Proof Theory of
Paraconsistent Weak Kleene Logic

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Abstract

Paraconsistent Weak Kleene Logic (PWK) is the 3-valued propositional logic defined on the weak Kleene tables and with two designated values. Most of the existing proof systems for PWK are characterised by the presence of linguistic restrictions on some of their rules. This feature can be seen as a shortcoming. We provide a cut-free calculus (a hybrid between a natural deduction calculus and a sequent calculus) for PWK that is devoid of such provisos. Moreover, we introduce a Priest-style tableaux calculus for PWK.

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1 Introduction

Paraconsistent Weak Kleene Logic (PWK) belongs to the family of 3-valued propositional logics introduced by S.C. Kleene [26] and by other authors [6, 25, 35] in the middle decades of last century. These logics differ from one another under two respects: the semantics they offer for the propositional connectives and the choice of designated values. On the one hand, connectives can be given either a "strong" or a "weak" interpretation, each corresponding to a different set of truth tables on $\{1, \frac{1}{2}, 0\}$; on the other, one can designate only the value 1 ("true") or else the value 1 together with the value $\frac{1}{2}$ (generally interpreted as "meaningless" or "nonsensical"). If we opt for the set $\{1, \frac{1}{2}\}$ of designated values and for the weak interpretation of connectives, on which the value $\frac{1}{2}$ is *infectious* — namely, if a formula φ is assigned the value $\frac{1}{2}$, any formula where φ occurs is also assigned the value $\frac{1}{2}$ — what we obtain is the logic PWK.

Until very recently, PWK has been straggling behind the other logics in the Kleene family (strong Kleene logic SK, Priest's Logic of Paradox LP, and Bochvar's logic B) in terms of recognition by the logical and philosophical communities. The reason for this state of affairs is twofold. For a start, it is not easy to motivate a logic containing a value that is attached to "nonsensical" sentences but is, at the same time, designated. A lively philosophical discussion about this problem is still ongoing [25, 10, 3, 21, 28, 39, 12]. Secondly, PWK is amenable to a *double-barrelled analysis* in the sense of Sylvan [38] — in other words, it is ultimately obtained from imposing a "linguistic strainer" (a variable-inclusion filter) on classical logic (see Theorem 4 below). Logics of this sort get bad press in many nonclassical circles. According to Sylvan, for example, no substantive philosophical property of correct entailments, like relevance or meaning containment, can be pinned down by sifting classically valid entailments through a linguistic sieve. For these reasons, philosophers of logic have considered PWK as less interesting than its other Kleene cousins.

Over the last few years, the situation has changed. There has been a revival of interest for PWK, both from a semantic and from a proof-theoretic point of view. PWK has been thoroughly investigated from the point of view of Abstract Algebraic Logic [7, 8, 30], as well as from related semantical perspectives [11, 12, 13, 14, 15, 21]. On the proof-theoretic side, it has been given Hilbert-style [7, 8], natural deduction [34] and sequent [17] formulations. A common shortcoming of many of these calculi, however, is the presence of linguistic restrictions on some of their rules[1]. For example, the left introduction rule for conjunction in the sequent calculus in [17]:

$$(\wedge L)\,\frac{\phi, \psi, \Gamma \Rightarrow \Delta}{\phi \wedge \psi, \Gamma \Rightarrow \Delta}$$

is subject to the proviso that the propositional variables in φ or ψ also occur in Δ. These rules are somewhat opaque — rather than shedding light on the operational meaning of PWK connectives, they give a formal clothing to a rule of thumb "Just apply classical logic so long as this does not conflict with the prescribed variable-inclusion requirement". Therefore, it would be desirable to find calculi that avoid the recourse to such side conditions.

After dispatching some preliminaries in § 2, in § 3 and § 4 we provide a calculus that meets the above requirement, a calculus that is somewhere in between a sequent calculus and a natural deduction calculus. The key idea is to tweak the classical sequent calculus without Cut, a version of which is sound and complete with respect to a certain 3-valued semantics based on the *strong* Kleene tables [24, 16, 37], into a calculus that is sound and complete with respect to the same semantics, except that valuations map formulas to the *weak* Kleene tables. The upshot will be a calculus whose external consequence relation in the sense of Avron [1] coincides exactly with PWK.

In § 5, we introduce a Priest-style tableaux calculus for PWK that uses signed formulas. As opposed to the existing calculi for SK or LP, its distinctive features are the presence of semi-branching rules alongside the usual simple rules and branching rules, as well as of a third sign $*$ (which stands for "classical") alongside the customary signs $+$ (read as "designated") and $-$ (read as "non-designated"). Observe that the annotation $*$ is not to be interpreted as a truth

[1]The Hilbert style calculi in [8] do not have such restrictions, yet contain infinitely many rule schemata.

value, as it is customary to do in other kinds of labelled proof systems, but essentially as a guarantor that a formula *does not* take a particular truth value, namely $\frac{1}{2}$.

2 Preliminaries

In this section we review some known concepts and results concerning Kleene logics and abstract proof systems, only to such an extent as it is necessary for following the rest of this paper. The reader is referred to [41, 32, 21] for background on Kleene logics and to [23] for undefined notions of Abstract Algebraic Logic (AAL).

2.1 Kleene Logics

We recap in the next definition some concepts and notational conventions that will be used throughout the paper.

Definition 1 *1. If $\mathcal{L}$ is a propositional language, $\mathbf{Fm}(\mathcal{L})$ is the absolutely free algebra of formulas of $\mathcal{L}$, with universe $Fm(\mathcal{L})$ and with generators $Var(\mathcal{L})$.*

2. A sequent *on $\mathcal{L}$ is an ordered pair (Γ, Δ) of finite, possibly empty subsets of $Fm(\mathcal{L})$, written $\Gamma \Rightarrow \Delta$ for ease of notation.*

3. $Seq(\mathcal{L})$ is the set of all sequents on $\mathcal{L}$.

4. $\mathcal{L}_0$ is the propositional language with connectives $\neg$ (unary), $\wedge$ (binary), $\vee$ (binary).

Kleene logics are most conveniently introduced via the 3-valued Kleene matrices. The next definition presents the algebras **SK** (Strong Kleene) and **WK** (Weak Kleene); subsequently, the Kleene logics SK, LP, B, and PWK are defined as the matrix logics obtained by variation along two degrees of freedom, according as we choose **SK** or **WK** as the algebra reduct of their unique defining matrix, and according as we choose $\{1\}$ or $\{1, \frac{1}{2}\}$ as the set of designated values (see our Introduction).

Definition 2 **SK** *is the algebra of language $\mathcal{L}_0$ with universe $\{0, \frac{1}{2}, 1\}$ and operations defined by the following tables:*

$\neg$	
0	1
1/2	1/2
1	0

$\wedge$	0	1/2	1
0	0	0	0
1/2	0	1/2	1/2
1	0	1/2	1

$\vee$	0	1/2	1
0	0	1/2	1
1/2	1/2	1/2	1
1	1	1	1

WK *is the algebra of language* $\mathcal{L}_0$ *with universe* $\left\{0, \frac{1}{2}, 1\right\}$ *and operations defined by the following tables:*

$\neg$	
0	1
1/2	1/2
1	0

$\wedge$	0	1/2	1
0	0	1/2	0
1/2	1/2	1/2	1/2
1	0	1/2	1

$\vee$	0	1/2	1
0	0	1/2	1
1/2	1/2	1/2	1/2
1	1	1/2	1

Definition 3

- *Strong Kleene Logic (*SK*) is the logic* $\left(\mathbf{Fm}\left(\mathcal{L}_0\right), \vDash_{(\mathbf{SK},\{1\})}\right)$*;*
- *The Logic of Paradox (*LP*) is the logic* $\left(\mathbf{Fm}\left(\mathcal{L}_0\right), \vDash_{\left(\mathbf{SK},\left\{1,\frac{1}{2}\right\}\right)}\right)$*;*
- *Bochvar's Logic (*B*) is the logic* $\left(\mathbf{Fm}\left(\mathcal{L}_0\right), \vDash_{(\mathbf{WK},\{1\})}\right)$*;*
- *Paraconsistent Weak Kleene Logic (*PWK*) is the logic* $\left(\mathbf{Fm}\left(\mathcal{L}_0\right), \vDash_{\left(\mathbf{WK},\left\{1,\frac{1}{2}\right\}\right)}\right)$.

Homomorphisms $v : \mathbf{Fm}\left(\mathcal{L}_0\right) \to \mathbf{WK}$ will be called *valuations*, with no further qualification. For $\Gamma \subseteq Fm\left(\mathcal{L}_0\right)$, we use the shorthand notation $v\left[\Gamma\right]$ to denote the set $\left\{v\left(\gamma\right) : \gamma \in \Gamma\right\}$. Curly brackets will be dropped when $v\left[\Gamma\right]$ is a singleton. If $\Gamma \cup \{\varphi\} \subseteq Fm\left(\mathcal{L}_0\right)$, we let

$$Var\left(\varphi\right) = \left\{x : x \in Var\left(\mathcal{L}_0\right), x \text{ occurs in } \varphi\right\};$$
$$Var\left(\Gamma\right) = \bigcup \left\{Var\left(\varphi\right) : \varphi \in \Gamma\right\}.$$

By the notation $\varphi\left(x_1, ..., x_n\right)$ we underscore the fact that $Var\left(\varphi\right) \subseteq \left\{x_1, ..., x_n\right\}$. Observe that, as a consequence of Definition 3, for $\Gamma \cup \{\varphi\} \subseteq Fm\left(\mathcal{L}_0\right)$, $\Gamma \vDash_{\mathrm{PWK}} \varphi$, iff, for any valuation v, if $v\left(\psi\right) \in \left\{1, \frac{1}{2}\right\}$ for all $\psi \in \Gamma$, then $v\left(\varphi\right) \in \left\{1, \frac{1}{2}\right\}$.

The next theorem, due to Ciuni and Carrara [11] (however, see also [41, 29, 17, 12]), exposes PWK as a "filter logic" in the sense of [31], namely, as the result of an attempt to single out a class of correct entailments (according to some intuitive criterion of correctness which may differ from one logic to another) by sifting classically valid entailments through some kind of linguistic sieve, in this case a variable-inclusion criterion. As a consequence of this result, PWK has the same theorems as classical logic (hereafter CL).

Theorem 4 *For any* $\Gamma \cup \{\varphi\} \subseteq Fm\left(\mathcal{L}_0\right)$*,* $\Gamma \vDash_{\mathrm{PWK}} \varphi$ *iff either* $\vDash_{\mathrm{CL}} \varphi$ *or there exists a nonempty* $\Delta \subseteq \Gamma$ *such that* $\Delta \vDash_{\mathrm{CL}} \varphi$ *and* $Var\left(\Delta\right) \subseteq Var\left(\varphi\right)$*.*

By Theorem 4, PWK invalidates Explosion: $x, \neg x \nvDash_{\mathrm{PWK}} y$, where x and y are propositional variables. This gives our logic a paraconsistent character, whence its name. PWK has more peculiar failures as well; for example, it invalidates Conjunctive simplification: $x \wedge y \nvDash_{\mathrm{PWK}} x$ and $x \wedge y \nvDash_{\mathrm{PWK}} y$.

2.2 Abstract Proof Systems

In this paper, we will deflect from standard practice in algebraic logic under two respects. On the one hand, we will consider consequence relations between syntactic units other than formulas, in particular between sequents; on the other, we will use a calculus that is a hybrid between natural deduction (in that it contains both introduction and elimination rules) and the sequent calculus (since it contains both left and right rules). For such reasons, we need a notion of consequence relation that can encompass the derivability relation (among sequents) of this calculus, and a notion of proof system that subsumes natural deduction calculi, as well as sequent calculi and hybrid calculi like the one we present hereby. All these tools are provided by the theory of abstract consequence relations introduced by Blok and Jónsson [5]. What follows is a short résumé of its core ideas.

Definition 5 *An* abstract consequence relation (acr) *over the set A is a relation $\vdash \subseteq \wp(A) \times A$ obeying the following conditions for all $a \in A$ and for all $X, Y \subseteq A$:*

1. *$X \vdash a$ if $a \in X$* (Reflexivity)*;*
2. *If $X \vdash a$ and $X \subseteq Y$, then $Y \vdash a$* (Monotonicity)*;*
3. *If $Y \vdash a$ and $X \vdash b$ for every $b \in Y$, then $X \vdash a$* (Cut)*.*

Henceforth, we write $X \vdash Y$ to mean that $X \vdash b$ for every $b \in Y$.

Standard consequence relations can be viewed as acr's over the set $Fm(\mathcal{L})$ of formulas of some propositional language $\mathcal{L}$. Other typical choices for the set A are the set $(Fm(\mathcal{L}))^2$ of *equations* and the set $Seq(\mathcal{L})$ of *sequents* of some propositional language $\mathcal{L}$. Moreover, exactly like the canonical example of a consequence relation in AAL is the derivability relation of a Hilbert-style calculus, the canonical example of an acr is the derivability relation of an *abstract proof system* [27], a notion we define below.

Definition 6 *Let A be a set. An* inference *for A is an ordered pair (X, a), where $X \subseteq A$ is a finite or empty set (of premisses) and $a \in A$ is the conclusion. Inferences with no premisses are called* axioms. *An* (inference) rule *r for A is a set of inferences for A, which are called* instances *of r.*

A rule r for $Fm(\mathcal{L})$ is *schematic* if there are $\Gamma \cup \{\varphi\} \subseteq Fm(\mathcal{L})$ s.t. r is the set of all inferences $(\sigma[\Gamma], \sigma(\varphi))$, for σ a substitution on $\mathbf{Fm}(\mathcal{L})$. By extension, a rule r for $Seq(\mathcal{L})$ is *schematic* if there are $\{\Gamma_i \Rightarrow \Delta_i\}_{i \leq n} \cup \{\Gamma \Rightarrow \Delta\} \subseteq Seq(\mathcal{L})$ s.t. r is the set of all inferences

$$\left(\{\sigma[\Gamma_i] \Rightarrow \sigma[\Delta_i]\}_{i \leq n}, \sigma[\Gamma] \Rightarrow \sigma[\Delta]\right),$$

for σ a substitution on $\mathbf{Fm}(\mathcal{L})$.

Definition 7 *An* abstract proof system (aps) *C is an ordered pair (A, R) consisting of a set A and a set R of rules for A.*

Definition 8 *Let $C = (A, R)$ be an aps. A C-*derivation *$\mathcal{D}$ of $a \in A$ from $X \subseteq A$ is a finite tree labelled by members of A such that:*

1. *a labels the root;*
2. *For each node labelled a_0, either $a_0 \in X$ or its child nodes are labelled $a_1, ..., a_n$ and $(\{a_1, ..., a_n\}, a_0)$ is an instance of a rule of C.*

*If there is a C-derivation $\mathcal{D}$ of a from X, then a is said to be C-*derivable *from X, written $X \vdash_C a$. a is a* theorem *of C iff $\emptyset \vdash_C a$.*

It is not hard to prove the following result:

Theorem 9 *Let $C = (A, R)$ be an aps. Then $\vdash_C$ is an acr over A.*

Both natural deduction calculi (in sequent format) and sequent calculi make instances of aps's on sequents, whence their derivability relations (again, on sequents) are acr's by Theorem 9. In particular:

Definition 10 *The sequent calculus LK for classical propositional logic can be defined as the aps $(Seq(\mathcal{L}_0), R)$, where R is the following set of rules (with the customary notational conventions):*

$$(Id)\ \frac{}{\phi \Rightarrow \phi} \qquad (Cut)\ \frac{\Gamma \Rightarrow \Delta, \phi \qquad \phi, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}$$

$$(WL)\ \frac{\Gamma \Rightarrow \Delta}{\phi, \Gamma \Rightarrow \Delta} \qquad (WR)\ \frac{\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \phi}$$

$$(\neg L)\ \frac{\Gamma \Rightarrow \Delta, \phi}{\neg\phi, \Gamma \Rightarrow \Delta} \qquad (\neg R)\ \frac{\phi, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \neg\phi}$$

$$(\wedge L)\ \frac{\phi, \psi, \Gamma \Rightarrow \Delta}{\phi \wedge \psi, \Gamma \Rightarrow \Delta} \qquad (\wedge R)\ \frac{\Gamma \Rightarrow \Delta, \phi \qquad \Gamma \Rightarrow \Delta, \psi}{\Gamma \Rightarrow \Delta, \phi \wedge \psi}$$

$$(\vee L)\ \frac{\phi, \Gamma \Rightarrow \Delta \qquad \psi, \Gamma \Rightarrow \Delta}{\phi \vee \psi, \Gamma \Rightarrow \Delta} \qquad (\vee R)\ \frac{\Gamma \Rightarrow \Delta, \phi, \psi}{\Gamma \Rightarrow \Delta, \phi \vee \psi}$$

Definition 11 *The sequent calculus LK^- is the aps obtained from LK by deleting the rule of Cut.*

Observe that, by Definition 8, $\{\Gamma_i \Rightarrow \Delta_i\}_{i \in I} \vdash_{LK} \Gamma \Rightarrow \Delta$ iff $\Gamma \Rightarrow \Delta$ is provable in the calculus obtained from LK by taking the $\Gamma_i \Rightarrow \Delta_i$'s as additional axioms. Also, observe that the aps LK^- is a *different* aps with respect to LK and gives rise to a *different* acr. For example, $x \Rightarrow y, y \Rightarrow z \vdash_{LK} x \Rightarrow z$, while $x \Rightarrow y, y \Rightarrow z \nvdash_{LK^-} x \Rightarrow z$, for propositional variables x, y, z. (This failure of transitivity at the level of sequents, which is a direct consequence of removing

Cut from LK, is a hallmark feature of LK^-.) In view of Gentzen's Hauptsatz, both acr's have the same *theorems*: $\vdash_{LK} \Gamma \Rightarrow \Delta$ iff $\vdash_{LK^-} \Gamma \Rightarrow \Delta$.

In [20] it is shown that the aps LK_S^-, which can be obtained from LK^- by adding to it the inverses of the logical rules for negation, conjunction and disjunction[2], is complete with respect to a 3-valued semantics based on strong Kleene tables, introduced by Girard and taken up again by Cobreros, Egré, Ripley and van Rooij (see the discussion in the Introduction and the references given therein). As a corollary, it follows that the external relation of this calculus, defined by $\Gamma \vdash^E_{LK_S^-} \varphi$ iff $\{\Rightarrow \gamma : \gamma \in \Gamma\} \vdash_{LK_S^-} \Rightarrow \varphi$, coincides with the consequence relation of LP, a result that had been proved independently by Pynko [36] and by Barrio et al. [2] (see also [32, p. 78]).

3 The Calculus for PWK and Its Semantics

There is nothing in Girard's 3-valued semantics that constrains its application to the *strong* Kleene tables. We will immediately see that all the relevant definitions can be recast in terms of valuations on the *weak* tables. In particular, given a sequent $\Gamma \Rightarrow \Delta$ on $\mathcal{L}_0$, we can stipulate that it is satisfied by one such valuation v in case $v(\delta) \in \left\{1, \frac{1}{2}\right\}$ for some $\delta \in \Delta$ whenever $v(\gamma) = 1$ for all $\gamma \in \Gamma$. We also require that $\Gamma \Rightarrow \Delta$ follows from the premiss-sequents in the set X if and only if it is satisfied (in the above sense) by all valuations that satisfy each member of X. The guiding idea is to tweak the calculus LK_S^- into a calculus that is complete with respect to this 3-valued semantics on the weak Kleene tables, in such a way as to obtain the consequence relation of PWK as its external relation. This task is nontrivial, since on the one hand some of the rules of LK_S^- (e.g. both inverses of the rules $\wedge R$) become unsound once we trade the strong semantics for the weak one; on the other hand, the calculus LK_S^- must be supplemented by some extra rules, because e.g. $\varphi, \neg\varphi \vee \psi \nvdash_{\mathrm{LP}} \varphi \wedge \psi$ (hence $\varphi, \neg\varphi \vee \psi \nvdash^E_{LK_S^-} \varphi \wedge \psi$), while $\varphi, \neg\varphi \vee \psi \vdash_{\mathrm{PWK}} \varphi \wedge \psi$.

After submitting this paper, we learnt[3] that other contributions to the literature have followed similar paths. Definition 12 below can be found, essentially, in Correia [18] and Szmuc and Ferguson [40]. Ferguson [22] observes that the rules $\wedge R_{1-2}$ and $\vee L_{1-2}$ are sound w.r.t. this semantics and conjectures that completeness can be attained by adding them to LK^-.

We now proceed to laying down the formal definitions.

Definition 12 *1. A valuation v* W-satisfies *a sequent $\Gamma \Rightarrow \Delta$ on $\mathcal{L}_0$ (in symbols, $v \vDash_W \Gamma \Rightarrow \Delta$) iff either there is $\gamma \in \Gamma$ s.t. $v(\gamma) \in \left\{0, \frac{1}{2}\right\}$ or there is $\delta \in \Delta$ s.t. $v(\delta) \in \left\{1, \frac{1}{2}\right\}$.*

2. A sequent $\Gamma \Rightarrow \Delta$ on $\mathcal{L}_0$ is W-valid *(in symbols, $\vDash_W \Gamma \Rightarrow \Delta$) if $v \vDash_W \Gamma \Rightarrow \Delta$ for all valuations v.*

[2] Recall that the inverse of the one-premiss schematic rule $r = (\{S\}, S')$ is the rule $r' = (\{S'\}, S)$ obtained from r by swapping premiss and conclusion. A two-premiss rule $(\{S_1, S_2\}, S)$ has two inverses, $(\{S\}, S_1)$ and $(\{S\}, S_2)$, respectively.

[3] We thank T. Ferguson, D. Szmuc and an anonymous referee for these pointers.

3. *A sequent* $\Gamma \Rightarrow \Delta$ *on* $\mathcal{L}_0$ *is a* W-consequence *of a set of sequents* X *(in symbols,* $X \vDash_W \Gamma \Rightarrow \Delta$*) if every valuation that W-satisfies all members of* X *also satisfies* $\Gamma \Rightarrow \Delta$*.*

Definition 13 LK_W^- *is the aps* $(Seq(\mathcal{L}_0), R')$*, where* R' *is following set of rules:*

$$(Id)\ \frac{}{\phi \Rightarrow \phi}$$

$$(WL)\ \frac{\Gamma \Rightarrow \Delta}{\phi, \Gamma \Rightarrow \Delta} \qquad (WR)\ \frac{\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \phi}$$

$$(\neg L)\ \frac{\Gamma \Rightarrow \Delta, \phi}{\neg\phi, \Gamma \Rightarrow \Delta} \qquad (\neg R)\ \frac{\phi, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \neg\phi}$$

$$(\neg L^{\partial})\ \frac{\Gamma \Rightarrow \Delta, \neg\phi}{\phi, \Gamma \Rightarrow \Delta} \qquad (\neg R^{\partial})\ \frac{\neg\phi, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \phi}$$

$$(\wedge L)\ \frac{\phi, \psi, \Gamma \Rightarrow \Delta}{\phi \wedge \psi, \Gamma \Rightarrow \Delta} \qquad (\wedge R_1)\ \frac{\Gamma \Rightarrow \Delta, \phi \qquad \phi, \Gamma \Rightarrow \Delta, \psi}{\Gamma \Rightarrow \Delta, \phi \wedge \psi}$$

$$(\wedge R_2)\ \frac{\Gamma \Rightarrow \Delta, \phi \qquad \phi, \Gamma \Rightarrow \Delta, \psi}{\Gamma \Rightarrow \Delta, \psi \wedge \phi}$$

$$(\wedge L^{\partial})\ \frac{\phi \wedge \psi, \Gamma \Rightarrow \Delta}{\phi, \psi, \Gamma \Rightarrow \Delta} \qquad (\wedge R_1^{\partial})\ \frac{\Gamma \Rightarrow \Delta, \phi \wedge \psi}{\Gamma \Rightarrow \Delta, \phi, \psi}$$

$$(\wedge R_2^{\partial})\ \frac{\Gamma \Rightarrow \Delta, \phi \wedge \psi}{\phi, \Gamma \Rightarrow \Delta, \psi} \qquad (\wedge R_3^{\partial})\ \frac{\Gamma \Rightarrow \Delta, \phi \wedge \psi}{\psi, \Gamma \Rightarrow \Delta, \phi}$$

$$(\vee L_1)\ \frac{\phi, \Gamma \Rightarrow \Delta, \psi \qquad \psi, \Gamma \Rightarrow \Delta}{\phi \vee \psi, \Gamma \Rightarrow \Delta}$$

$$(\vee L_2)\ \frac{\phi, \Gamma \Rightarrow \Delta, \psi \qquad \psi, \Gamma \Rightarrow \Delta}{\psi \vee \phi, \Gamma \Rightarrow \Delta} \qquad (\vee R)\ \frac{\Gamma \Rightarrow \Delta, \phi, \psi}{\Gamma \Rightarrow \Delta, \phi \vee \psi}$$

$$(\vee L_1^{\partial})\ \frac{\phi \vee \psi, \Gamma \Rightarrow \Delta}{\phi, \psi, \Gamma \Rightarrow \Delta} \qquad (\vee L_2^{\partial})\ \frac{\phi \vee \psi, \Gamma \Rightarrow \Delta}{\phi, \Gamma \Rightarrow \Delta, \psi}$$

$$(\vee L_3^{\partial})\ \frac{\phi \vee \psi, \Gamma \Rightarrow \Delta}{\psi, \Gamma \Rightarrow \Delta, \phi} \qquad (\vee R^{\partial})\ \frac{\Gamma \Rightarrow \Delta, \phi \vee \psi}{\Gamma \Rightarrow \Delta, \phi, \psi}$$

We now list a few easy facts about LK_W^- and its derivability relation.

Lemma 14 *1. The standard right introduction rule for conjunction and the* *standard left introduction rule for disjunction:*

$$(\wedge R)\ \frac{\Gamma \Rightarrow \Delta, \phi \qquad \Gamma \Rightarrow \Delta, \psi}{\Gamma \Rightarrow \Delta, \phi \wedge \psi}$$

$$(\vee L)\ \frac{\phi, \Gamma \Rightarrow \Delta \qquad \psi, \Gamma \Rightarrow \Delta}{\phi \vee \psi, \Gamma \Rightarrow \Delta}$$

are derivable in LK^-_W.

2. $\vdash_{LK^-} \subset \vdash_{LK^-_W} \subset \vdash_{LK}$.

3. *For any* $S \in Seq(\mathcal{L}_0)$, $\vdash_{LK^-_W} S$ *iff* $\vdash_{LK} S$.

4. *Cut is admissible in* LK^-_W.

5. $\wedge R_{1-2}$ *are jointly equivalent to the three-premiss* invertible *rule*

$$(\wedge R_3)\ \frac{\Gamma \Rightarrow \Delta, \phi, \psi \qquad \phi, \Gamma \Rightarrow \Delta, \psi \qquad \psi, \Gamma \Rightarrow \Delta, \phi}{\Gamma \Rightarrow \Delta, \phi \wedge \psi}$$

Proof. (1) Given the premisses $\Gamma \Rightarrow \Delta, \varphi$ and $\Gamma \Rightarrow \Delta, \psi$, apply WL to the latter to get $\varphi, \Gamma \Rightarrow \Delta, \psi$; an application of $\wedge R_1$ delivers the conclusion $\Gamma \Rightarrow \Delta, \varphi \wedge \psi$. For the other rule we argue analogously.

(2) By item (1), $\vdash_{LK^-} \subseteq \vdash_{LK^-_W}$. It is not hard to check that all the rules of LK^-_W are derivable in LK. For example, consider $\wedge R_1$. Assume in LK the premisses $\Gamma \Rightarrow \Delta, \varphi$ and $\varphi, \Gamma \Rightarrow \Delta, \psi$ and derive via WR and Cut $\Gamma \Rightarrow \Delta, \psi$. From this sequent and $\Gamma \Rightarrow \Delta, \varphi$, an application of $\wedge R$ yields $\Gamma \Rightarrow \Delta, \varphi \wedge \psi$. Therefore $\vdash_{LK^-_W} \subseteq \vdash_{LK}$. Moreover, these inclusions are strict. In fact, given three distinct propositional variables x, y, z, we have that $x \Rightarrow y, y \Rightarrow z \vdash_{LK} x \Rightarrow z$, while $x \Rightarrow y, y \Rightarrow z \nvdash_{LK^-_W} x \Rightarrow z$; on the other, $\Rightarrow x \vee y \vdash_{LK^-_W} \Rightarrow x, y$, while $\Rightarrow x \vee y \nvdash_{LK^-} \Rightarrow x, y$. Observe that we are presupposing Theorem 15 below in our non-derivability claim for LK^-_W, while the one for LK^- can be verified by inspection of the rules.

(3) All the rules of LK^-_W are classically sound, whence $\vdash_{LK^-_W} S$ only if $\vdash_{LK} S$. The converse direction follows by item (2). In fact, suppose that $\vdash_{LK} S$; then $\vdash_{LK^-} S$ (by Cut elimination for LK) and therefore $\vdash_{LK^-_W} S$.

(4) From item (3).

(5) Suppose $\wedge R_3$ is given and assume the premisses $\Gamma \Rightarrow \Delta, \varphi$ and $\varphi, \Gamma \Rightarrow \Delta, \psi$ of $\wedge R_1$. Then

$$\frac{\dfrac{\Gamma \Rightarrow \Delta, \phi}{\psi, \Gamma \Rightarrow \Delta, \phi} \qquad \phi, \Gamma \Rightarrow \Delta, \psi \qquad \dfrac{\Gamma \Rightarrow \Delta, \phi}{\Gamma \Rightarrow \Delta, \phi, \psi}}{\Gamma \Rightarrow \Delta, \phi \wedge \psi}.$$

The conclusion of $\wedge R_2$ is obtained similarly. For the other direction, suppose that $\wedge R_1$ and $\wedge R_2$ are given and assume the premisses (i) $\Gamma \Rightarrow \Delta, \varphi, \psi$, (ii) $\varphi, \Gamma \Rightarrow \Delta, \psi$ and (iii) $\psi, \Gamma \Rightarrow \Delta, \varphi$ of $\wedge R_3$. Then by $\wedge R_1$ we obtain that:

$$(\wedge R_1)\ \frac{\Gamma \Rightarrow \Delta, \phi, \psi \qquad \phi, \Gamma \Rightarrow \Delta, \psi}{\Gamma \Rightarrow \Delta, \phi \wedge \psi, \psi}.$$

Applying $\wedge R_2$ we have that:

$$(\wedge R_2)\ \frac{\Gamma \Rightarrow \Delta, \phi, \psi \qquad \psi, \Gamma \Rightarrow \Delta, \phi}{\Gamma \Rightarrow \Delta, \phi \wedge \psi, \phi}.$$

A final application of the derivable rule $\wedge R$ yields:

$$(\wedge R)\ \frac{\Gamma \Rightarrow \Delta, \phi \wedge \psi, \psi \qquad \Gamma \Rightarrow \Delta, \phi \wedge \psi, \phi}{\Gamma \Rightarrow \Delta, \phi \wedge \psi}.$$

■

Observe that, as a consequence of Lemma 14.(4), the calculus LK_W^- can be viewed as the result of adding to its fragment consisting of its structural rules and of its left and right introduction rules, all the *inverses* of such rules, in the same way as LK_S^- is obtained out of LK^-.

4 Soundness and Completeness

We first prove the soundness of LK_W^- with respect to our 3-valued semantics.

Theorem 15 *Let* $\{\Pi_i \Rightarrow \Lambda_i\}_{i \leq n} \cup \{\Gamma \Rightarrow \Delta\} \subseteq Seq(\mathcal{L}_0)$. *Then*

$$\{\Pi_i \Rightarrow \Lambda_i\}_{i \leq n} \vdash_{LK_W^-} \Gamma \Rightarrow \Delta \textit{ implies } \{\Pi_i \Rightarrow \Lambda_i\}_{i \leq n} \vDash_W \Gamma \Rightarrow \Delta.$$

Proof. Induction on the length of the LK_W^--derivation of $\Gamma \Rightarrow \Delta$ from $\{\Pi_i \Rightarrow \Lambda_i\}_{i \leq n}$. It is enough to check that any valuation W-satisfies the axiom of LK_W^- and that all rules preserve W-satisfaction by a given valuation. The weakening rules WL, WR are trivial and the disjunction rules are left to the reader. Let us check the remaining rules.

(Id) Given any valuation v and any $\varphi \in Fm(\mathcal{L}_0)$, it is not possible that both $v(\varphi) = 1$ and $v(\varphi) = 0$.

$(\neg L)$ Suppose that $v \vDash_W \Gamma \Rightarrow \Delta, \varphi$ and that $v[\{\neg\varphi\} \cup \Gamma] = 1$, $v[\Delta] = 0$. Then $v(\varphi) = 0$, whence $v[\Gamma] = 1$, $v[\Delta \cup \{\varphi\}] = 0$, a contradiction.

$(\neg R)$ Similar.

$(\neg R^\partial)$ Suppose that $v \vDash_W \neg\varphi, \Gamma \Rightarrow \Delta$ and that $v[\Gamma] = 1$, $v[\Delta \cup \{\varphi\}] = 0$. Then $v(\neg\varphi) = 1$, whence $v[\{\neg\varphi\} \cup \Gamma] = 1$, $v[\Delta] = 0$, a contradiction.

$(\neg L^\partial)$ Similar.

$(\wedge L)$ Suppose that $v \vDash_W \varphi, \psi, \Gamma \Rightarrow \Delta$ and that $v[\{\varphi \wedge \psi\} \cup \Gamma] = 1$, $v[\Delta] = 0$. Then $v(\varphi) = v(\psi) = 1$, whence $v[\{\varphi\} \cup \{\psi\} \cup \Gamma] = 1$, $v[\Delta] = 0$, a contradiction.

$(\wedge R_{1-2})$ Suppose that $v \vDash_W \Gamma \Rightarrow \Delta, \varphi$, that $v \vDash_W \varphi, \Gamma \Rightarrow \Delta, \psi$ and that $v[\Gamma] = 1$, $v[\Delta \cup \{\varphi \wedge \psi\}] = 0$. Then either $v(\varphi) = 0$, or ($v(\varphi) = 1$ and $v(\psi) = 0$). If the former, then $v[\Gamma] = 1$, $v[\Delta \cup \{\varphi\}] = 0$, a contradiction. If the latter, then $v[\{\varphi\} \cup \Gamma] = 1$, $v[\Delta \cup \{\psi\}] = 0$, also a contradiction. The rule $\wedge R_2$ is checked similarly.

$(\wedge L^\partial)$ Suppose that $v \vDash_W \varphi \wedge \psi, \Gamma \Rightarrow \Delta$ and that $v[\{\varphi\} \cup \{\psi\} \cup \Gamma] = 1$, $v[\Delta] = 0$. Then $v(\varphi \wedge \psi) = 1$, whence $v[\{\varphi \wedge \psi\} \cup \Gamma] = 1$, $v[\Delta] = 0$, a contradiction.

($\wedge R^{\partial}_{1-3}$) Suppose throughout that $v \vDash_W \Gamma \Rightarrow \Delta, \varphi \wedge \psi$. If $v[\Gamma] = 1$, $v[\Delta \cup \{\varphi\} \cup \{\psi\}] = 0$, then $v[\Gamma] = 1$, $v[\Delta \cup \{\varphi \wedge \psi\}] = 0$, a contradiction. The very same contradiction obtains if $v[\Gamma \cup \{\psi\}] = 1$, $v[\Delta \cup \{\varphi\}] = 0$ or if $v[\Gamma \cup \{\varphi\}] = 1$, $v[\Delta \cup \{\psi\}] = 0$. ■

Corollary 16 *For any* $\Gamma \cup \{\varphi\} \subseteq Fm(\mathcal{L}_0)$, $\{\Rightarrow \gamma : \gamma \in \Gamma\} \vdash_{LK^-_W} \Rightarrow \varphi$ *implies* $\Gamma \vDash_{\mathrm{PWK}} \varphi$.

Proof. By Theorem 15, $\{\Rightarrow \gamma : \gamma \in \Gamma\} \vdash_{LK^-_W} \Rightarrow \varphi$ implies $\{\Rightarrow \gamma : \gamma \in \Gamma\} \vDash_W \Rightarrow \varphi$, a condition which obtains exactly in case $\Gamma \vDash_{\mathrm{PWK}} \varphi$. ■

As usual, completeness is a bit trickier to establish. Actually, we will not prove the converse of Theorem 15 in full generality, but a weaker condition which will be sufficient to attain the converse of Corollary 16. For a start, we prove a number of lemmas whose upshot is a sort of "hourglass lemma" for LK^-_W: every sequent $\Gamma \Rightarrow \Delta$ can be derived from itself in LK^-_W by first decomposing it into atomic sequents whose formulas are exactly the propositional variables occurring in $\Gamma \cup \Delta$, and then reassembling $\Gamma \Rightarrow \Delta$ out of such atomic sequents.

Lemma 17 *Let* $\Gamma \Rightarrow \Delta \in Seq(\mathcal{L}_0)$ *be such that* $\varphi(x_1, \ldots, x_n) \in \Gamma \cup \Delta$. *Then there exists a derivation* $\mathcal{D}$ *of* $\Gamma \Rightarrow \Delta$ *from itself in* LK^-_W *such that:*

1. *if* $\varphi(x_1, \ldots, x_n) \in \Delta$, *then for every branch* $\mathcal{B}_i \in \mathcal{D}$ *there exists a node* $n_i \in \mathcal{B}_i$ *labelled by a sequent of the form* $\Gamma, V_i^1 \Rightarrow V_i^2, \Delta \smallsetminus \{\varphi(x_1, \ldots, x_n)\}$ *such that* $V_i^1 \cup V_i^2 = \{x_1 \ldots x_n\}$;
2. *if* $\varphi(x_1, \ldots, x_n) \in \Gamma$, *then for every branch* $\mathcal{B}_i \in \mathcal{D}$ *there exists a node* $n_i \in \mathcal{B}_i$ *labelled by a sequent of the form* $\Gamma \smallsetminus \{\varphi(x_1, \ldots, x_n)\}, V_i^1 \Rightarrow V_i^2, \Delta$ *such that* $V_i^1 \cup V_i^2 = \{x_1 \ldots x_n\}$;
3. *each branch* $\mathcal{B}_i$ *terminates with leaves labelled by* $\Gamma \Rightarrow \Delta$.

Proof. We prove the lemma by induction on the complexity of $\varphi(x_1, \ldots, x_n)$. We only consider the case $\varphi(x_1, \ldots, x_n) \in \Delta$ (case (1)), as the other case is symmetric. Let $\Sigma = \Delta \smallsetminus \{\varphi(x_1, \ldots, x_n)\}$.

Base. If $\varphi(x_1 \ldots x_n) = x$ the statement is obviously true. Let $\varphi(x_1 \ldots x_n) = \neg x$.

$$\dfrac{\dfrac{\Gamma \Rightarrow \neg x, \Sigma}{\color{red}{\Gamma, x \Rightarrow \Sigma}}}{\Gamma \Rightarrow \neg x, \Sigma}$$

The red node is what we need to prove claim (1) for the single branch in this tree, and, upon recalling that $\Delta = \Sigma \cup \{\varphi(x_1, \ldots, x_n)\}$, the derivation complies with claim (3). Let now $\varphi(x_1, \ldots, x_n) = x \wedge y$. The following derivation, where "contraction" steps are made explicit for the sake of clarity, establishes our claim.

$$\dfrac{\dfrac{\dfrac{\dfrac{\Gamma \Rightarrow x \wedge y, \Sigma}{\color{red}{\Gamma, x \Rightarrow y, \Sigma}} \quad \dfrac{\Gamma \Rightarrow x \wedge y, \Sigma}{\color{red}{\Gamma \Rightarrow x, y, \Sigma}}}{\Gamma \Rightarrow x \wedge y, y, \Sigma} \quad \dfrac{\dfrac{\Gamma \Rightarrow x \wedge y, \Sigma}{\color{red}{\Gamma, y \Rightarrow x, \Sigma}} \quad \dfrac{\Gamma \Rightarrow x \wedge y, \Sigma}{\color{red}{\Gamma \Rightarrow x, y, \Sigma}}}{\Gamma \Rightarrow x \wedge y, x, \Sigma}}{\Gamma, \Gamma \Rightarrow x \wedge y, x \wedge y, \Sigma, \Sigma}}{\Gamma \Rightarrow x \wedge y, \Sigma}$$

The case $\varphi(x_1, \dots, x_n) = x \vee y$ is left to the reader.

Inductive step. We confine ourselves to the case $\varphi(x_1, \dots, x_n) = \alpha(\vec{x}) \wedge \beta(\vec{y})$. Consider the following derivation of $\Gamma \Rightarrow \alpha(\vec{x}) \wedge \beta(\vec{y}), \Sigma$ from itself:[4]

$$
\dfrac{
\dfrac{
\dfrac{\dfrac{\mathcal{B}_1}{\Gamma \Rightarrow \alpha(\vec{x}) \wedge \beta(\vec{y}), \Sigma}}{\Gamma, \alpha(\vec{x}) \Rightarrow \beta(\vec{y}), \Sigma}
\quad
\dfrac{\dfrac{\mathcal{B}_2}{\Gamma \Rightarrow \alpha(\vec{x}) \wedge \beta(\vec{y}), \Sigma}}{\Gamma \Rightarrow \alpha(\vec{x}), \beta(\vec{y}), \Sigma}
}{\Gamma \Rightarrow \alpha(\vec{x}) \wedge \beta(\vec{y}), \beta(\vec{y}), \Sigma}
\quad
\dfrac{
\dfrac{\dfrac{\mathcal{B}_3}{\Gamma \Rightarrow \alpha(\vec{x}) \wedge \beta(\vec{y}), \Sigma}}{\Gamma, \beta(\vec{y}) \Rightarrow \alpha(\vec{x}), \Sigma}
\quad
\dfrac{\dfrac{\mathcal{B}_4}{\Gamma \Rightarrow \alpha(\vec{x}) \wedge \beta(\vec{y}), \Sigma}}{\Gamma \Rightarrow \alpha(\vec{x}), \beta(\vec{y}), \Sigma}
}{\Gamma \Rightarrow \alpha(\vec{x}), \alpha(\vec{x}) \wedge \beta(\vec{y}), \Sigma}
}{
\dfrac{\Gamma, \Gamma \Rightarrow \alpha(\vec{x}) \wedge \beta(\vec{y}), \alpha(\vec{x}) \wedge \beta(\vec{y}), \Sigma, \Sigma}{\Gamma \Rightarrow \alpha(\vec{x}) \wedge \beta(\vec{y}), \Sigma}
}
$$

Now, let us consider the branch $\mathcal{B}_1$ and the sequent $\Gamma, \alpha(\vec{x}) \Rightarrow \beta(\vec{y}), \Sigma$ in that branch. Since $\beta(\vec{y})$ has a strictly lower complexity than $\alpha(\vec{x}) \wedge \beta(\vec{y})$, by H.I. there exists a derivation $\mathcal{D}_1$ of $\Gamma, \alpha(\vec{x}) \Rightarrow \beta(\vec{y}), \Sigma$ from itself, fulfilling conditions (1)-(3). That is, $\mathcal{D}_1$ has the following structure:

$$
\dfrac{
\begin{array}{c}
\dfrac{\dfrac{\mathcal{B}_1^1}{\Gamma, \alpha(\vec{x}) \Rightarrow \beta(\vec{y}), \Sigma}}{\vdots} \\
\overline{\Gamma, \alpha(\vec{x}), V_1^1 \Rightarrow V_1^2, \Sigma} \\
\vdots
\end{array}
\quad \dots \quad
\begin{array}{c}
\dfrac{\dfrac{\mathcal{B}_k^1}{\Gamma, \alpha(\vec{x}) \Rightarrow \beta(\vec{y}), \Sigma}}{\vdots} \\
\overline{\Gamma, \alpha(\vec{x}), V_k^1 \Rightarrow V_k^2, \Sigma} \\
\vdots
\end{array}
}{\Gamma, \alpha(\vec{x}) \Rightarrow \beta(\vec{y}), \Sigma}
$$

where $V_i^1 \cup V_i^2 = \{\vec{y}\}$ for $1 \leq i \leq k$. On the other hand, since $\alpha(\vec{x})$ has complexity strictly less than the complexity of $\alpha(\vec{x}) \wedge \beta(\vec{y})$, there exists a derivation $\mathcal{D}_1^1$ of $\Gamma, \alpha(\vec{x}), V_1^1 \Rightarrow V_1^2, \Sigma$ from itself verifying conditions (1)-(3):

$$
\dfrac{
\begin{array}{c}
\dfrac{\Gamma, \alpha(\vec{x}), V_1^1 \Rightarrow V_1^2, \Sigma}{\vdots} \\
\overline{\Gamma, V_1^1, W_1^1 \Rightarrow V_1^2, W_1^2, \Sigma} \\
\vdots
\end{array}
\quad \dots \quad
\begin{array}{c}
\dfrac{\Gamma, \alpha(\vec{x}), V_1^1 \Rightarrow V_1^2, \Sigma}{\vdots} \\
\overline{\Gamma, V_1^1, W_r^1 \Rightarrow W_r^2, V_1^2, \Sigma} \\
\vdots
\end{array}
}{\Gamma, \alpha(\vec{x}), V_1^1 \Rightarrow V_1^2, \Sigma}
$$

where $W_i^1 \cup W_i^2 = \{\vec{x}\}$ for $1 \leq i \leq r$. Now, it is possible to combine these two proofs (replacing part of the branch $\mathcal{B}_1$) and to get the following derivation

[4]We thank an anonymous referee for patching an earlier inaccurate proof of the inductive step.

$$
\dfrac{
\dfrac{
\begin{array}{c}
\Gamma,\alpha(\vec{x}) \Rightarrow \beta(\vec{y}),\Sigma \\ \hline
\vdots \\ \hline
\Gamma,\alpha(\vec{x}),V_1^1 \Rightarrow V_1^2,\Sigma \\ \hline
\vdots \\ \hline
\Gamma,V_1^1,W_1^1 \Rightarrow W_1^2,V_1^2,\Sigma
\end{array}
\quad \cdots \quad
\begin{array}{c}
\Gamma,\alpha(\vec{x}) \Rightarrow \beta(\vec{y}),\Sigma \\ \hline
\vdots \\ \hline
\Gamma,\alpha(\vec{x}),V_1^1 \Rightarrow V_1^2,\Sigma \\ \hline
\vdots \\ \hline
\Gamma,V_1^1,W_r^1 \Rightarrow W_r^2,V_1^2,\Sigma
\end{array}
}{
\begin{array}{c}
\Gamma,\alpha(\vec{x}),V_1^1 \Rightarrow V_1^2,\Sigma \\ \hline
\vdots
\end{array}
}
\qquad \cdots \qquad
\begin{array}{c}
\mathcal{B}_k^1 \\ \hline
\Gamma,\alpha(\vec{x}) \Rightarrow \beta(\vec{y}),\Sigma \\ \hline
\vdots \\ \hline
\Gamma,\alpha(\vec{x}),V_k^1 \Rightarrow V_k^2,\Sigma \\ \hline
\vdots
\end{array}
}{
\Gamma,\alpha(\vec{x}) \Rightarrow \beta(\vec{y}),\Sigma
}
$$

with the property that $(W_j^1 \cup V_1^1) \cup (W_j^2 \cup V_1^2) = \{\vec{x}, \vec{y}\}$ for $1 \leq j \leq r$. By repeating this procedure for every branch, it is possible to construct a deduction enjoying the desired characteristics. This concludes the proof.
■

Theorem 18 *Let $\Gamma \Rightarrow \Delta \in Seq\,(\mathcal{L}_0)$ be such that $Var(\Gamma) \cup Var(\Delta) = \{x_1 \dots x_n\}$. Then there exists a derivation $\mathcal{D}$ of $\Gamma \Rightarrow \Delta$ in LK_W^- such that:*

1. *for every branch $\mathcal{B}_i \in \mathcal{D}$ there exists a node $n_i \in \mathcal{B}_i$ labelled by a sequent of the form $V_i^1 \Rightarrow V_i^2$, containing only propositional variables, and such that $V_i^1 \cup V_i^2 = \{x_1 \dots x_n\}$.*

2. *each branch $\mathcal{B}_i$ terminates with leaves labelled by $\Gamma \Rightarrow \Delta$.*

Proof. Preliminarly, observe that, since $Var(\Gamma) \cup Var(\Delta) = \{x_1 \dots x_n\}$, either $\Gamma \neq \emptyset$, or $\Delta \neq \emptyset$. By applying Lemma 17.(2) to each formula in Γ and Lemma 17.(1) to each formula in Δ, for each branch $\mathcal{B}_i$ there will be a node n_i labelled by a sequent of the form $V_i^1 \Rightarrow V_i^2$, whose formulas are exactly the variables occurring in $\Gamma \cup \Delta$. By Lemma 17.(3), every branch $\mathcal{B}_i$ will terminate with leaves labelled by $\Gamma \Rightarrow \Delta$. ■

PWK can be given a Hilbert-style axiomatisation [7] whose axioms are all the classical tautologies (or a suitable axiomatisation of CL in the language $\mathcal{L}_0$) and whose sole rule is a restricted form of modus ponens: ψ can be derived from $\varphi, \neg\varphi \vee \psi$ provided that $Var(\varphi) \subseteq Var(\psi)$. It is therefore to be expected that a restricted form of Cut, respecting some variable-inclusion constraint from premisses to conclusion, be *derivable* (not only admissible!) in LK_W^-. The next lemma confirms this insight.

Lemma 19 *The following rule is derivable in LK_W^- whenever $Var(\varphi) \subseteq Var(\Gamma \cup \Delta)$:*

$$\dfrac{\Gamma,\varphi \Rightarrow \Delta \qquad \Gamma \Rightarrow \varphi,\Delta}{\Gamma \Rightarrow \Delta}$$

Proof. We prove the statement by induction on the complexity of the cutformula φ.

Base. Let $\varphi(x_1, \ldots, x_n) = x$. So there is a formula $\varphi(x, \vec{y}) \in \Gamma \cup \Delta$, with x actually occurring in it. W.l.o.g. let $\varphi(x, \vec{y}) \in \Delta$. By Theorem 18 there is a LK_W^--derivation of the following form:

$$\dfrac{\dfrac{V_1^1 \Rightarrow V_1^2}{\vdots} \qquad \ldots \qquad \dfrac{V_k^1 \Rightarrow V_k^2}{\vdots}}{\Gamma \Rightarrow \Delta}$$

where each $V_i^1 \Rightarrow V_i^2$ $(1 \leq i \leq k)$ is a sequent containing only propositional variables and such that $V_i^1 \cup V_i^2 = Var(\Gamma \cup \Delta)$. This entails that, for each $1 \leq i \leq k$, the variable x actually occurs in $V_i^1 \cup V_i^2$. According as x occurs on the left-hand or on the right-hand side, we apply Theorem 18 in order to obtain the following derivation (here we assume w.l.o.g. $x \in V_1^1, V_k^2$)

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\Gamma, x \Rightarrow \Delta}{\vdots}}{V_1^1, x \Rightarrow V_1^2}}{V_1^1 \Rightarrow V_1^2}}{\vdots} \qquad \ldots \qquad \dfrac{\dfrac{\dfrac{\dfrac{\Gamma \Rightarrow x, \Delta}{\vdots}}{V_k^1 \Rightarrow x, V_k^2}}{V_k^1 \Rightarrow V_k^2}}{\vdots}}{\Gamma \Rightarrow \Delta}$$

as desired. Observe that the critical step

$$\dfrac{V_i^1, x \Rightarrow x, V_i^2}{V_i^1 \Rightarrow V_i^2}$$

where the displayed x may not occur in the sequent's succedent or antecedent according as $i = 1$ or $i = k$, relies on the assumption that $x \in Var(\Gamma \cup \Delta)$. The case $\varphi = \neg x$ is similar. Let now $\varphi(x_1, \ldots x_n) = x \wedge y$, so $x, y \in Var(\Gamma \cup \Delta)$. We adopt the same strategy as above, applying Theorem 18 to $\Gamma \Rightarrow \Delta$ in order to get a derivation of the form

$$\dfrac{\dfrac{V_1^1 \Rightarrow V_1^2}{\vdots} \qquad \ldots \qquad \dfrac{V_k^1 \Rightarrow V_k^2}{\vdots}}{\Gamma \Rightarrow \Delta}$$

where each $V_i^1 \Rightarrow V_i^2$ $(1 \leq i \leq k)$ is a sequent containing only propositional variables and such that $V_i^1 \cup V_i^2 = Var(\Gamma \cup \Delta)$. This entails that the variables x, y actually occur in each node $V_i^1 \Rightarrow V_i^2$ $(1 \leq i \leq k)$. There are four possibilities, namely (a) $x, y \in V_i^1$, (b) $x \in V_i^1, y \in V_i^2$, (c) $y \in V_i^1, x \in V_i^2$, (d) $x, y \in V_i^2$. Thanks to Theorem 18, in all cases (a)-(d) we can conclude the derivation as follows (here, for the sake of exemplification, we assume that the combinations (a)-(d) hold, from left to right, in the displayed branches):

$$\dfrac{\begin{array}{c}\dfrac{\dfrac{\dfrac{\dfrac{\Gamma, x\wedge y \Rightarrow \Delta}{\Gamma, x, y \Rightarrow \Delta}}{\vdots}}{V_1^1, x, y \Rightarrow V_1^2}}{V_1^1 \Rightarrow V_1^2} \\ \vdots \end{array} \quad \begin{array}{c}\dfrac{\dfrac{\dfrac{\dfrac{\Gamma \Rightarrow x\wedge y, \Delta}{\Gamma, x \Rightarrow y, \Delta}}{\vdots}}{V_2^1, x \Rightarrow y, V_2^2}}{V_2^1 \Rightarrow V_2^2} \\ \vdots \end{array} \quad \dots \quad \begin{array}{c}\dfrac{\dfrac{\dfrac{\dfrac{\Gamma \Rightarrow y\wedge x, \Delta}{\Gamma, y \Rightarrow x, \Delta}}{\vdots}}{V_{k-1}^1, y \Rightarrow x, V_{k-1}^2}}{V_{k-1}^1 \Rightarrow V_{k-1}^2} \\ \vdots \end{array} \quad \begin{array}{c}\dfrac{\dfrac{\dfrac{\dfrac{\Gamma \Rightarrow x\wedge y, \Delta}{\Gamma \Rightarrow x, y, \Delta}}{\vdots}}{V_k^1 \Rightarrow x, y, V_k^2}}{V_k^1 \Rightarrow V_k^2} \\ \vdots \end{array}}{\Gamma \Rightarrow \Delta}$$

Inductive step. We need a derivation of $\Gamma \Rightarrow \Delta$ all of whose branches terminate either with axioms or with the assumption $\Gamma \Rightarrow \Delta, \varphi$, or else with the assumption $\varphi, \Gamma \Rightarrow \Delta$. We only consider the case $\varphi = \alpha(x_1 \dots, x_n) \wedge \beta(y_1, \dots, y_m)$, where $\{x_1, \dots, x_n, y_1, \dots, y_m\} \subseteq Var(\Gamma \cup \Delta)$. By H.I., since $\{x_1, \dots, x_n\} \subseteq Var(\Gamma \cup \Delta)$ and the complexity of $\alpha(\vec{x})$ is smaller than the complexity of φ, there exists a derivation of $\Gamma \Rightarrow \Delta$ all of whose branches terminate either with axioms or with the assumption $\Gamma \Rightarrow \Delta, \alpha(\vec{x})$, or else with the assumption $\alpha(\vec{x}), \Gamma \Rightarrow \Delta$:

$$\dfrac{\begin{array}{c}\dfrac{\Gamma \Rightarrow \Delta, \alpha(\vec{x})}{\vdots}\end{array} \quad \dots \quad \begin{array}{c}\dfrac{\alpha(\vec{x}), \Gamma \Rightarrow \Delta}{\vdots}\end{array}}{\Gamma \Rightarrow \Delta}$$

We now apply the H.I. to all leaves labelled by $\Gamma \Rightarrow \Delta, \alpha(\vec{x})$. Since $\{y_1, \dots, y_m\} \subseteq Var(\Gamma \cup \Delta)$ and the complexity of $\beta(\vec{x})$ is smaller than the complexity of φ, every such branch can be extended as follows:

$$\dfrac{\begin{array}{c}\dfrac{\dfrac{\dfrac{\Gamma \Rightarrow \Delta, \alpha(\vec{x}), \beta(\vec{y})}{\vdots} \quad \dots \quad \dfrac{\beta(\vec{y}), \Gamma \Rightarrow \Delta, \alpha(\vec{x})}{\vdots}}{\Gamma \Rightarrow \Delta, \alpha(\vec{x})}}{\vdots}\end{array} \quad \dots \quad \begin{array}{c}\dfrac{\alpha(\vec{x}), \Gamma \Rightarrow \Delta}{\vdots}\end{array}}{\Gamma \Rightarrow \Delta}$$

Here, all of the new leaves are labelled either by axioms or by $\Gamma \Rightarrow \Delta, \alpha(\vec{x}), \beta(\vec{y})$ or by $\beta(\vec{y}), \Gamma \Rightarrow \Delta, \alpha(\vec{x})$. Symmetrically, we apply the H.I. to all leaves labelled by $\alpha(\vec{x}), \Gamma \Rightarrow \Delta$. The corresponding branches can be extended in such a way that the new leaves are labelled either by axioms or by $\alpha(\vec{x}), \Gamma \Rightarrow \Delta, \beta(\vec{y})$ or by $\alpha(\vec{x}), \beta(\vec{y}), \Gamma \Rightarrow \Delta$. Now apply to all the leaves that are not labelled by axioms the rules $\wedge L^{\partial}$ or $\wedge R^{\partial}_{1-3}$, as appropriate. The resulting proof-tree will be such that its leaves are labelled either by axioms or by $\Gamma \Rightarrow \Delta, \varphi$, or else by $\varphi, \Gamma \Rightarrow \Delta$. This concludes the proof. ■

Theorem 20 *The following are equivalent for any $\Gamma \cup \{\varphi\} \subseteq Fm(\mathcal{L}_0)$:*

1. $\{\Rightarrow \gamma : \gamma \in \Gamma\} \vdash_{LK_W^-} \Rightarrow \varphi$;

2. $\Gamma \vDash_{\mathrm{PWK}} \varphi$.

Proof. (1)⇒(2) follows from Corollary 16.

(2)⇒(1). Assume that $\Gamma \vDash_{\mathrm{PWK}} \varphi$. By Theorem 4, either $\vDash_{\mathrm{CL}} \varphi$ (whence our conclusion follows from Lemma 14.(3)) or there exists a nonempty, and necessarily finite, $\Gamma' \subseteq \Gamma$ such that $\Gamma' \vDash_{\mathrm{CL}} \varphi$ and $Var(\Gamma') \subseteq Var(\varphi)$. Let $\Gamma' = \{\gamma_1, \ldots \gamma_m\}$. As $Var(\Gamma') \subseteq Var(\varphi)$, Lemma 14.(3) and Lemma 19 ensure that we can obtain the following derivation:

$$\dfrac{\dfrac{\dfrac{\vdots}{\Gamma' \Rightarrow \varphi}}{\gamma_1 \wedge \cdots \wedge \gamma_m \Rightarrow \varphi} \qquad \dfrac{\Rightarrow \gamma_m \qquad \dfrac{\dfrac{\Rightarrow \gamma_3 \qquad \dfrac{\Rightarrow \gamma_2 \qquad \Rightarrow \gamma_1}{\Rightarrow \gamma_1 \wedge \gamma_2}}{\vdots}}{\Rightarrow \gamma_1 \wedge \cdots \wedge \gamma_{m-1}}}{\Rightarrow \gamma_1 \wedge \cdots \wedge \gamma_m}}{\Rightarrow \varphi}$$

and, therefore, $\{\Rightarrow \gamma : \gamma \in \Gamma\} \vdash_{LK_W^-} \Rightarrow \varphi$. ■

5 Tableaux

The tableaux system for PWK, modelled after the standard proof-theoretical format for many-valued logics to be found in [33] (although earlier versions can be traced back to [19] or [4]), uses *signed formulas* — namely, formulas prefixed either by the sign + (read as "designated") or by the sign − (read as "non-designated"). Priest's textbook has tableaux calculi both for LP and for SK, but does not provide one for B or PWK. The main concept behind our strategy is augmenting the expressive power of Priest's calculi by admitting the new sign $*$ (meaning "classical").

Definition 21 *By a* signed formula *we mean an ordered pair* (φ, s)*, where* $\varphi \in Fm(\mathcal{L}_0)$ *and* $s \in \{+, -, *\}$*. A member of this last set is called a* sign*.*

Hereafter, when referring to signed formulas, parentheses are omitted whenever this is possible.

Definition 22 *A* PWK-tableau *is a labelled tree* $\mathcal{T}$ *whose vertices are labelled by signed formulas. The signed formulas labelling the root of* $\mathcal{T}$ *(and, possibly, some of its immediate successors) are called* initial formulas *of* $\mathcal{T}$*. The other formulas occurring in* $\mathcal{T}$ *are called* non-initial*. Non-initial formulas in* $\mathcal{T}$ *are obtained from predecessor vertices by one of the rules below (where the usual* *typographical conventions are adopted):*

$$\frac{\varphi\wedge\psi,+}{\begin{array}{c}\varphi,+\\ \neg\varphi,+\end{array}\Big|\begin{array}{c}\psi,+\\ \neg\psi,+\end{array}\Big|\begin{array}{c}\neg\varphi,-\\ \neg\psi,-\end{array}}\qquad\frac{\varphi\vee\psi,+}{\varphi,+\mid\psi,+}\qquad\frac{\varphi\wedge\psi,-}{\begin{array}{c}\varphi,*\\ \psi,*\\ \varphi,-\mid\psi,-\end{array}}\qquad\frac{\varphi\vee\psi,-}{\begin{array}{c}\varphi,-\\ \psi,-\end{array}}$$

$$\frac{\neg\left(\varphi\wedge\psi\right),+}{\neg\varphi,+\mid\neg\psi,+}\qquad\frac{\neg\left(\varphi\vee\psi\right),+}{\begin{array}{c}\varphi,+\\ \neg\varphi,+\end{array}\Big|\begin{array}{c}\psi,+\\ \neg\psi,+\end{array}\Big|\begin{array}{c}\varphi,-\\ \psi,-\end{array}}\qquad\frac{\neg\left(\varphi\wedge\psi\right),-}{\begin{array}{c}\neg\varphi,-\\ \neg\psi,-\end{array}}\qquad\frac{\neg\left(\varphi\vee\psi\right),-}{\begin{array}{c}\varphi,*\\ \psi,*\\ \neg\varphi,-\mid\neg\psi,-\end{array}}$$

$$\frac{\neg\neg\varphi,+}{\varphi,+}\qquad\frac{\neg\neg\varphi,-}{\varphi,-}\qquad\frac{\varphi\wedge\psi,*}{\begin{array}{c}\varphi,*\\ \psi,*\end{array}}\qquad\frac{\varphi\vee\psi,*}{\begin{array}{c}\varphi,*\\ \psi,*\end{array}}$$

$$\frac{\neg\varphi,*}{\varphi,*}$$

The rules of Definition 22 divide into α-*rules* (simple), β-*rules* (branching) and γ-*rules* (semi-branching). In the next table, which provides the corresponding classification, each rule is labelled after the signed formula which constitutes its premiss.

α	β	γ
$\varphi\vee\psi,-$	$\varphi\wedge\psi,+$	$\varphi\wedge\psi,-$
$\neg\left(\varphi\wedge\psi\right),-$	$\varphi\vee\psi,+$	$\neg\left(\varphi\vee\psi\right),-$
$\neg\neg\varphi,+$	$\neg\left(\varphi\wedge\psi\right),+$	
$\neg\neg\varphi,-$	$\neg\left(\varphi\vee\psi\right),+$	
$\varphi\wedge\psi,*$		
$\varphi\vee\psi,*$		
$\neg\varphi,*$		

Definition 23 *If* $\varphi_1,...,\varphi_n,\psi\in Fm\left(\mathcal{L}_0\right)$*, we say that the* PWK*-tableau* $\mathcal{T}$ *is* for $\left(\left\{\varphi_1,...,\varphi_n\right\},\psi\right)$ *iff its root is labelled by* $(\varphi_1,+)$ *and followed by a chain of vertices respectively labelled by* $(\varphi_2,+),...,(\varphi_n,+),(\psi,-)$.

Definition 24 *A branch of a* PWK*-tableau* $\mathcal{T}$ *is called* closed *iff either (i) for some formula* φ*, it contains both* $(\varphi,+)$ *and* $(\varphi,-)$*; or (ii) for some formula* φ*, it contains both* $(\varphi,-)$ *and* $(\neg\varphi,-)$*; or else (iii) for some formula* φ*, it contains all of* $(\varphi,+)$*,* $(\neg\varphi,+)$ *and* $(\varphi,*)$.

Definition 25 *A branch* $\mathcal{B}$ *of a* PWK*-tableau* $\mathcal{T}$ *is called* complete *iff: i) whenever it contains the premiss of an* α*-rule, it contains all its conclusion(s); ii) whenever it contains the premiss of a* β*-rule, it contains at least one of its conclusion; iii) whenever it contains the premiss of a* γ*-rule, it contains all its conclusions of the form* $\langle\varphi,*\rangle$ *and at least one of its remaining conclusions.*

A PWK*-tableau* $\mathcal{T}$ *is said to be* completed *iff all of its branches are complete.*

Definition 26 *By the notation* $\varphi_1, ..., \varphi_n \vdash_{T-\mathrm{PWK}} \psi$ *we mean that there is a completed* PWK-*tableau* $\mathcal{T}$ *for* $(\{\varphi_1, ..., \varphi_n\}, \psi)$ *whose branches are all closed.*

We now provide some examples of PWK-tableaux.

Example 27 *(De Morgan Laws)*

$$
\begin{array}{c}
\neg(\varphi \wedge \psi), + \\
\neg\varphi \vee \neg\psi, - \\
\neg\varphi, - \\
\neg\psi, - \\
\begin{array}{c|c}
\neg\varphi, + & \neg\psi, + \\
\times & \times
\end{array}
\end{array}
$$

$$
\begin{array}{c}
\neg(\varphi \vee \psi), + \\
\neg\varphi \wedge \neg\psi, - \\
\neg\varphi, * \\
\neg\psi, * \\
\varphi, * \\
\psi, * \\
\begin{array}{c|c}
\begin{array}{c}
\neg\varphi, - \\
\begin{array}{c|c|c}
\varphi, + & \psi, + & \varphi, - \\
\neg\varphi, + & \neg\psi, + & \psi, - \\
\times & \times & \times
\end{array}
\end{array}
&
\begin{array}{c}
\neg\psi, - \\
\begin{array}{c|c|c}
\varphi, + & \psi, + & \varphi, - \\
\neg\varphi, + & \neg\psi, + & \psi, - \\
\times & \times & \times
\end{array}
\end{array}
\end{array}
\end{array}
$$

The next definitions will be useful in what follows.

Definition 28 *A valuation* v *is called* faithful *to the signed formula* (φ, s) *iff one of the following conditions hold:*

- (φ, s) *has the form* $(\psi, +)$ *and* $v(\psi) \in \{1, \frac{1}{2}\}$*;*
- (φ, s) *has the form* $(\neg\psi, +)$ *and* $v(\psi) \in \{0, \frac{1}{2}\}$*;*
- (φ, s) *has the form* $(\psi, -)$ *and* $v(\psi) = 0$*;*
- (φ, s) *has the form* $(\neg\psi, -)$ *and* $v(\psi) = 1$*;*
- (φ, s) *has either the form* $(\psi, *)$ *or the form* $(\neg\psi, *)$ *and* $v(\psi) \in \{0, 1\}$.

A set S *of signed formulas is called* satisfiable *iff there is a valuation* v *which is faithful to every signed formula in* S.

Definition 29 *A set* S *of signed formulas is called a* Hintikka set *iff the following conditions hold:*

- *for every formula* φ*, it is not the case that both* $(\varphi, +)$ *and* $(\varphi, -)$ *belong to* S*;*
- *for every formula* φ*, it is not the case that both* $\langle\varphi, -\rangle$ *and* $(\neg\varphi, -)$ *belong to* S*;*
- *for every formula* φ*, it is not the case that all of* $(\varphi, +), (\neg\varphi, +)$ *and* $(\varphi, *)$ *belong to* S*;*

- *if (φ, s) is a possible premiss for an α-rule of a* PWK*-tableau, the conclusion(s) that would result from the application of the rule to (φ, s) occur in S;*

- *if (φ, s) is a possible premiss for a β-rule of a* PWK*-tableau, at least one of the conclusions that would result from the application of the rule to (φ, s) occur in S;*

- *if (φ, s) is a possible premiss for a γ-rule of a* PWK*-tableau, all the conclusions of the form $(\psi, *)$ that would result from the application of the rule to (φ, s), and at least one of the remaining conclusions, occur in S.*

Theorem 30 *For every $\varphi_1, ..., \varphi_n, \psi \in Fm(\mathcal{L}_0)$, the following are equivalent:*

1. $\varphi_1, ..., \varphi_n \vdash_{T-\mathrm{PWK}} \psi$;
2. $\varphi_1, ..., \varphi_n \vdash_{\mathrm{PWK}} \psi$.

Proof. (1)⇒(2) Just remark that: i) any valuation which is faithful to the premiss of an α-rule is faithful to its conclusion(s) as well; ii) any valuation which is faithful to the premiss of a β-rule is faithful to at least one of its conclusions as well; iii) any valuation which is faithful to the premiss of a γ-rule is faithful to its conclusions of the form $(\psi, *)$, as well as to at least one of the remaining conclusions. Reasoning inductively, it follows that if v is faithful to $(\varphi_1, +), ..., (\varphi_n, +), (\psi, -)$, then in any PWK-tableau $\mathcal{T}$ there is at least a branch $\mathcal{B}$ s.t. v is faithful to all the signed formulas in $\mathcal{B}$. However, if $\mathcal{T}$ is a PWK-tableau all of whose branches are closed, no interpretation can be faithful to all the signed formulas in any of its branches. Therefore, whenever $v(\varphi_1), ..., v(\varphi_n) \in \left\{1, \frac{1}{2}\right\}$, it is also the case that $v(\psi) \in \left\{1, \frac{1}{2}\right\}$.

(2)⇒(1) We prove that every complete open branch of a PWK-tableau is satisfiable. Since the set of signed formulas occurring in a complete open branch of a PWK-tableau is a Hintikka set, it will be enough to show that any Hintikka set S is satisfiable. Construct a valuation v as follows, for any variable x occurring in some signed formula in S (the remaining variables may be assigned arbitrary values):

- $(x, +) \in S, (\neg x, +) \notin S$: $v(x) = 1$;
- $(x, -) \in S, (\neg x, -) \notin S$: $v(x) = 0$;
- $(x, +) \in S, (\neg x, +) \in S$: $v(x) = \frac{1}{2}$;
- $(\neg x, -) \in S, (x, -) \notin S$: $v(x) = 1$;
- $(\neg x, +) \in S, (x, +) \notin S$: $v(x) = 0$;
- $(x, +) \notin S, (x, -) \notin S, (\neg x, +) \notin S, (\neg x, -) \notin S, (x, *) \in S$: $v(x) = 1$ (alternatively: $v(x) = 0$).

In light of Definition 29, v is well-defined because the cases envisaged above do not clash and exhaust all the alternatives. Moreover, v is faithful to all the signed literals in S. Upon remarking that:

1. a valuation which is faithful to the possible conclusion(s) of an α-rule is also faithful to the respective premiss;

2. a valuation which is faithful to at least one of the possible conclusions of a β-rule is also faithful to the respective premiss;

3. a valuation which is faithful to the possible conclusions of of the form $(\psi, *)$ of a γ-rule, and to at least one of the remaining conclusions, is also faithful to the respective premiss;

we conclude by induction that v is faithful to all the signed formulas in S.

■

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