is comparatively scarce [30, 5, 7, 6, 21, 23, 8].

Since $\mathcal{BA}$ is a strongly irregular variety – just let $\varphi(x, y)$ be $x \wedge (x \vee y)$ – Theorem 5 immediately yields:

Theorem 6. *If* $\mathbf{A}$ *is an* $\mathcal{L}_1$*-algebra,* $\mathbf{A} \in \mathcal{GIB}$ *iff* $\mathbf{A}$ *is decomposable as a Płonka sum over a semilattice direct system of Boolean algebras.*

Boolean algebras can be seen as instances of generalised involutive bisemilattices, too. More precisely, $\mathbf{A} \in \mathcal{GIB}$ is a Boolean algebra iff it satisfies the absorption identity $x \wedge (x \vee y) \approx x$. The other nontrivial proper subvariety of $\mathcal{GIB}$ is the regularisation of the trivial variety of language $\mathcal{L}_1$, which is axiomatised relative to $\mathcal{GIB}$ by either one of the equivalent identities $x \wedge y \approx x \vee y$ and $\neg x \approx x$. Precisely for this reason, we can consider this variety $\mathcal{SL}$ as an incarnation in the type $\mathcal{L}_1$ of the variety of *semilattices*. $\mathcal{BA}$ and $\mathcal{SL}$ correspond to limiting cases of the representation in Theorem 6: in the case of Boolean algebras there is a single fibre, while in the case of semilattices all the fibres are trivial algebras. Finally, another notable example of a generalised involutive bisemilattice is the 3-element $\mathcal{L}_1$-algebra **WK**, with the following operation tables:

$\neg$	
0	1
n	n
1	0

$\wedge$	0	n	1
0	0	n	0
n	n	n	n
1	0	n	1

$\vee$	0	n	1
0	0	n	1
n	n	n	n
1	1	n	1

which is isomorphic to the unique Płonka sum of Boolean algebras of the form $\mathbf{B}_2 \oplus \mathbf{B}_1$.

If $\mathbf{A} \in \mathcal{GIB}$, we can define two semilattice orderings on $\mathbf{A}$, namely

$$x \leq_\wedge y \text{ iff } x \wedge y = x$$

and

$$x \leq_\vee y \text{ iff } x \vee y = y.$$

These orderings coincide if and only if $\mathbf{A}$ is a Boolean algebra, and are dual to each other if and only if $\mathbf{A}$ is a semilattice. The following proposition can either be proved by adapting a technique devised by Kalman [18], or obtained as a corollary of general facts about subdirectly irreducibles in regular varieties [20, p. 487]:

Proposition 7. *The only nontrivial subdirectly irreducible generalised involutive bisemilattices are* **WK***, the* 2*-element semilattice* $\mathbf{S}_2$*, and the* 2*-element Boolean algebra* $\mathbf{B}_2$*, up to isomorphism. Since* $\mathbf{B}_2$ *is a subalgebra of* **WK** *and* $\mathbf{S}_2$ *is a quotient of such,* $\mathcal{GIB} = HSP(\mathbf{WK})$*.*

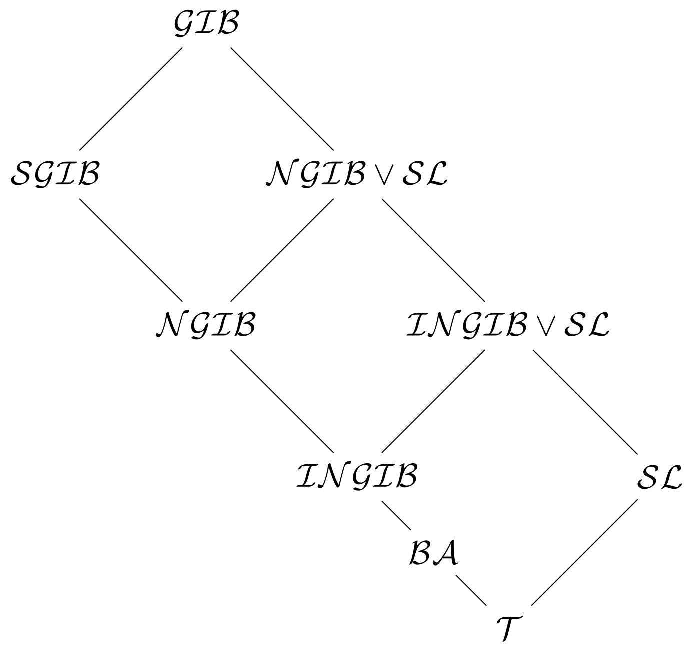
Corollary 8. *The only nontrivial proper subvarieties of* $\mathcal{GIB}$ *are the disjoint varieties* $\mathcal{BA}$ *of Boolean algebras and* $\mathcal{SL}$ *of semilattices.*

Given $\mathbf{A} \in \mathcal{GIB}$, we say that an element $a \in A$ is:

- *positive* if $\neg a \leq_\vee a$;
- *negative* if $a \leq_\wedge \neg a$;
- a *fixpoint*, if $\neg a = a$.

$P(\mathbf{A})$ and $N(\mathbf{A})$ will denote the sets of all positive and negative elements of $\mathbf{A}$, respectively. Clearly, fixpoints are both negative and positive. If a is a fixpoint and $a \leq_\vee b$, then b is also a fixpoint. Each element of the form $a \vee \neg a$ is positive, and coincides (in the Płonka sum representation of $\mathbf{A}$) with the top element of the fibre where a lives. Similarly, each element of the form $a \wedge \neg a$ is negative, and coincides (in the Płonka sum representation of $\mathbf{A}$) with the bottom element of the fibre where a lives. Therefore, each fibre $\mathbf{A}_i$ of the representation of $\mathbf{A}$ has the form $[a \wedge \neg a, a \vee \neg a]$ for an arbitrary $a \in A_i$, where the notation $[b, c]$ is short for $\{x \in A : b \leq_\vee x \leq_\vee c\}$ [5].

The lattice of sub*quasi*varieties of $\mathcal{GIB}$ has been completely described in [3] (see also [23]) in the context of a more general study of subquasivarieties of regular varieties. There are exactly nine subquasivarieties of $\mathcal{GIB}$, arranged as in the following Hasse diagram:



In the next table, we list relative quasiequational bases w.r.t. $\mathcal{GIB}$ and generators for each member of the lattice.

$\mathcal{GIB}$		$\mathbf{WK}, \mathbf{S}_2$
$\mathcal{SGIB}$	$x \approx \neg x \,\&\, y \approx \neg y \Rightarrow x \approx y$	$\mathbf{WK}$
$\mathcal{NGIB} \vee \mathcal{SL}$	$x \approx \neg x \Rightarrow y \approx \neg y$	$\mathbf{WK} \times \mathbf{B}_2, \mathbf{S}_2$
$\mathcal{NGIB}$	$x \approx \neg x \Rightarrow y \approx z$	$\mathbf{WK} \times \mathbf{B}_2$
$\mathcal{INGIB} \vee \mathcal{SL}$	$\neg x \vee \neg y \leq_\vee x \wedge y \Rightarrow \neg x \leq_\vee x$	$\mathbf{B}_2 \times \mathbf{S}_2, \mathbf{S}_2$
$\mathcal{INGIB}$	$\neg x \vee \neg y \leq_\vee x \wedge y \Rightarrow \neg x \leq_\vee x$	$\mathbf{B}_2 \times \mathbf{S}_2$
	$x \approx \neg x \Rightarrow y \approx z$	
$\mathcal{BA}$	$x \wedge (x \vee y) \approx x$	$\mathbf{B}_2$
$\mathcal{SL}$	$x \wedge y \approx x \vee y$	$\mathbf{S}_2$
$\mathcal{T}$	$x \approx y$	$\mathbf{S}_1$

We also recall from [23] that $\mathcal{INGIB} \vee \mathcal{SL}$ comprises all the generalised involutive bisemilattices whose Płonka sum representations consist in semilattice direct systems of Boolean algebras with injective homomorphisms.

2.4. **Logics of variable inclusion.** The theory of regular varieties and of Płonka sums of algebras (and matrices) is strictly connected to the investigation into *logics of variable inclusion*: logics one obtains from a given base logic by applying to its valid entailments appropriate variable inclusion strainers. More precisely, a logic can have a *left* and a *right* variable inclusion companion, all in the same language. For a logic $\mathrm{L} = \langle \mathbf{Fm}(\mathcal{L}), \vdash_{\mathrm{L}} \rangle$, these companions are denoted as L^l and L^r, respectively. Their consequence relations can be characterised as follows.

$$\Gamma \vdash_{\mathrm{L}^l} \varphi \iff \begin{cases} \exists \Delta \subseteq \Gamma : \\ \Delta \vdash_{\mathrm{L}} \varphi \text{ and } Var(\Delta) \subseteq Var(\varphi); \end{cases}$$

$$\Gamma \vdash_{\mathrm{L}^r} \varphi \iff \begin{cases} \Gamma \vdash_{\mathrm{L}} \varphi \text{ and } Var(\varphi) \subseteq Var(\Gamma), \text{ or} \\ \Gamma \text{ is an L-antitheorem.} \end{cases}$$

When $\mathrm{L} = \mathrm{CL}$, i.e., classical logic formulated in the language $\mathcal{L}_1$, known results to be found in [11, 34] imply that CL^r and CL^l are, respectively, the so-called *paracomplete* and *paraconsistent weak Kleene logics* B_3 and PWK—investigated also in [4, 16, 19], usually defined as follows via their characteristic matrices:

$$\mathrm{B}_3 = \langle \mathbf{Fm}(\mathcal{L}_1), \vdash_{\langle \mathbf{WK}, \{1\} \rangle} \rangle; \ \mathrm{PWK} = \langle \mathbf{Fm}(\mathcal{L}_1), \vdash_{\langle \mathbf{WK}, \{1,n\} \rangle} \rangle.$$

Observe that B_3 (resp. PWK) has the same antitheorems (resp. theorems) as CL.

The paper [7] and the monograph [8] contain a detailed investigation of logics of left variable inclusion. Specific results on PWK and its extensions are to be found in [5], [23] and [8]. We summarise hereafter the ones of interest for the present investigation; for the notion of an l-partition function see [7] and [8].

Theorem 9. *Let* L *be a logic of language* $\mathcal{L}$ *and* M *be a class of* $\mathcal{L}$*-matrices containing* $\langle \mathbf{1}, \{1\} \rangle$*. If* L *is complete w.r.t.* M*, then* L^l *is complete w.r.t.* $\mathcal{P}_{\mathfrak{k}}^l(\mathsf{M})$*.*

Corollary 10. *Let* L *be a logic. Its left variable inclusion companion* L^l *is complete w.r.t. any of the following classes of matrices:*

$$\mathcal{P}_{\mathfrak{k}}^l(\mathsf{Mod}(\mathrm{L})) \quad \mathcal{P}_{\mathfrak{k}}^l(\mathsf{Mod}^*(\mathrm{L})) \quad \mathcal{P}_{\mathfrak{k}}^l(\mathsf{Mod}^{\mathsf{Su}}(\mathrm{L})).$$

Theorem 11. *Let* L *be an equivalential and finitary logic whose language* $\mathcal{L}$ *does not contain constants, having an l-partition function and antitheorems. Let moreover* $\langle \mathbf{A}, F \rangle$ *be an* $\mathcal{L}$*-matrix. The following conditions are equivalent:*

(1) $\langle \mathbf{A}, F \rangle \in \mathsf{Mod}^{\mathsf{Su}}(\mathrm{L}^l)$.

(2) *There exists an l-direct system of matrices* $\mathbb{M} \subseteq \mathsf{Mod}^*(\mathrm{L})$ *with at most one trivial fibre such that* $\langle \mathbf{A}, F \rangle \cong \mathcal{P}_t^l(\mathbb{M})$.

Corollary 10 and Theorem 11 apply to CL and to its left variable inclusion companion PWK, for which we get both a completeness theorem with respect to Płonka sums of models of classical logic, and an exhaustive description of its Suszko reduced models and their algebra reducts. In particular:

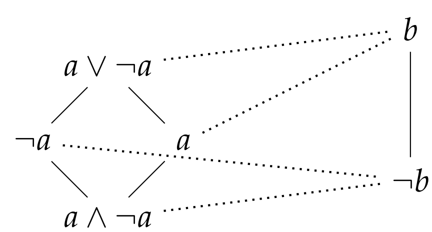
Theorem 12. PWK *is complete w.r.t. the Płonka sum of any class of* $\mathcal{L}_1$*-matrices containing* $\langle \mathbf{B}_2, \{1\} \rangle$ *and the trivial matrix* $\langle \mathbf{1}, \{1\} \rangle$*; in particular, it is complete w.r.t.* $\langle \mathbf{WK}, \{1, n\} \rangle$.

Theorem 13. *Let* $\mathbf{A}$ *be an algebra of type* $\mathcal{L}_1$*. The following are equivalent:*

(1) $\mathbf{A} \in \mathsf{Alg}(\mathrm{PWK})$;
(2) $\mathbf{A} \in \mathcal{SGIB}$;
(3) *there exists an l-direct system* $\mathbb{M}$ *of Leibniz reduced models of* CL *with at most one trivial fibre s.t.* $\langle \mathbf{A}, F \rangle \cong \mathcal{P}_t^l(\mathbb{M})$;
(4) $\mathbf{A} \in ISP(\mathbf{WK})$.

PWK is a non-protoalgebraic but truth-equational logic. Indeed, for all $\langle \mathbf{A}, F \rangle \in \mathsf{Mod}^*(\mathrm{PWK})$, we have that $a \in F$ iff $a \in Sol_\tau^{\mathbf{A}}$, where $\tau(x) = x \approx x \vee \neg x$. There is a unique logic strictly in between PWK and CL, the non-paraconsistent logic PWK_E obtained by adding to PWK the rule (EFQ). PWK_E has the single finite characteristic matrix $\langle \mathbf{WK}, \{1, n\} \rangle \times \langle \mathbf{B}_2, \{1\} \rangle$.

The algebra reduct of this matrix is isomorphic to the unique Płonka sum of the form $\mathbf{B}_4 \oplus \mathbf{B}_2$. When referring to this algebra, to avoid notational overload, we will denote by $a, \neg a$ the atoms of $\mathbf{B}_4$ and by b the top element of $\mathbf{B}_2$, so that the top and bottom element of $\mathbf{B}_4$ will be unambiguously denoted by $a \vee \neg a$ and $a \wedge \neg a$ respectively, while $\neg b$ will denote the bottom element of $\mathbf{B}_2$. In particular, a will be the atom of $\mathbf{B}_4$ such that $p_{ij}(a) = b$:



2.5. **Logics of pure variable inclusion.** The left and right variable inclusion companions of a logic, in a sense, are logics of variable inclusion only by courtesy. Let us explicate this remark by using the examples of B_3 and PWK. B_3 satisfies the rule (EFQ), while PWK satisfies (VEQ) – both of these rules have instances that do not abide by any variable inclusion constraint. Put otherwise, the variable inclusion strainer applies to B_3-valid entailments unless their set of premisses is an antitheorem; and to PWK-valid entailments unless their conclusion is a theorem.

In [25], we studied (together with D. Szmuc) logics where the variable inclusion patterns are unconstrained, referring to them as *pure variable inclusion logics*. Given a logic $\mathrm{L} = \langle \mathbf{Fm}(\mathcal{L}), \vdash_{\mathrm{L}} \rangle$, we denote by L^{pl} and L^{pr} the logics in the same language whose consequence relations are as follows:

$$\Gamma \vdash_{\mathrm{L}^{pl}} \varphi \iff \begin{cases} \exists \Delta \subseteq \Gamma, \Delta \neq \emptyset : \\ \Delta \vdash_{\mathrm{L}} \varphi \text{ and } Var(\Delta) \subseteq Var(\varphi); \end{cases}$$

$$\Gamma \vdash_{\mathrm{L}^{pr}} \varphi \iff \Gamma \vdash_{\mathrm{L}} \varphi \text{ and } Var(\varphi) \subseteq Var(\Gamma).$$

Clearly, the logics we just defined have no theorems or antitheorems. The pure variable inclusion companions of CL, namely, its pure left companion CL^{pl} and its pure right companion CL^{pr}, have especially interesting properties (see also [33]). CL^{pr} has no single characteristic matrix, but is determined by a *matrix bundle* consisting of two matrices with the common algebra reduct **WK**.

Theorem 14. *We have that:*

1. CL^{pr} *has no single characteristic matrix.*
2. CL^{pr} *is sound and complete w.r.t. the set of matrices*

$$\{\langle \mathbf{WK}, \{1\} \rangle, \langle \mathbf{WK}, \{1, 0\} \rangle\}.$$

CL^{pl} has a single 5-element characteristic matrix, but it can also be characterised, exactly like CL^{pr}, as the intersection of two logics (one of which is PWK), each determined by a single matrix based on **WK**.

Theorem 15. *We have that:*

1. CL^{pl} *has a 5-element characteristic matrix* $\langle \mathbf{PK}, \{1, n\} \rangle$, *where* **PK** *is described by the tables below:*

	$\neg$
0	1
p	m
n	n
m	p
1	0

$\wedge$	0	p	n	m	1
0	0	0	n	0	0
p	0	p	n	p	0
n	n	n	n	n	n
m	0	p	n	m	1
1	0	0	n	1	1

$\vee$	0	p	n	m	1
0	0	0	n	1	1
p	0	p	n	m	1
n	n	n	n	n	n
m	1	m	n	m	1
1	1	1	n	1	1

(2) CL^{pl} *is sound and complete w.r.t. the set of matrices*

$$\{\langle \mathbf{WK}, \{1, n\}\rangle, \langle \mathbf{WK}, \{n\}\rangle\}.$$

Recalling the notion of a semilattice-based logic from Subsection 2.1, it is interesting to observe that both CL^{pl} and CL^{pr} belong in this category.

Theorem 16. $\mathrm{CL}^{pl} = \mathrm{WK}^{\leq_\vee}$ *and* $\mathrm{CL}^{pr} = \mathrm{WK}^{\leq_\wedge}$.

In what follows, if $\mathrm{CL}^{pl} \leq \mathrm{L}$, then we will denote:

- by L_E, the logic $\mathrm{L} + (EFQ)$;
- by L_M, the logic $\mathrm{L} + (EFV)$;
- by L_{E^*}, the logic $\mathrm{L} + (EFVQ)$;
- by L_P, the logic $\mathrm{L} + (eMP)$.

3. Deductive filters and matrix models

In this short section, we describe the deductive filters of CL^{pl} on generalised involutive bisemilattices. We also observe that the algebra reducts of reduced matrix models of CL^{pl} must perforce be generalised involutive bisemilattices.

Lemma 17. $\mathsf{Alg}^*\left(\mathrm{CL}^{pl}\right) \subseteq \mathcal{GIB}$.

Proof. By Theorem 15.1, CL^{pl} is sound and complete w.r.t. $\langle \mathbf{PK}, \{1, n\}\rangle$, whose algebra reduct $\mathbf{PK}$ belongs to $\mathcal{GIB}$. By [15, Lemma 5.78], $\mathsf{Alg}^*\left(\mathrm{CL}^{pl}\right) \subseteq V(\mathbf{PK}) = \mathcal{GIB}$, where the last equality is justified by Proposition 7 and the fact that $\mathbf{WK}$ is a subalgebra of $\mathbf{PK}$. ⊠

Theorem 18. *Let* $\mathbf{A} \in \mathcal{GIB}$ *and let* $F \subseteq A$. *The following conditions are equivalent:*

(1) F *is a* CL^{pl}*-filter on* $\mathbf{A}$*;*
(2) *for all* $a, b \in A$*: (a) if* $a, b \in F$*, then* $a \wedge b \in F$*; (b) if* $a \in F$ *and* $a \leq_\vee b$*, then* $b \in F$*.*

Proof. (1) $\Rightarrow$ (2). This direction follows from the fact that $\varphi \vdash_{\mathrm{CL}^{pl}} \varphi \vee \psi$ and $\varphi, \psi \vdash_{\mathrm{CL}^{pl}} \varphi \wedge \psi$.

(2) $\Rightarrow$ (1). Let $\Gamma \vdash_{\mathrm{CL}^{pl}} \varphi$, and let $v \in Hom(\mathbf{Fm}(\mathcal{L}_1), \mathbf{A})$ be such that $v[\Gamma] \subseteq F$. Since $\Gamma \vdash_{\mathrm{CL}^{pl}} \varphi$, there exists a nonempty $\Delta \subseteq \Gamma$ such that $\Delta \vdash_{\mathrm{CL}} \varphi$ and $Var(\Delta) \subseteq Var(\varphi)$. As CL^{pl} and CL are finitary, we can assume $\Delta = \{\delta_1, ..., \delta_n\}$. Since $Var(\Delta) \subseteq Var(\varphi)$ and $\Delta \vdash_{\mathrm{CL}} \varphi$,

$$(\delta_1 \wedge ... \wedge \delta_n) \vee \varphi \approx \varphi$$

is a regular $\mathcal{L}_1$-identity valid in $\mathcal{BA}$, hence it holds in $\mathcal{GIB}$. Therefore,

$$v(\delta_1 \wedge ... \wedge \delta_n) \vee v(\varphi) = v(\varphi),$$

or, in other words,

$$v(\delta_1 \wedge ... \wedge \delta_n) \leq_\vee v(\varphi).$$

Now, by (2), part (a), $v(\delta_1 \wedge ... \wedge \delta_n) \in F$, whence by (2), part (b), $v(\varphi) \in F$. ⊠

4. The lattice of extensions of CL^{pl}

We now undertake the main task of the present paper, i.e., that of completely describing the lattice of extensions of CL^{pl}. For a start, this goal must somehow be qualified. As a theoremless logic, CL^{pl} has extensions that are uncomparable to CL, say the *almost inconsistent logic* AIL such that $\Gamma \vdash_{AIL} \varphi$ iff $\Gamma \neq \emptyset$. For many purposes, such logics are uninteresting. As a result, we will confine our study to the interval $\left[\mathrm{CL}^{pl}, \mathrm{CL}\right]$ in the lattice of extensions of CL^{pl}. We denote such an interval by $\mathcal{E}xt\left(\mathrm{CL}^{pl}\right)$.

Many methods and tools employed here are inspired by the paper [1], devoted to an enquiry into the lattice of super-Belnap logics of which the present approach is strongly reminiscent. Like Albuquerque and colleagues, in particular, we start by looking at the mutual deductive strength relationship between the inference rules in Subsection 2.1.

In what follows, for $\Gamma = \{\gamma_1, ..., \gamma_n\}$ a finite set of $\mathcal{L}_1$-formulas, by $\bigwedge_{\gamma \in \Gamma} \gamma$, or by $\bigwedge_{i=1}^{n} \gamma_i$, we mean the (generalised) conjunction of members of Γ, listed in an arbitrary but fixed order and associated to the left. An analogous notation is used for generalised disjunctions.

Lemma 19. *Over* CL^{pl}*, the rules (CS), (Res), (qEFQ), (MP) are pairwise equivalent.*

Proof. (CS) ⇒ (Res). Suppose $\mathrm{CL}^{pl} \leq \mathrm{L}$. Clearly, $\varphi \vee \psi, \chi \vee \neg\psi \vdash_{\mathrm{L}} (\varphi \vee \psi) \wedge (\chi \vee \neg\psi)$ and

$$(\varphi \vee \psi) \wedge (\chi \vee \neg\psi) \vdash_{\mathrm{L}} (\varphi \wedge \neg\psi) \vee (\varphi \wedge \chi) \vee (\psi \wedge \neg\psi) \vee (\psi \wedge \chi).$$

However, we also have that

$$(\varphi \wedge \neg\psi) \vee (\varphi \wedge \chi) \vee (\psi \wedge \neg\psi) \vee (\psi \wedge \chi) \vdash_{\mathrm{L}} (\varphi \vee \chi) \wedge (\psi \vee \neg\psi).$$

Hence $(\varphi \vee \psi) \wedge (\chi \vee \neg\psi) \vdash_{\mathrm{L}} (\varphi \vee \chi) \wedge (\psi \vee \neg\psi)$ and, by (CS), $(\varphi \vee \chi) \wedge (\psi \vee \neg\psi) \vdash_{\mathrm{L}} \varphi \vee \chi$, whereby (Res) follows.

(Res) $\Rightarrow$ (qEFQ). Suppose $\mathrm{CL}^{pl} \leq \mathrm{L}$. Observe that:

$$(\varphi \wedge \neg\varphi) \vee \psi \vdash_{\mathrm{L}} (\varphi \vee \psi) \wedge (\neg\varphi \vee \psi)\,;$$
$$(\varphi \vee \psi) \wedge (\neg\varphi \vee \psi) \vdash_{\mathrm{L}} \varphi \vee \psi, \neg\varphi \vee \psi.$$

By (Res), however, $\varphi \vee \psi, \neg\varphi \vee \psi \vdash_{\mathrm{L}} \psi \vee \psi$, which, together with $\psi \vee \psi \vdash_{\mathrm{L}} \psi$, yields (qEFQ).

(qEFQ) $\Rightarrow$ (MP). Suppose $\mathrm{CL}^{pl} \leq \mathrm{L}$. Then $\varphi, \neg\varphi \vee \psi \vdash_{\mathrm{L}} (\varphi \wedge \neg\varphi) \vee \psi$, while, by (qEFQ), $(\varphi \wedge \neg\varphi) \vee \psi \vdash_{\mathrm{L}} \psi$. It follows that $\varphi, \neg\varphi \vee \psi \vdash_{\mathrm{L}} \psi$.

(MP) $\Rightarrow$ (CS). Suppose $\mathrm{CL}^{pl} \leq \mathrm{L}$. Then $\varphi \wedge \psi \vdash_{\mathrm{L}} \neg(\varphi \wedge \psi) \vee \varphi$ and, by (MP), $\varphi \wedge \psi, \neg(\varphi \wedge \psi) \vee \varphi \vdash_{\mathrm{L}} \varphi$. It follows that $\varphi \wedge \psi \vdash_{\mathrm{L}} \varphi$, and in a similar guise $\varphi \wedge \psi \vdash_{\mathrm{L}} \psi$. ⊠

Observe that the four equivalent rules of Lemma 19 imply (EFQ). If $\mathrm{CL}^{pl} \leq \mathrm{L}$, indeed, $\varphi \wedge \neg\varphi \vdash_{\mathrm{L}} (\varphi \wedge \neg\varphi) \vee \psi$; by (qEFQ), moreover, $(\varphi \wedge \neg\varphi) \vee \psi \vdash_{\mathrm{L}} \psi$, whence (EFQ) follows by Cut. On the other hand, consider $\langle \mathbf{B}_4 \oplus \mathbf{B}_2, \{a \vee \neg a, b\}\rangle$, the characteristic matrix of PWK_E. Clearly, (EFQ) holds in that matrix. However, we have a counterexample to $x, \neg x \vee y \vdash_{\langle \mathbf{B}_4 \oplus \mathbf{B}_2, \{a \vee \neg a, b\}\rangle} y$ by letting $h \in Hom(\mathbf{Fm}(\mathcal{L}_1), \mathbf{B}_4 \oplus \mathbf{B}_2)$ be such that $h(x) = b$ and $h(y) = a$.

Lemma 20. *Over* CL^{pl}*, the rules (VEQ) and (qMP) are equivalent.*

Proof. Let us prove the nontrivial implication. Suppose $\mathrm{CL}^{pl} \leq \mathrm{L}$. Observe that $\varphi \vdash_{\mathrm{L}} \neg\varphi \vee \psi \vee \neg\psi$. By (qMP), $\varphi, \neg\varphi \vee \psi \vee \neg\psi \vdash_{\mathrm{L}} \psi \vee \neg\psi$, and by Cut, $\varphi \vdash_{\mathrm{L}} \psi \vee \neg\psi$. ⊠

Suszko logic SL (see e.g. [17, Ex. 2.11.6]) is the intersection of CL and the almost inconsistent logic AIL. In other words, for $\Gamma \cup \varphi \subseteq Fm(\mathcal{L}_1)$, we have that $\Gamma \vdash_{\mathrm{SL}} \varphi$ iff $\Gamma \vdash_{\mathrm{CL}} \varphi$ and $\Gamma \neq \emptyset$. Clearly, $\mathrm{SL} = \mathsf{Log}(\mathsf{K})$, where $\mathsf{K} = \{\langle \mathbf{B}_2, \{1\}\rangle, \langle \mathbf{B}_1, \emptyset\rangle\}$. We now show that SL is axiomatised relative to CL^{pl} by the modus ponens rule (MP).

Lemma 21. $\mathrm{SL} = \mathrm{CL}^{pl} + (MP)$.

Proof. For the nontrivial direction, let $\Gamma \vdash_{\mathrm{SL}} \varphi$. In particular, $\Gamma \neq \emptyset$. Since SL is finitary, Γ can be taken to be finite and hence $Var(\Gamma) = \{x_1, ..., x_n\}$. We have that:

$$\Gamma \vdash_{\mathrm{CL}^{pl}+(MP)} \varphi \vee \left(\bigwedge_{i=1}^{n} x_i \wedge \neg \left(\bigwedge_{i=1}^{n} x_i \right) \right).$$

By Lemma 19, $\mathrm{CL}^{pl}+(MP)$ satisfies (qEFQ), whence

$$\varphi \vee \left(\bigwedge_{i=1}^{n} x_i \wedge \neg \left(\bigwedge_{i=1}^{n} x_i \right) \right) \vdash_{\mathrm{CL}^{pl}+(MP)} \varphi.$$

It follows that $\Gamma \vdash_{\mathrm{CL}^{pl}+(MP)} \varphi$. ⊠

As we have observed, PWK is obtained from CL^{pl} by adding all classical theorems. It make sense to wonder whether we can get a weaker logic by adjoining to CL^{pl} a smaller set of classical theorems – in other words, whether PWK properly contains any axiomatic extension of CL^{pl}. The next result answers this question in the negative.

Theorem 22. PWK *is the smallest axiomatic extension of* CL^{pl}.

Proof. Let $\vdash_{\mathrm{CL}} \varphi(x_1, ..., x_n)$, and let ψ be an arbitrary formula. We have that $\varphi(x_1/\psi, ..., x_n/\psi) \vdash_{\mathrm{CL}^{pl}} \psi \vee \neg\psi$, whence $\vdash_{\mathrm{CL}^{pl}+\{\varphi\}} \psi \vee \neg\psi$. Therefore, if $\vdash_{\mathrm{CL}} \chi(y_1, ..., y_m)$, we have that $\bigwedge_{i=1}^{m} y_i \vee \neg \left(\bigwedge_{i=1}^{m} y_i \right) \vdash_{\mathrm{CL}^{pl}} \chi(y_1, ..., y_m)$ and thus $\vdash_{\mathrm{CL}^{pl}+\{\varphi\}} \chi(y_1, ..., y_m)$. ⊠

What is the smallest *antiaxiomatic* extension of CL^{pl}, that is to say, the smallest extension of such by a rule $\Gamma \vdash x$, where Γ is a CL-antitheorem and $x \notin Var(\Gamma)$? We show that it is the logic CL_E^{pl}, resulting from the addition of (EFQ) to CL_E^{pl}.

Lemma 23. CL_E^{pl} *is the smallest antiaxiomatic extension of* CL^{pl} .

Proof. Let Γ be a classical antitheorem. Without loss of generality, we can assume that it is finite. Let now $\varphi(x_1, ..., x_n) = \bigwedge_{\gamma \in \Gamma} \gamma$. By the disjunctive normal form theorem for CL, we have that for some formulas $\psi_1, ..., \psi_k$ and some set $\{y_1, ..., y_k\}$ of variables such that $\{x_1, ..., x_n\} = \{y_1, ..., y_k\} \cup Var(\psi_1, ..., \psi_k)$,

$$\varphi(x_1, ..., x_n) \dashv\vdash_{\mathrm{CL}} \bigvee_{i=1}^{k} (y_i \wedge \neg y_i \wedge \psi_i).$$

However, both of the following entailments hold, for arbitrary χ:

$$\bigwedge_{i=1}^{k}(y_i \wedge \psi_i) \wedge \neg\left(\bigwedge_{i=1}^{k}(y_i \wedge \psi_i)\right) \vdash_{\mathrm{CL}_E^{pl}} \chi;$$

$$\bigvee_{i=1}^{k}(y_i \wedge \neg y_i \wedge \psi_i) \vdash_{\mathrm{CL}_E^{pl}} \bigwedge_{i=1}^{k}(y_i \wedge \psi_i) \wedge \neg\left(\bigwedge_{i=1}^{k}(y_i \wedge \psi_i)\right).$$

Hence $\bigvee_{i=1}^{k}(y_i \wedge \neg y_i \wedge \psi_i) \vdash_{\mathrm{CL}_E^{pl}} \chi$. Due to the assumptions on variables we previously made,

$$\Gamma \vdash_{\mathrm{CL}_E^{pl}} \varphi(x_1, ..., x_n) \vdash_{\mathrm{CL}_E^{pl}} \bigvee_{i=1}^{k}(y_i \wedge \neg y_i \wedge \psi_i) \vdash_{\mathrm{CL}_E^{pl}} \chi.$$

⊠

CL_E^{pl} is not only the smallest antiaxiomatic extension of CL^{pl}, but also the *largest* one below CL we can obtain by only adding rules of the above form to CL^{pl}. In the terminology of [1], CL_E^{pl} is the *explosive part* of CL relative to CL^{pl}. Using results on explosive extensions from [1], we can identify a characteristic matrix for CL_E^{pl}.

Lemma 24. $\langle \mathbf{PK}, \{1, n\}\rangle \times \langle \mathbf{B}_2, \{1\}\rangle$ *is characteristic for* CL_E^{pl}.

Proof. We apply [1, Prop. 2.19] and Lemma 23. The logic determined by $\langle \mathbf{PK}, \{1, n\}\rangle \times \langle \mathbf{B}_2, \{1\}\rangle$ is the explosive part of the logic determined by $\langle \mathbf{B}_2, \{1\}\rangle$ relative to the logic determined by $\langle \mathbf{PK}, \{1, n\}\rangle$; that is to say, the explosive part of CL relative to CL^{pl}, which, as we have just observed, is CL_E^{pl}. ⊠

Next, we determine the intersections between some logics encountered so far. The intersection of PWK and CL_E^{pl} turns out to be the logic CL_M^{pl}, obtained by adding (EFV) to CL^{pl}. We also provide characteristic matrices and axiomatisations (relative to CL^{pl}) for PWK ∩ SL and $\mathrm{PWK}_E \cap \mathrm{SL}$.

Lemma 25. $\mathrm{CL}_M^{pl} = \mathrm{PWK} \cap \mathrm{CL}_E^{pl}$.

Proof. Recall that, by Theorem 22, PWK is axiomatised relative to CL^{pl} by (EM), while by definition CL_E^{pl} is axiomatised relative to CL^{pl} by (EFQ). Applying [1, Prop. 2.17], $\mathrm{PWK} \cap \mathrm{CL}_E^{pl}$ is axiomatised relative to CL^{pl} by (EFV) and thus coincides with CL_M^{pl}. ⊠

Theorem 26. *The following logics are mutually coincident:*

(1) $\mathrm{PWK} \cap \mathrm{SL}$*;*
(2) $\mathsf{Log}(\mathsf{K})$*, where* $\mathsf{K} = \{\langle \mathbf{WK}, \{1, n\}\rangle, \langle \mathbf{B}_1, \emptyset\rangle\}$*;*
(3) $\mathrm{CL}^{pl} + (VEQ)$*.*

Proof. (1) = (2). We have that:

$$\begin{array}{llll} \Gamma \vdash_{\mathrm{PWK}\cap\mathrm{SL}} \varphi & \text{iff} & \Gamma \vdash_{\mathrm{PWK}} \varphi & \text{and} \quad \Gamma \vdash_{\mathrm{SL}} \varphi \\ & \text{iff} & \Gamma \vdash_{\langle \mathbf{WK}, \{1,n\}\rangle} \varphi & \text{and} \quad \Gamma \vdash_{\{\langle \mathbf{B}_2, \{1\}\rangle, \langle \mathbf{B}_1, \emptyset\rangle\}} \varphi \\ & \text{iff} & \Gamma \vdash_{\langle \mathbf{WK}, \{1,n\}\rangle} \varphi & \text{and} \quad \Gamma \vdash_{\langle \mathbf{B}_1, \emptyset\rangle} \varphi, \end{array}$$

where the third equivalence is justified since $\langle \mathbf{B}_2, \{1\}\rangle$ is a submatrix of $\langle \mathbf{WK}, \{1, n\}\rangle$.

(1) ⊆ (3). Suppose that $\Gamma \vdash_{\mathrm{PWK}\cap\mathrm{SL}} \varphi$. Then $\Gamma \neq \emptyset$ and there is $\Delta \subseteq \Gamma$ s.t. $\Delta \vdash_{\mathrm{CL}} \varphi$ and $Var(\Delta) \subseteq Var(\varphi)$. If $\Delta \neq \emptyset$, then $\Gamma \vdash_{\mathrm{CL}^{pl}} \varphi$. Otherwise, $\varphi = \varphi(x_1, ..., x_n)$ is a classical theorem. Hence,

$$\bigwedge_{i=1}^{n} x_i \vee \neg \left(\bigwedge_{i=1}^{n} x_i \right) \dashv\vdash_{\mathrm{CL}^{pl}} \varphi.$$

Also,

$$\Gamma \vdash_{\mathrm{CL}^{pl}+(VEQ)} \bigwedge_{i=1}^{n} x_i \vee \neg \left(\bigwedge_{i=1}^{n} x_i \right).$$

So $\Gamma \vdash_{\mathrm{CL}^{pl}+(VEQ)} \varphi$.

(3) ⊆ (1). Clearly, $\mathrm{CL}^{pl} \subseteq \mathrm{PWK} \cap \mathrm{SL}$. Moreover, (VEQ) holds both in PWK and in SL, whence our conclusion follows. ⊠

Theorem 27. *The following logics are mutually coincident:*

(1) $\mathrm{PWK}_E \cap \mathrm{SL}$*;*
(2) $\mathsf{Log}(\mathsf{K})$*, where* $\mathsf{K} = \{\langle \mathbf{WK}, \{1, n\}\rangle \times \langle \mathbf{B}_2, \{1\}\rangle, \langle \mathbf{B}_1, \emptyset\rangle\}$*;*
(3) $\mathrm{CL}^{pl} + \{(VEQ), (EFQ)\}$*.*

Proof. Similar to Theorem 26. ⊠

Recall that the logic CL_P^{pl} is axiomatised relative to CL^{pl} by the rule (eMP). We now identify both a class of characteristic matrices and a variable inclusion criterion for the valid entailments of this logic. Hereafter, let m denote the unique element of the upper fibre of the 2-element semilattice $\mathbf{S}_2$, and let n denote the unique element of its lower fibre.

Theorem 28. *The following are equivalent:*

(1) $\Gamma \vdash_{\mathrm{CL}_P^{pl}} \varphi$*;*
(2) $\Gamma \vdash_{\mathrm{CL}} \varphi$ *and* $Var(\gamma) \subseteq Var(\varphi)$ *for some* $\gamma \in \Gamma$*;*

(3) $\Gamma \vdash_{\mathsf{K}} \varphi$, *where* K *is* $\{\langle \mathbf{B}_2, \{1\}\rangle, \langle \mathbf{S}_2, \{m\}\rangle\}$.

Proof. (3) $\Rightarrow$ (2). Suppose $\Gamma \vdash_{\mathsf{K}} \varphi$. Hence $\Gamma \vdash_{\mathrm{CL}} \varphi$, as $\langle \mathbf{B}_2, \{1\}\rangle$ is characteristic for CL. Observe that it always exists a $h \in Hom(\mathbf{Fm}(\mathcal{L}_1), \mathbf{S}_2)$ such that $h(\varphi) = n$ (for example, $h(x) = n$ for every variable x occurring in φ). Fix one such homomorphism h. Suppose now, towards a contradiction, that for no $\gamma \in \Gamma = \{\gamma_1, ..., \gamma_n\}$ it holds $Var(\gamma) \subseteq Var(\varphi)$. This means that for every $\gamma_i \in \Gamma$ $(1 \leq i \leq n)$ there exists $x_i \in Var(\gamma_i) \smallsetminus Var(\varphi)$. Define now $g : \mathbf{Fm}(\mathcal{L}_1) \to \mathbf{S}_2$ as follows:

$$g(x) := \begin{cases} h(x) & \text{if } x \in Var(\varphi) \\ m & \text{otherwise.} \end{cases}$$

Clearly, for every $1 \leq i \leq n$, $g(x_i) = m$, so $g(\gamma_i) = m$. However $g(\varphi) = h(\varphi) = n$, which entails $\Gamma \nvdash_{\mathsf{K}} \varphi$, a contradiction. So, there exists $\gamma \in \Gamma$ such that $Var(\gamma) \subseteq Var(\varphi)$, as desired.

(2) $\Rightarrow$ (1). Suppose that $\gamma, \gamma_1, ..., \gamma_n \vdash_{\mathrm{CL}} \varphi$, with $Var(\gamma) \subseteq Var(\varphi)$. Thus $\gamma \vdash_{\mathrm{CL}^{pl}} \neg(\gamma_1 \wedge ... \wedge \gamma_n) \vee \varphi$ and $\varphi \vee \neg\varphi, \gamma_1 \wedge ... \wedge \gamma_n, \neg(\gamma_1 \wedge ... \wedge \gamma_n) \vee \varphi \vdash_{\mathrm{CL}^{pl}_P} \varphi$, whence $\varphi \vee \neg\varphi, \gamma, \gamma_1, ..., \gamma_n \vdash_{\mathrm{CL}^{pl}_P} \varphi$. However, $\gamma \vdash_{\mathrm{CL}^{pl}} \varphi \vee \neg\varphi$, whereby $\gamma, \gamma_1, ..., \gamma_n \vdash_{\mathrm{CL}^{pl}_P} \varphi$.

(1) $\Rightarrow$ (3). It is immediate to check that the (eMP) is sound for K. ⊠

Recall that the logic $\mathrm{CL}^{pl}_{E^\star}$ is axiomatised relative to CL^{pl} by the rule (EFVQ). Once again, we identify both a class of characteristic matrices and a variable inclusion criterion for the valid entailments of this logic.

Theorem 29. *The following are equivalent:*

(1) $\Gamma \vdash_{\mathrm{CL}^{pl}_{E^\star}} \varphi$
(2) $(\Gamma \vdash_{\mathrm{CL}^{pl}} \varphi)$ *or* (Γ *is classically unsatisfiable and for some* $\gamma \in \Gamma$, $Var(\gamma) \subseteq Var(\varphi)$)
(3) $\Gamma \vdash_{\mathsf{K}} \varphi$, *where* K *is* $\{\langle \mathbf{PK}, \{1, n\}\rangle \times \langle \mathbf{B}_2, \{1\}\rangle, \langle \mathbf{S}_2, \{m\}\rangle\}$

Proof. (3) $\Rightarrow$ (2). Assume $\Gamma \vdash_{\mathsf{K}} \varphi$ and let $\Gamma \nvdash_{\mathrm{CL}^{pl}} \varphi$. Suppose towards a contradiction that Γ is classically satisfiable. If $\vdash_{\mathrm{CL}} \varphi$, we have that $Var(\gamma) \nsubseteq Var(\varphi)$ for every $\gamma \in \Gamma$. Then, there exists a valuation $h \in Hom(\mathbf{Fm}(\mathcal{L}_1), \mathbf{S}_2)$ such that $h(\varphi) = n$ and $h[\Gamma] \subseteq \{m\}$ (fix e.g. $h(x) = m$ for every variable $x \notin Var(\varphi)$, $h(x) = n$ otherwise). If, on the other hand, $\nvdash_{\mathrm{CL}} \varphi$, then $\Gamma \nvdash_{\mathrm{PWK}_E} \varphi$, so $\langle \mathbf{PK}, \{1, n\}\rangle \times \langle \mathbf{B}_2, \{1\}\rangle$ falsifies the inference, as the characteristic matrix of PWK_E can be embedded into it. It follows that Γ must be classically unsatisfiable. Finally, suppose $Var(\gamma) \nsubseteq Var(\varphi)$, for every $\gamma \in \Gamma$. In this case, by reasoning as above, $\langle \mathbf{S}_2, \{m\}\rangle$ falsifies the inference. So, we have proved that Γ is classically unsatisfiable and that $Var(\gamma) \subseteq Var(\varphi)$, for some $\gamma \in \Gamma$, as desired.

(2) $\Rightarrow$ (1). Let $\Gamma = \{\gamma, \gamma_1, ..., \gamma_n\}$ be classically unsatisfiable with $Var(\gamma) \subseteq Var(\varphi)$. Then $\Gamma \vdash_{\mathrm{CL}^{pl}} \varphi \vee \neg\varphi$. W.l.g., suppose there is $\delta \in \Gamma$ such that $\neg\delta \in \Gamma$. We also have $\varphi \vee \neg\varphi, \delta, \neg\delta \vdash_{\mathrm{CL}^{pl}_{E^\star}} \varphi$, whereby $\Gamma \vdash_{\mathrm{CL}^{pl}_{E^\star}} \varphi$.

(1) $\Rightarrow$ (3). Trivial.

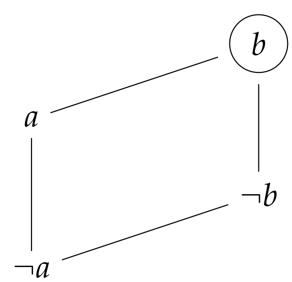
⊠

Lemma 30. *The following are equivalent:*

(1) $\Gamma \vdash_{\mathrm{CL}^{pl}_M \vee \mathrm{CL}^{pl}_P} \varphi$

(2) $\Gamma \vdash_{\langle \mathbf{B}_2, \{1\}\rangle \times \langle \mathbf{S}_2, \{m\}\rangle} \varphi$

Proof. In what follows, it will be expedient to keep in mind that the product matrix $\langle \mathbf{B}_2, \{1\}\rangle \times \langle \mathbf{S}_2, \{m\}\rangle$ is isomorphic to the matrix whose algebra reduct is the unique Płonka sum of the form $\mathbf{B}_2 \oplus \mathbf{B}_2$ and whose single designated element is the top element of its upper fibre:



(1) $\Rightarrow$ (2). Observe that $\vdash_{\mathrm{CL}^{pl}} \subseteq \vdash_{\langle \mathbf{B}_2, \{1\}\rangle \times \langle \mathbf{S}_2, \{m\}\rangle}$ and that the rules (EFV) and (eMP) are sound in $\langle \mathbf{B}_2, \{1\}\rangle \times \langle \mathbf{S}_2, \{m\}\rangle$.

(2) $\Rightarrow$ (1). Notice that $\mathrm{CL}^{pl}_M \vee \mathrm{CL}^{pl}_P = \mathrm{CL}^{pl}_E \vee \mathrm{CL}^{pl}_P$. Indeed, for the nontrivial inclusion, it is easy to derive (EFQ) from (eMP) together with (EFV).

Suppose now $\Gamma \vdash_{\langle \mathbf{B}_2, \{1\}\rangle \times \langle \mathbf{S}_2, \{m\}\rangle} \varphi$ and, towards a contradiction, that $\Gamma \nvdash_{\mathrm{CL}^{pl}_E \vee \mathrm{CL}^{pl}_P} \varphi$ with $\Gamma = \{\gamma_1, \ldots, \gamma_n\}$. By our assumption, we lose no generality in supposing that $\Gamma \vdash_{\mathrm{CL}} \varphi$. Then, by Theorem 29 and the definition of CL^{pl}_E, Γ is classically satisfiable and for every $\gamma_i \in \Gamma$ with $1 \leq i \leq n$ it holds $Var(\gamma_i) \nsubseteq Var(\varphi)$. Let $v \in Hom(\mathbf{Fm}(\mathcal{L}_1), \langle \mathbf{B}_2, \{1\}\rangle \times \langle \mathbf{S}_2, \{m\}\rangle)$ be a valuation such that $v(\gamma_i) = a$ for each $1 \leq i \leq n$. Cleary such a valuation exists, as Γ

is classically satisfiable. Moreover, $v(\varphi) = a$, because $\Gamma \vdash_{\mathrm{CL}} \varphi$. Let now Var^{-} be the set of variables of Γ that do not occur in φ. Consider now the map v' that differs from v only by the values of the variables in Var^{-}, as follows: for $x \in Var^{-}$, $v'(x) = b$ if $v(x) = a$, $v'(x) = \neg b$ if $v(x) = \neg a$. By the properties of Płonka sums it is easy to verify that v' is a valuation such that $v'(\Gamma) \subseteq \{b\}$, while $v'(\varphi) = v(\varphi) = a$. This is a contradiction, so $\Gamma \nvdash_{\mathrm{CL}_E^{pl} \vee \mathrm{CL}_P^{pl}} \varphi$. ⊠

4.1. **Splitting pairs in $\mathcal{E}xt\left(\mathrm{CL}^{pl}\right)$.** Let $\mathbf{L}$ be a lattice and let $a, b \in L$. Readers are reminded that the pair $\langle a, b\rangle$, with $a \nleq b$, is said to *split* $\mathbf{L}$ if for all $c \in L$, either $a \leq c$ or $c \leq b$. The study of splitting pairs in the lattice of extensions of a given base logic offers a fruitful technique to identify the covers, if any, of a given element, to determine whether such covers are unique and perhaps to prove that the lattice is finite. Here, we put to good use the study of the splitting pairs in $\mathcal{E}xt\left(\mathrm{CL}^{pl}\right)$ to yield an exhaustive description of such.

Theorem 31. *The following pairs split $\mathcal{E}xt\left(\mathrm{CL}^{pl}\right)$:*

(1) $\left\langle \mathrm{CL}_{E^\star}^{pl}, \mathrm{PWK} \right\rangle$.
(2) $\left\langle \mathrm{CL}_{M}^{pl}, \mathrm{CL}_{P}^{pl} \right\rangle$.
(3) $\left\langle \mathrm{PWK} \cap \mathrm{SL}, \mathrm{CL}_{P}^{pl} \vee \mathrm{CL}_{M}^{pl} \right\rangle$.
(4) $\langle \mathrm{PWK}, \mathrm{SL}\rangle$.
(5) $\left\langle \mathrm{CL}_{P}^{pl}, \mathrm{PWK}_E \right\rangle$;

Proof. Let $\mathrm{L} \in \mathcal{E}xt\left(\mathrm{CL}^{pl}\right)$. We will tacitly make use of Lemma 17 and Theorem 18 in the proof.

(1). Suppose that $\mathrm{CL}_{E^\star}^{pl} \nleq \mathrm{L}$. Therefore, there exist $\langle \mathbf{A}, F\rangle \cong \langle \mathcal{P}_{\nmid}(\mathbf{A}_i), \bigcup F_i\rangle_{i\in I} \in \mathsf{Mod}^*(\mathrm{L})$ and $h \in Hom(\mathbf{Fm}(\mathcal{L}_1), \mathbf{A})$ such that $h[y \vee \neg y, x, \neg x] \subseteq F$, $h(y) \notin F$. This entails that there are $i, j \in I$ such that $\mid A_j \mid = 1$ and $\mid A_i \mid \geqslant 2$, so $i < j$. Since the model is Leibniz-reduced, $F_i = \{h(y \vee \neg y)\}$ and $F_j = A_j$. It is immediate to check that $\langle \mathbf{A}_i \oplus \mathbf{A}_j, F_i \cup F_j\rangle \geqslant \langle \mathbf{WK}, \{1, n\}\rangle$ is a submatrix of $\langle \mathbf{A}, F\rangle$. It follows by Theorem 12 that $\mathrm{L} \leq \mathrm{PWK}$.

(2). Suppose that $\mathrm{CL}_{M}^{pl} \nleq \mathrm{L}$. Therefore, there exist $\langle \mathbf{A}, F\rangle \cong \langle \mathcal{P}_{\nmid}(\mathbf{A}_i), \bigcup F_i\rangle_{i\in I} \in \mathsf{Mod}^*(\mathrm{L})$ and $h \in Hom(\mathbf{Fm}(\mathcal{L}_1), \mathbf{A})$ such that $h[x, \neg x] \in F$, $h(y \vee \neg y) \notin F$. This entails there are $i, j \in I$ such that

$F_i = \emptyset$ and $F_j = A_j$. As the model is Leibniz reduced, $\mid A_j \mid = 1$ and so $i < j$. Two cases arise. Firstly, if $\mid A_i \mid = 1$, then $\langle \mathbf{A}, F\rangle$ contains $\langle \mathbf{S}_2, \{m\}\rangle$ as submatrix. Our conclusion follows from Theorem 28 upon noticing that if an inference $\Gamma \vdash \varphi$ holds in the matrix $\langle \mathbf{A}, F\rangle$ then it is not disproved by the matrix $\langle \mathbf{B}_2, \{1\}\rangle$. The remaining case is $\mid A_i \mid \geqslant 2$, and this, as $\mathbf{\Omega}^{\mathbf{A}} F = \Delta$, entails there exists $i \leq k \leq j$ with $\mid A_k \mid \geqslant 2$, $F_k = \{a \vee \neg a\}$, for a arbitrary in A_k. Therefore the matrix $\langle \mathbf{P}_5, \{1, n\}\rangle$ embeds into $\langle \mathbf{A}, F\rangle$, proving by Theorem 15 that $L \leq \mathrm{CL}^{pl}$ and, a fortiori $L \leq \mathrm{CL}^{pl}_{P}$.

(3). Suppose that PWK ∩ SL $\not\leq$ L. Therefore, by Theorem 26, there exist $\langle \mathbf{A}, F\rangle \cong \langle \mathcal{P}_{\text{ł}}(\mathbf{A}_i), \bigcup F_i\rangle_{i\in I} \in \mathsf{Mod}^{*}(\mathrm{L})$ and $h \in Hom(\mathbf{Fm}(\mathcal{L}_1), \mathbf{A})$ such that $h(x) \in F$, $h(y \vee \neg y) \notin F$. This amounts to saying that there exist $i, j \in I$ such that $F_i = \emptyset$, $F_j \neq \emptyset$. Since such a matrix must be arranged as an l-direct system, then $i < j$. There are three possible cases to consider. (A): $\mid A_j \mid = \mid A_i \mid = 1$; (B): $\mid A_j \mid = 1$ and $\mid A_i \mid \geqslant 2$; (C): $\mid A_j \mid \geqslant 2$. If (A) is the case, then $\langle \mathbf{S}_2, \{m\}\rangle$ embeds into $\langle \mathbf{A}, F\rangle$. Reasoning as in the first part of (1) above, we obtain by Theorem 28 that $\mathrm{L} \leq \mathrm{CL}^{pl}_{P} \leq \mathrm{CL}^{pl}_{P} \vee \mathrm{CL}^{pl}_{M}$. If (B) is the case, we can reason as in the second part of (2), proving $\mathrm{L} \leq \mathrm{CL}^{pl}$. Finally, if (C) occurs, $\langle \mathbf{B}_2, \{1\}\rangle \times \langle \mathbf{S}_2, \{m\}\rangle$ embeds into $\langle \mathbf{A}, F\rangle$, and our conclusion follows by Theorem 30.

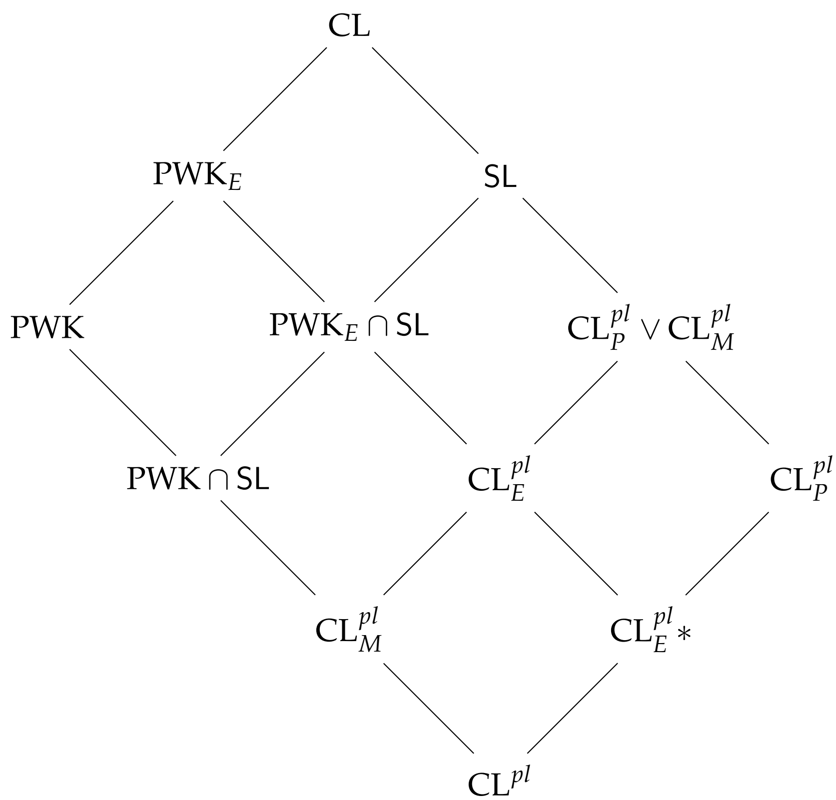
(4). Suppose that PWK $\not\leq$ L. Therefore, there exist $\langle \mathbf{A}, F\rangle \cong \langle \mathcal{P}_{\text{ł}}(\mathbf{A}_i), \bigcup F_i\rangle_{i\in I} \in \mathsf{Mod}^{*}(\mathrm{L})$ and $h \in Hom(\mathbf{Fm}(\mathcal{L}_1), \mathbf{A})$ such that $h(x \vee \neg x) \notin F$. Since $\langle \mathbf{A}, F\rangle$ is Leibniz reduced, if $\mathbf{A}$ has a unique fibre $\mathbf{A}_i$, then $\mid A_i \mid = \mid A \mid = 1$, and so it is isomorphic to $\langle \mathbf{1}, \emptyset\rangle$. This, and the fact that $\mathrm{L} \leq \mathrm{CL}$, entails $\mathrm{L} \leq \mathrm{SL}$. We can therefore assume that $\mathbf{A}$ is nontrivial, whence there exists $\langle \mathbf{A}_i, F_i\rangle$ s.t. $F_i = \emptyset$ and for each $j \neq i$ we have that $F_j \neq \emptyset$, for otherwise $\mathbf{\Omega}^{\mathbf{A}} F \neq \Delta$. If $\mid A_i \mid = 1$ then $\langle \mathbf{1}, \emptyset\rangle$ embeds into $\langle \mathbf{A}, F\rangle$ and the conclusion follows since $\mathrm{L} \leq \mathrm{CL}$. If $\mid A_i \mid \geqslant 2$ and $\mid A_j \mid = 1$, we reason as in (B) of point (3). The only remaining case is $\mid A_i \mid \geqslant 2$, $\mid A_j \mid \geqslant 2$. By point (C) of (3), $\langle \mathbf{B}_2, \{1\}\rangle \times \langle \mathbf{S}_2, \{m\}\rangle$ embeds into $\langle \mathbf{A}, F\rangle$, therefore by Theorem 30 $\mathrm{L} \leq \mathrm{CL}^{pl}_{E} \vee \mathrm{CL}^{pl}_{P} \leq \mathrm{SL}$.

(5.) Suppose that $\mathrm{CL}^{pl}_{P} \not\leq$ L. Therefore, there exist $\langle \mathbf{A}, F\rangle \cong \langle \mathcal{P}_{\text{ł}}(\mathbf{A}_i), \bigcup F_i\rangle_{i\in I} \in \mathsf{Mod}^{*}(\mathrm{L})$ and $h \in Hom(\mathbf{Fm}(\mathcal{L}_1), \mathbf{A})$ such that $h[x \vee \neg x, y, \neg y \vee x] \subseteq F$, $h(x) \notin F$. W.l.o.g. consider $h(x) = a$, $h(y) = b$ for $a \in A_i$, $b \in A_j$ and fix $k = i \vee j$. Observe that $F_i \neq \emptyset$ and $\mid A_i \mid \geqslant 2$, for otherwise $a \vee \neg a \in F$ entails $a \in F$, against our assumption. Similarly, $b \in F$ and $a \notin F$ ensure $p_{ik}(a) = \neg b \vee a \in F_k$

while $a \notin F_i$. So $i \lneq k$. Now, two cases may arise. (A): $\mid A_k \mid = 1$ or (B): $\mid A_k \mid \geqslant 2$. If (A) occurs, then $\langle \mathbf{A}_i \oplus \mathbf{A}_k, F_i \cup F_k \rangle \geqslant \langle \mathbf{WK}, \{1, n\} \rangle$, i.e. $\mathrm{L} \leq \mathrm{PWK} \leq \mathrm{PWK_E}$. If (B) is the case, the above considerations force that $\mid A_i \mid \geqslant 4$, so $\langle \mathbf{WK}, \{1, n\} \rangle \times \langle \mathbf{B}_2, \{1\} \rangle$ embeds into $\langle \mathbf{A}_i \oplus \mathbf{A}_k, F_i \cup F_k \rangle \leq \langle \mathbf{A}, F \rangle$, whence $\mathrm{L} \leq \mathrm{PWK}_E$ by the remarks following Theorem 13.

$\boxtimes$

Theorem 32. *The following diagram is a complete description of* $\mathcal{E}xt\left(\mathrm{CL}^{pl}\right)$*:*



Proof. The fact that these 12 logics are pairwise distinct, and their inclusion relationships, are clear from the above. What remains to be shown is that the logics appearing at each node (uniquely) cover the logic(s) appearing at the parent node(s). We already recalled that CL uniquely covers $\mathrm{PWK_E}$ and that $\mathrm{PWK_E}$ uniquely covers PWK, while it is well-known that CL uniquely covers SL. Also, it follows from Theorem 31.(2) that $\mathrm{CL}_P^{pl} \vee \mathrm{CL}_M^{pl}$ uniquely covers CL_P^{pl}, while by Theorem 31.(3) any proper extension of $\mathrm{CL}_P^{pl} \vee \mathrm{CL}_M^{pl}$ satisfies (eMP) and (VEQ), hence (MP). Thus, SL uniquely covers $\mathrm{CL}_P^{pl} \vee \mathrm{CL}_M^{pl}$.

- *The unique covers of* $\mathrm{PWK} \cap \mathrm{SL}$ *are* PWK *and* $\mathrm{PWK_E} \cap \mathrm{SL}$. By Theorem 26 and Theorem 31.(4), PWK covers $\mathrm{PWK} \cap \mathrm{SL}$. By Theorem 26 and Theorem 31.(1)-(4), any other cover of $\mathrm{PWK} \cap \mathrm{SL}$ must be above $\mathrm{CL}^{pl}_{E^\star}$. So it should satisfy both (EFVQ) and (VEQ), and hence (EFQ). So it must perforce be $\mathrm{PWK_E} \cap \mathrm{SL}$.
- *The unique covers of* $\mathrm{PWK_E} \cap \mathrm{SL}$ *are* $\mathrm{PWK_E}$ *and* SL. By Theorem 31.(4) and the fact that $\mathrm{PWK_E}$ uniquely covers PWK, any cover of $\mathrm{PWK_E} \cap \mathrm{SL}$ must be either $\mathrm{PWK_E}$ or below SL. By Theorem 31.(5) it must also be above CL^{pl}_P. So it satisfies both (VEQ) and (eMP) and thus (MP). It follows that it coincides with SL.
- *The unique covers of* CL^{pl}_M *are* $\mathrm{PWK} \cap \mathrm{SL}$ *and* CL^{pl}_E. By Lemma 25, any cover L of CL^{pl}_M cannot be both under PWK and CL^{pl}_E. If L is not under PWK, by Theorem 31.(1) it must be above $\mathrm{CL}^{pl}_{E^\star}$. So L should satisfy both (EFVQ) and (EFV), and hence (EFQ). Thus, it must perforce be CL^{pl}_E. If, on the other hand, L is under PWK but not under CL^{pl}_E, we show that it must be $\mathrm{PWK} \cap \mathrm{SL}$. If it is not, by Theorem 31.(3) it must be below $\mathrm{CL}^{pl}_P \vee \mathrm{CL}^{pl}_M$. Since L covers CL^{pl}_M, there must be a set $\Gamma \cup \{\varphi\}$ such that $\Gamma \vdash_{\mathrm{L}} \varphi$ but $\Gamma \nvdash_{\mathrm{CL}^{pl}_M} \varphi$, and a variable x such that $x \in Var(\Gamma) \setminus Var(\varphi)$. W.l.g. we can assume that $\Gamma = \{\gamma_1, ..., \gamma_n\}$. But then we can find a counterexample to $\gamma_1 \wedge ... \wedge \gamma_n \vdash \varphi$ in $\langle \mathbf{B}_2, \{1\} \rangle \times \langle \mathbf{S}_2, \{m\} \rangle$ via a homomorphism h such that $h(x) = \langle 1, m \rangle$ and $h(y) = \langle 0, n \rangle$ for each variable $y \neq x$. By Lemma 30, $\Gamma \nvdash_{\mathrm{CL}^{pl}_P \vee \mathrm{CL}^{pl}_M} \varphi$, which is a contradiction.
- *The unique covers of* CL^{pl}_E *are* $\mathrm{PWK_E} \cap \mathrm{SL}$ *and* $\mathrm{CL}^{pl}_P \vee \mathrm{CL}^{pl}_M$. Similar to the preceding item.
- *The unique covers of* CL^{pl} *are* CL^{pl}_M *and* $\mathrm{CL}^{pl}_{E^\star}$. By Theorem 31.(2), any extension of CL^{pl} that is not above CL^{pl}_M is below CL^{pl}_P. So, by Theorem 31.(1), any cover of CL^{pl} different from CL^{pl}_M is either $\mathrm{CL}^{pl}_{E^\star}$ or below PWK. By Theorems 15.(2) and 28, $\mathrm{PWK} \cap \mathrm{CL}^{pl}_P = \mathrm{CL}^{pl}$, whence our conclusion follows.
- *The unique covers of* $\mathrm{CL}^{pl}_{E^\star}$ *are* CL^{pl}_E *and* CL^{pl}_P. Similar to the preceding item.

⊠

In the following table, we recapitulate the logics addressed in the paper, indicating for each of them the rules that axiomatise them relative to CL^{pl} and a set of characteristic matrices.

Logic	Axioms	Char. matrices
CL^{pl}		$\langle \mathbf{PK}, \{1, n\} \rangle$
CL^{pl}_{M}	EFV	$\{\langle \mathbf{PK}, \{1, n\} \rangle \times \langle \mathbf{B}_2, \{1\} \rangle, \langle \mathbf{WK}, \{1, n\} \rangle\}$
$\mathrm{CL}^{pl}_{E^\star}$	EFVQ	$\{\langle \mathbf{PK}, \{1, n\} \rangle \times \langle \mathbf{B}_2, \{1\} \rangle, \langle \mathbf{S}_2, \{m\} \rangle\}$
$\mathrm{PWK} \cap \mathrm{SL}$	VEQ	$\{\langle \mathbf{WK}, \{1, n\} \rangle, \langle \mathbf{B}_1, \emptyset \rangle\}$
CL^{pl}_{E}	EFQ	$\langle \mathbf{PK}, \{1, n\} \rangle \times \langle \mathbf{B}_2, \{1\} \rangle$
CL^{pl}_{P}	eMP	$\{\langle \mathbf{B}_2, \{1\} \rangle, \langle \mathbf{S}_2, \{m\} \rangle\}$
PWK	EM	$\langle \mathbf{WK}, \{1, n\} \rangle$
$\mathrm{PWK}_E \cap \mathrm{SL}$	VEQ, EFQ	$\{\langle \mathbf{WK}, \{1, n\} \rangle \times \langle \mathbf{B}_2, \{1\} \rangle, \langle \mathbf{B}_1, \emptyset \rangle\}$
$\mathrm{CL}^{pl}_{P} \vee \mathrm{CL}^{pl}_{M}$	eMP, EFQ	$\langle \mathbf{B}_2, \{1\} \rangle \times \langle \mathbf{S}_2, \{m\} \rangle$
PWK_E	EM, EFQ	$\langle \mathbf{WK}, \{1, n\} \rangle \times \langle \mathbf{B}_2, \{1\} \rangle$
SL	MP	$\{\langle \mathbf{B}_2, \{1\} \rangle, \langle \mathbf{B}_1, \emptyset \rangle\}$
CL	EM, MP	$\langle \mathbf{B}_2, \{1\} \rangle$

5. Algebraic counterparts

In this final section, we determine the class $\mathsf{Alg}(\mathrm{L})$ for all the logics L in the previous diagram. It is well-known that $\mathsf{Alg}(\mathrm{CL}) = \mathsf{Alg}(\mathrm{SL})$ is $\mathcal{BA}$, the variety of Boolean algebras. It is also known (see [5, 23]) that $\mathsf{Alg}(\mathrm{PWK}) = \mathcal{SGIB}$ and $\mathsf{Alg}(\mathrm{PWK_E}) = \mathcal{NGIB}$. We now address the problem for the remaining logics.

In what follows, given an algebra $\mathbf{A} \in \mathcal{GIB}$, viewed as usual in its Płonka sum representation, and a subset $F \subseteq A_i$, we set

$$\uparrow F = \bigcup_{j \geqslant i} p_{ij}[F].$$

Theorem 33. *(i)* $\mathsf{Alg}(\mathrm{CL}^{pl}) = \mathcal{GIB}$;
(ii) $\mathsf{Alg}(\mathrm{CL}^{pl}_{E^*}) = \mathcal{NGIB} \vee \mathcal{SL}$;
(iii) $\mathsf{Alg}(\mathrm{CL}^{pl}_{M}) = \mathcal{SGIB}$;
(iv) $\mathsf{Alg}(\mathrm{PWK} \cap \mathrm{SL}) = \mathcal{SGIB}$;
(v) $\mathsf{Alg}(\mathrm{CL}^{pl}_{E}) = \mathcal{NGIB}$;
(vi) $\mathsf{Alg}(\mathrm{PWK_E} \cap \mathrm{SL}) = \mathcal{NGIB}$;
(vii) $\mathsf{Alg}(\mathrm{CL}^{pl}_{P}) = \mathcal{INGIB} \vee \mathcal{SL}$;
(viii) $\mathsf{Alg}(\mathrm{CL}^{pl}_{P} \vee \mathrm{CL}^{pl}_{M}) = \mathcal{INGIB}$.

Proof. We only prove items (i), (ii), (vii), as the others rely on similar, but simpler arguments.

(i). ($\subseteq$). By Lemma 17, $\mathsf{Alg}^*(\mathrm{CL}^{pl}) \subseteq \mathcal{GIB}$, so by Proposition 3.(i) $\mathsf{Alg}(\mathrm{CL}^{pl}) = P_{SD}(\mathsf{Alg}^*(\mathrm{CL}^{pl})) \subseteq P_{SD}(\mathcal{GIB}) = \mathcal{GIB}$, because $\mathcal{GIB}$ is a variety.

($\supseteq$). Let $\mathbf{A} \cong \mathcal{P}_{\text{ł}}(\mathbf{A}_i)_{i\in I} \in \mathcal{GIB}$ and consider $\langle \mathbf{A}, \emptyset \rangle$, which clearly is a model of CL^{pl}. If $\mathbf{A}$ is a (possibly trivial) Boolean algebra, our conclusion is obvious. Otherwise, fix distinct $a, b \in A$ with $a \in A_i$, $b \in A_j$ and w.l.o.g. assume $i \nleq j$. Clearly $G := Fg^{\mathbf{A}}_{\mathrm{CL}^{pl}}(a) = p_{ij}[Fg^{\mathbf{A}}_{\mathrm{CL}}(a)]$ is a filter containing a and not containing b. Moreover $\emptyset \subseteq G$ and this proves $\widetilde{\mathbf{\Omega}}^{\mathbf{A}}\emptyset = \Delta$, i.e. $\langle \mathbf{A}, F \rangle \in \mathsf{Mod}^{\mathsf{Su}}(\mathrm{CL}^{pl})$, as desired.

(ii). Firstly observe that, in light of point (i), any logic L in $\mathcal{E}xt\left(\mathrm{CL}^{pl}\right)$ is such that $\mathsf{Alg}(\mathrm{L}) \subseteq \mathcal{GIB}$.

($\subseteq$). Let $\mathbf{A} \in \mathsf{Alg}(\mathrm{CL}^{pl}_{E^*})$ and suppose, towards a contradiction, that $\mathbf{A}$ has a non trivial fibre $\mathbf{A}_i$ and a trivial fibre $\mathbf{A}_j$. This entails that for any $k > j$, $\mathbf{A}_k$ is trivial. Notice that for any $\mathrm{CL}^{pl}_{E^*}$-filter G, if $A_i \cap G \neq \emptyset$, then $G = A$. This is true because $y \vee \neg y, x, \neg x \vdash y$ is a rule of $\mathrm{CL}^{pl}_{E^*}$. Since $\mathbf{A} \in \mathsf{Alg}(\mathrm{CL}^{pl}_{E^*})$, by Lemma 1 and the definition of Suszko congruence there exist a filter F on $\mathbf{A}$ and a unary polynomial function $p(x)$ such that, for arbitrary $a \in A_i$,

$$p(a \vee \neg a) \in F \iff p(a \wedge \neg a) \notin F.$$

This, by the above argument, entails $F = A$, a contradiction. So, $\mathbf{A}$ cannot contain both a trivial and a non trivial fibre.

($\supseteq$). Let $\mathbf{A} \in \mathcal{NGIB} \vee \mathcal{SL}$ and fix distinct elements $a \in A_i, b \in A_j$, assuming that it is not the case that $j < i$. If $i = j$ then $\langle \mathbf{A}_i, \{a \vee \neg a\} \rangle \in \mathsf{Mod}^*(\mathrm{CL})$. Since $F = \uparrow a \vee \neg a$ is a $\mathrm{CL}^{pl}_{E^*}$- filter on A, $\langle a, b \rangle \notin \mathbf{\Omega}^{\mathbf{A}}F$. If $i < j$, it is immediate to verify that $\uparrow \mathrm{Fg}^{\mathbf{A}_j}_{\mathrm{CL}}(b)$ is a $\mathrm{CL}^{pl}_{E^*}$-filter containing b and not containing a. This entails that $\langle \mathbf{A}, \emptyset \rangle \in \mathsf{Mod}^*(\mathrm{CL}^{pl}_{E^*})$, as desired.

(vii). ($\subseteq$). Let $\mathbf{A} \in \mathsf{Alg}(\mathrm{CL}^{pl}_{P})$ and suppose, in view of a contradiction, that there exists a non-injective homomorphism $p_{ij} : \mathbf{A}_i \to \mathbf{A}_j$ where $\mathbf{A}_i$ is nontrivial. So there exists $a \in A_i$ such that $p_{ij}(a) = p_{ij}(a \vee \neg a) = b \vee \neg b$, for b arbitrary in A_j. By assumption, there is a filter F on A and a unary polynomial function $p(x)$ such that

$$p(a \vee \neg a) \in F_i \iff p(a) \notin F_i.$$

Since F is a filter and F_i is non empty, it contains $a \vee \neg a$; moreover, also $a \in F_i$, otherwise $a \vee \neg a, b \vee \neg b, \neg(b \vee \neg b) \vee a \in F$ while $a \notin F$. This entails that $\langle \mathbf{A}_i, F_i \rangle \in \mathsf{Mod}(\mathrm{CL})$ is such that $a, \top_i \in F_i$ and $\langle a, \top_i \rangle \notin \boldsymbol{\Omega}^{\mathbf{A}_i} F_i$, a contradiction. So, p_{ij} is injective or $\mathbf{A}_i$ is trivial. By the remarks at the end of § 2.3, we are done.

($\supseteq$). This inclusion is provable as in point (ii).

$\boxtimes$

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