ABSTRACT FACTORIZATION THEOREMS
WITH APPLICATIONS TO IDEMPOTENT FACTORIZATIONS

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ABSTRACT. Let $\preceq$ be a preorder on a monoid H with identity 1_H and s be an integer ≥ 2. The $\preceq$-height of an element $x \in H$ is the supremum of the integers $k \geq 1$ for which there is a (strictly) $\preceq$-decreasing sequence $x_1, \ldots, x_k$ of $\preceq$-non-units of H with $x_1 = x$, where $u \in H$ is a $\preceq$-unit if $u \preceq 1_H \preceq u$ and a $\preceq$-non-unit otherwise. We say H is $\preceq$-artinian if there is no infinite $\preceq$-decreasing sequence of elements of H, and strongly $\preceq$-artinian if the $\preceq$-height of each element is finite.

We establish that, if H is $\preceq$-artinian, then each $\preceq$-non-unit $x \in H$ factors through the $\preceq$-irreducibles of degree s, where a $\preceq$-irreducible of degree s is a $\preceq$-non-unit $a \in H$ that cannot be written as a product of s or fewer $\preceq$-non-units each of which is (strictly) smaller than a with respect to $\preceq$. In addition, we show that, if H is strongly $\preceq$-artinian, then x factors through the $\preceq$-quarks of H, where a $\preceq$-quark is a $\preceq$-minimal $\preceq$-non-unit. In the process, we obtain upper bounds for the length of a shortest factorization of x into $\preceq$-irreducibles of degree s (resp., $\preceq$-quarks) in terms of its $\preceq$-height.

Next, we specialize these results to the case in which H is the multiplicative submonoid of a ring R formed by the zero divisors of R (and the identity 1_R) and $a \preceq b$ if and only if the right annihilator of $1_R - b$ is contained in that of $1_R - a$. If H is $\preceq$-artinian (resp., strongly $\preceq$-artinian), then every zero divisor of R factors as a product of $\preceq$-irreducibles of degree s (resp., $\preceq$-quarks); and we prove that, for a variety of right Rickart rings, either the $\preceq$-quarks or the $\preceq$-irreducibles of degree 2 or 3 are coprimitive idempotents (an idempotent $e \in R$ is coprimitive if $1_R - e$ is primitive). In the latter case, we also derive sharp upper bounds for the length of a shortest idempotent factorization of a zero divisor $x \in R$ in terms of the $\preceq$-height of x and the uniform dimension of R_R. In particular, we can thus recover and improve on classical theorems of J. A. Erdos (1967), R. J. H. Dawlings (1981), and J. Fountain (1991) on idempotent factorizations in the endomorphism ring of a free module of finite rank over a skew field or a commutative DVD (e.g., we find that every singular n-by-n matrix over a commutative DVD, with $n \geq 2$, is a product of $2n - 2$ or fewer idempotent matrices of rank $n - 1$).

1. INTRODUCTION

In many branches of mathematics, one is often faced with the problem of proving that every "large element" of a monoid is a (finite) product of other elements that are regarded as "elementary building blocks" since they cannot be "broken down into smaller pieces". One way to make these ideas precise is to combine the language of monoids with the language of preorders, as recently done by the second author in [53]. This is part of a broader program [52, 5, 19] whose ultimate goal is to enlarge as much as possible the current boundaries of the "classical theory of factorization", whose focus is on commutative or "nearly cancellative" monoids and where the building blocks are either atoms in the sense of [12] or irreducibles in the sense of [4, Definition 2.4]. We refer the reader to the papers [26, 24, 11, 50, 25, 7], the

surveys [27, 22, 8, 9], and the volumes [20, 23, 2] for an overview of some trends and typical questions in this field. Here, we concentrate on the case where the building blocks are idempotent elements.

More specifically, let $\mathcal{E}(H)$ be the submonoid generated by the idempotents of a (commutative or non-commutative) monoid H. An element $x \in H$ has an idempotent factorization if $x \in \mathcal{E}(H)$, and H is idempotent-generated if $H = \mathcal{E}(H)$. Over the years, a great deal of work has been done to characterize idempotent-generated monoids. The first result in this direction is usually credited to J.H. Howie [35], who proved in 1966 that the monoid of singular transformations of a finite set X is idempotent-generated. One year later, J.A. Erdos showed in [18] that, for every positive integer n, the monoid of singular n-by-n matrices over a field is generated by idempotent matrices. Afterwards, research in the area has focused on generalizations of Erdos' theorem from fields to broader classes of rings. In parallel, various authors have studied the problem of bounding the minimum length of (i.e., the minimum number of factors in) an idempotent factorization of an idempotent-generated element [40, 31, 30, 10].

Most notably, it was shown by T.J. Laffey [39, Theorems 1 and 2] that Erdos' theorem carries over to the singular monoid of the ring of n-by-n matrices over R when R is either a skew field or a commutative euclidean domain. Later, J. Fountain [21, Theorems 4.1 and 4.6] proved that not only the same is true when R is either the integers or a commutative discrete valuation domain (DVD), but also in each of these circumstances the idempotent factors can be taken to be matrices of rank $n - 1$. Laffey's results were subsequently generalized by A. Alahmadi et al. [1] to the case where R is a right- and left-quasi-euclidean domain. On the other hand, R.J.H. Dawlings established in [17] that any singular endomorphism of an n-dimensional vector space is a product of at most n idempotent linear maps of rank $n - 1$, so providing a "quantitative version" of Erdos' theorem. We refer the reader to [38] for a more thorough account of the literature in this area and to [15, 16] for some recent developments.

In the present work, we address both aspects of the study of idempotent factorizations (that is, the existence and the bounds) by an approach that is apparently unprecedented. More precisely, assume $\preceq$ is a preorder on a monoid H and let s be an integer ≥ 2. After introducing the notions of $\preceq$-non-unit, degree-s $\preceq$-irreducible, $\preceq$-quark, and $\preceq$-height (Definitions 3.1 and 3.2), we show that, if $\preceq$ is artinian, (i.e., there is no (strictly) $\preceq$-decreasing sequence $x_1, x_2, \ldots$ in H), then each $\preceq$-non-unit $x \in H$ factors into a (finite) product of $\preceq$-irreducibles of degree s (Theorem 3.4). If $\preceq$ is strongly artinian (i.e., the $\preceq$-height of every element is finite) and further conditions are satisfied, then x factors into a product of $\preceq$-quarks (Theorem 3.5). In the process, we also obtain upper bounds for the length of a shortest factorization of x into $\preceq$-irreducible of degree s (resp., $\preceq$-quarks) in terms of its $\preceq$-height.

Next, we specialize these abstract results to the case where H is the monoid of zero divisors of a ring R, while $\preceq$ is the preorder defined by $a \preceq b$ if and only if the right annihilator of $1_R - b$ is contained in that of $1_R - a$ (Sect. 3.1). Accordingly, we prove that, for a variety of rings, either the $\preceq$-quarks or the $\preceq$-irreducibles of degree 2 or 3 of H are "coprimitive idempotents" (Remarks 5.1 and 5.4, Theorems 5.2 and 5.7, and Proposition 5.18(ii)). Moreover, we derive sharp upper bounds for the minimum length of an idempotent factorization of an element $x \in H$ in terms of the $\preceq$-height of x and the uniform dimension of R_R (Proposition 5.9 and Theorem 5.19). In particular, we can recover and improve on some of the classical theorems reviewed in the above. That is, the Erdos-Dawlings-Laffey theorem on singular matrices over a skew field (Corollary 5.10) and Fountain's theorem on singular matrices over a commutative DVD (Corollary 5.20). As an aside, we also obtain a characterization of the (multiplicative)

monoid generated by the idempotents of a semisimple ring and a tight upper bound for the minimum length of an idempotent factorization of an element from the same (Remark 5.11).

Loosely speaking, Theorems 3.4 and 3.5 work as a sort of black box for a wide range of problems. The inputs of the black box are a monoid H and an artinian or strongly artinian preorder $\preceq$ on H; the output is the existence of certain factorizations for every "large element" of H (cf. the first paragraph of this section), where an element is taken to be "large" if it is not $\preceq$-equivalent to the identity of H (two elements $x, y \in H$ are $\preceq$-equivalent if $x \preceq y \preceq x$). Here, we test this approach against the study of idempotent factorizations; further applications (e.g., to the classical theory of factorization) are discussed in [53, Sect. 4.1], while prospects for future research are outlined in Sect. 6.

We note in passing that the study of *commutative* rings with non-trivial zero divisors from the point of view of the classical theory of factorization is a topic with a long tradition (see [5, Sect. 2.4 and Remark 4.5] and references therein). However, the classical theory of factorization corresponds, from the point of view of the underlying philosophy of this work, to the case where H is the multiplicative monoid of a ring and the preorder $\preceq$ in the foregoing discussion is the divisibility preorder on H (so that $x \preceq y$ if and only if $y \in HxH$), which has no bearing on the "non-classical factorizations" considered herein.

2. Preliminaries

Below, we collect some elementary but fundamental results that will often come into play in later sections, and we review notations and terminology used all through the paper. Further notations and terminology, if not explained when first introduced, are standard or should be clear from context.

2.1. **Generalities.** We denote by $\mathbb{Z}$ the (set of) integers, by $\mathbb{N}$ the non-negative integers, and by $\mathbb{N}^+$ the positive integers. For all $a, b \in \mathbb{Z} \cup \{\pm\infty\}$, we let $[\![a, b]\!] := \{x \in \mathbb{Z} : a \leq x \leq b\}$ be the discrete interval between a and b. Unless noted otherwise, we reserve the letters m and n (with or without subscripts or superscripts) for positive integers; and the letters i, j, k, and s for non-negative integers.

2.2. **Monoids.** We take a monoid to be a semigroup with an identity. Unless stated otherwise, monoids will typically be written multiplicatively and need not have any special property (e.g., commutativity). We refer to [36, Ch. 1] for basic aspects of semigroup theory.

Let H be a monoid with identity 1_H. A unit of H is an element $u \in H$ such that $uv = vu = 1_H$ for a provably unique element $v \in H$, named the inverse of u (in H) and denoted by u^{-1}. An idempotent of H is an element $e \in H$ such that $e^2 = e$. In particular, an idempotent is *proper* if it is not the identity.

An element $a \in H$, on the other hand, is right-cancellative (resp., left-cancellative) if $xa \neq ya$ (resp., $ax \neq ay$) for all $x, y \in H$ with $x \neq y$; otherwise, a is right-singular (resp., left-singular). Accordingly, a is cancellative if it is right- and left-cancellative, and is singular if it is right- and left-singular.

We will write $\mathcal{E}(H)$ for the subsemigroup (in fact, a submonoid) of H generated by the idempotents; $H^{\#}$ for the set made up of the singular elements of H and the identity 1_H; and $H^\times$ for the set of units of H. It is readily seen that $H^{\#}$ is a submonoid and $H^\times$ is a subgroup of H, henceforth called the singular monoid and the group of units of H, resp. Similar considerations apply to the left-singular elements of H: Together with the identity, they too form a submonoid of H, which we refer to as the monoid of left-singular elements of H but for which we introduce no special notation. (The case with right-singular elements is completely analogous.) It is straightforward that, if $1_H \neq x \in \mathcal{E}(H)$, then x is singular. For, if

x factors as a non-empty product $e_1 \cdots e_n$ of idempotents of H and we assume without loss of generality that $e_i \neq 1_H$ for each $i \in [\![1, n]\!]$, then $e_1 x = 1_H x = x 1_H = x e_n$ (i.e., x is singular). Hence $\mathcal{E}(H) \subseteq H^{\#}$.

We say that H is von Neumann regular if, for each $x \in H$, there is $y \in H$ such that $x = xyx$; and Dedekind-finite if, for all $x, y \in H$, xy is a unit if and only if at least one (and hence all) of x, y, and yx is a unit (equivalently, $xy = 1_H$ if and only if $yx = 1_H$). Moreover, we say an element $x \in H$ *factors through* (the elements of) a set $A \subseteq H$ if x can be expressed as a finite product of elements from A (with the usual understanding that an empty product is equal to the identity 1_H), and a set $X \subseteq H$ factors through (the elements of) A if each element of X factors through A.

2.3. **Rings and modules.** We refer the reader to [3] for basic aspects on rings and modules. Most notably, a ring will always mean an associative (commutative or non-commutative) non-zero ring with identity; and a module will mean a *unitary* module, and more precisely a *right* unitary module unless stated otherwise.

Let R be a ring and M be a module over R. We denote by 0_R (resp., by 1_R) the additive (resp., multiplicative) identity of R, by 0_M the zero of M, and by R_R the regular right module over R (i.e., R viewed in the usual way as a right module over itself). For $X, Y \subseteq M$, $v \in M$, and $A \subseteq R$, we set

$$X + Y := \{x + y \colon x \in X, \ y \in Y\} \quad \text{and} \quad vA := \{va \colon a \in A\}.$$

In particular, we refer to the sumset $X + Y$ as a direct sum, and we write $X \oplus Y$ in place of $X + Y$, if X and Y *intersect trivially*, that is, $X \cap Y = \{0_M\}$. We use $\mathcal{M}_n(R)$ for the ring of n-by-n matrices over R and $\mathrm{End}(M)$ for the endomorphism ring of M, i.e., the ring obtained by endowing the set of all (right) R-linear functions on M with the operations of pointwise addition and functional composition (the latter playing the role of ring multiplication): The additive identity of $\mathrm{End}(M)$ is the function $M \to M \colon x \mapsto 0_M$, while the multiplicative identity is the identity map id_M on M.

A submodule N of M is indecomposable if N is neither the zero module $\{0_M\}$ nor the direct sum of two non-zero submodules; and is a direct summand (of M) if $M = N \oplus N'$ for some (not-necessarily-unique) submodule N', herein named an additive complement of N (relative to M). If N is a direct summand of M and N' is an additive complement of N, we have a well-defined endomorphism $p \in \mathrm{End}(M)$, called the projection of M on N along N', that maps a vector $v \in M$ to the unique $x \in N$ such that $v - x \in N'$.

We take the uniform (or Goldie) dimension $\mathrm{udim}(M)$ of M to be the supremum of the set of all integers $k \geq 1$ such that $X_1 \oplus \cdots \oplus X_k \subseteq M$ for some submodules $X_1, \ldots, X_k$ of M, with $\sup \varnothing := 0$. Accordingly, we let the right uniform dimension of a right ideal $\mathfrak{i}$ of R, herein denoted by $\mathrm{r.udim}_R(\mathfrak{i})$, be the uniform dimension of $\mathfrak{i}$ viewed as a submodule of R_R. It is seen from [41, Definition (6.2) and Corollaries (6.6) and (6.10)] that the right uniform dimension is *monotone* and *additive*, in the sense that

$$\mathrm{r.udim}_R(\mathfrak{i}) \leq \mathrm{r.udim}_R(\mathfrak{j}), \qquad \text{for all right ideals } \mathfrak{i}, \mathfrak{j} \subseteq R \text{ with } \mathfrak{i} \subseteq \mathfrak{j} \tag{1}$$

and

$$\mathrm{r.udim}_R(\mathfrak{i} + \mathfrak{j}) = \mathrm{r.udim}_R(\mathfrak{i}) + \mathrm{r.udim}_R(\mathfrak{j}), \qquad \text{for all right ideals } \mathfrak{i}, \mathfrak{j} \subseteq R \text{ with } \mathfrak{i} \cap \mathfrak{j} = \{0_R\}. \tag{2}$$

Given $a \in R$, we write $\mathrm{r.Ann}_R(a)$ for the right annihilator and $\mathrm{l.Ann}_R(a)$ for the left annihilator of a, i.e.,

$$\mathrm{r.Ann}_R(a) := \{x \in R \colon ax = 0_R\} \quad \text{and} \quad \mathrm{l.Ann}_R(a) := \{x \in R \colon xa = 0_R\}.$$

In addition, we define

$$\mathrm{r.Fix}_R(a) := \mathrm{r.Ann}_R(1_R - a) = \{x \in R \colon ax = x\}. \tag{3}$$

As a matter of fact, $\mathrm{r.Ann}_R(a)$ is a right ideal of R, and hence so also is $\mathrm{r.Fix}_R(a)$. Moreover, it is obvious that $ax = 0_R \neq x$ for every non-zero $x \in \mathrm{r.Ann}_R(a)$, which shows that

$$\mathrm{r.Ann}_R(a) + \mathrm{r.Fix}_R(a) = \mathrm{r.Ann}_R(a) \oplus \mathrm{r.Fix}_R(a). \tag{4}$$

We call R a **right Rickart** ring if $\mathrm{r.Ann}_R(a)$ is a direct summand of R_R for every $a \in R$. As will be seen in Sect. 5, this class of rings is of utmost importance in the applications of the abstract theorems of Sect. 3.

We say that R is a **von Neumann regular** (resp., **Dedekind-finite**) ring if so is the multiplicative monoid $(R, \cdot)$ of R. By a unit of R we always understand a *multiplicative* unit, that is, a unit of $(R, \cdot)$. The same goes for idempotents and singular, left-singular, and right-singular elements. Accordingly, we define $R^\# := (R, \cdot)^\#$ and $R^\times := (R, \cdot)^\times$, and we denote by $\mathcal{E}(R)$ the subsemigroup of $(R, \cdot)$ generated by the idempotents of R. Note that an element $a \in R$ is left-singular (resp., right-singular) if and only if a is a right (resp., left) zero divisor of R. Accordingly, we will simply refer to $R^\#$ and $R^\times$, resp., as the **monoid of zero divisors** and the **group of units** of the ring R. Likewise, we will talk of the "monoid of left (resp., right) zero divisors" of R to mean the monoid of left-singular (resp., right-singular) elements of $(R, \cdot)$.

In particular, we let an idempotent $e \in R$ be **coprimitive** if $\mathrm{r.Ann}(e)$ is an indecomposable submodule of R_R. The notion is left-right symmetric. It is a basic fact (see, e.g., [37, p. 48]) that

$$R = eR \oplus (1_R - e)R = Re \oplus R(1_R - e), \tag{5}$$

and this implies at once that

$$\mathrm{r.Ann}_R(e) = \mathrm{r.Fix}_R(1_R - e) = (1_R - e)R \quad \text{and} \quad \mathrm{l.Ann}_R(e) = R(1_R - e). \tag{6}$$

Consequently, we conclude from the equivalence between conditions (1) and (1)′ in [42, Proposition (21.8)] that e is coprimitive if and only if $\mathrm{l.Ann}_R(e)$ is indecomposable as a *left* module over R.

Lastly, we let a **Peirce basis** of R be a non-empty, finite tuple $(e_1, \ldots, e_n)$ of pairwise orthogonal idempotents of R such that $1_R = e_1 + \cdots + e_n$, where $x, y \in R$ are **orthogonal** if $xy = yx = 0_R$.

The following result provides a characterization of idempotents that, while elementary, is central for this paper (note that we henceforth drop the subscript 'R' from the notation '$\mathrm{r.Ann}_R$', '$\mathrm{l.Ann}_R$', and '$\mathrm{r.Fix}_R$' when the ring R is clear from context).

Proposition 2.1. Let R be a ring. The following are equivalent for an element $a \in R$:

(a) a is an idempotent of R.
(b) $R = \mathrm{r.Ann}(a) \oplus \mathrm{r.Fix}(a)$.
(c) $aR = \mathrm{r.Fix}(a)$.

Proof. (a) $\Rightarrow$ (b): Let $x \in R$. Since a is idempotent, we have $ax = a^2 x$ and hence $a(x - ax) = 0_R$; that is, $ax \in \mathrm{r.Fix}(a)$ and $x - ax \in \mathrm{r.Ann}(a)$. It follows that $x = (x - ax) + ax \in \mathrm{r.Ann}(a) + \mathrm{r.Fix}(a)$, which shows in turn that $R \subseteq \mathrm{r.Ann}(a) + \mathrm{r.Fix}(a)$. The opposite inclusion is obvious, so by Eq. (4) we are done.

(b) $\Rightarrow$ (c): Let $b \in R$. By the hypothesis that $R = \mathrm{r.Ann}(a) \oplus \mathrm{r.Fix}(a)$, we can write $b = x + y$ for some $x \in \mathrm{r.Ann}(a)$ and $y \in \mathrm{r.Fix}(a)$. This implies $ab = ax + ay = y \in \mathrm{r.Fix}(a)$ and hence $aR \subseteq \mathrm{r.Fix}(a)$. On the other hand, it is clear that $\mathrm{r.Fix}(a) \subseteq aR$, because $z = az$ for each $z \in \mathrm{r.Fix}(a)$. So, $aR = \mathrm{r.Fix}(a)$.

(c) $\Rightarrow$ (a): Since $aR = \mathrm{r.Fix}(a)$, we have $a \in \mathrm{r.Fix}(a)$ and hence $a^2 = a$ (namely, a is idempotent). $\blacksquare$

The next proposition, on the other hand, puts together a few more basic properties of idempotents that we will often come in handy in Sects. 4 and 5.

Proposition 2.2. Let R be a ring and $e, f \in R$ be idempotents. The following hold:

(i) If $R = \mathfrak{i}_1 \oplus \cdots \oplus \mathfrak{i}_n$ for certain right ideals $\mathfrak{i}_1, \ldots, \mathfrak{i}_n \subseteq R$, then there is a uniquely determined Peirce basis $(e_1, \ldots, e_n)$ of R such that $\mathfrak{i}_1 = e_1 R, \ldots, \mathfrak{i}_n = e_n R$.

(ii) $eR = \mathfrak{i}_1 \oplus \mathfrak{i}_2$, for some right ideals $\mathfrak{i}_1, \mathfrak{i}_2 \subseteq R$, if and only if there exist orthogonal idempotents $e_1, e_2 \in R$ with $e = e_1 + e_2$ such that $\mathfrak{i}_1 = e_1 R$ and $\mathfrak{i}_2 = e_2 R$.

(iii) If eR is contained in fR, then $fR = eR \oplus f_0 R$ for some idempotent $f_0 \in R$.

(iv) If $ef = fe = 0_R$ and $z \in eRf \cup fRe$, then $e + z$ is an idempotent of R.

Proof. For (the left analogue of) item (i), see Proposition 7.2 in [3], and for item (ii), see Proposition (21.8) in [42] (the case $e = 0_R$ is trivial).

(iii): Since $eR \subseteq fR$, we have from [46, Lemma 4(i)] that there exists an idempotent $e_0 \in R$ such that $e_0 R = eR$ and $e_0 = e_0 f = f e_0$ (in fact, we may take $e_0 := ef$). Set $f_0 := f - e_0 = f(1_R - e_0)$. Then

$$f_0^2 = f^2 - f e_0 - e_0 f + e_0^2 = f - e_0 = f_0,$$

namely, f_0 is idempotent. It thus remains to see that $fR = e_0 R \oplus f_0 R$. For, note first that $e_0 R$ and $f_0 R$ intersect trivially. Indeed, if $x = e_0 r = f_0 s$ for some $r, s \in R$, then

$$x = e_0^2 r = e_0 x = e_0 f_0 s = e_0 (f - e_0) s = (e_0 f - e_0^2) s = (e_0 - e_0) s = 0_R.$$

Moreover, $e_0 R + f_0 R = f e_0 R \oplus f(1_R - e_0)R \subseteq fR$, and on the other hand, $fR \subseteq e_0 R + f_0 R$ because $f = e_0 + f_0$. So putting it all together, we find $fR = e_0 R \oplus f_0 R$ (as wished).

(iv): If $z \in eRf \cup fRe$ and $ef = fe = 0_R$, then it is clear that $z^2 = 0_R$ and $ez + ze = z$, which shows in turn that $(e + z)^2 = e^2 + ez + ze + z^2 = e + z$ (i.e., z is an idempotent of R). ∎

3. PREORDERS AND ABSTRACT FACTORIZATION THEOREMS

To start with, we recall that a preorder on a set X is a reflexive and transitive relation $\preceq$ on X. In particular, we write $x \prec y$ to mean that $x \preceq y$ and $y \not\preceq x$, and we recall from the introduction that $\preceq$ is an artinian preorder if there is no sequence $(x_k)_{k \geq 0}$ of elements of X with $x_{k+1} \prec x_k$ for each $k \in \mathbb{N}$. Here, we are mostly interested in the interplay between preorders and monoids. This leads to the following.

Definition 3.1. Let $\preceq$ be a preorder on a monoid H. An element $u \in H$ is a $\preceq$-unit (of H) if $u \preceq 1_H \preceq u$, and is a $\preceq$-non-unit if it is not a $\preceq$-unit. Accordingly, we say that a $\preceq$-non-unit $a \in H$ is

- a $\preceq$-irreducible of degree s or degree-s $\preceq$-irreducible (of H), for some integer $s \geq 2$, if for every $k \in [\![2, s]\!]$ there exist no $\preceq$-non-units $b_1, \ldots, b_k$ with $b_1 \prec a, \ldots, b_k \prec a$ such that $a = b_1 \cdots b_k$;
- a $\preceq$-quark (of H) if there does not exist any $\preceq$-non-unit b with $b \prec a$.

In particular, we will simply refer to a $\preceq$-irreducible of degree 2 as a $\preceq$-irreducible (of H); and say that H is $\preceq$-factorable if every $\preceq$-non-unit of H factors as a product of $\preceq$-irreducibles.

It is straightforward that, if $\preceq$ is a preorder on a monoid H, then a $\preceq$-irreducible of degree $s \geq 2$ is also a $\preceq$-irreducible of degree k for every $k \in [\![2, s]\!]$, and each $\preceq$-quark is a degree-s $\preceq$-irreducible for every $s \geq 2$ (for a partial converse to this latter statement, see Theorem 3.5 below).

Definition 3.2. Given a preorder $\preceq$ on a monoid H and an element $x \in H$, we denote by $\operatorname{ht}^H_\preceq(x)$ the supremum of the set of all integers $n \geq 1$ for which there are $\preceq$-non-units $x_1, \ldots, x_n \in H$ with $x_1 = x$ and $x_{k+1} \prec x_k$ for each $k \in [\![1, n-1]\!]$, with $\sup \varnothing := 0$. We call $\operatorname{ht}^H_\preceq(x)$ the $\preceq$-height of x (relative to H); and we say $\preceq$ is a strongly artinian preorder (on H), or H is a strongly $\preceq$-artinian monoid, if the $\preceq$-height of every element is finite. We will refer to the map $H \to \mathbb{N} \cup \{\infty\}\colon y \mapsto \operatorname{ht}^H_\preceq(y)$ as the $\preceq$-height (function) of H, and write $\operatorname{ht}(x)$ in place of $\operatorname{ht}^H_\preceq(x)$ if no confusion can arise.

The following observations, although elementary, will prove useful in several occasions.

Remarks 3.3. (1) Let K be a submonoid of a monoid H and $\preceq_K$ be the restriction to K of a preorder $\preceq$ on H. If 1_H is the only $\preceq$-unit of H, then 1_H is also the only $\preceq_K$-unit of K (recall that $1_K = 1_H$). In particular, the $\preceq_K$-height of an element $a \in K$ is no larger than the $\preceq$-height of a (relative to H), which shows in turn that, if H is strongly $\preceq$-artinian, then K is strongly $\preceq_K$-artinian.

(2) Let $\preceq$ be a preorder on a monoid H, and for a fixed $a \in H$ denote by $\Lambda(a)$ the set of all integers $n \geq 1$ for which there exist $\preceq$-non-units $x_1, \ldots, x_n \in H$ with $x_1 = a$ ed $x_{k+1} \prec x_k$ for each $k \in [\![1, n-1]\!]$ (so, if a is a $\preceq$-unit, then $\Lambda(a) = \varnothing$). It is immediate that $\Lambda(a) = [\![1, \operatorname{ht}(a)]\!]$. For, note that, if an integer n is in $[\![1, \operatorname{ht}(a)]\!]$ but not in $\Lambda(a)$, then $\Lambda(a) \cap [\![n, \infty]\!] = \varnothing$ and hence $\operatorname{ht}(a) = \sup \Lambda(a) \leq n - 1 < \operatorname{ht}(a)$ (which is absurd).

With the exception of $\preceq$-irreducibles of degree $s \geq 3$, the notions introduced in Definitions 3.1 and 3.2 were first considered in [53, Sect. 3]. Their significance is linked to the wide range of applications of the next results, which generalize Theorem 3.10 and Proposition 3.14 of [53] from the case where $s = 2$.

Theorem 3.4. Let $\preceq$ be an artinian preorder on a monoid H and s be an integer ≥ 2. Then every $\preceq$-non-unit $x \in H$ factors as a non-empty, finite product of $s^{\operatorname{ht}(x)-1}$ or fewer $\preceq$-irreducibles of degree s.

Proof. We divide the proof into two parts: In PART 1, we prove that every $\preceq$-non-unit $x \in H$ factors as a finite product of $\preceq$-irreducibles of degree s; and in PART 2, we establish that the minimum length of such a factorization is no larger than $s^{\operatorname{ht}(x)-1}$.

PART 1: Let Ω be the set of $\preceq$-non-units of H that do not factor as a finite product of $\preceq$-irreducibles of degree s, and suppose for the sake of a contradiction that Ω is non-empty. Since $\preceq$ is an artinian preorder, it is then a routine matter to show that Ω has a $\preceq$-minimal element $\bar{x}$, meaning that there exists no $x \in \Omega$ with $x \prec \bar{x}$ (see [53, Remark 3.9(3)] for further details). Clearly, $\bar{x}$ is a $\preceq$-non-unit but not a $\preceq$-irreducible of degree s. So, there are $k \in [\![2, s]\!]$ and $\preceq$-non-units $x_1, \ldots, x_k \in H$ such that $x_i \prec \bar{x}$ for every $i \in [\![1, k]\!]$ and $\bar{x} = x_1 \cdots x_k$. On the other hand, the $\preceq$-minimality of $\bar{x}$ as an element of Ω yields $x_1, \ldots, x_k \notin \Omega$. Thus, each of $x_1, \ldots, x_k$ factors as a product of $\preceq$-irreducibles of degree s, which however implies that the same holds for the product $x_1 \cdots x_k$ and hence contradicts that $x_1 \cdots x_k = \bar{x} \in \Omega$.

PART 2: Pick a $\preceq$-non-unit $x \in H$, denote by $\ell(x)$ the minimum length of a factorization of x into $\preceq$-irreducibles of degree s (observe that, by the first part of the proof, $\ell(x)$ is a well-defined positive integer), and set $N := \operatorname{ht}(x)$. We need to prove that $\ell(x) \leq s^{N-1}$. If $N = \infty$, the conclusion is immediate. So, suppose $1 \leq N < \infty$ (note that the $\preceq$-height of an element $u \in H$ is zero if and only if u is a $\preceq$-unit). We proceed by induction on N.

If $N = 1$, then x is a $\preceq$-quark and hence $\ell(x) = 1 = s^{N-1}$, since a $\preceq$-quark is a degree-t $\preceq$-irreducible for every integer $t \geq 2$. Thus, let $N \geq 2$ and assume inductively that every $\preceq$-non-unit of $\preceq$-height $h \leq N-1$

factors as a product of s^{h-1} or fewer $\preceq$-irreducibles of degree s. If x is a $\preceq$-irreducible of degree s, then $\ell(x) = 1 \le s^{N-1}$ and we are done. Otherwise, there are $k \in [\![2, s]\!]$ and $\preceq$-non-units $x_1, \ldots, x_k \in H$ such that $x = x_1 \cdots x_k$ and $x_i \prec x$ for every $i \in [\![2, k]\!]$. In particular, we have $h_i := \mathrm{ht}(x_i) \le N - 1$ for each $i \in [\![1, k]\!]$, which, by the inductive hypothesis, tells us that x_i is a product of ℓ_i degree-s $\preceq$-irreducibles for some $\ell_i \in [\![1, s^{h_i-1}]\!]$. Consequently, $\ell(x) \le \ell_1 + \cdots + \ell_k \le ks^{N-2} \le s^{N-1}$, as wished. ∎

The upper bound derived in Theorem 3.4 for the minimum length of a factorization of a fixed element of H into a product of $\preceq$-irreducibles of degree s is rather weak and can be greatly improved under further assumptions on the preorder $\preceq$, as in the next result.

Theorem 3.5. Let $\preceq$ be a strongly artinian preorder on a monoid H and s be an integer ≥ 2 such that, for every $x \in H$ that is neither a $\preceq$-unit nor a $\preceq$-quark, there exists $k \in [\![2, s]\!]$ such that $x = y_1 \cdots y_k$ for some $\preceq$-non-units $y_1, \ldots, y_k \in H$, with the additional property that $y_i \preceq x$ for each $i \in [\![1, k]\!]$ and $\mathrm{ht}(y_1) + \cdots + \mathrm{ht}(y_k) \le \mathrm{ht}(x) + k - 2$. Then the following hold:

(i) The preorder $\preceq$ is artinian, the $\preceq$-non-units of H factor through the $\preceq$-irreducibles of degree s, and every $\preceq$-irreducible of degree s is a $\preceq$-quark.

(ii) Each $\preceq$-non-unit $x \in H$ is a product of $(s-1)\,\mathrm{ht}(x) - (s-2)$ or fewer $\preceq$-quarks.

Proof. (i): By the assumption that H is strongly $\preceq$-artinian, it is clear that the function $\lambda\colon H \to \mathbb{N}\colon x \mapsto \mathrm{ht}(x)$ is well defined. On the other hand, it is obvious from the definition of the $\preceq$-height that $\lambda(x) < \lambda(y)$ whenever $x \prec y$. Consequently, we see that $\preceq$ is an artinian preorder, or else there would exist a sequence $(N_k)_{k \ge 0}$ of non-negative integers such that $N_{k+1} < N_k$ for each $k \in \mathbb{N}$ (a contradiction). We thus get from Theorem 3.4 that every $\preceq$-non-unit of H factors as a (finite) product of $\preceq$-irreducibles of degree s, and it only remains to show that each $\preceq$-irreducible of degree s is in fact a $\preceq$-quark.

Let $x \in H$ be neither a $\preceq$-unit nor a $\preceq$-quark; we need to check that x is not a $\preceq$-irreducible of degree s. By hypothesis, there exist $k \in [\![2, s]\!]$ and $\preceq$-non-units $y_1, \ldots, y_k \in H$ with $y_1 \preceq x, \ldots, y_k \preceq x$ such that $x = y_1 \cdots y_k$ and $\mathrm{ht}(y_1) + \cdots + \mathrm{ht}(y_k) \le \mathrm{ht}(x) + k - 2$. If $x \preceq y_j$ for some j, then $\mathrm{ht}(y_j) = \mathrm{ht}(x)$ and

$$k - 1 \le (\mathrm{ht}(y_1) + \cdots + \mathrm{ht}(y_k)) - \mathrm{ht}(y_j) \le \mathrm{ht}(x) - \mathrm{ht}(y_j) + k - 2 = k - 2,$$

which is of course impossible. (Recall that the $\preceq$-height of an element $u \in H$ is zero if and only if u is a $\preceq$-unit.) Therefore, $y_1 \prec x, \ldots, y_k \prec x$ and x is not a $\preceq$-irreducible of degree s (as wished).

(ii): Fix a $\preceq$-non-unit $x \in H$, set $N := \mathrm{ht}(x)$, and denote by $q(y)$ the minimum length of a factorization of a $\preceq$-non-unit $y \in H$ into $\preceq$-quarks (note that, by item (i), $q(x)$ is a well-defined positive integer and $1 \le N < \infty$). We need to prove $q(x) \le (s-1)N - (s-2)$, and we proceed by induction on N.

If $N = 1$, then x is a $\preceq$-quark and the conclusion is trivial. So, assume $N \ge 2$ and suppose inductively that $q(y) \le (s-1)\,\mathrm{ht}(y) - (s-2)$ for every $\preceq$-non-unit $y \in H$ of $\preceq$-height $\le N - 1$. Since x is neither a $\preceq$-unit nor a $\preceq$-quark, there exists $k \in [\![1, s]\!]$ such that $x = y_1 \cdots y_k$ for some $\preceq$-non-units $y_1, \ldots, y_k \in H$ with $y_1 \preceq x, \ldots, y_k \preceq x$ and $\mathrm{ht}(y_1) + \cdots + \mathrm{ht}(y_k) \le N + k - 2$.

As in the proof of item (i), it follows that $1 \le \mathrm{ht}(y_i) \le N - 1$ for each $i \in [\![1, k]\!]$, which, by the inductive hypothesis, implies that y_i factors as a product of $(s-1)\,\mathrm{ht}(y_i) - (s-2)$ or fewer $\preceq$-quarks. Thus

$$q(x) \le \sum_{i=1}^{k} q(y_i) \le \sum_{i=1}^{k} \big((s-1)\,\mathrm{ht}(y_i) - (s-2)\big) = (s-1)\sum_{i=1}^{k} \mathrm{ht}(y_i) - k(s-2),$$

which, using that $k \le s$, yields $q(x) \le (s-1)(N + k - 2) - k(s-2) \le (s-1)N - (s-2)$, as wished. ∎

While the artinianity of a preorder $\preceq$ on a monoid H is sufficient for each $\preceq$-non-unit of H to factor as a product of $\preceq$-irreducibles (Theorem 3.4), the same condition is far from being necessary. For, let H be the multiplicative monoid of the non-zero elements of the integral domain constructed by A. Grams in [29, Sect. 1] and $\preceq$ be the divisibility preorder on H (see the comments at the end of [53, Sect. 3] for further details). This raises the question of whether the previous results can be further generalized to the point of proving a (non-trivial) characterization of when the monoid H is $\preceq$-factorable. Moreover, it points out how the characterization of $\preceq$-irreducibles (of a fixed degree) and $\preceq$-quarks is an interesting problem in its own right, independent of the artinianity of $\preceq$.

3.1. **The r.Fix-preorder.** Theorems 3.4 and 3.5 lie at the heart of our approach to the study of idempotent factorizations. Another essential ingredient in this direction is provided by the following.

Definition 3.6. Given a monoid H and an element $a \in H$, we set $\mathrm{r.Fix}_H(a) := \{x \in H : ax = x\}$ and let the r.Fix-preorder on H be the binary relation $\preceq$ on H defined by $b \preceq c$ if and only if $\mathrm{r.Fix}_H(c) \subseteq \mathrm{r.Fix}_H(b)$.

In the language of [53, Example 3.3], the r.Fix-preorder on a monoid H is the dual of the pullback of the inclusion order on the power set $\mathcal{P}(H)$ of H through the function $\phi \colon H \to \mathcal{P}(H) \colon a \mapsto \mathrm{r.Fix}_H(a)$; in particular, it is a preorder (as suggested by the name).

Definition 3.7. A submonoid K of a monoid H is $\mathrm{r.Fix}_H$-artinian (resp., strongly $\mathrm{r.Fix}_H$-artinian) if the restriction $\preceq_K$ to K of the r.Fix-preorder on H is artinian (resp., strongly artinian). Moreover, we refer to an element $u \in K$ as an $\mathrm{r.Fix}_H$-unit (resp., $\mathrm{r.Fix}_H$-non-unit) of K if u is a $\preceq_K$-unit (resp., a $\preceq_K$-non-unit). We set a degree-s $\mathrm{r.Fix}_H$-irreducible (resp., an $\mathrm{r.Fix}_H$-quark) of K to be a degree-s $\preceq_K$-irreducible (resp., a $\preceq_K$-quark) of K, and call the $\preceq_K$-height of K the $\mathrm{r.Fix}_R$-height of K (see Definition 3.2).

Note that, given a monoid H and a submonoid K of H, the restriction to K of the r.Fix-preorder on H is not, in general, the r.Fix-preorder on K (the idea is that, in many situations, we can exploit the preorder defined on the larger monoid to probe the smaller one).

Corollary 3.8. Let H be a monoid and K be an $\mathrm{r.Fix}_H$-artinian submonoid of H. Then each $a \in K$ factors as a finite product of degree-s $\mathrm{r.Fix}_H$-irreducibles of K, for every integer $s \geq 2$.

Proof. It is clear that $\mathrm{r.Fix}_H(x) = H$, for some $x \in K$, if and only if $x = 1_K = 1_H$. Consequently, 1_H is the only $\mathrm{r.Fix}_H$-unit of K and the conclusion follows at once from Theorem 3.4 (applied to K), upon considering that 1_H is an empty (and hence finite) product of elements from any subset of H. ∎

The next proposition puts together some basic properties of the r.Fix-preorder that, in addition to being of independent interest, will be useful to deal with endomorphism rings in Sect. 5.2.

Proposition 3.9. Let H be a submonoid of the multiplicative monoid of a ring R such that $R^\times H R^\times = H$ and let $\preceq$ be the $\mathrm{r.Fix}_R$-preorder on H. Given $a \in R$ and $u \in R^\times$, the following hold:

   (i) $\mathrm{r.Ann}_R(u^{-1}au) = u^{-1} \mathrm{r.Ann}_R(a) u$ and hence $\mathrm{r.Fix}_R(u^{-1}au) = u^{-1} \mathrm{r.Fix}_R(a) u$.
   (ii) a is a left zero divisor of R if and only if so is $u^{-1}au$.
   (iii) a is an idempotent (resp., a coprimitive idempotent) of R if and only if so is $u^{-1}au$.
   (iv) a is an $\mathrm{r.Fix}_R$-irreducible of H of some degree $s \geq 2$ if and only so is $u^{-1}au$.

Proof. (i): Since $a = u(u^{-1}au)u^{-1}$, it is enough to check that $u^{-1}\,\mathrm{r.Ann}(a)\,u \subseteq \mathrm{r.Ann}(u^{-1}au)$, which is rather easy. In fact, we have $u^{-1}au(u^{-1}xu) = ax = 0_R$ and hence $u^{-1}xu \in \mathrm{r.Ann}(u^{-1}au)$ for all $x \in \mathrm{r.Ann}(a)$. Thus $u^{-1}\,\mathrm{r.Ann}(a)\,u \subseteq \mathrm{r.Ann}(u^{-1}au)$.

As for the second part, it is enough to note that $1_R - u^{-1}au = u^{-1}(1_R - a)u$, so we derive from the first part that $\mathrm{r.Fix}_R(u^{-1}au) = \mathrm{r.Ann}(1_R - u^{-1}au) = u^{-1}\,\mathrm{r.Ann}(1_R - a)\,u = u^{-1}\,\mathrm{r.Fix}_R(a)\,u$.

(ii): This is clear from item (i), since $b \in R$ is a left zero divisor if and only if $\mathrm{r.Ann}(b) \neq \{0_R\}$.

(iii): The first part is obvious, since $a^2 = a$ implies $(u^{-1}au)^2 = u^{-1}a^2u = u^{-1}au$. For the second part, it suffices to check that, if a is a coprimitive idempotent, then so also is $u^{-1}au$ (cf. the proof of item (i)).

For, suppose $1_R - u^{-1}au = e + f$ for some orthogonal idempotents $e, f \in R$. Then $1_R - a = ueu^{-1} + ufu^{-1}$; and since $ueu^{-1}ufu^{-1} = uefu^{-1} = 0_R$ and similarly $ufu^{-1}ueu^{-1} = 0_R$, we gather from the first part that $1_R - a$ is the sum of two orthogonal idempotents. Consequently, we see that a is a coprimitive idempotent of R only if so is $u^{-1}au$.

(iv): Suppose $u^{-1}au$ is not a degree-s $\mathrm{r.Fix}_R$-irreducible of H for a certain $s \geq 2$; as in the proof of item (i), it will be enough to show that neither is a. Since the only $\mathrm{r.Fix}_R$-unit of H is the identity 1_R and, on the other hand, $u^{-1}bu = 1_R$, for some $b \in R$, if and only if $a = 1_R$, we may assume that $u^{-1}au$ is an $\mathrm{r.Fix}_R$-non-unit of H. There then exist $k \in [\![2, s]\!]$ and $\mathrm{r.Fix}_R$-non-units $a_1, \ldots, a_k$ of H such that $u^{-1}au = a_1 \cdots a_k$ and $\mathrm{r.Fix}_R(u^{-1}au) \subsetneq \mathrm{r.Fix}_R(a_i)$ for each $i \in [\![1, k]\!]$. It is therefore immediate that $a = ua_1u^{-1} \cdots ua_ku^{-1}$. In addition, ua_iu^{-1} is a non-identity element of H for each $i \in [\![1, k]\!]$, by the assumption that $R^\times H R^\times = H$; and it is straightforward from (i) that $\mathrm{r.Fix}_R(a) \subsetneq \mathrm{r.Fix}_R(ua_iu^{-1}) \subsetneq R$ (note that, if $X \subsetneq Y \subsetneq R$, then $vXw \subsetneq vYw \subsetneq R$ for all $v, w \in R^\times$). As a result, a is not an $\mathrm{r.Fix}_R$-irreducible of H, because it factors into a product of s or fewer $\mathrm{r.Fix}_R$-non-units of H each of which is (strictly) smaller than a with respect to the $\mathrm{r.Fix}_R$-preorder on H. ∎

In the end, Corollary 3.8 shows that, for an $\mathrm{r.Fix}_H$-artinian submonoid K of a monoid H to be idempotent-generated, it is sufficient to prove that every $\mathrm{r.Fix}_H$-irreducible of some degree s of K is idempotent. We will see this principle at work in Sect. 5. Necessary conditions, on the other hand, are discussed in the following remark. It turns out that the monoid of zero divisors of a ring R can only be idempotent-generated if the multiplicative monoid of R is directly irreducible, that is, does not factor into the direct product of two non-trivial monoids.

Remark 3.10. Let $H := \prod_{i \in I} H_i$ be the direct product of a family $(H_i)_{i \in I}$ of monoids indexed by the non-empty set I (we refer to [36, pp. 5–6] for notation and terminology), and let $S := \bigcup_{i \in I} H_i$. Given $j \in I$ and $a \in H_j$, we will write $\delta_j^{(a)}$ for the function $I \to S$ that maps j to a and an index $i \neq j$ to 1_{H_i}.

It is fairly clear that an idempotent of H is a function $e \colon I \to S$ such that $e(i)$ is an idempotent of H_i for all $i \in I$. In addition, every $f \in \prod_{i \in I} H_i^\#$ is also in $H^\#$: This is obvious if $f(i) = 1_{H_i}$ for each $i \in I$. Otherwise, $f(j)$ is a singular element of H_j for some $j \in I$, i.e., there exist $x, y, z, w \in H_j$ with $x \neq y$ and $z \neq w$ such that $xf(j) = yf(j)$ and $f(j)z = f(j)w$. So, we find $\delta_j^{(x)}f = \delta_j^{(y)}f$ and $f\delta_j^{(z)} = f\delta_j^{(w)}$, which shows that f is in $H^\#$, because $\delta_j^{(x)} \neq \delta_j^{(y)}$ and $\delta_j^{(z)} \neq \delta_j^{(w)}$.

Since an element that factors into a product of proper idempotents is necessarily singular (as already mentioned in Sect. 2.2), it thus follows that

$$\mathcal{E}(H) \subseteq \prod_{i \in I} \mathcal{E}(H_i) \subseteq \prod_{i \in I} H_i^\# \subseteq H^\#, \tag{7}$$

with $\prod_{i \in I} \mathcal{E}(H_i) = \prod_{i \in I} H_i^{\#}$ if and only if $\mathcal{E}(H_i) = H_i^{\#}$ for each $i \in I$. We will look for conditions under which each of the left-most and right-most inclusions in Eq. (7) holds as an equality (trivially, this is the case if H is the multiplicative monoid of a domain R, for then the only elements of $H^{\#}$ are 0_R and 1_R).

To start with, we recall from [34, Definitions 1.1] that the idempotent depth of an idempotent-generated monoid M, herein denoted by $\Delta(M)$, is the infimum of the set of integers $k \geq 1$ such that every element of M factors into a product of k or fewer idempotents, with the understanding that $\inf \varnothing := \infty$ (note that, since the identity 1_M is an idempotent, an element of M factors as a product of m idempotents if and only if it factors as a product of n idempotents for every $n \geq m$).

Claim A. $\mathcal{E}(H) = \prod_{i \in I} \mathcal{E}(H_i)$ if and only if $\Delta(\mathcal{E}(H_i)) = \infty$ for finitely many $i \in I$.

Proof. To ease notation, set $\Delta_i := \Delta(\mathcal{E}(H_i))$ for each $i \in I$ and then $J := \{j \in I : \Delta_j = \infty\}$. We need to check that $\mathcal{E}(H) = \prod_{i \in I} \mathcal{E}(H_i)$ if and only if J is a finite set.

Suppose first that J is finite and let f be a function in $\prod_{i \in I} \mathcal{E}(H_i)$. Then $\Delta := 1 + \sup_{i \in I \smallsetminus J} \Delta_i$ is a positive integer (with $\sup \varnothing := 0$) and, for each $j \in J$, $f(j)$ factors into a product of k_j idempotents of H_j for some $k_j \in \mathbb{N}^+$. Hence, there exists an integer $k \geq 1$ such that, for every $i \in I$, $f(i)$ is a product of k idempotents $e_i^{(1)} \ldots, e_i^{(k)} \in H_i$ (some of which are possibly equal to the identity 1_{H_i}); for instance, we may take k to be the maximum of the (non-empty, finite) set $\{\Delta\} \cup \{k_j : j \in J\} \subseteq \mathbb{N}^+$. As a result, f is a product of k idempotents $e_1, \ldots, e_k \in H$, where, for each $j \in [\![1, k]\!]$, e_j is the function $I \to S : i \mapsto e_i^{(j)}$. Therefore, we gather from Eq. (7) that $\prod_{i \in I} \mathcal{E}(H_i) = \mathcal{E}(H)$.

Assume, on the other hand, that J is infinite. There is then a sequence $i_1, i_2, \ldots$ of pairwise distinct indices in I with $\Delta_{i_j} \geq j$ for all $j \in \mathbb{N}^+$. Consequently, we have an S-valued sequence $x_1, x_2, \ldots$ such that, for each $j \in \mathbb{N}^+$, x_j is an element of $\mathcal{E}(H_{i_j})$ whose shortest idempotent-factorization in H_{i_j} has at least j factors. Accordingly, we set $I' := \{i_1, i_2, \ldots\}$ and define g to be the function

$$I \to S : i \mapsto \begin{cases} x_i & \text{if } i \in I' \\ 1_{H_i} & \text{if } i \in I \smallsetminus I' \end{cases}.$$

By construction, $g \in \prod_{i \in I} \mathcal{E}(H_i)$. However, $g \notin \mathcal{E}(H)$ and hence $\mathcal{E}(H) \neq \prod_{i \in I} \mathcal{E}(H_i)$. To see this, suppose to the contrary that g factors as a product of n idempotents of H for some $n \in \mathbb{N}^+$. Then $g(i)$ is a product of n idempotents of H_i for every $i \in I$, which is impossible because we have $g(i_{n+1}) = x_{n+1}$ and, again by construction, x_{n+1} does not factor into less than $n + 1$ idempotents of $H_{i_{n+1}}$. $\blacksquare$

We note for later reference that the minimum length $\ell(f)$ of an idempotent factorization of a function $f \in \mathcal{E}(H)$ is bounded above by $\max_{i \in I} \ell_i(f)$, where we denote by $\ell_i(f)$ the minimum length of an idempotent factorization of $f(i)$ in $\mathcal{E}(H_i)$. In fact, $f = e_1 \cdots e_n$ for some idempotents $e_1, \ldots, e_n \in H$, with the result that $\ell_i(f) \leq n$ for every $i \in I$. Therefore, the (non-empty) set $\{\ell_i(f) : i \in I\} \subseteq \mathbb{N}$ has a maximum, say m; and similarly as in the proof of Claim A, we see that, for each $i \in I$, $f(i)$ factors as a product of m idempotents of H_i (some of which may be equal to 1_{H_i}). This, in turn, shows that f can be written as a product of m idempotents of H. That is to say, $\ell(f) \leq m$ (as wished).

Claim B. If $H_j^{\#} \subsetneq H_j$ and $H_k^{\#} \neq \{1_{H_k}\}$ for some $j, k \in I$ with $j \neq k$, then $\prod_{i \in I} H_i^{\#} \subsetneq H^{\#}$.

Proof. Assume $H_j^{\#} \subsetneq H_j$ and $H_k^{\#} \neq \{1_{H_k}\}$ for some $j, k \in I$ with $j \neq k$, so that we can pick an element $a \in H_j \smallsetminus H_j^{\#}$ and a singular element $b \in H_k^{\#}$. There then exist $x, y, z, w \in H_k$ with $x \neq y$ and $z \neq w$ such that $xb = yb$ and $bz = bw$. Therefore, the function $f := \delta_j^{(a)} \delta_k^{(b)}$ is in $H^{\#}$, because $\delta_k^{(x)} f = \delta_k^{(y)} f$ and

$f\delta_k^{(z)} = f\delta_k^{(w)}$ with $\delta_k^{(x)} \neq \delta_k^{(y)}$ and $\delta_k^{(z)} \neq \delta_k^{(w)}$. Yet, f is not in $\prod_{i \in I} H_i^\#$, by the fact that $f(j) = a \notin H_j^\#$. Consequently, we gather from Eq. (7) that $H^\#$ is properly contained in $\prod_{i \in I} H_i^\#$. ∎

If I is a finite set or $\Delta(\mathcal{E}(H_i)) < \infty$ for each $i \in I$, we obtain from Eq. (7) and Claim A that $\mathcal{E}(H) = \prod_{i \in H_i} H_i^\#$ if and only if $H_i^\#$ is idempotent-generated for every $i \in I$. If, on the other hand, each of the monoids H_i is the multiplicative monoid of a (non-zero) ring R_i, then $H_i^\#$ is a non-trivial monoid for all $i \in I$ and hence we derive from Claim B that $H^\# = \prod_{i \in H_i} H_i^\#$ only if, for all $i \in I$, any element of R_i other than the identity is a zero divisor. Rings with this property are known as "$\mathcal{O}$-rings" and were first considered by P.M. Cohn in [14] (though only in the commutative case). While it was proved by H.G. Moore et al. [48] that any right (or left) artinian $\mathcal{O}$-ring is commutative, it appears to be open [33] whether non-commutative $\mathcal{O}$-rings do actually exist (see also https://mathoverflow.net/questions/395787/).

4. IDEMPOTENT FACTORIZATIONS IN ARBITRARY RINGS

We are going to apply the abstract results of Sect. 3 to the study of idempotent factorizations in the multiplicative monoid of certain rings (it may help to review Sects. 2.2 and 2.3 before reading further).

Throughout this section (unless noted otherwise), we let R be an arbitrary ring.

Our goal is to prove a series of results that will be used in Sect. 5 to characterize or partially characterize the r.Fix$_R$-irreducibles and r.Fix$_R$-quarks of the monoid of zero divisors of R when R is a right Rickart ring (Proposition 4.4, Theorem 5.2, and Remark 5.4), a von Neumann regular ring satisfying the comparability axiom (Theorem 5.7), or the endomorphism ring of a free module of finite rank over a commutative discrete valuation domain (Proposition 5.18). We start with a lemma that is perhaps of independent interest.

Lemma 4.1. If r.Ann(a) is a direct summand of R_R for some $a \in R$, then there is an idempotent $e \in R$ such that r.Ann$(a) = eR$ and r.Fix$(a) \subseteq (1_R - e)R$.

Proof. Let r.Ann(a) be a direct summand of R_R. By Proposition 2.2(i), there exists an idempotent $f \in R$ such that r.Ann$(a) = fR$. Set $e := f - fa$. Since $fe = f^2(1_R - a) = e$ and $ef = f^2 - faf = f$ (note that $af = 0_R$), we have $eR = fR$. Moreover, we check that $e^2 = fe - faf(1_R - a) = e$, i.e., e is an idempotent of R. Lastly, if $x \in R$ and $(1_R - a)x = 0_R$, then $(1_R - e)x = x - f(1_R - a)x = x - 0_R = x$, which shows that r.Fix$(a)$ is contained in $(1_R - e)R$. ∎

We continue with a couple of technical lemmas whose role will be clarified by Proposition 4.4.

Lemma 4.2. Assume there exist $a \in R$ and right ideals $\mathfrak{i}, \mathfrak{j} \subseteq R$ such that $R = \mathfrak{i} \oplus \mathfrak{j}$ and $\mathfrak{i} \subseteq$ r.Ann(a), and set $\mathfrak{p} :=$ r.Ann$(a^2) \cap \mathfrak{j}$ and $\mathfrak{q} := \{x + ax \colon x \in \mathfrak{p}\}$. The following hold:
   (i) r.Ann$(a) \cap \mathfrak{j} = \{0_R\}$ if and only if $\mathfrak{i} =$ r.Ann(a).
   (ii) $\mathfrak{p}$ and $\mathfrak{q}$ are right ideals of R with $\mathfrak{q} \cap$ r.Fix$(a) = \{0_R\}$.
   (iii) $\mathfrak{p} = \{0_R\}$ if and only if $\mathfrak{q} = \{0_R\}$, if and only if $\mathfrak{i} =$ r.Ann$(a) =$ r.Ann(a^2).

Proof. (i): Since $\mathfrak{i}$ and $\mathfrak{j}$ are ideals of R with $\mathfrak{i} \oplus \mathfrak{j} = R$, it is clear that $\mathfrak{i} =$ r.Ann(a) implies r.Ann$(a) \cap \mathfrak{j} = \{0_R\}$. As for the converse, assume $\mathfrak{i} \subsetneq$ r.Ann(a) and, accordingly, let x be an element in r.Ann(a) but not in $\mathfrak{i}$ (recall that, by hypothesis, r.Ann(a) contains $\mathfrak{i}$). Then $x = y + z$ for some $y \in \mathfrak{i}$ and $z \in \mathfrak{j} \smallsetminus \{0_R\}$, with the result that $0_R = ax = ay + az = az$. This proves $0_R \neq z \in$ r.Ann$(a) \cap \mathfrak{j}$, and we are done.

(ii): The intersection of two right ideals is still a right ideal; therefore, $\mathfrak{p}$ is a right ideal, because so are r.Ann(a^2) and $\mathfrak{j}$. On the other hand, if $x, y \in \mathfrak{p}$ and $b, c \in R$, then

$$(x + ax)b + (y + ay)c = (xb + yc) + a(xb + yc) \in \mathfrak{q},$$

by the fact that $\mathfrak{p}$ is a right ideal and hence $xb + yc \in \mathfrak{p}$. So, $\mathfrak{q}$ is a right ideal too.

To complete the argument, let $x \in \mathfrak{q} \cap$ r.Fix(a). Then $ax = x$ and $x = y + ay$ for some $y \in \mathfrak{p} \subseteq$ r.Ann(a^2), so that

$$y + ay = a(y + ay) = ay + a^2y = ay + 0_R = ay.$$

It follows that $y = 0_R$ and hence $x = y + ay = 0_R$. As a result, $\mathfrak{q}$ and r.Fix(a) intersect trivially.

(iii): Suppose first that r.Ann$(a^2) \smallsetminus$ r.Ann(a) is a non-empty set and let x be one of its elements. Since $R = \mathfrak{i} \oplus \mathfrak{j}$ and $\mathfrak{i} \subseteq$ r.Ann(a), we have $x = y + z$ for some $y \in$ r.Ann(a) and $z \in \mathfrak{j} \smallsetminus \{0_R\}$. Consequently,

$$0_R = a^2x = a^2(y + z) = a(ay + az) = a(0_R + az) = a^2z,$$

from which we gather that z is a non-zero element of $\mathfrak{p} = \mathfrak{j} \cap$ r.Ann(a^2).

Since r.Ann$(a) \subseteq$ r.Ann(a^2), we obtain from the above that, if $\mathfrak{p} = \{0_R\}$, then r.Ann$(a) =$ r.Ann(a^2) and hence $\{0_R\} = \mathfrak{p} = \mathfrak{j} \cap$ r.Ann(a). On the other hand, we have from item (i) that $\mathfrak{j} \cap$ r.Ann$(a) = \{0_R\}$ if and only if $\mathfrak{i} =$ r.Ann(a). So putting it all together, we conclude that if $\mathfrak{p} = \{0_R\}$ then $\mathfrak{i} =$ r.Ann(a) $=$ r.Ann(a^2); the converse is trivial, because $\mathfrak{i} \cap \mathfrak{j} = \{0_R\}$.

It remains to show that $\mathfrak{q} = \{0_R\}$ if and only if $\mathfrak{p} = \{0_R\}$. Since the "if" part is obvious, it suffices to check that if $\mathfrak{p} \neq \{0_R\}$ then $\mathfrak{q} \neq \{0_R\}$. For, assume $\mathfrak{p}$ contains an element $x \neq 0_R$. Then either $ax = 0_R$ and hence $x = x + ax \in \mathfrak{q} \smallsetminus \{0_R\}$; or $0_R \neq ax = ax + a^2x = a(x + ax)$ and hence $x + ax \in \mathfrak{q} \smallsetminus \{0_R\}$. ∎

Lemma 4.3. Let $e \in R$ be an idempotent in the right annihilator r.Ann(a) of an element $a \in R$, and set $\mathfrak{p} :=$ r.Ann$(a^2) \cap (1_R - e)R$ and $\mathfrak{q} := \{x + ax : x \in \mathfrak{p}\}$. The following hold:

(i) $a = bc$ and $c^2 = c$, where $b := e + (1_R - e)a$ and $c := 1_R - e + ea$.

(ii) r.Ann$(a) \cap (1_R - e)R \subseteq$ r.Ann(b) and $eR \oplus$ r.Fix$(a) \subseteq$ r.Fix(b).

(iii) $eR =$ r.Ann(c) and r.Fix$(a) \oplus ($r.Ann$(a) \cap (1_R - e)R) \subseteq$ r.Fix(c).

(iv) If $eR =$ r.Ann(a), then $\mathfrak{p} \subseteq$ r.Ann(b) and $\mathfrak{q} \oplus$ r.Fix$(a) \subseteq$ r.Fix(c).

Proof. (i): Since e is an idempotent element of R from the right annihilator of a, we have

$$e(1_R - e) = (1_R - e)e = ae = 0_R \quad \text{and} \quad a(1_R - e) = a - ae = a \in R(1_R - e). \tag{8}$$

Consequently, it is clear that $ea \in eR(1_R - e)$ and we get from Proposition 2.2(iv) (applied with $z = ea$) that c is an idempotent of R. Moreover, a simple calculation shows that

$$bc = e(1_R - e) + e^2a + (1_R - e)a(1_R - e) + (1_R - e)aea = 0_R + ea + (1_R - e)a + 0_R = a.$$

(ii) and (iii): It is easily checked that, for every $x \in$ r.Fix(a),

$$bx = ex + (1_R - e)ax = ex + (1_R - e)x = x \quad \text{and} \quad cx = (1_R - e)x + eax = x; \tag{9}$$

while it follows from Eq. (8) (and e being idempotent) that

$$be = e^2 + (1_R - e)ae = e + 0_R = e \quad \text{and} \quad ce = (1_R - e)e + eae = 0_R + 0_R = 0_R. \tag{10}$$

Likewise, if $x \in$ r.Ann$(a) \cap (1_R - e)R$, then $x = (1_R - e)y$ for some $y \in R$ and hence

$$bx = e(1_R - e)y + (1_R - e)ax = 0_R + 0_R = 0_R \quad \text{and} \quad cx = x - e(1_R - e)y + eax = x. \tag{11}$$

So, recalling that $e \in \mathrm{r.Ann}(a)$, we conclude from Eqs. (4) and (9)–(11) that

$$\mathrm{r.Ann}(a) \cap (1_R - e)R \subseteq \mathrm{r.Ann}(b) \quad \text{and} \quad eR \oplus \mathrm{r.Fix}(a) \subseteq \mathrm{r.Fix}(b)$$

and

$$eR \subseteq \mathrm{r.Ann}(c) \quad \text{and} \quad (\mathrm{r.Ann}(a) \cap (1_R - e)R) \oplus \mathrm{r.Fix}(a) \subseteq \mathrm{r.Fix}(c),$$

where, in particular, we have used that if $\mathfrak{h}$, $\mathfrak{i}$, and $\mathfrak{j}$ are ideals of R with $\mathfrak{i} \cup \mathfrak{j} \subseteq \mathfrak{h}$, then $\mathfrak{i} + \mathfrak{j} \subseteq \mathfrak{h}$.

It remains to see that $\mathrm{r.Ann}(c) \subseteq eR$. For, let $x \in \mathrm{r.Ann}(c)$. Since $R = eR \oplus (1_R - e)R$, we can write $x = y + z$, where $y \in eR$ and $z \in (1_R - e)R$. Then $y = eu$ and $z = (1_R - e)v$ for some $u, v \in R$, which implies, by Eq. (8), that $(1_R - e)y = ay = 0_R$ and hence $cy = 0_R$. As a result,

$$0_R = cx = cz = (1_R - e)z + eaz = (1_R - e)^2 v + eaz = (1_R - e)v + eaz = z + eaz.$$

This, however, is only possible if $z = eaz = 0_R$, because $eR \cap (1_R - e)R = \{0_R\}$ and, on the other hand, $z \in (1_R - e)R$ and $eaz \in eR$. Therefore, we obtain $x = y + z = y \in eR$ (as wished).

(iv): Suppose $eR = \mathrm{r.Ann}(a)$, and pick $x \in \mathfrak{p}$ and $y \in \mathfrak{q}$. By the very definition of $\mathfrak{p}$ and $\mathfrak{q}$ (and, more specifically, by the fact that $\mathfrak{p}$ is contained in $(1_R - e)R$), there then exist $\bar{x}, \bar{y} \in R$ with $x = (1_R - e)\bar{x}$ and $y = z + az$, where $z := (1_R - e)\bar{y} \in \mathfrak{p}$. Moreover, $a^2 x = a^2 z = 0_R$ (because $\mathfrak{p}$ is also contained in the right annihilator of a^2) and hence $ax, az \in \mathrm{r.Ann}(a) = eR$, implying that $ax = eu$ and $az = ev$ for some $u, v \in R$. Consequently, a simple calculation shows that

$$bx = ex + (1_R - e)ax = e(1_R - e)\bar{x} + (1_R - e)eu = 0_R + 0_R = 0_R;$$

and on the other hand, we have

$$(1_R - e)z = (1_R - e)^2 \bar{y} = (1_R - e)\bar{y} = z \quad \text{and} \quad eaz = e^2 v = ev = az,$$

which, in turn, yields

$$cy = (1_R - e)z + (1_R - e)az + eaz + ea^2 z = z + 0_R + az + 0_R = y.$$

We thus see that $\mathfrak{p} \subseteq \mathrm{r.Ann}(b)$ and $\mathfrak{q} \subseteq \mathrm{r.Fix}(c)$; and by Lemma 4.2(ii) and the observation already made on the last line of the proof of items (ii) and (iii), we conclude that $\mathfrak{q} \oplus \mathrm{r.Fix}(a) \subseteq \mathrm{r.Fix}(c)$. ∎

The lemmas we have hitherto proved make it already possible to put non-trivial constraints on the $\mathrm{r.Fix}_R$-irreducibles and $\mathrm{r.Fix}_R$-quarks of the monoid of left zero divisors of R. Sharper conditions will be obtained in Sect. 5 (in particular, see Theorem 5.2).

Proposition 4.4. Let H be the monoid of left zero divisors of a ring R and a be an element of R such that $\mathrm{r.Ann}(a)$ is a direct summand of R_R. The following hold:

(i) If a is an $\mathrm{r.Fix}_R$-irreducible of H, then $\mathrm{r.Ann}(a)$ is an indecomposable submodule of R_R with the additional property that $\mathrm{r.Ann}(a) = \mathrm{r.Ann}(a^2)$ and $\mathrm{r.Ann}(a) \cap aR = \{0_R\}$.

(ii) If a is an $\mathrm{r.Fix}_R$-quark of H, then a is a coprimitive idempotent of R.

Proof. (i): Let a be an $\mathrm{r.Fix}_R$-irreducible of H. Then a is a left zero divisor of R. Since $\mathrm{r.Ann}(a)$ is a direct summand of R_R (by hypothesis), we get from Proposition 2.2(i) that there are an idempotent $e \in R$ and a right ideal $\mathfrak{l} \subseteq R$ such that

$$R = \mathrm{r.Ann}(a) \oplus \mathfrak{l} \quad \text{and} \quad \mathrm{r.Ann}(a) = eR \neq \{0_R\}. \tag{12}$$

We divide the rest of the proof into two parts. First, we prove $\mathrm{r.Ann}(a)$ is an indecomposable submodule of R_R. Next, we check $\mathrm{r.Ann}(a) = \mathrm{r.Ann}(a^2)$. This will suffice to prove the statement, for it implies at once that $\mathrm{r.Ann}(a)$ and aR intersect trivially. (If $y \in \mathrm{r.Ann}(a) \cap aR$, then $ay = 0_R$ and $y = az$ for some $z \in R$, so that $0_R = ay = a^2 z$. It follows that $z \in \mathrm{r.Ann}(a^2) = \mathrm{r.Ann}(a)$ and hence $y = az = 0_R$.)

PART 1: Assume to the contrary that $\mathrm{r.Ann}(a)$ is not an indecomposable submodule of R_R. Since $\mathrm{r.Ann}(a) \neq \{0_R\}$, it follows that $\mathrm{r.Ann}(a) = \mathfrak{g} \oplus \mathfrak{h}$ for some non-zero right ideals $\mathfrak{g}, \mathfrak{h} \subseteq R$, which, by Eq. (12), yields $R = \mathfrak{g} \oplus \mathfrak{h} \oplus \mathfrak{l}$. In view of Lemma 4.2(i) (applied with $\mathfrak{i} = \mathfrak{g}$ and $\mathfrak{j} = \mathfrak{h} \oplus \mathfrak{l}$) and Proposition 2.2(i), there then exists an idempotent $f \in R$ such that

$$\{0_R\} \neq \mathfrak{g} = fR \subsetneq \mathrm{r.Ann}(a) \quad \text{and} \quad \mathfrak{n} := \mathrm{r.Ann}(a) \cap (1_R - f)R \neq \{0_R\}.$$

Consequently, we conclude from Lemma 4.3 that $a = bc$ for some $b, c \in R$ with

$$\{0_R\} \neq \mathfrak{n} \subseteq \mathrm{r.Ann}(b) \quad \text{and} \quad \{0_R\} \neq fR = \mathrm{r.Ann}(c)$$

and

$$\mathrm{r.Fix}(a) \subsetneq fR \oplus \mathrm{r.Fix}(a) \subseteq \mathrm{r.Fix}(b) \quad \text{and} \quad \mathrm{r.Fix}(a) \subsetneq \mathrm{r.Fix}(a) \oplus \mathfrak{n} \subseteq \mathrm{r.Fix}(c).$$

This however means that b and c are left zero divisors of R each of which is (strictly) smaller than a with respect to the $\mathrm{r.Fix}_R$-preorder on H. Therefore, a is not an $\mathrm{r.Fix}_R$-irreducible of H (which is absurd), because the only $\mathrm{r.Fix}_R$-unit of H is the identity 1_H (which is not a left zero divisor of R).

PART 2: Suppose for the sake of a contradiction that $\mathrm{r.Ann}(a) \neq \mathrm{r.Ann}(a^2)$ and hence $\mathrm{r.Ann}(a) \subsetneq \mathrm{r.Ann}(a^2)$. Then we get from items (ii) and (iii) of Lemma 4.2 (applied with $\mathfrak{i} = eR$ and $\mathfrak{j} = (1_R - e)R$) that

$$\mathfrak{p} := \mathrm{r.Ann}(a^2) \cap (1_R - e)R \quad \text{and} \quad \mathfrak{q} := \{x + ax \colon x \in \mathfrak{p}\}$$

are non-zero right ideals of R. So, we are guaranteed by Lemma 4.3 that $a = bc$ for some $b, c \in R$ with

$$\{0_R\} \neq \mathfrak{p} \subseteq \mathrm{r.Ann}(b) \quad \text{and} \quad \{0_R\} \neq \mathrm{r.Ann}(a) = \mathrm{r.Ann}(c)$$

and

$$\mathrm{r.Fix}(a) \subsetneq \mathrm{r.Ann}(a) \oplus \mathrm{r.Fix}(a) \subseteq \mathrm{r.Fix}(b) \quad \text{and} \quad \mathrm{r.Fix}(a) \subsetneq \mathfrak{q} \oplus \mathrm{r.Fix}(a) \subseteq \mathrm{r.Fix}(c).$$

But similarly as in PART 1, this entails that a is not an $\mathrm{r.Fix}_R$-irreducible of H (which is absurd).

(ii): Assume by way of contradiction that a is an $\mathrm{r.Fix}_R$-quark of H but not a coprimitive idempotent of R. Then a is an $\mathrm{r.Fix}_R$-irreducible of H (see the comments after Definition 3.1); and by item (i), this is only possible if $\mathrm{r.Ann}(a)$ is an indecomposable submodule of R_R. On the other hand, we have from Eq. (12) and Lemma 4.1 that $\mathrm{r.Ann}(a) = e_0 R \neq \{0_R\}$ and $\mathrm{r.Fix}(a) \subsetneq (1_R - e_0)R$ for a certain idempotent $e_0 \in R$. In particular, $\mathrm{r.Fix}(a)$ is not equal to $(1_R - e_0)R$, or else we would gather from the above and Proposition 2.1 that a is a coprimitive idempotent (which is absurd). So, e_0 is a non-zero element of R and we conclude from Eq. (6) that $1_R - e_0$ is a zero divisor of R with $\mathrm{r.Fix}(a) \subsetneq (1_R - e_0)R = \mathrm{r.Fix}(1_R - e_0) \subsetneq R$, contradicting that a is an $\mathrm{r.Fix}_R$-quark of H. ∎

There is more that we can gain from the last proof, but we will come back to this point in Remark 5.3, after proving in Theorem 5.2 that, for a right Rickart ring, the conclusions of Proposition 4.4(ii) can be turned into an "if and only if" statement.

5. Idempotent factorizations in right Rickart rings

We recall from Sect. 2.3 that a ring R is a right Rickart ring if r.Ann(a) is a direct summand of the regular right module R_R for every $a \in R$. Some remarks are in order before proceeding.

Remarks 5.1. (1) First considered by S. Maeda [47] and A. Hattori [32] and known also as "principal projective rings" or "p.p. rings" (see [44, Sect. 1] for further historical details), right Rickart rings form a wide class of rings, including von Neumann regular rings and, more generally, right semi-hereditary rings, see [28, Corollary 1.2(d)] and the comments on the bottom of [41, p. 261]. In addition, we gather from [44, Proposition 3.2] that the same class also includes the endomorphism ring of a Rickart (right) module, by which we mean a (right) module M where the kernel of each endomorphism is a direct summand of M, see [44, Definition 2.2]. Most notably, we have from [45, Theorem 3.6] that every finitely generated, projective (or free) right module M over a right semi-hereditary ring is a Rickart module.

(2) If R is a right Rickart ring and r.Fix$(a) \subseteq$ r.Fix(b) for some $a, b \in R$, then we get from Eq. (3) that each of r.Fix(a) and r.Fix(b) is a principal right ideal of R generated by an idempotent, which, by Proposition 2.2(iii), implies the existence of a right ideal $\mathfrak{j} \subseteq R$ such that r.Fix$(v) = \mathfrak{j} \oplus$ r.Fix(u).

(3) Let R be a ring. The ring $T_n(R)$ of n-by-n upper triangular matrices over R is finitely generated as a left (resp., right) R-module; it thus follows from [42, Proposition (1.21)] (and its right analogue) and the Hopkins-Levitzki theorem that, if R is an artinian ring, then $T_n(R)$ is artinian and hence noetherian. On the other hand, we gather from [41, Example (2.36)] that, if R is a skew field, then $T_n(R)$ is a hereditary ring and hence, by Remark 5.1(1), a right Rickart ring. Yet, $T_n(R)$ is not a von Neumann regular ring for $n \geq 2$. For, if $A \in T_n(R)$ is a matrix whose $(1, n)$-entry is non-zero and all of whose other entries are zero, then $AXA = 0_{T_n(R)} \neq A$ for every $X \in T_n(R)$.

The role of right Rickart rings in our approach to the study of idempotent factorizations should be already clear from the centrality of right annihilators in the results of Sect. 2.1, and it will become even clearer (we hope) with the next theorem.

Theorem 5.2. Let H be the monoid of left zero divisors of a right Rickart ring R. An element $a \in R$ is an r.Fix$_R$-quark of H if and only if a is a coprimitive idempotent.

Proof. The "only if" part is immediate from Proposition 4.4(ii). For the "if" part, let a be a coprimitive idempotent of R and suppose by way of contradiction that a is not an r.Fix-quark of H (note that every idempotent of R is an element of H). There then exists $b \in H$ with $b \neq 1_R$ such that r.Fix$(a) \subsetneq$ r.Fix(b), and so we see from Remark 5.1(2) that $R = \mathfrak{i} \oplus$ r.Fix(b) and r.Fix$(b) = \mathfrak{j} \oplus$ r.Fix(a) for some non-zero right ideals $\mathfrak{i}, \mathfrak{j} \subseteq R$. In particular, $\mathfrak{i}$ is non-zero because b is a left zero divisor of R and hence r.Fix(b) is properly contained in R. It thus follows by Proposition 2.1 that

$$\text{r.Ann}(a) \oplus \text{r.Fix}(a) = R = \mathfrak{i} \oplus \text{r.Fix}(b) = \mathfrak{i} \oplus \mathfrak{j} \oplus \text{r.Fix}(a).$$

But this implies by [3, Exercise 5.4(1), p. 76] that r.Ann(a) is isomorphic, as a right R-module, to $\mathfrak{i} \oplus \mathfrak{j}$, which is impossible because r.Ann(a) is an indecomposable submodule of R_R (by the hypothesis that a is a coprimitive idempotent) and, on the other hand, $\mathfrak{i}$ and $\mathfrak{j}$ are both non-zero right ideals of R. ∎

Remark 5.3. Let H be the monoid of left zero divisors of a right Rickart ring R and $\preceq$ be the restriction to H of the r.Fix-preorder on (the multiplicative monoid of) R, and pick $a \in H$. Since R is right Rickart,

r.Ann(a) is a direct summand of R_R. So, we obtain from Proposition 4.4(i) that a is a $\preceq$-irreducible of H only if r.Ann(a) is an indecomposable submodule of R_R with r.Ann$(a) =$ r.Ann(a^2). If, on the other hand, r.Ann$(a) = \mathfrak{g} \oplus \mathfrak{h}$ for some non-zero right ideals $\mathfrak{g}, \mathfrak{h} \subseteq R$ such that $\mathfrak{g}$ is an indecomposable R-module (which, by [41, Corollary (6.7)(1)], is guaranteed when R is a right noetherian ring) or r.Ann(a) is an indecomposable R-module with r.Ann$(a) \neq$ r.Ann(a^2), then the proof of the same Proposition 4.4(i) shows that a factors as the product of two left zero divisors $b, c \in R$ with $b \prec a$ and $c \prec a$ such that r.Ann(c) is an indecomposable R-module; that is, c is a coprimitive idempotent of R and hence a $\preceq$-quark (Theorem 5.2), which gives in turn that $\mathrm{ht}(b) + \mathrm{ht}(c) \leq \mathrm{ht}(a)$, where $\mathrm{ht}(\cdot)$ is the $\preceq$-height (function) of H.

The next remarks show that, in certain classes of rings, the monoid of left zero divisors coincides with the monoid of zero divisors, which makes Proposition 4.4 and Theorem 5.2 smoother.

Remarks 5.4. (1) A ring R is strongly π-regular if there is no $a \in R$ with $a^{k+1}R \subsetneq a^k R$ for all $k \in \mathbb{N}^+$ (e.g., every left or right artinian ring is strongly π-regular). By the comments under the solution to Exercise 4.17 in [43], the elements of a strongly π-regular ring are either units or zero divisors. In particular, this is true of the ring of n-by-n upper triangular matrices with entries in a skew field (Remark 5.1(3)).

(2) Every non-unit element of a Dedekind-finite, von Neumann regular monoid H is singular. In fact, let $x \in H$ be a non-unit. Since H is von Neumann regular, we have $x = xyx$ for some $y \in H$. Therefore, if x is left- or right-cancellative (namely, x is not a zero divisor), then $xy = 1_H$ or $yx = 1_H$, which, by the Dedekind-finiteness of H, implies that x is a unit (which is absurd).

(3) A (right) module M is said to be hopfian if every surjective endomorphism of M is an automorphism. Consequently, an element f in the endomorphism ring of a hopfian module is a zero divisor if and only if f is a *left* zero divisor, if and only if f is a non-injective map. Most notably, we have from (the right analogue of) [43, Exercise 1.12(1)] that every noetherian module M is hopfian, and by (the right analogue of) [42, Proposition (1.21)], this is especially the case when M is a finitely generated module over a right noetherian ring.

We continue by proving that, in a right Rickart ring R such that the regular right module R_R has finite uniform dimension, the r.Fix$_R$-preorder is strongly artinian.

Proposition 5.5. Let R be a right Rickart ring with finite uniform dimension. Then the r.Fix$_R$-height of an element $a \in R$ relative to the multiplicative monoid of R is finite and bounded above by $\mathrm{udim}(R_R) - \mathrm{r.udim}_R(\mathrm{r.Fix}(a))$. In particular, every multiplicative submonoid of R is strongly r.Fix$_R$-artinian.

Proof. Set $N := \mathrm{udim}(R_R)$ and fix $a \in R$. It is enough to check that $\alpha + h \leq N$, where α is the right uniform dimension of r.Fix(a) and h is the r.Fix$_R$-height of a relative to the monoid $H := (R, \cdot)$. Since N is finite (by hypothesis) and the only r.Fix$_R$-unit of H is the identity 1_R, this will in fact prove that H is a strongly r.Fix$_R$-artinian monoid and, by Remark 3.3(1), so are also the submonoids of H.

To start with, it is clear from Eq. (1) that $\alpha + h \leq N$ when a is an r.Fix$_R$-unit of H and hence $h = 0$. Therefore, we assume from now on that a is an r.Fix$_R$-non-unit. It follows that either $h \in \mathbb{N}^+$ or $h = \infty$. Accordingly, pick an integer n in the (non-empty) interval $[\![1, h]\!]$. By Remark 3.3(2) and Definition 3.6 (applied to the r.Fix$_R$-preorder on H), there then exist $x_1, \ldots, x_n \in R$ with $x_1 = a$ and r.Fix$(x_k) \subsetneq$ r.Fix$(x_{k+1}) \subsetneq R$ for every $k \in [\![1, n-1]\!]$ (here again we use that the only r.Fix$_R$-unit of H is 1_R). So, letting $x_{n+1} := 1_R$, we get from Remark 5.1(2) that there are non-zero right ideals $\mathfrak{i}_1, \ldots, \mathfrak{i}_n$ of R such

that

$$\mathrm{r.Fix}(x_{k+1}) = \mathrm{r.Fix}(x_k) \oplus \mathfrak{i}_k, \qquad \text{for each } k \in [\![1, n]\!].$$

Consequently, we see (by induction) that $R = \mathrm{r.Fix}(x_k) \oplus \mathfrak{i}_k \oplus \cdots \oplus \mathfrak{i}_n$ for all $k \in [\![1, n]\!]$; in particular,

$$R = \mathrm{r.Fix}(x_1) \oplus \mathfrak{i}_1 \oplus \cdots \oplus \mathfrak{i}_n = \mathrm{r.Fix}(a) \oplus \mathfrak{i}_1 \oplus \cdots \oplus \mathfrak{i}_n. \tag{13}$$

Since $\mathfrak{i}_1, \ldots, \mathfrak{i}_n$ are non-zero right ideals of R, we therefore conclude from Eqs. (2) and (13) that

$$N = \mathrm{r.udim}_R(R) = \mathrm{r.udim}_R(\mathrm{r.Fix}(a)) + \sum_{k=1}^{n} \mathrm{r.udim}_R(\mathfrak{i}_k) \geq \mathrm{r.udim}_R(\mathrm{r.Fix}(a)) + n = \alpha + n,$$

which shows in turn that $\alpha + h \leq N$ (as wished), because n was an arbitrary integer in $[\![1, h]\!]$ (note, in particular, that h must be finite). ∎

5.1. **Von Neumann regular rings.** By Theorem 5.2 and Remark 5.4, the r.Fix-quarks of the monoid of zero divisors of large classes of right Rickart rings are idempotent. Given a Rickart ring R, it is then natural to ask whether also the $\mathrm{r.Fix}_R$-irreducibles of $R^{\#}$ of some degree $s \geq 2$ are idempotent. In the present subsection, we address this question in the class of von Neumann regular rings and show that the answer is positive under certain conditions (Theorem 5.7). We start with some general considerations.

Remarks 5.6. (1) Let e and f be idempotents of a ring R for which there exist right R-linear maps $\alpha \colon eR \to fR$ and $\beta \colon fR \to eR$ with the property that $\beta(\alpha(e)) = e$, and set $z := \alpha(e)$ and $y := \beta(f)$. We claim that $z = fze$, $y = eyf$, and $yz = e$: Incidentally, this results in a generalization of Proposition 4 in [37, Ch. III, Sect. 7], where eR and fR are rather assumed to be isomorphic (as right R-modules).

For the claim, it is clear that $z \in fR$ and $y \in eR$, which gives in turn that $z = fz$ and $y = ey$ (by the assumption that $e^2 = e$ and $f^2 = f$). Moreover, we have from the right R-linearity of α and β that

$$ze = \alpha(e)e = \alpha(e^2) = \alpha(e) = z \quad \text{and} \quad yf = \beta(f)f = \beta(f^2) = \beta(f) = y.$$

It follows that $z = fz = ze = fze$ and $y = ey = yf = eyf$; and since $\beta(\alpha(e)) = e$ (by hypothesis), it is then immediate to check that $yz = \beta(f)z = \beta(fz) = \beta(z) = \beta(\alpha(e)) = e$.

(2) Let R be a von Neumann regular ring, and let $a \in R$ be such that $\mathrm{r.Ann}(a) \cap aR = \{0_R\}$. Since, by Remark 5.1(1), R is a right Rickart ring, $\mathrm{r.Ann}(a)$ and $\mathrm{r.Fix}(a)$ are both direct summands of R_R, viz., $\mathrm{r.Ann}(a) = eR$ and $\mathrm{r.Fix}(a) = fR$ for some idempotents $e, f \in R$. Then $\mathfrak{i} := \mathrm{r.Ann}(a) + aR = eR \oplus aR$; in particular, $\mathfrak{i}$ is a finitely generated right ideal of R, which, by [28, Theorem 1.1], implies that $R = \mathfrak{i} \oplus \mathfrak{j} = \mathrm{r.Ann}(a) \oplus aR \oplus \mathfrak{j}$ for a certain right ideal $\mathfrak{j} \subseteq R$. Therefore, $fR = \mathrm{r.Fix}(a) \subseteq aR = gR$ for some idempotent $g \in R$, and by Proposition 2.2(iii), this proves that $aR = \mathrm{r.Fix}(a) \oplus g_0 R$ for some other idempotent $g_0 \in R$. So, putting it all together, we see that there is a Peirce basis (e_1, e_2, e_3) of R with $\mathrm{r.Ann}(a) = e_1 R$, $\mathrm{r.Fix}(a) = e_2 R$, and $aR \subseteq e_2 R \oplus e_3 R = (1_R - e_1)R$.

Following K.R. Goodearl and D. Handelman (see the unnumbered definition on p. 80 and the notes on p. 94 of [28]), we say that a von Neumann regular R satisfies the comparability axiom if, for all $u, v \in R$, there is an injective homomorphism of (right) R-modules either from uR to vR or from vR to uR.

Theorem 5.7. Let a be an $\mathrm{r.Fix}_R$-irreducible of the monoid of zero divisors of a Dedekind-finite, von Neumann regular ring R satisfying the comparability axiom. Then a is a coprimitive idempotent.

Proof. Since R is Dedekind-finite and von Neumann regular, we are guaranteed by Remark 5.4(2) that every left or right zero divisor of R is a zero divisor. By Proposition 4.4(i) and Remark 5.1(1), it follows that r.Ann(a) is an indecomposable R-module with r.Ann(a)$\cap aR = \{0_R\}$, and it is left to see that $a = a^2$. To start with, we have from Remark 5.6(2) that there is a Peirce basis $\bar{E} = (\bar{e}_1, \bar{e}_2, \bar{e}_3)$ of R with

$$\text{r.Ann}(a) = \bar{e}_1 R \neq \{0_R\}, \quad \text{r.Fix}(a) = \bar{e}_2 R, \quad \text{and} \quad aR \subseteq (1_R - \bar{e}_1)R = \text{r.Fix}(a) \oplus \bar{e}_3 R. \tag{14}$$

Suppose by way of contradiction that $a \neq a^2$. By Proposition 2.1 and Eq. (14), this means that $\bar{e}_3 \neq 0_R$. We claim that there exists a Peirce basis $E = (e_1, e_2, e_3)$ of R with $e_1 = \bar{e}_1$, r.Fix(a) $\subseteq e_2 R$, and $e_3 \neq 0_R$ and a pair of right R-linear maps $\alpha\colon e_3 R \to e_1 R$ and $\beta\colon e_1 R \to e_3 R$ such that $\beta(\alpha(e_3)) = e_3$.

For, since R satisfies the comparability axiom, there is an injective homomorphism of R-modules φ either from $\bar{e}_3 R$ to $\bar{e}_1 R$ or from $\bar{e}_1 R$ to $\bar{e}_3 R$. In fact, we may assume that φ is actually a map $\bar{e}_1 R \to \bar{e}_3 R$, or else we are done by taking $E := \bar{E}$, $\alpha := \varphi$, and $\beta := \varphi^{-1}$, where φ^{-1} is the inverse of the corestriction of φ to its own image. Consequently, we have from [28, Theorem 1.1] and Proposition 2.2(iii) that $\{0_R\} \neq \varphi(\bar{e}_1)R = \varphi(\bar{e}_1 R) = \bar{e}_{3,2}R \subseteq \bar{e}_{3,1}R \oplus \bar{e}_{3,2}R = \bar{e}_3 R$ for some orthogonal idempotents $\bar{e}_{3,1}, \bar{e}_{3,2} \in R$, which finishes the proof of the claim upon taking $E := (\bar{e}_1, \bar{e}_2 + \bar{e}_{3,1}, \bar{e}_{3,2})$, $\alpha := \varphi^{-1}$, and $\beta := \varphi$.

Now, it follows from the above and Remark 5.6(1) that there are $y \in e_3 R e_1$ and $z \in e_1 R e_3$ such that $yz = e_3$. Accordingly, define $b := e_3 + ae_2 + (a - 1_R)y$ and $c := 1_R - e_1 + z$. Since E is a Peirce basis of R and, by construction, $y \in Re_1$ and $z \in e_1 Re_3 \subseteq e_1 R(1_R - e_1)$, we get from Proposition 2.2(iv) that $c = c^2$ and hence $c \in R^\#$. Moreover, $e_3 z = ae_2 z = y(1_R - e_1) = 0_R$. So, using that $yz = e_3$, we find

$$bc = e_3(1_R - e_1) + ae_2(1_R - e_1) + (a - 1_R)y(1_R - e_1) + e_3 z + ae_2 z + (a - 1_R)yz$$

$$= e_3 + ae_2 + (a - 1_R)e_3 = 0_R + a(e_2 + e_3) = a(e_1 + e_2 + e_3) = a.$$

We claim that b and c are both zero divisors of R, and each is strictly smaller than a with respect to the r.Fix$_R$-preorder on $R^\#$. This will contradict that a is an r.Fix$_R$-irreducible of $R^\#$ and complete the proof of the theorem. To begin, let $x \in$ r.Fix(a). Since r.Fix(a) $\subseteq e_2 R$ (by construction), we have $x = e_2 u$ for some $u \in R$. Moreover, $ye_2 = ze_2 = 0_R$ (recall that $y \in Re_1$ and $z \in Re_3$). Therefore,

$$bx = e_3 e_2 u + ae_2^2 u + (a - 1_R)ye_2 u = ae_2 u = ax = x \quad \text{and} \quad cx = x - e_1 e_2 u + ze_2 u = x. \tag{15}$$

In a similar fashion, it is found that $ye_3 = ze_1 = 0_R$, which implies

$$be_3 = e_3^2 + ae_2 e_3 + (a - 1_R)ye_3 = e_3 \quad \text{and} \quad ce_1 = (1_R - e_1)e_1 + ze_1 = 0_R. \tag{16}$$

In particular, we get from Eqs. (15) and (16) that

$$\text{r.Fix}(a) \cup e_3 R \subseteq \text{r.Fix}(b), \quad \text{r.Fix}(a) \subseteq \text{r.Fix}(c), \quad \text{and} \quad \text{r.Ann}(a) = e_1 R \subseteq \text{r.Ann}(c). \tag{17}$$

In fact, it is not difficult to verify that r.Ann(a) = r.Ann(c), which will come in handy in Remark 5.8(1). For, let $v \in$ r.Ann(c). Since $v = e_1 v + e_2 v + e_3 v$ and $e_2 z = e_3 z = 0_R$ (recall that $z \in e_1 R$), we see that $0_R = (1_R - e_1)v + zv = e_2 v + e_3 v + zv$. But this is only possible if $0_R = e_2^2 v + e_2 e_3 v + e_2 zv = e_2 v$ and, likewise, $0_R = e_3 v$, so that $v = e_1 v \in e_1 R$ and hence r.Ann(c) $\subseteq e_1 R =$ r.Ann(a).

Also, r.Fix(a) $\cap e_3 R = \{0_R\}$, because $e_2 r = e_3 r'$ for some $r, r' \in R$ only if $e_2^2 r = e_2 e_3 r' = 0_R$ (recall that r.Fix(a) $\subseteq e_2 R$). Thus, we conclude from Eq. (17), similarly as in the proof of Lemma 4.3(iii), that r.Fix(a) $\subsetneq$ r.Fix(a) $\oplus e_3 R \subseteq$ r.Fix(b), where in the first inclusion we use that $e_3 \neq 0_R$. It only remains to check that $b \in R^\# \smallsetminus \{1_R\}$ and r.Fix(a) $\subsetneq$ r.Fix(c).

For, we have from Eq. (14) that $aR \subseteq (1_R - e_1)R$ (recall that $e_1 = \bar{e}_1$) and hence $e_1 a = 0_R$. It follows that $e_1 b = e_1 e_3 + e_1 a e_2 + e_1 (a - 1_R) y = 0_R$ (recall that $y \in e_3 R e_1$) and hence $0_R \neq e_1 \in \mathrm{l.Ann}(b)$. As a result, b is in $R^\# \smallsetminus \{1_R\}$, for we have already noted that every left zero divisor of R is a zero divisor.

Finally, assume $\mathrm{r.Fix}(a) = \mathrm{r.Fix}(c)$. Since $c = c^2$ (as proved earlier), we get from Proposition 2.1 that $(1_R - e_1 + z)R = \mathrm{r.Fix}(a) \subseteq e_2 R$. In particular, this yields $1_R - e_1 + z = e_2 w$ for some $w \in R$ and hence $0_R = e_1 e_2 w = e_1(1_R - e_1) + e_1 z = z$ (because $z \in e_1 R$). Thus, $0_R \neq e_3 = yz = 0_R$ (which is absurd). ∎

Remarks 5.8. Let R be a Dedekind-finite, von Neumann regular ring satisfying the comparability axiom.

(1) Given a zero divisor $a \in R$ such that $\mathrm{r.Ann}(a)$ is an indecomposable R-module and a is not idempotent, we gather from the proof of Theorem 5.7 that a factors as the product bc of two zero divisors $b, c \in R$ with $c = c^2$, $\mathrm{r.Fix}(a) \subsetneq \mathrm{r.Fix}(b)$, $\mathrm{r.Fix}(a) \subsetneq \mathrm{r.Fix}(c)$, and $\mathrm{r.Ann}(a) = \mathrm{r.Ann}(c)$; in particular, $\mathrm{ht}(b) + \mathrm{ht}(c) \leq \mathrm{ht}(a)$, where $\mathrm{ht}(\cdot)$ is the $\mathrm{r.Fix}_R$-height on $R^\#$ (cf. Remark 5.3).

(2) We have from Corollary 3.8 and Theorem 5.7 that, if $R^\#$ is an $\mathrm{r.Fix}_R$-artinian monoid, then each zero divisor $x \in R$ factors as a finite product of coprimitive idempotents of R. But, since we are guaranteed by Remark 5.4(2) that every non-unit of R is a zero divisor, the artinianity of the $\mathrm{r.Fix}_R$-preorder on $R^\#$ is equivalent to the ascending chain condition on the principal right ideals of R, which, by [28, Corollary 2.16], is in turn equivalent to R being semisimple, meaning that every right ideal of R is a direct summand of R_R. This narrows down the class of von Neumann regular rings R to which we can apply Corollary 3.8 and leaves open the question whether a refinement of the techniques developed in the present work can handle a greater variety of cases. An affirmative answer is perhaps not so unreasonable, especially when considering that, by Example 8.1 in [28], there exist Dedekind-finite, von Neumann regular rings satisfying the comparability axiom that are not artinian and hence not semisimple (incidentally, the same example also shows that the characterization provided by Theorem 5.7 applies to a class of von Neumann regular rings that is *strictly larger* than the class of semisimple rings).

Motivated by Remark 5.8(2), we conclude this subsection by restricting our focus on idempotent factorizations in semisimple rings. We recall that semisimple rings are Dedekind-finite (see [43, Exercise 3.10]), von Neumann regular (as is obvious from the definitions and [28, Theorem 1.1]), and noetherian (see [42, Corollaries (2.6) and (3.7)]). Moreover, the ring of n-by-n matrices over a skew field is semisimple (see, e.g., [42, Theorem 3.3 and Corollary (3.7)]) and satisfies the comparability axiom (see [28, Theorem 9.12 and Corollary 9.16]). We will use these basic facts without further mention in what follows.

Proposition 5.9. Set $R := \mathcal{M}_n(K)$, where K is a skew field. Every singular matrix $A \in R$ factors as a product of at most $\mathrm{ht}(A)$ idempotent matrices of rank $n - 1$, where $\mathrm{ht}(\cdot)$ is the $\mathrm{r.Fix}_R$-height of $R^\#$.

Proof. It follows from Theorem 5.2 and Remarks 5.3 and 5.8(1) that a singular matrix $A \in R$ either is a coprimitive idempotent of R or factors as the product BC of two singular matrices $B, C \in R$ such that C is an $\mathrm{r.Fix}_R$-quark of $R^\#$ and $\mathrm{ht}(B) + \mathrm{ht}(C) \leq \mathrm{ht}(A)$. On the other hand, we have from Proposition 5.5 that $R^\#$ is a strongly $\mathrm{r.Fix}_R$-artinian monoid, for the right uniform dimension of a noetherian ring is finite (see, e.g., [41, Corollary (6.7)(1)]). We can therefore apply Theorem 3.5 with $s = 2$ and conclude that every $A \in R^\#$ factors as a product of at most $\mathrm{ht}(A)$ coprimitive idempotents, which are in turn the idempotent matrices of R of rank $n - 1$ (Proposition 5.13(i)). ∎

The next corollary is now an immediate consequence of Propositions 5.5 and 5.9, when considering that, for a skew field K, the uniform dimension of $\mathcal{M}_n(K)$ as a right $\mathcal{M}_n(K)$-module equals to n (see, e.g., [51, Corollary 2.8 and Proposition 2.15]).

Corollary 5.10. Every singular n-by-n matrix with entries in a skew field is a product of at most n idempotent matrices of rank $n - 1$.

On the one hand, Corollary 5.10 strengthens a result of Laffey [39] (in turn, a generalization of Erdos' classical theorem [18]) on the existence of an idempotent factorization for singular matrices over a skew field; on the other, it provides a non-commutative extension of Dawlings' theorem [17] on the existence of a factorization into idempotent factors of rank $n-1$ for the singular linear transformations of a vector space of finite dimension n over a field.

The bound n for the minimum length of a factorization into coprimitive idempotents of a singular n-by-n matrix over a skew field is sharp, as it is known from the unnumbered corollary on the bottom of [10, p. 81] that, for any integer $n \geq 2$ and any field K, there are n-by-n singular matrices with entries in K that cannot be written as a product of $n - 1$ idempotents of $\mathcal{M}_n(K)$.

Remark 5.11. Let R be a semisimple ring. By the Wedderburn-Artin theorem, there are a unique integer $k \geq 1$ and a unique tuple $(n_1, \ldots, n_k) \in \mathbb{N}^{\times k}$ with $1 \leq n_1 \leq \cdots \leq n_k$ such that R is isomorphic (as a ring) to a direct product of the form $\prod_{i=1}^k \mathcal{M}_{n_i}(D_i)$, where the D_i's are skew fields. Since ring homomorphisms map idempotents to idempotents, we can actually assume without loss of generality that $R = \prod_{i=1}^k R_i$, where for ease of notation $R_i := \mathcal{M}_{n_i}(D_i)$. It then follows from Claim A in Remark 3.10 that $\mathcal{E}(R) = \prod_{i=1}^k \mathcal{E}(R_i)$. So we find $\mathcal{E}(R) = \prod_{i=1}^k R_i^{\#}$, because it is guaranteed by Corollary 5.10 that $\mathcal{E}(R_i) = R_i^{\#}$ for each $i \in [\![1, k]\!]$. On the other hand, we gather from the comments under the proof of the aforementioned Claim A that the minimum length $\ell(x)$ of an idempotent factorization of an element $x = (x_1, \ldots, x_k) \in \prod_{i=1}^k R_i^{\#}$ is bounded above by $\max_{1 \leq i \leq k} \ell_i(x)$, where $\ell_i(x)$ is the minimum length of an idempotent factorization of x_i in $R_i^{\#}$. In particular, Corollary 5.10 gives $\ell(x) \leq \max(n_1, \ldots, n_k) = n_k$.

As a complement to Remark 5.11, note that, by [28, Corollary 8.6], a semisimple ring R satisfies the axiom of comparability only if $R = \mathcal{M}_n(K)$ for a skew field K, which explains why we cannot just apply Theorems 3.5 and 5.7 and conclude in general that the idempotents generate the zero divisors of R.

5.2. **Endomorphism rings.** Let M be a (right) module over a ring D. In this subsection, we apply some of the results of Sect. 4 to the monoid of zero divisors of the endomorphism ring $\mathrm{End}(M)$.

Given $f \in \mathrm{End}(M)$, we denote by $\ker(f) := \{x \in M : f(x) = 0_M\}$ the kernel of f (as usual) and set

$$\mathrm{Fix}(f) := \ker(\mathrm{id}_M - f) = \{x \in M : f(x) = x\}. \tag{18}$$

Since M is canonically a left module over $\mathrm{End}(M)$, with multiplication given by the map $\mathrm{End}(M) \times M \to M : (g, x) \mapsto g(x)$, we have $\ker(g) = \{x \in M : gx = 0_M\}$ for every $g \in \mathrm{End}(M)$. As a result, we gather from (the right analogue of) [49, Lemma C.1(1)] that

$$\ker(f - f^2) = \ker(f) \oplus \ker(\mathrm{id}_M - f) = \ker(f) \oplus \mathrm{Fix}(f). \tag{19}$$

In particular, note that, if f is idempotent, then $\ker(f - f^2) = M$. This leads to the next proposition, whose proof is along the same lines of the proof of Proposition 2.1 (we leave the details to the reader).

Proposition 5.12. An endomorphism f of M is idempotent in $\text{End}(M)$ if and only if $\ker(f) \oplus \text{Fix}(f) = M$, if and only if $\text{Fix}(f) = f(M)$.

Building on these premises, we collect some elementary results that are folklore in module theory but for which we have not been able to find any convenient reference.

Proposition 5.13. Let R be the endomorphism ring of M and let $f \in R$. Suppose that $M = \ker(f) \oplus N$ for some submodule $N \subseteq M$ and denote by p the projection of M on $\ker(f)$ along N. The following hold:

(i) $\text{r.Ann}(f) = pR$, $\ker(f) = p(M)$, and $R = pR \oplus (\text{id}_M - p)R$.

(ii) $\text{r.Ann}(f) \subseteq R_R$ is an indecomposable module if and only if so is $\ker(f)$ as a submodule of M.

Proof. (i): Every projection of M is an idempotent of R, see, e.g., [3, Corollary 5.8]. Moreover, it is clear that $\ker(f) = p(M)$. In view of Eq. (5), it is therefore enough to show that $\text{r.Ann}(f) = pR$.

Since $\ker(f) = p(M)$, we have $fp(x) = f(p(x)) = 0_M$ for each $x \in M$, which implies $fp = 0_R$ and hence $pR \subseteq \text{r.Ann}(f)$. As for the opposite inclusion, let $g \in \text{r.Ann}(f)$. Then $fg = 0_R$ and hence $g(M) \subseteq \ker(f) = p(M)$. Since $p(y) = y$ for all $y \in \ker(f)$, it follows that $g = pg \in pR$ and hence $\text{r.Ann}(f) \subseteq pR$.

(ii): Assume first that $\text{r.Ann}(f)$ is an indecomposable submodule of R_R and suppose for a contradiction that $\ker(f) = N_1 \oplus N_2$ for some non-zero submodules $N_1, N_2 \subseteq M$. It follows from our hypotheses that $M = \ker(f) \oplus N = N_1 \oplus N_2 \oplus N$. Let p_1 be the projection of M on N_1 along $N_2 \oplus N$, and p_2 be the projections of M on N_2 along $N_1 \oplus N$. It is straightforward that $p = p_1 + p_2$ and $p_1 p_2 = p_2 p_1 = 0_R$, i.e., p_1 and p_2 are orthogonal idempotents of R (recall [3, Corollary 5.8]). So, putting it all together, we conclude from item (i) and Proposition 2.2(ii) that $\text{r.Ann}(f) = pR = p_1 R \oplus p_2 R$. But this is impossible, because $p_1 R$ and $p_2 R$ are *non-zero* right ideals of R (and $\text{r.Ann}(f)$ is assumed to be indecomposable).

Conversely, let $\ker(f)$ be an indecomposable submodule of M and suppose for a contradiction that $\text{r.Ann}(f) = \mathfrak{i}_1 \oplus \mathfrak{i}_2$ for some non-zero right ideals $\mathfrak{i}_1, \mathfrak{i}_2 \subseteq R$. By item (i), pR is a direct summand of R_R with $pR = \text{r.Ann}(f)$ and $p(M) = \ker(f)$. So, again by Proposition 2.2(ii), there exist *non-zero* orthogonal idempotents $e_1, e_2 \in R$ with $p = e_1 + e_2$ such that $\mathfrak{i}_1 = e_1 R$ and $\mathfrak{i}_2 = e_2 R$. This is however absurd, for it yields $\ker(f) = p(M) = e_1(M) \oplus e_2(M)$ and hence contradicts that $\ker(f)$ is indecomposable. ∎

Proposition 5.14. Let M be a Rickart module and R be its endomorphism ring, and let $f, g \in R$. Then $f \preceq g$ if and only if $\text{Fix}(g) \subseteq \text{Fix}(f)$, where $\preceq$ is the r.Fix preorder on R.

Proof. Since M is a Rickart module, it follows from Eq. (18) and Remark 5.1(1) that $M = \text{Fix}(f) \oplus N_f = \text{Fix}(g) \oplus N_g$ for some submodules $N_f, N_g \subseteq M$. Let p be the projection of M on $\text{Fix}(f)$ along N_f, and q be the projection of M on $\text{Fix}(g)$ along N_g. By Definition 3.6 and Proposition 5.13(i), $f \preceq g$ if and only if $\text{r.Fix}(g) \subseteq \text{r.Fix}(f)$, if and only if $qR \subseteq pR$. We claim that $qR \subseteq pR$ if and only if $\text{Fix}(g) \subseteq \text{Fix}(f)$.

Assume first that $qR \subseteq pR$, i.e., $q = p\alpha$ for some $\alpha \in R$. Then $pq = p\alpha = q$. It follows that $q(x) = p(q(x)) \in p(M)$ for each $x \in M$, whence $\text{Fix}(g) = q(M) \subseteq p(M) = \text{Fix}(f)$. If, on the other hand, $\text{Fix}(g) \subseteq \text{Fix}(f)$, then $q(M) \subseteq p(M)$, i.e., for every $x \in M$ there exists $y \in M$ such that $q(x) = p(y)$. But then $pq(x) = p(q(x)) = p(p(y)) = p(y) = q(x)$, whence $q = pq$ and $qR \subseteq pR$. ∎

Assume from now on (unless noted otherwise) that D is a commutative PID and M is a free module of finite rank n over D, and denote by R the endomorphism ring $\text{End}(M)$. If D is not a field, then R is not von Neumann regular, hence the study of idempotent factorizations in $R^{\#}$ cannot be reduced to the

cases covered by Sect. 5.1. On the other hand, since every commutative PID is a right semi-hereditary ring, we are guaranteed by Remark 5.1(1) that M is a Rickart module and R is a right Rickart ring. It follows by Eq. (18) that $\mathrm{Fix}(f)$ is a direct summand of M. So, if $f, g \in R$ and $\mathrm{Fix}(f) \subseteq \mathrm{Fix}(g)$, we get from [3, Exercise 5.4(2), p. 76] that $\mathrm{Fix}(g) = \mathrm{Fix}(f) \oplus N$ for a certain submodule $N \subseteq M$. From here and Proposition 5.14, arguing as in Proposition 5.5, we find

$$\mathrm{ht}(f) \leq n - \mathrm{rk}(\mathrm{Fix}(f)), \tag{20}$$

where $\mathrm{ht}(\cdot)$ is the r.Fix_R-height of R and $\mathrm{rk}(\cdot)$ denotes the rank of a free module over D. Therefore, we gather from Remark 3.3(1) that every (multiplicative) submonoid of R is strongly r.Fix_R-artinian.

In addition, since M is a noetherian module, we have from Remark 5.4(3) that $f \in R$ is a zero divisor if and only f is a non-injective endomorphism of M, i.e., $\mathrm{rk}(\ker(f)) \geq 1$. We thus conclude from the rank-nullity theorem that $R^{\#} = \{f \in H : \mathrm{rk}(f(M)) \leq n - 1\} \cup \{\mathrm{id}_M\}$.

Remark 5.15. Given $f \in R$, $\ker(f)$ is an indecomposable free D-module if and only if $\mathrm{rk}(\ker(f)) = 1$. This shows, in light of Proposition 5.12, that f is a coprimitive idempotent if and only if $\mathrm{Fix}(f) = f(M)$ and $\mathrm{rk}(\ker(f)) = 1$, if and only if $\mathrm{rk}(f(M)) = \mathrm{rk}(\mathrm{Fix}(f)) = n - 1$ (by the rank-nullity theorem).

We can now recover the characterizations of r.Fix_R-quarks and r.Fix_R-irreducibles obtained at the beginning of Sect. 5 for a general right Rickart ring. We rephrase the results in terms of $\mathrm{End}(M)$.

Proposition 5.16. Let f be a non-injective endomorphism of M. The following hold:

(i) f is an r.Fix_R-quark of $R^{\#}$ if and only if f is a coprimitive idempotent.
(ii) f is an r.Fix_R-irreducible of $R^{\#}$ only if $\mathrm{rk}(\ker(f)) = 1$, and if $\mathrm{rk}(\ker(f)) > 1$ then $f = gh$ for some non-injective endomorphisms $g, h \in R$ with $\mathrm{Fix}(f) \subsetneq \mathrm{Fix}(g)$ and $\mathrm{Fix}(f) \subsetneq \mathrm{Fix}(h)$ such that h is a coprimitive idempotent.

Proof. Since R is a right Rickart ring in which every left zero divisor is a zero divisor, (i) follows immediately from Theorem 5.2, and (ii) follows from Propositions 4.4(i) and 5.14 and Remark 5.3. ∎

Note that one cannot expect that every r.Fix_R-irreducible of a certain degree $s \geq 2$ of $R^{\#}$ is idempotent. In fact, it is well known from classical results in matrix theory [13, 16] that, for some choices of the PID D, there is at least one element $f \in R^{\#}$ that does not factor as a product of idempotents of R. In the remainder of this section, we will however see that $R^{\#}$ is idempotent-generated when D is a discrete valuation domain (Theorem 5.19), by proving that, in such case, the degree-3 r.Fix_R-irreducibles of $R^{\#}$ are coprimitive idempotents (Proposition 5.18). Here we follow [6, Ch. 9] and let a discrete valuation domain (DVD) be a commutative PID with a unique maximal ideal generated by a non-zero prime element, where an element $p \in D$ is prime if p is a non-unit and $ab \in pD$, for some $a, b \in D$, if and only if $a \in pD$ or $b \in pD$.

Lemma 5.17. If D is a DVD, N is a torsion-free D-module, and X is a non-empty finite set of linearly dependent vectors of N, then there is an $x \in X$ that lies in the submodule of N generated by $X \smallsetminus \{x\}$.

Proof. Let $x_1, \ldots, x_m$ be the elements of X. By hypothesis, $x_1 a_1 + \cdots + x_m a_m = 0_N$ for some non-zero scalars $a_1, \ldots, a_m \in D$. So, it suffices to show that x_j lies in the submodule of N generated by $X \smallsetminus \{x_j\}$ for a certain $j \in [\![1, m]\!]$. Since D is a DVD, we gather from [6, Proposition 9.2] that there is a non-zero $p \in D$ such that every non-zero $x \in D$ is of the form up^s for some $s \in \mathbb{N}$ and $u \in D^{\times}$. Consequently,

there exist $s_1, \ldots, s_m \in \mathbb{N}$ and $u_1, \ldots, u_m \in D^\times$ such that $x_i = u_i p^{s_i}$ for every $i \in [\![1, m]\!]$. Accordingly, we define $s := \min(s_1, \ldots, s_m)$ and, for each $i \in [\![1, m]\!]$, we set $b_i := u_i p^{s_i - s}$. It follows that

$$0_N = x_1 u_1 p^{s_1} + \cdots + x_m u_m p^{s_m} = (x_1 b_1 + \cdots + x_m b_m) p^s,$$

which is only possible if $x_1 b_1 + \cdots + x_m b_m = 0_N$ (because N is a torsion-free module). On the other hand, we have $s_j = s$, and hence $b_j = u_j \in D^\times$, for some $j \in [\![1, m]\!]$. So letting $I := [\![1, m]\!] \smallsetminus \{j\}$, we obtain that $x_j = -\sum_{i \in I} x_i b_i u_j^{-1}$, i.e., x_j lies in the submodule of N generated by $X \smallsetminus \{x_j\}$. $\blacksquare$

We already know from Theorem 5.2 and Remark 5.4(3) that every r.Fix_R-quark of $R^\#$ is a coprimitive idempotent. Now, we aim at characterizing the degree-3 r.Fix_R-irreducibles of $R^\#$ when D is a DVD.

Note that if $n = 1$, then R is ring-isomorphic to D, whence $R^\# = \{0_R, 1_R\}$ and 0_R is a coprimitive idempotent (see also the comments under Eq. (7)). This explains the formulation of the next result.

Proposition 5.18. Let D be a DVD and M be a free D-module of finite rank $n \geq 2$, and set $R := \mathrm{END}(M)$. The following hold for a non-injective endomorphism $\alpha \in R$:

    (i) If $\mathrm{rk}(\ker(\alpha)) = 1$ and $\mathrm{Fix}(\alpha) = \{0_M\}$, then $\alpha = \beta\gamma$ for some non-identity $\beta, \gamma \in R^\#$ such that β is a coprimitive idempotent and $\mathrm{Fix}(\gamma) \neq \{0_M\}$.

    (ii) If α is a degree-3 r.Fix_R-irreducible of $R^\#$, then α is a coprimitive idempotent of R.

Proof. Denote by d the rank of $\mathrm{Fix}(\alpha)$ (recall that any submodule of a free module over a commutative PID is free). We have already observed that every degree-3 r.Fix_R-irreducible of $R^\#$ is, in particular, an r.Fix_R-irreducible. So, by Proposition 5.16(ii), we may assume $\mathrm{rk}(\ker(\alpha)) = 1$ both in (i) and in (ii).

Since M is a Rickart module (as noted in the comments under the proof of Proposition 5.14), we get from Eq. (19) that $\mathrm{Fix}(\alpha) \oplus \ker(\alpha)$ is a direct summand of M, namely, $M = \mathrm{Fix}(\alpha) \oplus Q \oplus \ker(\alpha)$ for a certain submodule Q of M. Using that $\mathrm{rk}(M) = n \geq 2$, there is hence a basis $(e_1, \ldots, e_n)$ of M with $\mathrm{Fix}(\alpha) = \bigoplus_{i=1}^{d} e_i D$, $Q = \bigoplus_{i=d+1}^{n-1} e_i D$, and $\ker(\alpha) = e_n D$ (note that $0 \leq d < n$, with $d = 0$ if and only if $\mathrm{Fix}(\alpha) = \{0_M\}$). So, there are uniquely determined coefficients $a_{i,j} \in D$ ($1 \leq i, j \leq n$) such that

$$\alpha(e_j) = \sum_{i=1}^{n} e_i a_{i,j}, \qquad \text{for every } i \in [\![1, n]\!]. \tag{21}$$

Accordingly, we define, for each $i \in [\![d+1, n]\!]$,

$$v_i := \sum_{j=d+1}^{n-1} e_j a_{i,j} \in Q. \tag{22}$$

Clearly, $v_{d+1}, \ldots, v_n$ form a non-empty set of linear dependent vectors of Q (with $v_n = 0_M$ in the limit case when $d = n - 1$). So, by Lemma 5.17, there exists an index $\ell \in [\![d+1, n]\!]$ such that $v_\ell = \sum_{i \in I} v_i b_i$, where $I := [\![d+1, n]\!] \smallsetminus \{\ell\}$ and $b_i \in D$. As a result, we conclude from Eq. (22) that

$$a_{\ell,j} = \sum_{i \in I} a_{i,j} b_i, \qquad \text{for every } j \in [\![d+1, n-1]\!]. \tag{23}$$

With these premises in place, we can now focus on the proof of items (i) and (ii).

(i): Suppose that $\mathrm{Fix}(\alpha) = \{0_M\}$. It then follows from the above that $d = 0$, and we distinguish two cases depending on whether $\ell \neq n$ or $\ell = n$.

CASE 1: $\ell \neq n$. Let β and γ be the right R-linear maps $M \to M$ uniquely defined by taking

$$\beta(e_j) := \begin{cases} 0_M & \text{if } j = \ell \\ e_\ell b_j + e_j & \text{if } j \in I \end{cases} \quad \text{and} \quad \gamma(e_j) := \begin{cases} e_\ell + \sum_{i \in I} e_i a_{i,\ell} & \text{if } j = \ell \\ \sum_{i \in I} e_i a_{i,j} & \text{if } j \in I \smallsetminus \{n\} \\ 0_M & \text{if } j = n \end{cases}.$$

It is fairly obvious that β and γ are both singular endomorphisms of M. On the other hand, we get from Eqs. (21) and (23) (specialized to $d = 0$) that, for every $j \in [\![1, n-1]\!]$,

$$\beta(\gamma(e_j)) = \beta\left(\sum_{i \in I} e_i a_{i,j}\right) = \sum_{i \in I} \beta(e_i) a_{i,j} = \sum_{i \in I} (e_\ell b_i + e_i) a_{i,j}$$

$$= e_\ell \sum_{i \in I} b_i a_{i,j} + \sum_{i \in I} e_i a_{i,j} \overset{(23)}{=} e_\ell a_{\ell,j} + \sum_{i \in I} e_i a_{i,j} = \sum_{i=1}^{n} e_i a_{i,j} \overset{(21)}{=} \alpha(e_j).$$

Since $\beta(\gamma(e_n)) = 0_M = \alpha(e_n)$, it follows that $\alpha = \beta\gamma$. Moreover, β is a coprimitive idempotent of R (by Remark 5.15), since $\{e_\ell b_i + e_i : i \in I\}$ is a basis of $\text{Fix}(\beta)$ and $\beta(e_\ell) = 0_M$; and $\text{Fix}(\gamma) \neq \{0_M\}$, since $(\text{id}_M - \gamma)(M) \subseteq \bigoplus_{i \in I} e_i D$ and hence $\text{rk}(\text{Fix}(\gamma)) \geq 1$ (by the rank-nullity theorem).

CASE 2: $\ell = n$. Eq. (23) becomes $a_{n,j} = \sum_{i=1}^{n-1} a_{i,j} b_i$ for every $j \in [\![1, n-1]\!]$. By Proposition 3.9, we may assume that $\alpha(M) \subseteq \bigoplus_{i=1}^{n-1} e_i D$ (otherwise, let η be the automorphism of M uniquely determined by sending e_i to $e_i + e_n b_i$ for $i \in [\![1, n-1]\!]$ and to e_n for $i = n$, and replace α with $\eta^{-1}\alpha\eta$). Accordingly, we take β and γ to be the endomorphisms of M uniquely defined by

$$\beta(e_j) := \begin{cases} e_j & \text{if } j \in [\![1, n-1]\!] \\ \alpha(e_1) - e_1 & \text{if } j = n \end{cases} \quad \text{and} \quad \gamma(e_j) := \begin{cases} e_1 + e_n & \text{if } j = 1 \\ \alpha(e_j) & \text{if } j \in [\![2, n]\!] \end{cases}.$$

It is evident that $\beta(M) \subseteq \bigoplus_{i=1}^{n-1} e_i D$ and $\gamma(e_n) = 0_M$, so that β and γ are r.Fix-non-units of M. Moreover, $\beta(e_j) = e_j$ for $j \in [\![1, n-1]\!]$ (hence β is a coprimitive idempotent) and $\gamma(e_1 + e_n) = \gamma(e_1) + \gamma(e_n) = e_1 + e_n$. Lastly, it is routine to check that $\alpha = \beta\gamma$.

(ii): Let α be a degree-3 r.Fix$_R$-irreducible of $R^\#$ and assume for the sake of a contradiction that $d := \text{rk}(\text{Fix}(\alpha)) < n - 1$ (implying that $\text{Fix}(\alpha) \subsetneq \text{Im}(\alpha)$). Again, we distinguish two cases.

CASE 1: $\ell \neq n$. Let β and γ be the endomorphisms of M uniquely determined by

$$\beta(e_j) := \begin{cases} e_j & \text{if } j \in [\![1, d]\!] \\ 0_M & \text{if } j = \ell \\ e_j + e_\ell b_j & \text{if } j \in I \end{cases} \quad \text{and} \quad \gamma(e_j) := \begin{cases} e_j & \text{if } j \in [\![1, d]\!] \cup \{n\} \\ \alpha(e_j) - e_\ell a_{\ell,j} & \text{if } j \in [\![d+1, n-1]\!] \end{cases};$$

and denote by δ the projection of M on $\text{Fix}(\alpha) \oplus Q$ along $\ker(\alpha)$. Each of β, γ, and δ is a non-injective map, because $e_\ell \in \ker(\beta)$, $e_n \in \ker(\delta)$, and $\gamma(M) \subseteq \bigoplus_{j \in J} e_j D$ with $J := [\![1, n]\!] \smallsetminus \{\ell\}$. Moreover, it is readily seen that $\text{Fix}(\alpha) \cup \{e_j + e_\ell b_j : j \in I\}$ is a basis of $\text{Fix}(\beta)$, $\text{Fix}(\alpha) \cup \{e_n\}$ is contained in $\text{Fix}(\gamma)$, and $\text{Fix}(\alpha) \cup \{e_{d+1}, \ldots, e_{n-1}\}$ is a basis of $\text{Fix}(\delta)$. It follows that β, γ, and δ are non-injective endomorphisms of M each of which is (strictly) smaller than α with respect to the restriction to $R^\#$ of the r.Fix$_R$-preorder on R. Moreover, arguing as in the proof of (i), we gather from Eq. (23) that $\beta\gamma\delta(e_j) = \alpha(e_j)$ for every $j \in [\![1, n]\!]$. But this is in contradiction to the fact that α is a degree-3 r.Fix$_R$-irreducible of $R^\#$. (Incidentally, note that β and δ are both coprimitive idempotents of R.)

Case 2: $\ell = n$. Similarly as in Case 2 from the proof of item (i), we may assume $\alpha(M) \subseteq \bigoplus_{i=1}^{n-1} e_i D$. Accordingly, we take β and γ to be the endomorphisms of M uniquely defined by

$$\beta(e_j) := \begin{cases} e_j & \text{if } j \in [\![1, n-1]\!] \\ \alpha(e_{d+1}) - e_{d+1} & \text{if } j = n \end{cases} \quad \text{and} \quad \gamma(e_j) := \begin{cases} \alpha(e_j) & \text{if } j \in [\![1, n]\!] \smallsetminus \{d+1\} \\ e_{d+1} + e_n & \text{if } j = d+1 \end{cases}.$$

The desired conclusion follows arguing as in the last part of the proof of item (i). ∎

A closer look at Propositions 5.16 and 5.18 leads to the next result.

Theorem 5.19. Every non-injective endomorphism α of a free module M of finite rank $n \geq 2$ over a commutative DVD is a product of $2n - 2$ or fewer coprimitive idempotents of $\text{End}(M)$.

Proof. Let $\text{ht}(\cdot)$ be the r.Fix$_R$-height of $R^\#$ and $\ell(f)$ be the minimum number of factors in a factorization of a non-injective endomorphism f of M into coprimitive idempotents, where $R := \text{End}(M)$. We have from Propositions 5.16 and 5.18(i) and the proof of Proposition 5.18(ii) that either f is an r.Fix$_R$-quark of $R^\#$ or it factors as a product of 2 or 3 r.Fix$_R$-non-units of $R^\#$ the sum of whose r.Fix$_R$-heights is no larger than $\text{ht}(f)$ or $\text{ht}(f) + 1$, resp. It follows by Theorem 3.5 (applied with $s = 3$) that $\ell(f) \leq 2\,\text{ht}(f) - 1$.

Now, if $\text{rk}(\text{Fix}(\alpha)) \geq 1$, we get from Eq. (20) that $\ell(\alpha) \leq 2(n-1) - 1 = 2n - 3$. If, on the other hand, $\text{rk}(\text{Fix}(\alpha)) = 0$, then we see from the proof of Proposition 5.18 that $\alpha = \beta\gamma$, where β is a coprimitive idempotent and γ is an r.Fix$_R$-non-unit of $R^\#$ with $\text{ht}(\gamma) \leq n - 1$; therefore, we conclude from the previous case that $\ell(\alpha) \leq 1 + \ell(\gamma) \leq 1 + 2n - 3 = 2n - 2$. ∎

Corollary 5.20. Every singular n-by-n matrix over a commutative DVD, with $n \geq 2$, is a product of $2n - 2$ or fewer idempotent matrices of rank $n - 1$.

Proof. This is simply a reformulation of Theorem 5.19 in the language of matrices. ∎

Corollary 5.20 incorporates a classical result of Fountain [21, Theorem 4.1] on the *existence* of a factorization into idempotent matrices of rank $n-1$ for every n-by-n singular matrix over a commutative DVD. But the upper bound on the minimum length of such a factorization offered by the same corollary is apparently new, and it is sharp for $n = 2$ (it suffices to note that, for $n \geq 2$, the ring of n-by-n matrices over an arbitrary ring contains at least one singular non-idempotent matrix, namely, the strictly upper triangular matrix whose non-diagonal entries are all equal to the identity).

6. Closing remarks

Many questions remain open. On the one hand, it would be interesting to study various arithmetic invariants that can be naturally attached to the "minimal idempotent factorizations" of the elements of an idempotent-generated monoid along the lines of what is done in [5, Sect. 4] for atomic factorizations in an arbitrary monoid. On the other hand, it might be rewarding to try and extend the techniques developed in Sects. 3 and 4 so as to prove that the monoid of zero divisors of a broader class of right Rickart rings than covered in Sect. 5 is idempotent-generated.

For instance, let $R = \mathcal{M}_2(\mathbb{Z})$ be the ring of 2-by-2 matrices over the integers. It is well known that every singular matrix over a commutative euclidean domain is a product of idempotent matrices [39, Theorem 2]. In particular, it was proved by Fountain in [21, Theorem 4.6] that every singular $A \in R$ is a product of non-zero idempotent matrices. However, the r.Fix$_R$-preorder on the monoid $R^\#$ of zero

divisors of R cannot be the right preorder to consider if one's goal is to recover Fountain's result as a corollary of Theorem 3.4. In fact, if for some integer $s \geq 2$ the degree-s r.Fix$_R$-irreducibles of $R^\#$ were idempotent, then we would get from Eq. (20) that every singular matrix in R factors as a product of at most s idempotents. But this is impossible, since it was proved by Laffey [40, Sect. 1] that, for any integer $N \geq 1$, there exists a singular matrix $A \in R$ which does not factor as the product of fewer than N idempotents. (Surprisingly enough, this does not happen in $\mathcal{M}_n(\mathbb{Z})$ for $n \geq 3$, see [40, Sect. 2].)

Acknowledgements

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