Corrections to “Probabilistic state estimation for labeled continuous time Markov models with applications to attack detection”

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Abstract

This short note provides a correction for a flaw in the proof of Lemma 2 in [1]. The statement of Lemma 2 is correct by itself but its proof requires a slightly different definition of the e-transition probability matrix given in Definition 5 of [1].

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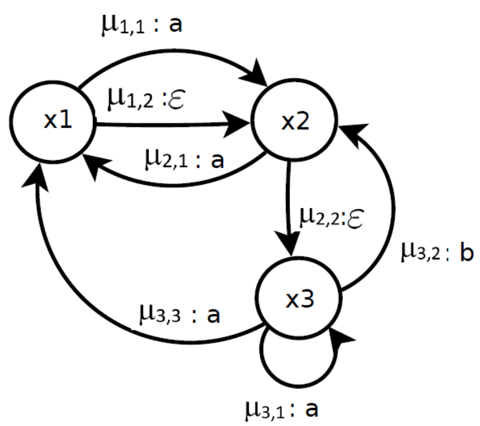

Figure 1: A labeled continuous-time Markov model.

In [1], there was a flaw in the proof of Lemma 2. The statement of Lemma 2 is correct by itself but its proof requires a slightly different definition of the e-transition probability matrix given in Definition 5. This note provides the corrections and adjust Examples 3 and 4 accordingly. The other results, proofs, and examples in [1] remain unchanged.

Correction to Definition 5 in [1]: Given an LCTMM $G = (X, E, \Lambda, \boldsymbol{\pi}_0)$, for each event $e \in E$ its *e-transition probability matrix* $Q_e = (q_{e,i,j}) \in \mathbb{R}_{\geq 0}^{n \times n}$ (where $q_{e,i,j}$ is the element of matrix Q_e in row i and column j) is defined by $q_{e,i,j} = \mu(x_i, e, x_j)$, where for $x_i \in X$, $e \in E$, and $x_j \in Post(x_i)$, we denote by $\mu(x_i, e, x_j)$ the sum of the firing rates of the e-transitions from state x_i to x_j ($\mu(x_i, e, x_j) = 0$ if no e-transition exists from state x_i to x_j). ▲

Correction to Example 3 in [1]: The a-transition and b-transition probability matrices of the LCTMM in Figure 1 with alphabet $E = \{a, b\}$ are the matrices Q_a and Q_b detailed below: ◇

$$Q_a = \begin{bmatrix} 0 & \mu_{1,1} & 0 \\ \mu_{2,1} & 0 & 0 \\ \mu_{3,3} & 0 & \mu_{3,1} \end{bmatrix}, \qquad Q_b = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & \mu_{3,2} & 0 \end{bmatrix}.$$

◇

Correction to Lemma 2 in [1]: Consider an LCTMM $G = (X, E, \Lambda, \boldsymbol{\pi}_0)$ and its e-transition probability matrices as in the revised Definition 5 above. Given an observation $\sigma = (e, t)$ with $e \in E$, it holds that:

$$\boldsymbol{\pi}(t \mid \boldsymbol{\pi}_0, \sigma) = \frac{\boldsymbol{\pi}(t^- \mid \boldsymbol{\pi}_0) \cdot Q_e}{\boldsymbol{\pi}(t^- \mid \boldsymbol{\pi}_0) \cdot Q_e \cdot \mathbf{1}_{n \times 1}}, \qquad (1)$$

where $\mathbf{1}_{n\times 1}$ is the all ones column vector of dimension n.

Proof. For each state x_j of the LCTMM it holds that

$$\begin{aligned}\pi_j(t \mid \boldsymbol{\pi}_0,(e,t)) &= \lim_{dt\to 0} Pr(x(t)=x_j \mid (e,(t-dt,t])) \\ &= \lim_{dt\to 0} \frac{Pr(x(t)=x_j \cap (e,(t-dt,t]))}{Pr((e,(t-dt,t]))} \\ &= \lim_{dt\to 0} \sum_{i=1}^{n} \frac{Pr((x(t)=x_j \cap (e,(t-dt,t])) \mid x(t-dt)=x_i)\cdot Pr(x(t-dt)=x_i)}{Pr((e,(t-dt,t]))}\end{aligned}$$

The numerator and denominator of the previous expression are reformulated.

- Given an infinitesimal interval dt, the quantity $q_{e,i,j}\cdot dt$ represents the probability that a transition to $x(t)=x_j$ occurs when event e is observed in interval $(t-dt,t]$ given that $x(t-dt)=x_i$. More formally, $Pr(x(t)=x_j \cap (e,(t-dt,t]) \mid x(t-dt)=x_i) = q_{e,i,j}\cdot dt$.
- On the other hand,

$$\begin{aligned}Pr((e,(t-dt,t])) &= \sum_{i=1}^{n} Pr((e,(t-dt,t]) \mid x(t-dt)=x_i)\cdot Pr(x(t-dt)=x_i) \\ &= \sum_{i=1}^{n} \left(\sum_{j=1}^{n} q_{e,i,j}.dt\right)\cdot Pr(x(t-dt)=x_i)\end{aligned}$$

Considering that $\lim_{dt\to 0} Pr(x(t-dt)=x_i) = \pi_i(t^- \mid \boldsymbol{\pi}_0)$, we have

$$\pi_j(t \mid \boldsymbol{\pi}_0,(e,t)) = \frac{\sum_{i=1}^{n} q_{e,i,j}\cdot \pi_i(t^- \mid \boldsymbol{\pi}_0)}{\sum_{j=1}^{n}\left(\sum_{i=1}^{n} q_{e,i,j}\cdot \pi_i(t^- \mid \boldsymbol{\pi}_0)\right)}$$

or equation (1) in matrix form. Observe that the denominator in equation (1) is nonzero because the event e has been observed at time t, i.e., there must exist a state x_i from which a transition labeled e may occur and such that $\pi_i(t- \mid \boldsymbol{\pi}_0) > 0$.

□

Correction to Example 4 in [1]: Consider the LCTMM in Figure 1 with sequence of observations $\sigma = (a,1)(b,3)(a,4)(a,5)$ within the time interval $[0,7]$. The state probabilities are reported in Figure 2.

In order to illustrate that the time stamps of the observations influence the probabilities of the states, consider also the sequence of observations $\sigma = (a,t_1)$ with several values of t_1 within the time interval $[0,4]$. Observe in Figure 3 that the probability of x_3 at time $t=4$ changes depending on the value of t_1.

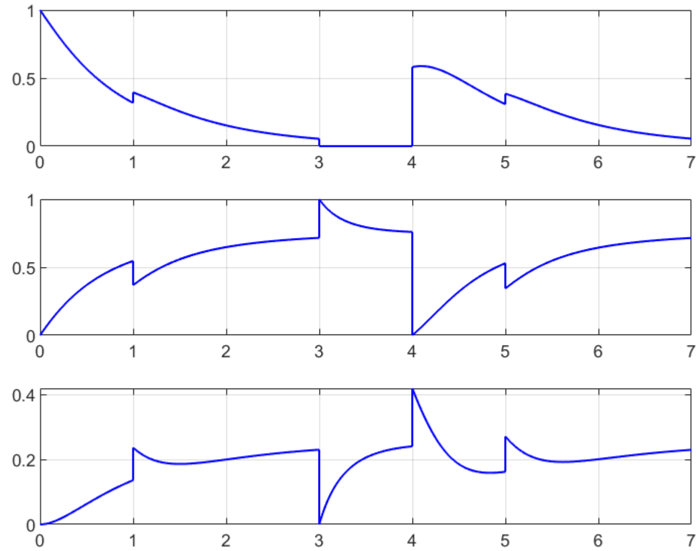

Figure 2: State probabilities with respect to $\sigma = (a, 1)(b, 3)(a, 4)(a, 5)$, x_1: top, x_2: center, x_3: bottom.

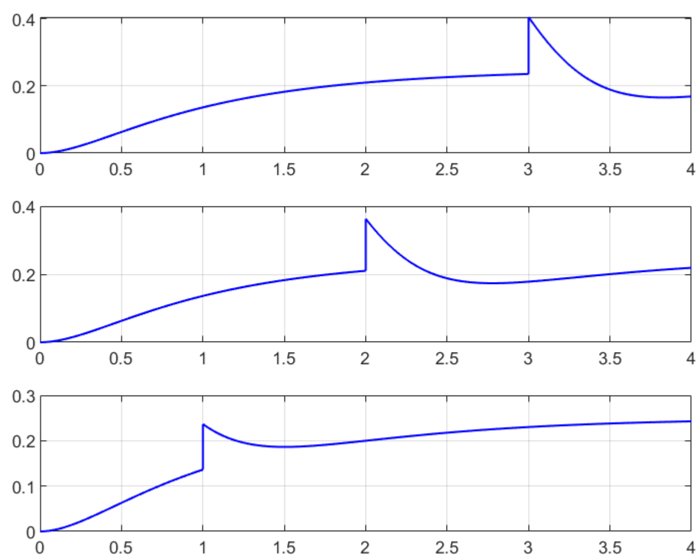

Figure 3: Probability of x_3 with respect to $\sigma = (a, t_1)$ with $t_1 = 3$ (top), $t_1 = 2$ (center) and $t_1 = 1$ (bottom).

Comments on the corrections : Let us consider some basic cases that explain and illustrate Definition 5 and Lemma 2.

Consider the LCTMM in Figure 4(a) with $\boldsymbol{\pi}_0 = [1\ 0\ 0]$ where a and b are two observable labels. As far as no label is observed at all up to time t, we have $\boldsymbol{\pi}(t^- \mid \boldsymbol{\pi}_0) = [1\ 0\ 0]$ because there exists no silent evolution from state x_1. When a label a is observed at t we will obtain $\boldsymbol{\pi}(t \mid \boldsymbol{\pi}_0, \sigma) = [0\ 1\ 0]$ with $\sigma = (a, t)$. According to Equation (1) this can be written as

$$\boldsymbol{\pi}(t \mid \boldsymbol{\pi}_0, \sigma) = \frac{\begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 & \mu & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}}{\begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 & \mu & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}} = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}. \tag{2}$$

Note that the probability $Pr((a, (t - dt, t])$ to observe a within $(t - dt, t]$ assuming that nothing was observed before time $t - dt$ (and consequently that the system stays at x_1 before $t - dt$) is equal to the probability that the delay of a is smaller than dt (which is μdt) and that the delay of b is greater than dt (which is $1 - \mu' dt$). Since the events a and b are independent and dt is an infinitesimal duration, we have: $Pr(a, dt) = (\mu dt) \cdot (1 - \mu' dt) = \mu dt - \mu\mu' dt^2 \approx \mu dt$.

Consider the LCTMM in Figure 4(b) with $\boldsymbol{\pi}_0 = [1\ 0\ 0]$ where a is the single observable label. As far as no label a is observed, we have $\boldsymbol{\pi}(t^- \mid \boldsymbol{\pi}_0) = [1\ 0\ 0]$. When a label a is observed at t we will obtain $\boldsymbol{\pi}(t \mid \boldsymbol{\pi}_0, \sigma)$ with $\sigma = (a, t)$ according to Equation (1):

$$\boldsymbol{\pi}(t \mid \boldsymbol{\pi}_0, \sigma) = \frac{\begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 & \mu & \mu' \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}}{\begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 & \mu & \mu' \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}} = \begin{bmatrix} 0 & \frac{\mu}{\Delta} & \frac{\mu'}{\Delta} \end{bmatrix} \tag{3}$$

with $\Delta = \mu + \mu.'$

Consider finally the LCTMM in Figure 4(c) with $\boldsymbol{\pi}_0 = [1\ 0\ 0]$. This example evolves exactly as the example in Figure 4(b) up to the first observation of the label a at time t. From that time, and despite the fact that no silent transition exists in this system, the probability of the states x_2 and x_3 will change depending on the values of μ and μ' and according to the extended ε sub chain of the system (Definition 4 in [1]). In particular, for a given value of time $t' \geq t$, there exists $\alpha_{t'} \in [0, 1]$ such that $\boldsymbol{\pi}(t'^- \mid \boldsymbol{\pi}_0, (a, t)) = [0\ \alpha_{t'}\ 1 - \alpha_{t'}]$. When a second label a is observed

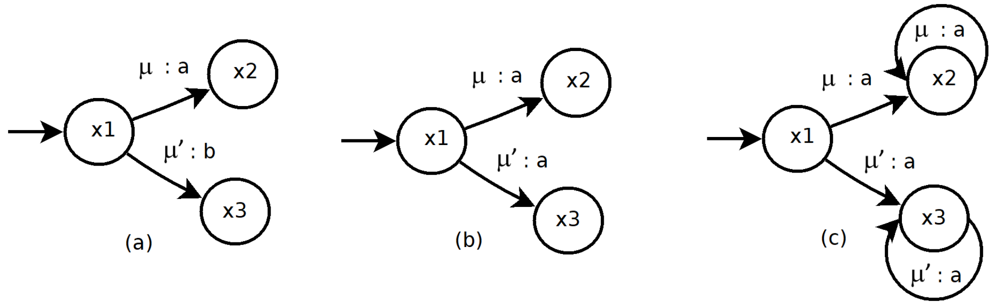

Figure 4: Three simple examples.

at t' we will obtain $\boldsymbol{\pi}(t' \mid \boldsymbol{\pi}_0, \sigma)$ with $\sigma = (a,t)(a,t')$ that can be written as

$$\boldsymbol{\pi}(t \mid \boldsymbol{\pi}_0, \sigma) = \frac{\begin{bmatrix} 0 & \alpha_{t'} & 1-\alpha_{t'} \end{bmatrix} \cdot \begin{bmatrix} 0 & \mu & \mu' \\ 0 & \mu & 0 \\ 0 & 0 & \mu' \end{bmatrix}}{\begin{bmatrix} 0 & \alpha_{t'} & 1-\alpha_{t'} \end{bmatrix} \cdot \begin{bmatrix} 0 & \mu & \mu' \\ 0 & \mu & 0 \\ 0 & 0 & \mu' \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}} = \begin{bmatrix} \frac{\mu\alpha_{t'}}{\Delta'} & 0 & \frac{\mu'(1-\alpha_{t'})}{\Delta'} \end{bmatrix} \tag{4}$$

with $\Delta' = \mu\alpha_{t'} + \mu'(1-\alpha_{t'})$.

Conflict of Interest: The authors declare that they have no conflict of interest.

References

[1] D. Lefebvre, C. Seatzu, C.N. Hadjicostis, A. Giua, "Probabilistic state estimation for labeled continuous time Markov models with applications to attack detection," *Discrete Event Systems*, Vol. 32, pp. 65-88, 2022.