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Boundary Sliding Mode Control of a Diffusion Process under Sampling and Hold: Chattering Analysis

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Abstract—This paper investigates discretization chattering effects in a distributed-parameter process governed by the diffusion PDE and equipped with a sliding-mode-based boundary controller. The collocated measurement is sampled in time and the boundary control signal is applied through a zero-order-hold element. The limit cycle modes constituting the chattering effects are determined exactly by exploiting the difference equation method, yielding their period and spatially dependent amplitude. Simulation results show that the theoretically determined limit cycle modes occur in the closed-loop system.

Index Terms—Chattering effect, distributed-parameter systems, diffusion process, boundary control, zero-order-hold method

I. INTRODUCTION

Sliding mode control (SMC) is well-known for its robust stabilization capabilities of systems affected by matched disturbances [1, 2]. In real-world applications, however, its main drawback becomes apparent when the controller is implemented in discrete time. The resulting finite switching frequency is one reason for the *chattering* phenomenon where the state trajectory oscillates in the vicinity of the sliding manifold with potentially harmful effects such as low control accuracy and high wear of moving mechanical parts [3, 4].

Hence, extensive research has been made regarding the analysis and avoidance of the chattering phenomenon [5–11]. Several methods exist regarding the determination of limit cycles in sampled-data systems. Some of these methods rely on the describing function approximation, whereas different approaches such as the difference equation method or the transform method allow the exact determination of limit cycle modes [12]. Existing results, however, mostly refer to chattering analysis in finite-dimensional systems. The recent achievements made in applying SMC to infinite-dimensional systems (see e.g. [13–17]) motivates the need for investigating the discretization chattering also in the infinite-dimensional systems setting.

In the present paper a sliding-mode based boundary controller is applied to a diffusion process by performing sampled measurements and applying the control signal via a zero-order-hold element. A similar scenario is addressed in [18], however, the considered control laws require the measurement of the whole state, whereas in the present

paper, only a collocated measurement at the controlled boundary is used. The discrete-time implementation results in discretization chattering, manifesting as periodic steady-state oscillations. It is shown that the considered closed-loop system indeed possesses such a periodic solution, and its shape and time evolution are exactly determined.

The paper is structured as follows: The considered system and feedback controller are presented in Section II. In Section III, the analysis of the chattering oscillations is performed. The obtained results are verified in Section IV by computer simulations, and conclusions are given in Section V.

A. NOTATION

Partial derivatives are denoted by indices, i.e. $\frac{\partial z}{\partial x} = z_x$, $\frac{\partial^2 z}{\partial x^2} = z_{xx}$, and time derivatives by $\frac{df}{dt} = \dot{f}$. The set of natural numbers is denoted by $\mathbb{N}_0$ if zero is included and $\mathbb{N}_1$ otherwise. The function identically equal to one is written as $\mathbb{1}(x)$.

II. PROBLEM FORMULATION

Consider the space- and time-varying scalar field $z(x, t)$ with the spatial variable $x \in [0, 1]$ and time variable $t \geq 0$. Let it be governed by the boundary-value problem (BVP)

$$z_t(x, t) = \theta z_{xx}(x, t), \quad (1a)$$

$$z_x(0, t) = -u(t), \quad (1b)$$

$$z_x(1, t) = 0 \quad (1c)$$

of Neumann type, where $\theta > 0$ is the diffusion coefficient and with initial condition

$$z(x, 0) = z_0(x). \quad (1d)$$

The variable $u(t)$ is the manipulable boundary control signal, applied through the Neumann-type boundary condition (1b). A continuous-time sliding controller capable of stabilizing the zero solution $z = 0$ was investigated in [19] in a more general setting where matched and unmatched disturbances also affect the controlled PDE. In the present paper, the same control law, formed by the parallel interconnection of a proportional controller and a relay controller, is implemented in discrete time and applied to the unperturbed continuous-time plant (1a)–(1c) via a zero-order-hold element. The control signal $u(t)$ thus takes the piecewise-constant form

$$u(t) = u(kT) =: u_k, \quad t \in [kT, (k+1)T) \quad (2)$$

where $T > 0$ is the discretization time and $k \in \mathbb{N}_0$.

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The sliding mode controller, consisting of a proportional and a discontinuous part, is given by

$$u_k = -k_c z_k(0) - M \operatorname{sign}(z_k(0)), \quad (3)$$

where z_k is the sampled state

$$z_k(x) := z(x, kT). \quad (4)$$

The constants

$$k_c \geq 0, \quad (5)$$

$$M > 0 \quad (6)$$

are tuning parameters for the proportional and discontinuous part, respectively.

III. CHATTERING ANALYSIS

First, the zero-order-hold discretization of the plant dynamics (1a)–(1c) in the time-domain is performed and based on this representation the chattering analysis is carried out.

A. Zero-Order-Hold Discretization

For performing zero-order-hold discretization, the solution of the BVP (1a)–(1c) subject to a piecewise-constant input is needed. To that end, a modal representation of the plant is considered. This representation demands homogeneous boundary conditions, hence, the state transformation

$$v(x, t) = z(x, t) - h(x)u(t) \quad (7)$$

with

$$h(x) = \frac{x^2}{2} - x \quad (8)$$

is applied. In the new coordinates the partial differential equation (1a) is given by

$$v_t(x, t) = \theta v_{xx}(x, t) + \theta \mathbb{1}(x)u(t) - h(x)\dot{u}(t), \quad (9a)$$

with homogeneous boundary conditions

$$v_x(0, t) = 0, \quad v_x(1, t) = 0. \quad (9b)$$

According to [20, Theorem 3.3.3], the classical solutions of (1a)–(1d) are related to the solutions of (9a) and (9b) by (7) when

$$v(x, 0) = z_0(x) - h(x)u(0). \quad (9c)$$

Next, the system (9a) and (9b) with (9c) is represented as an abstract initial value problem by introducing

$$\mathcal{A} = \frac{d^2}{dx^2} \quad (10a)$$

with the domain

$$D(\mathcal{A}) = \left\{ f \in L_2(0, 1) \mid f, \frac{df}{dx} \text{ are absolutely continuous, } \frac{d^2 f}{dx^2} \in L_2(0, 1) \text{ and } \frac{df}{dx}(0) = 0 = \frac{df}{dx}(1) \right\} \quad (10b)$$

Note that (10a) with (10b) is a self-adjoint Riesz-spectral operator with the eigenvalues

$$\lambda_i = -i^2 \pi^2, \quad i \in \mathbb{N}_0 \quad (11)$$

and eigenfunctions

$$\Phi_i(x) = \begin{cases} 1 & \text{for } i = 0 \\ \sqrt{2} \cos(i\pi x) & \text{for } i \in \mathbb{N}_1 \end{cases}. \quad (12)$$

In view of this, $\mathcal{A}$ has the representation

$$\mathcal{A}v = \sum_{i=0}^{\infty} \lambda_i v_i^*(t) \Phi_i \quad (13)$$

with $v_i^*(t) = \langle v, \Phi_i \rangle$ (see [20, Example 2.3.7]). Since the eigenfunctions Φ_i form a Riesz basis, the state v can be represented by

$$v(x, t) = \sum_{i=0}^{\infty} v_i^*(t) \Phi_i(x) \quad (14)$$

as stated in [20, Lemma 2.3.2]. Similarly, the input shape functions of u and $\dot{u}$ are written as

$$\mathbb{1}(x) = \sum_{i=0}^{\infty} \langle \mathbb{1}, \Phi_i \rangle \Phi_i(x) \quad (15)$$

and

$$h(x) = \sum_{i=0}^{\infty} \langle h, \Phi_i \rangle \Phi_i(x), \quad (16)$$

respectively. Combining (9a), (11) and (13)–(16) gives

$$\dot{v}_i^*(t) = -i^2 \pi^2 \theta v_i^*(t) + \theta \langle \mathbb{1}, \Phi_i \rangle u(t) - \langle h, \Phi_i \rangle \dot{u}(t). \quad (17)$$

Furthermore, writing

$$z(x, t) = \sum_{i=0}^{\infty} z_i^*(t) \Phi_i(x) \quad (18)$$

with $z_i^*(t) = \langle z, \Phi_i \rangle$ and considering (7), (14) and (16) yields

$$z_i^*(t) = v_i^*(t) + \langle h, \Phi_i \rangle u(t), \quad i \in \mathbb{N}_0. \quad (19)$$

Computing the time derivative of (19) and inserting (17) gives

$$\dot{z}_i^*(t) = -i^2 \pi^2 \theta v_i^*(t) + \theta \langle \mathbb{1}, \Phi_i \rangle u(t). \quad (20)$$

By considering (19) in (20)

$$\dot{z}_i^*(t) = -i^2 \pi^2 \theta z_i^*(t) + \theta \left(i^2 \pi^2 \langle h, \Phi_i \rangle + \langle \mathbb{1}, \Phi_i \rangle \right) u(t) \quad (21)$$

is obtained. Evaluating the scalar products $\langle h, \Phi_i \rangle$ and $\langle \mathbb{1}, \Phi_i \rangle$ results in

$$z_i^*(t) = \alpha_i z_i^*(t) + \beta_i u(t), \quad i \in \mathbb{N}_0 \quad (22)$$

where

$$\alpha_i = -i^2 \pi^2 \theta \quad \text{for } i \in \mathbb{N}_0 \quad (23)$$

$$\beta_i = \begin{cases} \theta & \text{for } i = 0 \\ \sqrt{2} \theta & \text{for } i \in \mathbb{N}_1 \end{cases}. \quad (24)$$

The solutions for the decoupled ODEs (22) are given by

$$z_i^*(t) = e^{\alpha_i t} \langle z_0, \Phi_i \rangle + \int_0^t e^{\alpha_i(t-\tau)} \beta_i u(\tau) d\tau \quad (25)$$

for $i \in \mathbb{N}_0$. By considering (2) and (4) zero-order-hold discretization provides

$$\langle z_{k+1}, \Phi_i \rangle = a_i \langle z_k, \Phi_i \rangle + b_i u_k, \quad i \in \mathbb{N}_0 \quad (26)$$

where

$$a_i = e^{-i^2 \pi^2 \theta T} \quad \text{for } i \in \mathbb{N}_0, \quad (27)$$

$$b_i = \begin{cases} \theta T & \text{for } i = 0 \\ \frac{\sqrt{2}}{i^2 \pi^2} (1 - a_i) & \text{for } i \in \mathbb{N}_1 \end{cases}. \quad (28)$$

By virtue of (18), equations (26) are combined into

$$z_{k+1}(x) = \mathcal{F} z_k(x) + b(x) u_k \quad (29)$$

where

$$\mathcal{F} = \sum_{i=0}^{\infty} a_i \langle \cdot, \Phi_i \rangle \Phi_i, \quad b(x) = \sum_{i=0}^{\infty} b_i \Phi_i. \quad (30)$$

Thereby the discrete-time plant description (29) is obtained via zero-order-hold discretization of the plant (1a)–(1c).

B. Main Result

The discrete-time plant description (29) enables representing the closed-loop dynamics with the controller (3) by the following difference equation

$$z_{k+1}(x) = \mathcal{F} z_k(x) - b(x) k_c z_k(0) - b(x) M \operatorname{sign}(z_k(0)) \quad (31)$$

It is now shown that (31) possesses a periodic solution with period two, and the parameters of this solution are calculated. The result is summarized in the following

Theorem 1: Let

$$|\delta(0)| \neq \kappa. \quad (32)$$

Then, the sequence

$$z_k(x) = (-1)^k \delta(x) + \kappa \quad (33)$$

is a solution of (31) if and only if

$$\delta(x) = \pm \frac{M \sum_{i=0}^{\infty} \frac{\bar{b}_i}{1+a_i} \Phi_i(x)}{1 - k_c \eta}, \quad (34)$$

$$|\kappa| \in \begin{cases} \{0\} & \text{for } 0 < k_c < \frac{1}{\eta} \\ [0, M\eta) & \text{for } k_c = 0 \end{cases} \quad (35)$$

where

$$\eta = \sum_{i=0}^{\infty} \frac{\bar{b}_i}{1+a_i}, \quad (36)$$

$$\bar{b}_i = \begin{cases} b_0 & \text{for } i = 0 \\ \sqrt{2} b_i & \text{for } i \in \mathbb{N}_1 \end{cases}. \quad (37)$$

Proof: The reference solution (33) is inserted into the closed loop (31) from which (34) and (35) will be

derived. The plant (29) is first brought back into its spectral representation by considering (18) and (29), yielding

$$\sum_{i=0}^{\infty} (a_i \langle z_k, \Phi_i \rangle + b_i u_k - \langle z_{k+1}, \Phi_i \rangle) \Phi_i = 0, \quad (38)$$

which is equivalent to (26). Inserting the reference solution (33) into (26) gives

$$\begin{aligned} & (-1)^{k+1} \langle \delta, \Phi_i \rangle + \langle \kappa, \Phi_i \rangle \\ & = (-1)^k a_i \langle \delta, \Phi_i \rangle + a_i \langle \kappa, \Phi_i \rangle + b_i u_k \end{aligned} \quad (39)$$

which simplifies to

$$\langle \delta, \Phi_i \rangle = -a_i \langle \delta, \Phi_i \rangle + (-1)^{k+1} b_i u_k \quad (40)$$

since

$$\langle \kappa, \Phi_i \rangle = \begin{cases} \kappa & \text{for } i = 0 \\ 0 & \text{for } i \in \mathbb{N}_1 \end{cases} \quad (41)$$

and $a_0 = 1$. Rearranging for $\langle \delta, \Phi_i \rangle$ yields

$$\langle \delta, \Phi_i \rangle = (-1)^{k+1} \frac{b_i}{1+a_i} u_k. \quad (42)$$

Applying (18) gives

$$\delta(x) = (-1)^{k+1} \sum_{i=0}^{\infty} \frac{b_i}{1+a_i} \Phi_i(x) u_k. \quad (43)$$

Evaluating (43) at $x = 0$ and inserting control law (3) results in

$$\delta(0) = (-1)^{k+1} \eta [-k_c z_k(0) - M \operatorname{sign}(z_k(0))]. \quad (44)$$

Inserting (33) and rearranging yields

$$\delta(0) = (-1)^k \frac{\eta}{1 - k_c \eta} [M \operatorname{sign}(z_k(0)) + k_c \kappa]. \quad (45)$$

Equation (45) has to hold for every $k \in \mathbb{N}_0$, hence

$$\delta(0) = (-1)^{k+1} \frac{\eta}{1 - k_c \eta} [M \operatorname{sign}(z_{k+1}(0)) + k_c \kappa]. \quad (46)$$

Addition of (45) and (46) gives

$$2\delta(0) = (-1)^k \frac{\eta}{1 - k_c \eta} M [\operatorname{sign}(z_k(0)) - \operatorname{sign}(z_{k+1}(0))] \quad (47)$$

and together with (33)

$$\begin{aligned} 2\delta(0) &= \frac{\eta}{1 - k_c \eta} M [\operatorname{sign}(\delta(0) + (-1)^k \kappa) \\ &\quad + \operatorname{sign}(\delta(0) - (-1)^k \kappa)]. \end{aligned} \quad (48)$$

From (44) it is obtained that

$$\delta(0) \neq 0, \quad (49)$$

since $\delta(0) = 0$ would render (44) to

$$0 = k_c \kappa + M \operatorname{sign}(\kappa) \quad (50)$$

which contradicts (5) and (6). Considering (49), it has to hold that

$$\operatorname{sign}(\delta(0) + (-1)^k \kappa) = \operatorname{sign}(\delta(0) - (-1)^k \kappa), \quad (51)$$

implying

$$|\kappa| < |\delta(0)|. \quad (52)$$

It also follows that

$$|\delta(0)| = \frac{\eta}{1 - k_c \eta} M. \quad (53)$$

Subtracting (45) and (46) and considering (51) leads to

$$0 = \frac{\eta}{1 - k_c \eta} k_c \kappa, \quad (54)$$

hence

$$k_c \kappa = 0 \quad (55)$$

since $\eta \neq 0$. Condition (35) follows directly from (52), (53) and (55). Inserting (3) and (33) into (43) and considering (55) and (53) multiplied by $\text{sign}(\delta(0))$ yields

$$\delta(x) = \frac{M \sum_{i=0}^{\infty} \frac{b_i}{1+a_i} \Phi_i(x)}{1 - k_c \eta} \text{sign}(\delta(0)). \quad (56)$$

Condition (34) constitutes the two possible cases of (56). The necessity of (34) and (35) is therefore shown. When assuming that (34) and (35) hold, the above calculations make it obvious that (33) is a solution of (31), showing the sufficiency of (34) and (35). ■

Remark 1: The provided analysis concerns only the *existence* of a periodic solution of period two. Whether or not such a solution can actually be observed depends on the stability of the limit cycle, for which no statements are made here. However, simulation studies—such as the one in Section IV—show that the closed-loop system state converges onto the reference solution (33) from an arbitrary initial state (if the controller gains are tuned according to (6) and $0 \leq k_c < \frac{1}{\eta}$).

Remark 2: The existence of η can be verified by writing (36) as

$$\eta = \frac{1}{2} \theta T + \frac{2}{\pi^2} \sum_{i=1}^{\infty} \frac{1}{i^2} \tanh\left(\frac{1}{2} \pi^2 \theta T i^2\right) \quad (57)$$

with the definitions in (27), (28) and (37). Since $0 < \tanh\left(\frac{1}{2} \pi^2 \theta T i^2\right) \leq 1$ for $i \in \mathbb{N}_1$ and $\sum_{i=1}^{\infty} \frac{1}{i^2}$ is convergent, the direct comparison test can be applied to show that the series in (57) is convergent as well.

Remark 3: Condition (32) ensures that $z_k(0) \neq 0$, $k \in \mathbb{N}_0$, holds for the sequence (33).

As (35) suggests, the bias κ of the oscillation (33) is zero in the case $k_c \in (0, \frac{1}{\eta})$, i.e. the oscillation takes place symmetrically around zero. If there is no proportional feedback present in (3), i.e. $k_c = 0$, the eigenvalue $a_0 = 1$ of the plant (29) is not shifted inside the unit circle. Therefore, the integrating behavior of the plant is preserved, allowing a possible bias level $\kappa \neq 0$ in (33) as it is also mentioned in [12]. The value of κ depends on the initialization (1d), however, it cannot exceed the bound given in (35).

The upper bound $\bar{k}_c := \frac{1}{\eta}$ for k_c in (35) becomes apparent when considering that (53) is positive and has a non-zero denominator. For an interpretation of the case $k_c = \bar{k}_c$, consider proportional feedback only. The resulting closed loop is

$$\langle z_{k+1}, \Phi_i \rangle = a_i \langle z_k, \Phi_i \rangle - b_i k_c z_k(0), \quad i \in \mathbb{N}_0, \quad (58)$$

obtained by inserting (3) with $M = 0$ into (26). The definition of the sequence $\mathbf{z}_k^* = (\langle z_k, \Phi_i \rangle)$, $i \in \mathbb{N}_0$, enables writing (58) as

$$\mathbf{z}_{k+1}^* = \mathbf{M} \mathbf{z}_k^* \quad (59)$$

with

$$\mathbf{M} = \mathbf{A} - \mathbf{b} \otimes k_c \mathbf{c}, \quad (60)$$

$$\mathbf{A} = \text{diag}(a_i), \quad i \in \mathbb{N}_0, \quad (61)$$

$$\mathbf{b} = (b_i), \quad i \in \mathbb{N}_0, \quad (62)$$

$$\mathbf{c} = (c_i), \quad c_i = \begin{cases} 1 & \text{for } i = 0 \\ \sqrt{2} & \text{for } i \in \mathbb{N}_1 \end{cases} \quad (63)$$

where $\otimes$ is the dyadic product. For analyzing the eigenvalues of $\mathbf{M}$,

$$\det(z\mathbf{I} - \mathbf{M}) = \det(z\mathbf{I} - \mathbf{A} + \mathbf{b} \otimes k_c \mathbf{c}) \quad (64)$$

is calculated, which, according to [21, Fact 2.14.2.], is given by

$$\begin{aligned} \det(z\mathbf{I} - \mathbf{M}) &= \det(z\mathbf{I} - \mathbf{A}) + k_c \mathbf{c} \cdot \text{adj}(z\mathbf{I} - \mathbf{A}) \mathbf{b} \\ &= \det(z\mathbf{I} - \mathbf{A}) \left(1 + k_c \mathbf{c} \cdot (z\mathbf{I} - \mathbf{A})^{-1} \mathbf{b}\right) \\ &= \left(\prod_{i=0}^{\infty} (z - a_i)\right) \left(1 + k_c \sum_{i=0}^{\infty} \frac{\bar{b}_i}{z - a_i}\right) \end{aligned} \quad (65)$$

where $\cdot$ denotes the inner product. By selecting $k_c = \bar{k}_c$ in (65) it holds that $\det(-\mathbf{I} - \mathbf{M}) = 0$, thus the matrix $\mathbf{M}$ in (59) possesses an eigenvalue at -1 , indicating a stability border. In the case with the original control law, i.e. $M > 0$, the closed loop (31) can also be written as

$$\mathbf{z}_{k+1}^* = \mathbf{M} \mathbf{z}_k^* - \mathbf{b} M \text{sign}(\mathbf{c} \cdot \mathbf{z}_k^*). \quad (66)$$

When viewing the term $-\mathbf{b} M \text{sign}(\mathbf{c} \cdot \mathbf{z}_k^*)$ as a bounded input, the stability of (66) is determined by the eigenvalues of $\mathbf{M}$ only. Thus, $k_c = \bar{k}_c$ remains a stability border.

From Theorem 1 two more statements are deduced in the following

Corollary 1: For the sequence (33) with (32), (34) and (35) it holds that

$$|z_k(0)| \in \begin{cases} \left\{ \frac{M\eta}{1 - k_c \eta} \right\} & \text{for } 0 < k_c < \frac{1}{\eta} \\ (0, 2M\eta) & \text{for } k_c = 0 \end{cases}. \quad (67)$$

Furthermore, it holds that

$$|z_k(x) - \kappa| \leq |z_k(0) - \kappa|. \quad (68)$$

Proof: Evaluating (33) at $x = 0$ and taking the absolute value yields

$$|z_k(0)| = |(-1)^k \delta(0) + \kappa|. \quad (69)$$

In the case $0 < k_c < \frac{1}{\eta}$, due to (53) and (55), it holds that

$$|z_k(0)| = \frac{M\eta}{1 - k_c\eta}, \quad (70)$$

whereas if $k_c = 0$, the triangle inequality gives

$$|z_k(0)| \leq |\delta(0)| + |\kappa|. \quad (71)$$

Statement (67) is obtained by considering (70) and (71) together with (35) and (53).

To prove (68), insert (33) to obtain

$$|\delta(x)| \leq |\delta(0)|. \quad (72)$$

For $|\delta(x)|$ it holds that, according to (34),

$$|\delta(x)| \leq \frac{M}{1 - k_c\eta} \sum_{i=0}^{\infty} \frac{b_i}{1 + a_i} |\Phi_i|. \quad (73)$$

Furthermore, by virtue of (12), (27), (28), (36) and (53) it holds that

$$|\delta(x)| \leq \frac{M\eta}{1 - k_c\eta} = |\delta(0)| \quad (74)$$

which proves (72) and therefore (68). ■

Statement (67) gives an exact value or a bound for the absolute value of the state at $x = 0$, depending on the parameter k_c . Furthermore, according to (68) the chattering amplitude is largest at $x = 0$. Hence, from (67) also a bound for $|z_k(x)|$ is found, i.e., for $x \in [0, 1]$

$$|z_k(x)| \in \begin{cases} \left[0, \frac{M\eta}{1 - k_c\eta}\right] & \text{for } 0 < k_c < \frac{1}{\eta} \\ [0, 2M\eta] & \text{for } k_c = 0 \end{cases}. \quad (75)$$

IV. NUMERICAL VERIFICATION

The analytically obtained results are verified numerically with $T = 0.3$ s and $\theta = 0.25 \frac{1}{s}$. In a first simulation the gains are chosen as $M = 1$, $k_c = 0$, i.e. without proportional feedback. System (31) is initialized according to (33) for $k = 0$ and $\kappa = 0.5M\eta$, fulfilling (35). Only a finite number of modes can be considered for the implementation, hence the difference equations are evaluated for $i = 0, \dots, 100$. The simulation result presented in Fig. 1 agrees with Theorem 1, stating that the solution of the initial value problem (31) is (33). Fig. 2 shows a simulation result where $z_0(x) = 1 - 0.1 \cos(2\pi x)$, i.e. the system is not initialized according to (33). However, for $k \rightarrow \infty$ it is observed that a chattering oscillation described by (33) takes place. According to Theorem 1 if such an oscillation occurs, it is subject to (34) and (35) and, furthermore, Corollary 1 holds. Statement (68) is obviously true in Fig. 1, and in Fig. 2 for $k \rightarrow \infty$. The evolution of $z_k(0)$ is visualized in Fig. 3a. Additionally, Fig. 3b shows the same evolution, but for a simulation where $M = 1$, $k_c = 1$, i.e. with proportional feedback. It can be verified that $k_c < \frac{1}{\eta}$ in this example. In both cases it is observed that (67) holds for $k \rightarrow \infty$.

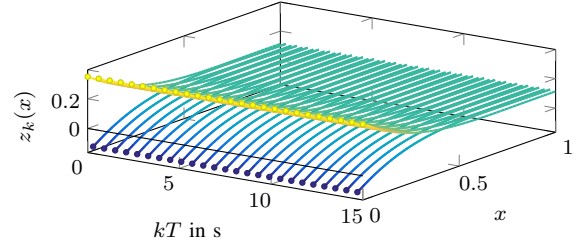


Fig. 1: If initialized accordingly, the trajectory stays on the reference solution. Since $k_c = 0$, a bias $\kappa \neq 0$ is possible.

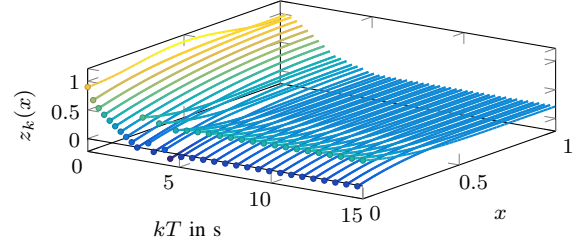


Fig. 2: If not initialized on the reference solution, a transient phase occurs after which the chattering oscillation takes place. The chattering amplitude is largest at $x = 0$.

For more insight as to why the oscillation occurs symmetrically around zero when $k_c \in (0, \frac{1}{\eta})$, see Fig. 4, which shows the time evolution of the first three modes, i.e. $\langle z_k, \Phi_i \rangle$ for $i \in \{0, 1, 2\}$. It is observed that the proportional feedback in Fig. 4b leads to a decay of the mode $\langle z_k, \Phi_0 \rangle$ whereas $\langle z_k, \Phi_0 \rangle$ remains with a bias in Fig. 4a. Note that this result is obtained in (55) without making any considerations about stability.

V. SUMMARY AND CONCLUSIONS

The effects of the discrete-time implementation of a sliding-mode controller, applied as boundary control to a dif-

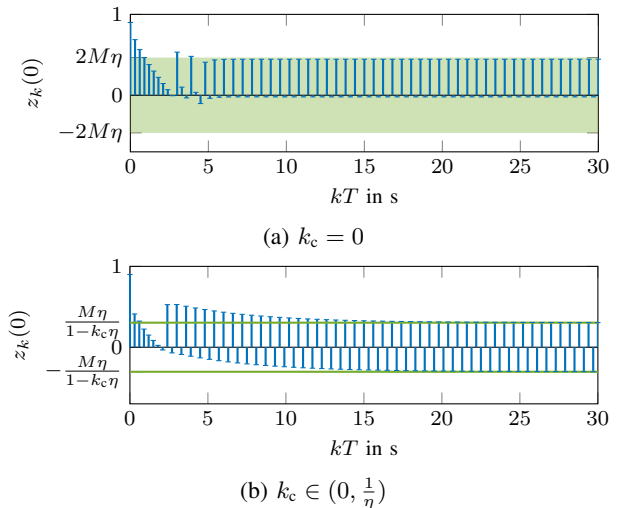


Fig. 3: The chattering oscillation obeys (67) in both cases $k_c = 0$ and $k_c \in (0, \frac{1}{\eta})$. In the latter case the oscillation takes place around zero.

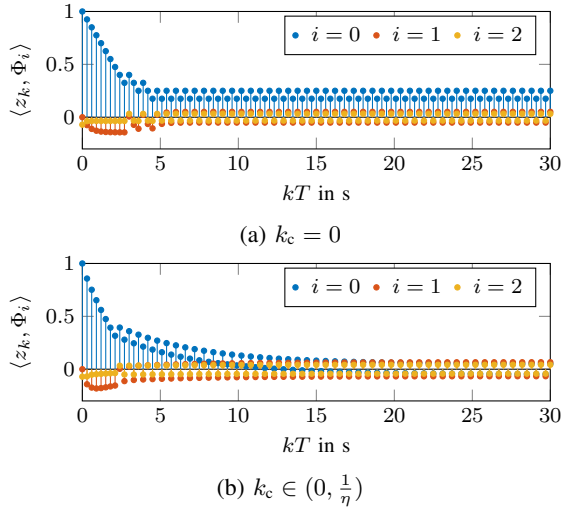


Fig. 4: For $k_c \in (0, \frac{1}{\eta})$ all modes $\langle z_k, \Phi_i \rangle$, $i \in \mathbb{N}_0$, oscillate around zero for $k \rightarrow \infty$ whereas for $k_c = 0$ this is only true for the modes $\langle z_k, \Phi_i \rangle$, $i \in \mathbb{N}_1$.

fusion process, were analyzed. Zero-order-hold discretization was performed on the plant in order to calculate the steady-state chattering oscillation of the closed loop. The analysis shows that the oscillation takes place symmetrically around zero if there is a proportional part present in the control law. This result is obtained without considerations about the stability of the limit cycles. Furthermore, a stability threshold for the proportional control parameter was identified by arithmetic considerations. With discontinuous feedback only, the oscillation can be offset by a constant, for which a bound is given. The impact of the controller parameters on the chattering amplitude is determined.

This paper discussed the existence of a periodic solution typically referred to as chattering, however, stability properties like the attractiveness of this solution are of great interest for future research. Moreover, taking a step beyond the analysis by considering chattering *avoidance* is an intriguing topic to address.

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