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Introduction

In this work I have collected the results of my research over the expansion of weak Kleene logics with external operators. The focus will be shared between two sides: a logical part, where I will investigate various weak Kleene logics (internal, external, modal), and a algebraic part, where I will study the algebraic counterparts of the logics introduced.

The communication between logic and algebra is the cornerstone of abstract algebraic logic (see [Font, 2016]), a field within which I try to collocate my thesis. We study algebraic structures inasmuch as they are a source of information about some logics that are, in a technical sense, naturally related to them. The notion of algebraic counterparts (which can recevie a precise formal definition with that of equivalent algebraic seantics) is pivotal in abstract algebraic logic. The relative equational consequence of algebraic counterpart of a logic is such that the logic in question can be translated, back and forth, inside the latter without losing any information (all these notions are precisely clarified in the following notational preliminaries). When a logic is amenable to such a treatment it is called algebraizable and the theory of algebrazable logics has already obtained plenty of general results.

I will consider the case of weak Kleene logics, Bochvar's logic **B** and paraconsistent weak Kleene **PWK**. Traditionally these three-valued logics are known for their infective non-classical truth-value. The infectivity amounts to the fact that a formula is evaluated as non-classical iff there is a variable in it which is non-classical, therefore the third truth-value propagates infectiously throughout all superformulae.

In recent years we have witnessed a resurgence in the interest for these logics, as they can be characterized as logics of variable inclusion ([Bonzio et al., 2022]): weak Kleene logics can be obtained by imposing a linguistic filter over classical logic, by requiring that variable are preserved in one or the other direction of the logical consequence relation ([Urquhart, 2001], [Ciuni and Carrara, 2016]).

On the algebraic perspective, weak Kleene logics have very poor algebraic properties though. If we consider the logics at the origin of what would have become weak Kleene, the scenario changes. D.A. Bochvar ([Bochvar, 1981]) and S. Halldén ([Halldén, 1949]) presented their logics in a language richer than the classical one; we denote these logics respectively $\mathbf{B}_{\mathrm{e}}$ and $\mathbf{PWK}_{\mathrm{e}}$. In the case of Bochvar, the motivation behind his formal system was an attempt to reason about set-theoretic and semantical paradoxes. In this way, paradoxical sentences are assigned the non-classical truth-value. Besides this, he introduced in the language so-called external unary operators, which we will denote J_i's, which express, in a purely classical distinction, whether a formula is true, false, or non-classical. Hence we have an external language (the fragment of Bochvar's logic where every formula is prefixed by a J_i) which is capable to talk about the internal one (the J_i-free fragment), without resorting to any hierarchy of metalanguages. Whether this is a viable way to deal with paradoxes, what we are interested in is that the more expressive language of Bochvar's external logic is powerful enough to establish a strong connection between this logic and what can properly be considered its algebraic counterpart. In fact, in [Bonzio et al., 2023] it was proved that $\mathbf{B}_{\mathrm{e}}$ is an algebraizable logic, and later in [Bonzio and Zamperlin, 2024] it was proved the same for $\mathbf{PWK}_{\mathrm{e}}$. These results motivate the decision to treat external weak Kleene logics within the framework of abstract algebraic logic. Interestingly, $\mathbf{B}_{\mathrm{e}}$ and $\mathbf{PWK}_{\mathrm{e}}$ have the same equivalent algebraic semantics, namely the class of Bochvar algebras BCA ([Finn and Grigolia, 1993]).

In a recent work, [Bonzio and Pra Baldi, 2024] provided an interesting structure theorem for Bochvar algebras, proving that they can be represented in terms of particular pastings of Boolean algebras, using an expanded version of the technique of Płonka sums ([Płonka, 1967], [Płonka and Romanowska, 1992]). The latter is an algebraic construction employed, in particular, in the study of regular varieties, and this lead to the intuition to extended it to the field of logics of variable inclusion, where it has revealed its deep usefulness in the investigation around these logics lately (see

[Bonzio et al., 2022]).

[Bonzio and Pra Baldi, 2024] showed how in any Bochvar algebra we can individuate a Boolean subalgebra which encodes, in a precise technical sense, all the information needed to rebuild the entire Bochvar algebra. This Boolean algebra is at the base of a Bochvar system, a notion introduced by [Bonzio et al., 2024], who proved how BCA and Bochvar systems are essentially the same structures, in the form of a categorical equivalence.

So far, Bochar algebras have a Płonka-style representation and one in terms of Bochvar systems. In chapter 2 I illustrate a further representation, developed with Gandolfo Vergottini ([Paoli et al., 202x]), which starts from Bochvar systems and shows how to build Bochvar algebras from them as special twist products and, most importantly, I show how every BCA is representable as a product of this kind. This product construction has shown its versatility in clarifying the categorical relation between Bochvar algebras and regular double Stone algebras ([Paoli et al., 202x]), while here it is use to offer an alternative representation theorem of BCA, which can highlight some aspects left implicit in the Płonka sum approach, like the way in which operations are computed only in terms of the elements of the Boolean subalgebra at the foundation of a BCA (its bottom fibre, using the jargon of Płonka sums). Finally, I show how passing through the product construction it is possible to reobtain the categorical equivalence between BCA and Bochvar systems.

In the following chapter 3, I move to the modal expansion of external weak Kleene logics, enriching the language already comprising the external operators with a further $\Box$ unary modal operator, which can be taken as an alethic operator with the minimal requirements to be considered normal. The resulting modal logics, modal Bochvar logic $\mathbf{B}_{\mathrm{e}}^{\Box}$ and modal external paraconsistent weak Kleeen $\mathbf{PWK}_{\mathrm{e}}^{\Box}$, which I have investigated with Stefano Bonzio (partially pulished in [Bonzio and Zamperlin, 2024]), are three-valued modal logics with classical recapture operators. The interaction between modality and external operators are the focus of this study.

I have provided two complete semantics for modal external weak Kleene logics. I started from a Kripke-style possible world semantics, based on standard Kripke frames upon which two different classes of three-valued valuations can be considered, differing only for the behaviour of the $\Box$. One determines

$\mathbf{B}_{\mathrm{e}}^{\Box}$, the other $\mathbf{PWK}_{\mathrm{e}}^{\Box}$, depending on whether we read a true box as its argument being true at every accessible point or as non-false. Furthermore, this reading of the modality consider the non-classical truth-value as locally infectious, in the sense that a formula evaluated as non-classical must have one of its variable non-classical at the point of evaluation.

By making a distinction between local and global modal external weak Kleene logic, I can move to he second semantic framework I consider, an algebraic one. By considering only global modal systems, it is possible to obtain algebraizable logics, therefore we can produce their equivalent algebraic semantics. Contrary to what happened in the non-modal case, these are two different classes of algebras, which I called modal Bochvar algebras $\mathsf{MBCA_B}$ and modal Halldén algebras $\mathsf{MBCA_H}$. The way in which the modal operator $\Box$ interacts with the Płonka sum decomposition of these expanded Bochvar algebras is not trivial. I study the resulting fibres and show how they relate with standard Boolean algebras with operators.

Despite the addition of the modal operator cannot be entirely extended to the Płonka sum representation, I will show how the core property of BCA, namely having a subreduct containing all the information of the full algebra, is preserved in the case of $\mathsf{MBCA_B}$. This feature makes possible to apply the product representation to modal Bochvar algebras, which can be represented as expanded Bochvar systems based on Boolean algebras with operators. This concludes this study of modal systems with a weak Kleene basis.

0.1 Notational preliminaries

In the following we are going to employ some basilar notions of universal algebra and abstract algebraic logic (for additional information, see [Burris and Sankappanavar, 2012] and [Font, 2016], respectively).

A similarity type, or language, is a pair $\langle \tau, f \rangle$, where τ is a finite list of symbols and $f : \tau \to \mathbb{N}$; for $\lambda \in \tau, f(\lambda)$ is the arity of λ and it determines the usual grammatical rules for well-formed formulae. When the context is clear, we identify a similarity type with the set τ. Given a set of variables X, by $\mathbf{Fm}_{\tau}(X)$ we denote the absolutely free algebra of type τ generated by X, also called the formula algebra of type τ, whose elements are called τ-formulae.

Since we will always assume X to be countable, we omit it and write just $\mathbf{Fm}_\tau$ for the formula algebra of type τ, and when the type is clear from the context we omit the subscript τ as well.

An τ-algebra (or simply algebra, when the type is clear) is a tuple $\langle A, \tau \rangle$ s.t. for each $\lambda \in \tau, \tau^A : A^n \to A$, with n the arity of λ. Algebras in the same similarity type are called similar. We use the notational convention to denote with $\mathbf{A}$ and algebra and A its support. A τ-equation $\alpha \approx \beta$ is a pair $\langle \alpha, \beta \rangle$ of τ-formulae; Eq_τ denotes the set of all τ-equations. A τ-equation $\alpha \approx \beta$ is valid or holds in a τ-algebra if for all homomorphisms $h : \mathbf{Fm}_\tau \to \mathbf{A}$, $h(\alpha) = h(\beta)$. A τ-quasiequation $\alpha_1 \approx \beta_1 \mathbin{\&} \alpha_2 \approx \beta_2 \mathbin{\&} \cdots \mathbin{\&} \alpha_n \approx \beta_n \Rightarrow \alpha_{n+1} \approx \beta_{n+1}$ is a $n+1$-tuple of τ-equations, with $n \in \mathbb{N}$. A τ-quasiequation $\alpha_1 \approx \beta_1 \mathbin{\&} \alpha_2 \approx \beta_2 \mathbin{\&} \cdots \mathbin{\&} \alpha_n \approx \beta_n \Rightarrow \alpha_{n+1} \approx \beta_{n+1}$ is valid or holds in a τ-algebra if for all homomorphisms $h : \mathbf{Fm}_\tau \to \mathbf{A}$, if $h(\alpha_i) = h(\beta_i)$ for all $0 \leq i \leq n$, then $h(\alpha_{n+1}) = h(\beta_{n+1})$.

A variety is a class of similar algebras that is closed under homomorphic images, subalgebras, and products. By Birkhoff's well-known theorem, a class K is a variety iff there is an equational theory Σ (in the same language) such that K is the class of algebras that make valid all the equations in Σ. If K is a class of algebras, $\mathbb{V}(\mathsf{K})$ is the variety generated by K, that is the smallest variety containing K. Similarly, a quasivariety is a class of similar algebras that is closed with respect to isomorphic images, subalgebras, products and ultraproducts. A class K of similar algebras is a quasivariety iff there is a quasiequational theory Σ (in the same language) such that K is the class of algebras that make valid all the quasiequations in Σ. $\mathbb{Q}(\mathsf{K})$ is the quasivariety generated by K.

For a class K of similar algebras, the relative equational consequence relation $\Theta \vDash_{\mathsf{K}} \varphi \approx \psi$ holds iff for all $\mathbf{A} \in \mathsf{K}$ and all homomorphisms (from the formulae in the same type) $h : \mathbf{Fm} \to \mathbf{A}$, if $h(\delta) = h(\epsilon)$ for all $\delta \approx \epsilon \in \Theta$, then $h(\varphi) = h(\psi)$.

A logic $\mathbf{\Lambda}$ in a language τ is a pair $\langle \tau, \vdash_{\mathbf{\Lambda}} \rangle$ such that $\vdash_{\mathbf{\Lambda}} \subseteq \mathscr{P}(Fm_\tau) \times Fm_\tau$ is a closure relation which is substitution-invariant, i.e. for all endomorphisms $\sigma : \mathbf{Fm}_\tau \to \mathbf{Fm}_\tau, \Gamma \vdash_{\mathbf{\Lambda}} \varphi$ implies $\sigma[\Gamma] \vdash_{\mathbf{\Lambda}} \sigma(\varphi)$. We most often identify a logic with its closure relation. For $\Delta \subseteq Fm$, by writing $\Gamma \vdash_{\mathbf{\Lambda}} \Delta$ we mean that for every $\delta \in \Delta, \Gamma \vdash_{\mathbf{\Lambda}} \delta$ (similarly for $\Theta \vDash_{\mathsf{K}} \Sigma$ when $\Sigma \subseteq Eq$).

A matrix is a pair $\langle \mathbf{A}, F\rangle$ such that $F \subseteq A$. The logic induced by a matrix $\langle \mathbf{A}, F\rangle$ (in the same language), denoted by $\vdash_{\langle \mathbf{A}, F\rangle}$, is defined as follows:

$$\Gamma \vdash_{\langle \mathbf{A}, F\rangle} \varphi \text{ iff for all homomorphisms } h\colon \mathbf{Fm} \to \mathbf{A},$$
$$h[\Gamma] \subseteq F \text{ implies } h(\varphi) \in F.$$

A matrix $\langle \mathbf{A}, F\rangle$ is a model of a logic Λ if $\vdash_{\mathbf{\Lambda}} \subseteq \vdash_{\langle \mathbf{A}, F\rangle}$, while $\mathbf{\Lambda}$ is complete w.r.t. to a class M of matrices if $\vdash_{\mathbf{\Lambda}} = \bigcap_{\mathbf{M} \in \mathsf{M}} \vdash_{\mathbf{M}}$.

The following definition is going to be used abundantly.

Definition 1. [[Font, 2016], definition 3.11, proposition 3.12, corollary 3.18] A finitary logic $\mathbf{\Lambda}$ is *algebraizable* w.r.t. a class of algebras K if there exist two maps (called transformers) $\tau\colon Fm \to \mathscr{P}(Eq)$ and $\rho\colon Eq \to \mathscr{P}(Fm)$, respectively from formulae to sets of equations and from equations to sets of formulae, both commuting with substitutions, such that:

$$\text{(ALG1)} \quad \Gamma \vdash_{\mathbf{\Lambda}} \varphi \iff \tau[\Gamma] \vDash_{\mathsf{K}} \tau(\varphi),$$
$$\text{(ALG4)} \quad \varphi \approx \psi \mathrel{=\!\!\mid\vDash}_{\mathsf{K}} \tau \circ \rho(\varphi \approx \psi).$$

The above conditions are equivalent to:

$$\text{(ALG2)} \quad \Theta \vDash_{\mathsf{K}} \epsilon \approx \delta \iff \rho[\Theta] \vdash_{\mathbf{\Lambda}} \rho(\epsilon \approx \delta),$$
$$\text{(ALG3)} \quad \varphi \dashv\vdash_{\mathbf{\Lambda}} \rho \circ \tau(\varphi).$$

If a logic $\mathbf{\Lambda}$ is algebraizable w.r.t. a class $\mathsf{K}, \mathbb{Q}(\mathsf{K})$ is called its *equivalent algebraic semantics* and it is the largest quasivariety w.r.t. which $\mathbf{\Lambda}$ is algebraizable.

The definition of algebraziability is actually not limited to finitary logics (see [Font, 2016, definition 3.11]), but in the current work we will only deal with finitary logics, therefore we avoid the further complications of the general definition.

Chapter 1

Weak Kleene logics and external operators

Kleene three-valued logics are traditionally divided into two families, depending on the intended interpretation of the connectives: *strong Kleene* logics, counting strong Kleene ([Kleene, 1952]) and the logic of paradox ([Priest, 1979]), and *weak Kleene* logics, namely Bochvar logic ([Bochvar, 1981], originally published in 1938) and paraconsistent weak Kleene ([Halldén, 1949])[1]. All the mentioned four logics are traditionally intended, and thus defined, over the algebraic language of classical logic.

The distinction between strong and weak was introduced by S.C. Kleene in [Kleene, 1952, chapter XII, section 63-64]. There his introduction to recursion theory covers introduces partial recursive function, and in order to account for the outcome of a partial function for an undefined input, he introduces the third truth-value (there denoted with $\mathfrak{u}$). Kleene proposes two readings of the standard classical connectives within this many-valued setting: strong and weak. The former are displayed in figure 1.1, the latter in figure 1.2.

It is to be noticed that this distinction concerns two sets of truth-tables, not logics in the modern sense of the word. The strength of these tables regards the number of classical occurrences, once we put two further requirements: over classical inputs the tables should coincide with the classical ones, and whenever in a column (row) a classical value occurs in the row (column) of

[1] Historically, both Bochvar and Halldén formulated their logics in richer languages than the one presented above. We will return to this point in section 1.2 and provide a more faithful presentation there.

	¬
1	0
1/2	1/2
0	1

∨	1	1/2	0
1	1	1	1
1/2	1	1/2	1/2
0	1	1/2	0

∧	1	1/2	0
1	1	1/2	0
1/2	1/2	1/2	0
0	0	0	0

Figure 1.1: Strong Kleene truth-tables.

the non-classical value, that classical values appears on all the entries of that column (row). The strong tables then correspond to the only possible ones with the maximum number of classical occurrences, viceversa for the weak tables.

Strong Kleene logics received a vastly larger popularity than their weak counterparts, finding plenty of applications in the literature. To name just a few, on the more computational-oriented side they have been used to model partial information ([Abdallah, 1995]) and nonmonotonic reasoning ([Turner, 1984]), while in philosophy they have been used for extremely influential proposals, like Kripke's theory of truth ([Kripke, 1975]) and Priest's dialetheism ([Priest, 2006]).

On the contrary, this tendency has only marginally involved weak Kleene logics, until recently. In the recent years thre have been an increasing attention over these systems from several points of view: semantical ([Ciuni and Carrara, 2016]), algebraic ([Bonzio et al., 2017], [Bonzio et al., 2022]), epistemic ([Szmuc, 2019], [Bonzio et al., 2023], [Boem and Bonzio, 2022]), computer-theoretic ([Carrara and Zhu, 2021], [Ciuni et al., 2019],) topic-theoretic ([Beall, 2016]), and belief revision ([Carrara et al., 2023]).

An essential landmark in the current development of weak Kleene logics has been their characterization variable inclusion logics, that is they can be obtained from classical logic by imposing a syntactic filter. The result for **B** by [Urquhart, 2001] was already obtained in 1986, but we had to wait for [Ciuni and Carrara, 2016] for the case of **PWK**. This motivated the employment of weak Kleene logics as the exemplar case of logics of variable inclusion ([Bonzio et al., 2022]). The recent theory of variable inclusion logics has developed in unison with the applicaiton of Płonka sums to logical systems. In the course of this chapter all these notions will be formally clarified.

1.1 Internal weak Kleene logics

Weak Kleene logics are expressed in the classical language $\mathscr{L}: \langle \vee, \neg, 0, 1\rangle$ of type $\langle 2, 1, 0, 0\rangle$, whose (algebraic) interpretation is given via the 3-element algebra $\mathbf{WK} = \langle \{0, 1, {}^1\!/\!_2\}, \neg, \vee, 0, 1\rangle$ whose operations tables are reproduced in the following table:

	$\neg$
1	0
$1/2$	$1/2$
0	1

$\vee$	1	$1/2$	0
1	1	$1/2$	1
$1/2$	$1/2$	$1/2$	$1/2$
0	1	$1/2$	0

Figure 1.2: The algebra **WK**.

The third value $1/2$ is traditionally read as "meaningless" (see e.g. [Ferguson, 2017], [Szmuc and Ferguson, 2021] and [Bonzio et al., 2022, Ch. 1]) due to its infectious behavior ([Ciuni and Carrara, 2016]).
The standardly defined connectives of conjunction and material implication have the expected tables:

$\wedge$	1	$1/2$	0
1	1	$1/2$	0
$1/2$	$1/2$	$1/2$	$1/2$
0	0	$1/2$	0

$\rightarrow$	1	$1/2$	0
1	1	$1/2$	0
$1/2$	$1/2$	$1/2$	$1/2$
1	1	$1/2$	1

Definition 2. Bochvar (internal) logic **B** is the logic $\langle \mathscr{L}, \vdash_{\langle \mathbf{WK}, \{1\}\rangle}\rangle$. Paraconsistent weak Kleene (internal) logic **PWK** is the logic $\langle \mathscr{L}, \vdash_{\langle \mathbf{WK}, \{1, 1/2\}\rangle}\rangle$.

We will see in the next session why these logics are described as internal.

While on the semantical side the non-classical truth-value of weak Kleene logics resulted to be tough to be explained with a unanymously convincing interpretation, on the syntactic part these logics display an elegant characterization. Let $Var(\varphi)$ be the set of variables occuring in φ:

Theorem 3 ([Urquhart, 2001]). *$\Gamma \vdash_{\mathbf{B}} \varphi$ iff $\Delta \vdash_{\mathbf{CL}} \varphi$ and either Γ is an antitheorem of* **CL** *or $Var(\varphi) \subseteq Var(\Gamma)$.*

Theorem 4 ([Ciuni and Carrara, 2016]). *$\Gamma \vdash_{\mathbf{PWK}} \varphi$ iff there exists $\Delta \subseteq \Gamma$ s.t. $\Delta \vdash_{\mathbf{CL}} \varphi$ and $Var(\Delta) \subseteq Var(\varphi)$.*

Weak Kleene logics are therefore logics of variable inclusion, logics that can be obtained from another one (classical logic) by putting a syntactic filter over the latter. The characterization theorems quoted above state that weak Kleene logics are therefore variable inclusion companions of classical logic, more precisely **B** is the right variable inclusion companion of **CL** ([Bonzio et al., 2022, definition 6.1.1]), while **PWK** is the left variable inclusion companion of **CL** ([Bonzio et al., 2022, definition 5.1.3]).

In recent years a general framework has been developed to study logics of variable inclusion, especially how to obtain in a large number of important cases a variable inclusion companion starting from a certain logic (see [Bonzio et al., 2022]). The core of this work is based on an ingenious application of the technique of Płonka sums. A Płonka sum ([Płonka, 1967], see also [Płonka, 1968], [Płonka, 1969] and the survey in [Płonka and Romanowska, 1992]) is a pasting of similar algebras arranged in a special enriched semilattice.

Definition 5. A *semilattice direct system* of algebras of type τ is a triple $\mathbb{A} = \langle \{\mathbf{A}_i\}_{i \in I}, \langle I, \leq \rangle, \{p_{ij} | i, j \in I, i \leq j\} \rangle$ consisting of:

- a semilattice $\langle I, \leq \rangle$ with join $\vee$;
- a family $\{\mathbf{A}_i\}_{i \in I}$ of τ-algebras with pairwise disjoint universes;
- a homomorphism $p_{ij} : \mathbf{A}_i \to \mathbf{A}_j$, for every $i, j \in I$ such that $i \leq j$. Moreover, if $i \leq j, j \leq k, p_{ik} = p_{jk} \circ p_{ij}$, and $p_{ii} = id_{A_i}$.

When τ contains constants, $\langle I, \leq \rangle$ is a semilattice with bottom element 0.

Definition 6. For a semilattice direct system $\mathbb{A} = \langle \{\mathbf{A}_i\}_{i \in I}, \langle I, \leq \rangle, \{p_{ij} | i, j \in I, i \leq j\} \rangle$, the *Płonka sum* $\mathcal{P}_{\ell}(\mathbb{A})$ over $\mathbb{A}$ is the algebra, also of type τ, such that:

- its universe is $\bigcup_{i \in I} A_i$ (each $\mathbf{A}_i$ is called a *fibre* of the sum);

- for every n-ary operation $g \in \tau$, and $a_1 \in A_{i_1}, \ldots, a_n \in A_{i_n}$;

$$g^{\mathcal{P}_{\l}(\mathbb{A})}(a_1, \ldots, a_n) := g^{A_j}(p_{i_1 j}(a_1), \ldots, p_{i_n j}(a_n))$$

  where $j = a_{i_1} \vee \ldots \vee a_{i_n}$;

- with constants in the type: $c^{\mathcal{P}_{\l}(\mathbb{A})} = c^{A_0}$.

The way in which operations are computed once we paste a semilattice direct system into a Płonka sum is such that if all the arguments of the operation belong to the same fibre, the output is the same as if it would have been computed in the original algebra that constitutes the fibre; if the elements belong to different fibres, they are mapped by the homomorphisms p_{ij} to the fibre which is their least upper bound and the operation is computed there on the respective images.

Płonka sums resulted greatly helpful in the study of *regular varieties*, those varieties whose valid equations are only of the form $\alpha \approx \beta$ with $Var(\alpha) = Var(\beta)$ (called *regular equations*).

Theorem 7 ([Płonka, 1967], theorem 1)**.** *If $\mathbb{A}$ is a semilattice direct system containing at least two fibres, the equations valid in $\mathcal{P}_{\l}(\mathbb{A})$ are precisely the regular equations that hold in all algebras of $\mathbb{A}$.*

The *regularization* $\mathcal{R}(\mathsf{V})$ of a variety V is the class of algebras whose equational theory is composed by all and only the regular equations that are valid in V. A variety V is called *strongly irregular* if there is a term $\varphi(x, y)$ in exactly two variables s.t. $\mathsf{V} \vDash \varphi(x, y) \approx x$; e.g., the variety of lattices satisfies the equation $x \wedge (x \vee y) \approx x$ and as such is strongly irregular, as are all the varieties of algebras with a lattice reduct, e.g., the variety BA of Boolean algebras.

Theorem 8 ([Płonka and Romanowska, 1992], theorem 7.1)**.** *Let V be a strongly irregular variety. Then $\mathbf{A} \in \mathcal{R}(\mathsf{V})$ iff $\mathbf{A}$ is decomposable as a Płonka sum over a semilattice direct system of algebras in V.*

The regularization of BA, formulated in the language $\langle \vee, \wedge, \neg, 0, 1 \rangle$ of type $\langle 2, 2, 1, 0, 0 \rangle$, is the class of involutive bisemilattices IBSL (see [Bonzio et al., 2022, def. 2.4.1]), axiomatised as follows:

1. $x \vee x \approx x$;

2. $x \vee y \approx y \vee x$;

3. $(x \vee y) \vee z \approx x \vee (y \vee z)$;

4. $\neg\neg x \approx x$;

5. $x \wedge y \approx \neg(\neg x \vee \neg y)$;

6. $x \wedge (\neg x \vee y) \approx x \wedge y$;

7. $0 \vee x \approx 0$;

8. $1 \approx \neg 0$;

Using the above characterization of $\mathsf{IBSL} = \mathcal{R}(\mathsf{BA})$, it is easy to visualize an involutive bisemilattice, since it is just a semilattice direct system of Boolean algebras.

We have seen how to produce a Płonka sum from a semilattice direct system, but the other process is possible as well. In order to find a decompose an algebra to a suitable underlying semilattice direct system we introduce partition functions ([Płonka, 1967]).

Definition 9. Let $\mathbf{A}$ be a τ-algebra. A function $\cdot : A^2 \to A$ is a partition function for A if the following conditions are satisfied for all $a, b, c, a_1, \ldots, a_n \in A$:

- $a \cdot b \approx a$;
- $(a \cdot b) \cdot c \approx a \cdot (b \cdot c)$;
- $a \cdot (b \cdot c) \approx a \cdot (c \cdot b)$;

for $\lambda \in \tau$ s.t. its arity is $n > 1$:

- for $\lambda \in \tau$ s.t. its arity is $n > 1$:
- $\lambda(a_1, \ldots a_n) \cdot b \approx \lambda(a_1 \cdot b, \ldots a_n \cdot b)$;
- $b \cdot \lambda(a_1, \ldots a_n) \approx b \cdot a_1 \cdot \ldots \cdot a_n$;

for $\kappa \in \tau$ constant:

- $a \cdot \kappa = a$.

Given a partition function for an algebra **A**, this allows to arrange A in a semilattice direct system $\mathbb{A}$ such that **A** and $\mathcal{P}_{\ell}(\mathbb{A})$ are isomorphic ([Płonka, 1967, theorem 2]). Furthermore, there is a bijection between partition functions for a given algebra and Płonka sum representations of the same algebra ([Płonka, 1967, theorem 3]).

Partition functions give a tool to actually compute the homomorphism in a semilattice direct system and, therefore, the operations in the resulting Płonka sum.

Returning to the main topic of weak Kleene logics, the reason behind this algebraic detour is that the algebra **WK** which induces these logics is (term-equivalent to) an involutive bisemilattice. Moreover **WK** is not just any involutive bisemilattice, it is the algebra generating the entire variety: $\mathsf{IBSL} = \mathbb{V}(\mathbf{WK})$ ([Bonzio et al., 2017]).

1.2 External weak Kleene logics

In the previous section we described weak Kleene logics, without any further specification. Historically though, the intent of the first developer of the *weak Kleene* formalism, Dmitri Anatolyevich Bochvar ([Bochvar, 1981]), was to work within an enriched language allowing to express all classical two-valued formulae – which he referred to as *external formulae* – beside the genuinely three-valued ones.

The result of this choice is the language $\mathscr{L}_J \colon \langle \neg, \vee, J_2, 0, 1\rangle$ of type $\langle 1, 2, 1, 0, 0\rangle$, which is obtained by enriching the classical language by an additional unary connective J_2 (and the constants $0, 1$), where the formula $J_2\varphi$ is to be read as "φ is true". The language $\mathscr{L}_J$ will be referred to as *external language*, while its J_2-reduct will be called *internal.* Until chapter 3, let us refer to **Fm** as the formula algebra over $\mathscr{L}_J$. We will employ the following abbreviations[2]:

- $\varphi \wedge \psi \coloneqq \neg(\neg\varphi \vee \neg\psi)$;

[2]The arrow connective $\to$ is material implication. Since we will not employ any different intensional implication, we will just use the symbol $\to$.

- $\varphi \to \psi := \neg\varphi \vee \psi$;
- $\varphi \leftrightarrow \psi := (\varphi \to \psi) \wedge (\psi \to \varphi)$;
- $J_0\varphi := J_2\neg\varphi$;
- $J_1\varphi := \neg(J_2\varphi \vee J_2\neg\varphi)$;
- $+\varphi := \neg J_1\varphi$;
- $\varphi \equiv \psi := (J_2\varphi \leftrightarrow J_2\psi) \wedge (J_0\varphi \leftrightarrow J_0\psi)$.

The algebraic interpretation of the language $\mathscr{L}_J$ is given via the 3-element algebra $\mathbf{WK}^e = \langle\{0, 1, 1/2\}, \neg, \vee, J_2, 0, 1\rangle$, which is an expansion of $\mathbf{WK}$ with the unary operator J_2 obeying the following table:

φ	$J_2\varphi$
1	1
1/2	0
0	0

Via J_2 one can define other external connectives: J_0 expressing the drastic negation of a formula, and J_1 expressing its non-classicality. Moreover, $\equiv$ is the proper logical equivalence for the external language[3]. Their interpretation in $\mathbf{WK}^e$ is displayed in the following tables:[4]

φ	$J_0\varphi$
1	0
1/2	0
0	1

φ	$J_1\varphi$
1	0
1/2	1
0	0

$\equiv$	0	1/2	1
0	1	0	0
1/2	0	1	0
1	0	0	1

In the interpretation of $\mathscr{L}_J$ some formulae are evaluated into $\{0, 1\}$ *only* (which is the universe of a Boolean subalgebra of $\mathbf{WK}^e$) by every homomorphism $h\colon \mathbf{Fm} \to \mathbf{WK}^e$: these are called *external* formulae (see definition 11, for a syntactic definition).

[3]We use the symbol $\equiv$ that in the previous chapter denoted material equivalence, but notice that in the current chapter we have redefined it as $\varphi \equiv \psi := (J_2\varphi \leftrightarrow J_2\psi) \wedge (J_0\varphi \leftrightarrow J_0\psi)$.

[4]We are adopting here the notation due to [Finn and Grigolia, 1993]; [Bochvar, 1981] and [Segerberg, 1965] instead used t, f, − for J_2, J_0, J_1, respectively.

1.2.1 Bochvar external logic

As **WK** induced two different logics by taking different filters of designated values, the same happens for $\mathbf{WK}^e$.

Definition 10. Bochvar (external) logic $\mathbf{B}_\mathrm{e}$ is the logic $\langle \mathscr{L}_J, \vdash_{\langle \mathbf{WK}^e, \{1\}\rangle} \rangle$. Paraconsistent weak Kleene (external) logic $\mathbf{PWK}_\mathrm{e}$ is the logic $\langle \mathscr{L}_J, \vdash_{\langle \mathbf{WK}^e, \{1, 1/2\}\rangle} \rangle$.

Therefore, $\mathbf{B}_\mathrm{e}$ is the logic preserving only the value 1 (for truth), while $\mathbf{PWK}_\mathrm{e}$ preserves both 1 and $1/2$ (thus, non-falsity). The latter, originally introduced by [Halldén, 1949], has been later on studied by [Segerberg, 1965], who named it **C**. Both $\mathbf{B}_\mathrm{e}$ and $\mathbf{PWK}_\mathrm{e}$ are finitary logics, as they are defined by a single finite matrix.

Hilbert-style axiomatizations of the two logics are given by [Finn and Grigolia, 1993] and [Segerberg, 1965], respectively. In order to introduce them, some technicalities are needed.

Definition 11. An occurrence of a variable x in a formula φ is *open* if it does not fall under the scope of J_k, for every $k \in \{0, 1, 2\}$. A variable x in φ is *covered* if all of its occurrences are not open, namely if for every occurrence of x in φ falls under the scope of J_k, for some $k \in \{0, 1, 2\}$. A formula $\varphi \in Fm$ is called *external* if all its variables are covered.

The following is the Hilbert-style axiomatization of $\mathbf{B}_\mathrm{e}$ provided by [Finn and Grigolia, 1993][5]:

Axioms

(A1) $(\varphi \vee \varphi) \equiv \varphi$;

(A2) $(\varphi \vee \psi) \equiv (\psi \vee \varphi)$;

(A3) $((\varphi \vee \psi) \vee \chi) \equiv (\varphi \vee (\psi \vee \chi))$;

(A4) $(\varphi \wedge (\psi \vee \chi) \equiv ((\varphi \wedge \psi) \vee (\varphi \wedge \chi))$;

[5] To be precise, in [Finn and Grigolia, 1993] the following definition of the connective is given: $\varphi \equiv \psi := \bigwedge_{i=0}^{2} J_i\varphi \leftrightarrow J_i\psi$. Nonetheless, since the operator J_1 is entirely determined by J_2 and J_0, Finn and Grigolia's definition can be safely substituted with ours, obtaining an equivalent calculus (see e.g. [Bonzio and Pra Baldi, 2024]).

(A5) $\neg(\neg\varphi) \equiv \varphi$;

(A6) $\neg 1 \equiv 0$;

(A7) $\neg(\varphi \vee \psi) \equiv (\neg\varphi \wedge \neg\psi)$;

(A8) $0 \vee \varphi \equiv \varphi$;

(A9) $J_i \neg\varphi \equiv J_{2-i}\varphi$, for any $i \in \{0, 1, 2\}$;

(A10) $J_i\varphi \equiv \neg(J_j\varphi \vee J_k\varphi)$, with $i \neq j \neq k \neq i$;

(A11) $(J_i\varphi \vee \neg J_i\varphi) \equiv 1$, with $i \in \{0, 1, 2\}$;

(A12) $((J_i\varphi \vee J_k\psi) \wedge J_i\varphi) \equiv J_i\varphi$, with $i, k \in \{0, 1, 2\}$;

(A13) $(\varphi \vee J_i\varphi) \equiv \varphi$, with $i \in \{1, 2\}$;

(A14) $J_0(\varphi \vee \psi) \equiv J_0\varphi \wedge J_0\psi$;

(A15) $J_2(\varphi \vee \psi) \equiv (J_2\varphi \wedge J_2\psi) \vee (J_2\varphi \wedge J_2\neg\psi) \vee (J_2\neg\varphi \wedge J_2\psi)$.

Let α, β, γ denote external formulae only:

(A16) $J_2\alpha \equiv \alpha$;

(A17) $J_0\alpha \equiv \neg\alpha$;

(A18) $J_1\alpha \equiv 0$;

(A19) $\alpha \to (\beta \to \alpha)$;

(A20) $(\alpha \to (\beta \to \gamma)) \to ((\alpha \to \beta) \to (\alpha \to \gamma))$;

(A21) $(\neg\alpha \to \neg\beta) \to (\beta \to \alpha)$.

Rule

(MP) $\varphi, \varphi \to \psi \vdash \psi$.

Notice that there is nothing special about the choice of axioms (A19)-(A21): it is only important that, together with *modus ponens*, yield a complete axiomatization for classical logic, but only relative to external formulae.[6]

The fact that $\mathbf{B}_\mathrm{e}$ coincides with the logic induced by the above introduced axiomatization has been proved in [Bonzio et al., 2023] ([Finn and Grigolia, 1993, theorem 3.4] only proved a weak completeness theorem for $\mathbf{B}_\mathrm{e}$). We will henceforth indicate by $\vdash_{\mathbf{B}_\mathrm{e}}$ both the consequence relation induced by the matrix $\langle \mathbf{WK}^e, \{1\}\rangle$ and the one induced by the above axiomatization.

Contrasting the very poor algebraic behaviour of internal Kleene logics[7], the introduction of external operators make the logic $\mathbf{B}_\mathrm{e}$ expressive enough to become algebraizable, with the quasivariety of Bochvar algebras as its equivalent algebraic semantics.[8] The algebraizability of $\mathbf{B}_\mathrm{e}$ is witnessed by the transformers $\tau(\varphi) \coloneqq \{\varphi \approx 1\}$ and $\rho(\varphi \approx \psi) \coloneqq \{\varphi \equiv \psi\}$ (see [Bonzio et al., 2023] for details) and allows to provide a more standard Hilbert-style axiomatization, whose axioms and rules make no difference bewteen external and non-external formulae.

Using the equational description of the quasivariety of Bochvar algebras presented in [Bonzio and Pra Baldi, 2024] (see in particular [Bonzio and Pra Baldi, 2024, Theorem 7]), we can apply the algorithm described in [Font, 2016, Proposition 3.47], obtaining the following calculus.

Definition 12. A Hilbert-style axiomatization of $\mathbf{B}_\mathrm{e}$ is given by the following axioms and rules:

Axioms

(ρ-B1) $\varphi \vee \varphi \equiv \varphi$;

(ρ-B2) $\varphi \vee \psi \equiv \psi \vee \varphi$;

[6]Actually the original presentation in [Finn and Grigolia, 1993] contains a longer but equivalent axiomatization for the part capturing classical logic for external formulae.

[7]Neither **B** ([Bonzio et al., 2022, theorem 6.1.14]) nor **PWK** ([Bonzio et al., 2022, theorem 5.5.1]) are even protoalgebraic, which is the most general class within the so-called Leibniz hierarchy, a classification of logics according to the strength of their relation with their algebraic counterpart. Intuitively, this property can be intended as the minimal condition which makes meaningful talking about the algebraic coutnerpart of a logic.

[8]This quasivariety has been introduced by [Finn and Grigolia, 1993], while its structural properties are studied in [Bonzio and Pra Baldi, 2024], [Bonzio et al., 2024].

(ρ-B3) $(\varphi \vee \psi) \vee \chi \equiv \varphi \vee (\psi \vee \chi)$;

(ρ-B4) $\varphi \vee 0 \equiv \varphi$;

(ρ-B5) $\neg\neg\varphi \equiv \varphi$;

(ρ-B6) $\neg(\varphi \vee \psi) \equiv \neg\varphi \wedge \neg\psi$;

(ρ-B7) $\neg 1 \equiv 0$;

(ρ-B8) $\varphi \wedge (\psi \vee \chi) \equiv (\varphi \wedge \psi) \vee (\varphi \wedge \chi)$;

(ρ-B9) $J_0 J_2 \varphi \leftrightarrow \neg J_2 \varphi$;

(ρ-B10) $J_2 \varphi \leftrightarrow \neg(J_0 \varphi \vee J_1 \varphi)$;

(ρ-B11) $J_2 \varphi \vee \neg J_2 \varphi \leftrightarrow 1$;

(ρ-B12) $J_2(\varphi \vee \psi) \leftrightarrow (J_2\varphi \wedge J_2\psi) \vee (J_2\varphi \wedge J_0\psi) \vee (J_0\varphi \wedge J_2\psi)$.

Rules

(ρ-B13) $J_2\varphi \leftrightarrow J_2\psi, J_0\varphi \leftrightarrow J_0\psi \vdash \varphi \equiv \psi$;

(B-Alg3) $\varphi \dashv\vdash J_2\varphi \leftrightarrow 1$.

The axiomatization is equivalent to Finn and Grigolia's calculus, moreover it consists of a proper set of axiom schemata. In fact Finn and Grigolia impose a syntactic restriction on axioms (A19)-(A21), as a result those are not schemata, instead each point represents a countable set of schemata. On the contrary, the above axiomatization makes no distinction between external and non-external formulae, hence it enjoys a proper closure under substitution.

The following results recaps some basic properties of $\mathbf{B}_{\mathrm{e}}$ which will be used in the following sections.

Lemma 13. *The following facts hold in* $\mathbf{B}_{\mathrm{e}}$*:*

1. $\varphi, \varphi \to \psi \vdash \psi$;

2. $\vdash \alpha \leftrightarrow J_2\alpha$, *for* α *external formula;*

3. $\varphi \vdash J_2\varphi$;

4. *If* α *is an external formula and a classical theorem then* $\vdash \alpha$*;*

5. $\neg\varphi \vdash J_0\varphi$*;*

6. $\vdash J_0\varphi \to \neg J_2\varphi$*;*

7. $\vdash J_0\varphi \wedge J_2\varphi \to 0$*;*

8. $\vdash \neg J_2\varphi \to J_1\varphi \vee J_0\varphi$*;*

9. $\vdash \neg J_1\varphi \to J_2\varphi \vee J_0\varphi$*;*

10. $\vdash J_1\varphi \leftrightarrow J_1\neg\varphi$*;*

11. $\vdash J_1\alpha \leftrightarrow 0$*, for* α *external formula;*

12. $\neg J_2\neg\varphi \vdash J_2\varphi \vee J_1\varphi$*;*

13. $\Gamma \vdash J_i\varphi \vee J_j\varphi$ *and* $\Gamma \vdash J_i\varphi \vee J_k\varphi$ *imply* $\Gamma \vdash J_i\varphi$*, with* $i \neq j \neq k \neq i$*.*

Finally, let us recall that $\mathbf{B}_{\mathrm{e}}$ has a deduction theorem, in the following form.

Theorem 14 (Deduction theorem for $\mathbf{B}_{\mathrm{e}}$)**.** *It holds that* $\Gamma \vdash_{\mathbf{B}_{\mathrm{e}}} \varphi$ *iff there exist some formulae* $\gamma_1, \ldots, \gamma_n \in \Gamma$ *such that* $\vdash_{\mathbf{B}_{\mathrm{e}}} J_2\gamma_1 \wedge \cdots \wedge J_2\gamma_n \to J_2\varphi$*.*

1.2.2 External paraconsistent weak Kleene logic

The Hilbert-style axiomatization for $\mathbf{PWK}_{\mathrm{e}}$, introduced by [Segerberg, 1965] is the following:

Axioms

(A1) $(\varphi \vee \varphi) \to \varphi$;

(A2) $\varphi \to (\varphi \vee \psi)$;

(A3) $(\varphi \vee \psi) \to (\psi \vee \varphi)$;

(A4) $(\varphi \to \psi) \to ((\gamma \vee \varphi) \to (\gamma \vee \psi))$;

(A5) $(\varphi \wedge \psi) \to \neg(\neg\varphi \vee \neg\psi)$;

(A6) $\neg(\neg\varphi \vee \neg\psi) \to (\varphi \wedge \psi)$;

(A7) $\varphi \to 1$;

(A8) $0 \to \varphi$;

(A9) $\varphi \to J_2\varphi$;

(A10) $J_2\varphi \to \neg J_0\varphi$;

(A11) $J_2(\varphi \wedge \psi) \leftrightarrow J_2\varphi \wedge J_2\psi$;

(A12) $J_2(\varphi \vee \psi) \leftrightarrow ((J_2\varphi \wedge J_2\psi) \vee (J_2\varphi \wedge J_0\psi) \vee (J_0\varphi \wedge J_2\psi))$.

Rule

(RMP) $\varphi, \varphi \to \psi \vdash \psi$ provided that no variable is open in φ and covered in ψ.

As before, there is nothing special behind the choice of the axioms (A1) to (A8): one can simply choose any set of axioms which, together with the (usual) rule of *modus ponens* yields an axiomatization of (propositional) classical logic. Observe that the rule (RMP) consists of a linguistic restriction of the standard rule of *modus ponens*: a fact that shall not surprise, as a very similar restricted rule has been introduced for an axiomatization of **PWK** (in the language of classical logic) [Bonzio et al., 2022, Ch. 7]. However, providing the same logic with a standard calculus presenting no linguistic restriction is preferable (for instance, for internal **PWK**, such Hilbert-style axiomatizations can be found in [Greati et al., 2024], [Bonzio et al., 2022, Ch. 7]). As in the case of Bochvar, also for $\mathbf{PWK}_\mathrm{e}$, we will indicate by $\vdash_{\mathbf{PWK}_\mathrm{e}}$ both the consequence relation induced by the matrix $\langle \mathbf{WK}^e, \{1, 1/2\}\rangle$ and the one induced by the above Hilbert-style axiomatization.

Like $\mathbf{B}_\mathrm{e}$, also the logic $\mathbf{PWK}_\mathrm{e}$ is algebraizable with the quasivariety of Bochvar algebras as its equivalent algebraic semantic. The transformers that witness the algebraizability are $\tau(\varphi) \coloneqq \{\neg J_0\varphi \approx 1\}$ and $\rho(\varphi \approx \psi) \coloneqq \{\varphi \equiv \psi\}$. We leave the proof of this result to appendix 1.4. Observe that, although it is not very common, the same class of algebras can play the role of equivalent algebraic semantics for different logics[9] (clearly, the algebraizability is given by

[9][Font, 2016, pp. 121-122] lists of Łukasiewicz's three-valued logic $Ł_3$ and Da Costa and D'Ottaviano's paraconistent logic $\mathbf{J}_3$ as examples. Although artificial pairs of logics with the same equivalent algebraic semantics can be produced, the example of $\mathbf{B}_\mathrm{e}$ and $\mathbf{PWK}_\mathrm{e}$ is a nice addition to this classification since it arises naturally from two logics already provided with an independent motivation in the literature.

different transformers). In this sense, $\mathbf{B}_e$ and $\mathbf{PWK}_e$ are deductively equivalent ([Font, 2016, pp. 178-181]), in the sense that composing the transformers that map the logics to and from their common equivalent algebraic semantics it is possible to translate one logic into the other and to reverse the process without losing any information ([Blok and Jónsson, 2006]).

This algebraizability allows us to apply the algorithm of [Font, 2016] already employed for $\mathbf{B}_e$, obtaining a Hilbert-style axiomatization alternative to Segerberg's.

Definition 15. A Hilbert-style axiomatization of $\mathbf{PWK}_e$ is given by the following axioms and rules:

Axioms

(ρ-B1) $\varphi \vee \varphi \equiv \varphi$;

(ρ-B2) $\varphi \vee \psi \equiv \psi \vee \varphi$;

(ρ-B3) $(\varphi \vee \psi) \vee \chi \equiv \varphi \vee (\psi \vee \chi)$;

(ρ-B4) $\varphi \vee 0 \equiv \varphi$;

(ρ-B5) $\neg\neg\varphi \equiv \varphi$;

(ρ-B6) $\neg(\varphi \vee \psi) \equiv \neg\varphi \wedge \neg\psi$;

(ρ-B7) $\neg 1 \equiv 0$;

(ρ-B8) $\varphi \wedge (\psi \vee \chi) \equiv (\varphi \wedge \psi) \vee (\varphi \wedge \chi)$;

(ρ-B9) $J_0 J_2 \varphi \leftrightarrow \neg J_2 \varphi$;

(ρ-B10) $J_2 \varphi \leftrightarrow \neg(J_0 \varphi \vee J_1 \varphi)$;

(ρ-B11) $J_2 \varphi \vee \neg J_2 \varphi \leftrightarrow 1$;

(ρ-B12) $J_2(\varphi \vee \psi) \leftrightarrow (J_2\varphi \wedge J_2\psi) \vee (J_2\varphi \wedge J_0\psi) \vee (J_0\varphi \wedge J_2\psi)$.

Rules

(ρ-B13) $J_2\varphi \leftrightarrow J_2\psi, J_0\varphi \leftrightarrow J_0\psi \vdash \varphi \equiv \psi$;

(PWK-Alg3) $\varphi \dashv\vdash \neg J_0 \varphi \leftrightarrow 1$.

The only difference between the new axiomatizations proposed for $\mathbf{B}_{\mathrm{e}}$ and $\mathbf{PWK}_{\mathrm{e}}$ relies on the rules (PWK-Alg3) and (B-Alg3), which is expected since the two logics have the same equivalent algebraic semantics and differ only for the τ transformer.

Lemma 16. *The following facts hold for the logic* $\mathbf{PWK}_{\mathrm{e}}$*:*

1. $\alpha, \alpha \to \beta \vdash \beta$*, for every* α, β *external formulae;*
2. $\varphi \vdash \neg J_0\varphi$*;*
3. *every theorem of classical logic is a theorem of* $\mathbf{PWK}_{\mathrm{e}}$*;*
4. $\vdash \neg J_0 0 \to 0$*;*
5. $\vdash \neg\varphi \to J_0\varphi$*;*
6. $\vdash \alpha \leftrightarrow J_2\alpha$*, for* α *external formula;*
7. $\varphi \vdash J_2\varphi \vee J_1\varphi$*;*
8. $\vdash J_2\neg\varphi \leftrightarrow J_0\varphi$*;*
9. $\vdash J_1\varphi \leftrightarrow J_1\neg\varphi$*;*
10. $\vdash \neg J_0 1$*;*
11. $\vdash \varphi \to J_2\varphi$*;*
12. $\vdash J_2\varphi \to +\varphi$*.*

$\mathbf{PWK}_{\mathrm{e}}$ has a deduction theorem very similar to $\mathbf{B}_{\mathrm{e}}$, by just adapting the statement of theorem 60 (from truth) to non-falsity, in the obvious way suggested by external connectives.

Theorem 17 (Deduction theorem for $\mathbf{PWK}_{\mathrm{e}}$)**.** *It holds that* $\Gamma \vdash_{\mathbf{PWK}_{\mathrm{e}}} \varphi$ *iff there exist some formulae* $\gamma_1, \ldots, \gamma_n \in \Gamma$ *such that* $\vdash_{\mathbf{PWK}_{\mathrm{e}}} \neg J_0\gamma_1 \wedge \cdots \wedge \neg J_0\gamma_n \to \neg J_0\varphi$*.*

1.3 The structure of Bochvar algebras

The algebra $\mathbf{WK}^e$ which induces external weak Kleene logic is an example of a Bochvar algebra. It is actually very representative of this class of structures, since it is the quasivariety generated by $\mathbf{WK}^e$. We denote this quasivariety by BCA; note that this is a proper quasivariety, that is it cannot be described as an equational theory without any quasiequation, which is equivalent to say that BCA is not closed under the formation of quotients of its members.

Bochvar algebras were originally axiomatized by [Finn and Grigolia, 1993]. As before, we employ the abbreviations $J_0\varphi \coloneqq J_2\neg\varphi$ and $J_1\varphi \coloneqq \neg(J_2\varphi \vee J_2\neg\varphi)$.

Definition 18. A Bochvar algebra is an structure $\mathbf{A} = \langle A, \neg, \vee, \wedge, J_2, 0, 1\rangle$ of type $\langle 1, 2, 2, 1, 0, 0\rangle$ satisfying the following equations and quasiequations[10]:

1. $x \vee x \approx x$;
2. $x \vee y \approx y \vee x$;
3. $(x \vee y) \vee z \approx x \vee (y \vee z)$;
4. $x \wedge (y \vee z) \approx (x \wedge y) \vee (x \wedge z)$;
5. $\neg\neg x \approx x$;
6. $\neg 1 \approx 0$;
7. $\neg(x \vee y) \approx \neg x \wedge \neg y$;
8. $0 \vee x \approx x$;
9. $J_0 J_2 x \approx \neg J_2 x$;
10. $J_2 x \approx \neg(J_0 x \vee J_1 x)$;
11. $J_2 x \vee \neg J_2 x \approx 1$;
12. $J_2(x \vee y) \approx (J_2 x \wedge J_2 y) \vee (J_2 x \wedge J_2\neg y) \vee (J_2\neg x \wedge J_2 y)$;

[10] In [Finn and Grigolia, 1993] Bochvar algebras are expressed in an extended language including J_1 and J_0. These two operations are term-definable in the language $\langle A, \vee, \wedge, \neg, J_2, 0, 1\rangle$, as already observed. Besides, we are using here the much shorter equivalent axiomatization introduced in [Bonzio and Pra Baldi, 2024, Theorem 7].

13. $J_0x \approx J_0y$ & $J_2x \approx J_2y \;\Rightarrow\; x \approx y$.

Bochvar algebras form a proper quasivariety ([Finn and Grigolia, 1980]), which we denote by BCA, and it is generated by $\mathbf{WK}^e$. An immediate example of Bochvar algebras is obtained by any Boolean algebra $\mathbf{B}$ expanded with a new operator $J_2 = id_B$; in this context J_0 coincides with the Boolean negation and J_1 is the constant 0.

Axioms (1-8) tells us that (the J_2-free reduct of) Bochvar algebras have the structure of involutive bisemattices, as expected since they are generated by $\mathbf{WK}^e$, which has $\mathbf{WK}$ as reduct. We can further restrict the class of such involutive bisemattices, noticing that the quasiequation (FP) $x \approx \neg x$ & $y \approx \neg y \Rightarrow x \approx y$ holds in $\mathbf{WK}$, therefore it transfers to $\mathbb{Q}(\mathbf{WK}^e)$. The quasiequation states that there can be at most one single point of a Bochvar algebra with is a fixpoint for negation. The subquasivariety of IBSL characterized by (FP) is SIBSL, single-fixpoint involutive bisemilattices ([Bonzio et al., 2017]).

Proposition 19 ([Bonzio and Pra Baldi, 2024], proposition 3.5)**.** *The J_2-free reduct of a Bochvar algebra is a* SIBSL.

With a little abuse of notation, sometimes we are going to talk about the Płonka sum representation of a Bochvar algebra, but what we mean is the representation of its J_2-free reduct.

Examples of SIBSL comes from taking any Boolean algebra $\mathbf{B}$ (which is obviously an IBSL) and adding an absorbing element $k \notin B$, that is an element s.t. for every operation λ of the type of $\mathbf{B}$:

$$\lambda^{\mathbf{B}^*}(a_1, \dots, a_n) = \begin{cases} k & \text{if } k \in \{a_1, \dots, a_n\}; \\ \lambda^{\mathbf{B}}(a_1, \dots, a_n) & \text{otherwise.} \end{cases}$$

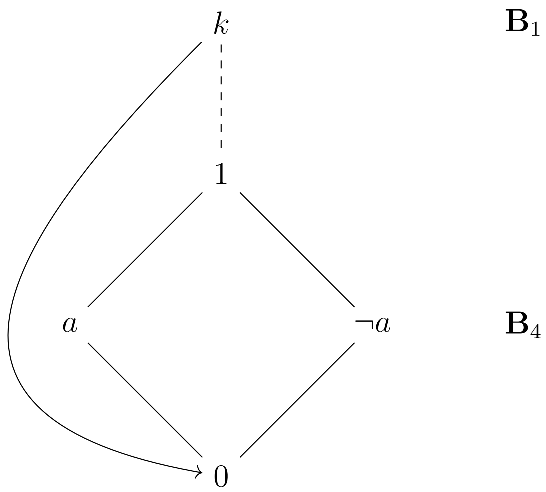
where $\mathbf{B}^*$ is the algebra similar to $\mathbf{B}$ whose support is $B^* = B \cup \{k\}$. The absorbing element is clearly the fixpoint for negation. In order to turn $\mathbf{B}^*$ into a Bochvar algebra $\mathbf{B}^*_J$ it is enough to define:

$$J_2^{\mathbf{JB}^*}(a) = \begin{cases} a & \text{if } a \neq k; \\ 0 & \text{otherwise.} \end{cases}$$

As an involutive bisemillatice, the J_2-free reduct of a Bochvar algebras can be represented as a Płonka sum of Boolean algebras, guiding us towards a better understanding of the structure of these algebras.

Proposition 20 ([Bonzio and Pra Baldi, 2024], lemma 3.9, remark 3.16)**.** *Let $\mathbf{A} \in \mathsf{BCA}$ be s.t. $\mathcal{P}_{\ell}(\mathbb{A})$ is the Płonka sum decomposition of its IBSL-reduct, with $\mathbb{A} = \langle\{\mathbf{A}_i\}_{i\in I}, \langle I, \leq\rangle, \{p_{ij} | i, j \in I, i \leq j\}\rangle$. For $i < j$, all the homomorphisms p_{ij} are surjective and not injective.*

Notice though that in general there is no way to represent a Bochvar algebra (in the full type) as a Płonka sum (unless its underlying semilattice direct system is trivial, hence the single resulting fibre is just a Boolean algebra). Once the external operator J_2 is added to the language, the structure of Płonka sum is broken, since by definition 6 the constants always inhabit the lowest fibre and the result of applying J_2 is to map any element to one of the two constants, therefore projecting it to the lowest fibre, which is not how operations should be computed in a Płonkas sum. Consider the algebra $\mathbf{JB}_4^*$, whose J_2-free reduct is the Płonka sum of the 4-element Boolean algebra $\mathbf{B}_4$ and the trivial algebra $\mathbf{B}_1$:



The two algebras are ordered according to the 2-element join-semilattice with the only possibile homomorphism $p : \mathbf{B}_4 \to \mathbf{B}_1$. The arrow represents the only non-trivial application of the J_2 operator (restricted to $\mathbf{B}_4$ it becomes just the identity).

It can be checked that $\mathbf{JB}_4^*$ is a Bochvar algebra and that the above is a picture of its IBSL-reduct seen as a Płonka sum. Observe how J_2 maps an element from an upper fibre, $\mathbf{B}_1$, to a lower one, $\mathbf{B}_4$, while definition 6 requires for a unary operator to map an element to the same fibre where said element belongs.

Despite this negative result, the J_2 operator has a strong interaction with the Płonka sum representation of the IBSL-reduct of a Bochvar algebra[11]. The application of J_2 to a fibre of the Płonka sum representation of a Bochvar algebra produces an isomorphic copy of that fibre inside the bottom fibre. Let us denote for a fibre $\mathbf{A}_i$, $1_i := a \vee \neg a$ for any element $a \in A_i$. 1_i represents the top element of the Boolean fibre $\mathbf{A}_i$, in this sense we will also call it the local top, to distinguish it from the algebraic constant 1.

Theorem 21 ([Bonzio and Pra Baldi, 2024], lemmas 3.9-3.14, theorems 3.12-3.15)**.** *, Let $\mathbf{A} \in$ BCA be s.t. $\mathcal{P}_{\ell}(\mathbb{A})$ is the Płonka sum decomposition of its IBSL-reduct, with $\mathbb{A} = \langle \{\mathbf{A}_i\}_{i\in I}, \langle I, \leq \rangle, \{p_{ij} | i, j \in I, i \leq j\}\rangle$. The following hold:*

(i) for every $i \in I$ and $a \in A_i$, $J_2(a) \in p_{0i}^{-1}(a)$; in particular for $a \in A_0, J_2(a) = a$;

(ii) for every $i \in I$, $J_2 : \mathbf{A}_i \to [\mathbf{0}, \mathbf{J_2 1_i}]$ is a (Boolean) isomorphism (with inverse the restriction of p_{0i} to $J_2[\mathbf{A}_i]$), with $[\mathbf{0}, \mathbf{J_2 1_i}] := \langle [0, J_2 1_i], \vee, \wedge, ^*, 0, J_2 1_i \rangle$ the interval Boolean algebra, where negation * is defined as $a^* := \neg a \wedge J_2 1_i$;

(iii) for every $i \neq j$, $J_2 1_i \neq J_2 1_j$; in particular, if $i < j$ then $J_2 1_j < J_2 1_i$.

The above results state that the bottom fibre of the IBSL-reduct of a Bochvar algebra contains all the information of the upper fibres, in the form of isomorphic copies of those fibres within the bottom one. The external operator J_2 maps a fibre A_i into the isomorphic interval Boolean algebra $[\mathbf{0}, \mathbf{J_2 1_i}]$ contained in the bottom fibre A_0, and the Płonka homomorphisms of the underlying semilattice direct systems isomorphically sends the interval algebra back to the original fibre A_i.

[11] Because of this structural interaction we are going to talk about Płonka-style representation of BCA, despite the fact that this is not, strictly speaking, a Płonka sum representation of the full Bochvar algebra, which cannot be performed. With some abuse of notation, we are simply going to talk about the Płonka sum representation of a Bochvar algebra instead of specifying that it is actually the representation of its J_2-free reduct.

Furthermore J_2 keeps track of the position of the fibres within the semilattice direct system, since given two different but comparable indexes $i < j$, the top element of the interval algebra $J_2[\mathbf{A}_j]$ will be strictly below the top of $J_2[\mathbf{A}_i]$ (notice that while in a Bochvar algebra there are two, possibly different, semilattice orderings to consider, it makes sense to refer to a single ordering within A_0 since this is a Boolean algebra).

The particular structure theory of Bochvar algebras will be the motivation behind the construction that we are going to introduce in section 2.1. The strategy will be to represent a structure in term of a certain subreduct of it and some special points of this reduct. In our case, we will show how a Bochvar algebra can be represented in term of its largest Boolean subalgebra, i.e. its bottom fibre in the Płonka decomposition, and a meet-subsemilattice of that, corresponding to the images through J_2 of the local ones of the fibres composing the algebra seen as a Płonka sum. This construction works precisely for the structural properties listed in theorem 21, which will yield the representation of any point of a Bochvar algebra in term of two Boolean components.

The process opposite to decomposition is possible as well, that is finding the properties a SIBSL must possess in order to be expanded to a Bochvar algebra[12].

Theorem 22 ([Bonzio and Pra Baldi, 2024], theorem 3.17)**.** *Let* $\mathbf{A}^- = \langle A, \vee, \wedge, \neg, 0, 1\rangle$ *be an involutive bisemilattice whose Płonka sum representation is such that:*

1. *all homomorphisms are surjective and* p_{0i} *is not injective for every* $0 \neq i \in I$*;*
2. *for each* $i \in I$ *there exists an element* $a_i \in A_0$ *such that* $p_{0i} \colon [\mathbf{0}, \mathbf{a}_i] \to \mathbf{A}_i$ *is an isomorphism, with* $a_i \neq a_j$ *for* $i \neq j$ *and, in particular,* $a_j < a_i$ *for each* $i < j$*.*

Define, for every $a \in A_i$ *and* $i \in I$*,* $J_2 a = p_{0i}^{-1}(a) \in [0, a_i]$*. Then* $\mathbf{A} = \langle A, \vee, \wedge, \neg, J_2, 0, 1\rangle$ *is a Bochvar algebra.*

[12] [Bonzio and Pra Baldi, 2024, example 3.19] shows a single-fixpoint involutive bisemilattice which cannot be transformed in a Bochvar algebra.

Theorem 23 ([Bonzio and Pra Baldi, 2024], corollary 3.18)**.** *Let* $\mathbf{A} \in \mathsf{SIBSL}$ *be such that its Płonka sum representation has surjective and non-injective homomorphisms. The following are equivalent:*

1. $\mathbf{A}$ *is the reduct of a Bochvar algebra;*

2. *for each* $i \in I$*,* $1/ker_{p_{0i}}$ *is a principal filter, with generator* $a_i \in A_0$*. Moreover, if* $i \neq j$ *then* $a_i \neq a_j$ *and* $a_j < a_i$ *for each* $i < j$*;*

3. *for each* $i \in I$ *there exists an element* $a_i \in A_0$ *such that* $p_{0i} \colon [\mathbf{0}, \mathbf{a}_i] \to \mathbf{A}_i$ *is an isomorphism. Moreover, if* $i \neq j$ *then* $a_i \neq a_j$ *and* $a_j < a_i$ *for each* $i < j$*.*

The relation between Bochvar algebras and their Płonka sum representation as involutive bisemilattices can be better appreciated in light of the categorical work performed in [Bonzio et al., 2024], where the algebraic category of Bochvar algebras is shown to be equivalent to a category whose objects are pairs consisting of a Boolean algebra and a meet-subsemilattice with 1 of such (called *Bochvar systems* in the same paper). The result refines the representation in theorem 21, building on the idea that all the relevant information about a Bochvar algebra is encoded in the Boolean algebra that inhabits the bottom fibre of its Płonka representation, together with the meet-semilattice of generators of the principal filters whose corresponding quotients inhabit the fibres of the same representation.

Definition 24. A *Bochvar system* is a pair $\mathbb{B} = \langle \mathbf{B}, \mathbf{S} \rangle$ such that $\mathbf{B} = \langle B, \vee, \wedge, \vee, \neg, 0, 1 \rangle$ is a Boolean algebra and $\mathbf{S} = \langle S, \wedge, 1 \rangle$ is a meet-subsemilattice with unit of $\mathbf{B}$.

Hereafter, for $b \in B$, by $[b)$ we denote the principal lattice filter generated by b in $\mathbf{B}$, and for F a filter of $\mathbf{B}$, by $\mathbf{B}/F$ we denote the quotient $\mathbf{B}/\theta_F$, where:

$$\theta_F = \{\langle a, b \rangle \in B^2 : (\neg a \vee b) \wedge (\neg b \vee a) \in F\}.$$

If $\mathbb{B} = \langle \mathbf{B}, \mathbf{S} \rangle$ is a Bochvar system, let:

$$\mathbb{A} = \langle \{\mathbf{A}_i\}_{i \in I}, \mathbf{S}^{\partial}, \{p_{ij} : i \leq_{\mathbf{S}} j\} \rangle,$$

be such that:

- for all $i \in S$, $\mathbf{A}_i := \mathbf{B}/[i)$;
- $\mathbf{S}^\partial$ is the semilattice dual to $\mathbf{S}$;
- for all $i, j \in I$ such that $i \leq_{\mathbf{I}^\partial} j$, $p_{ij}(a/[i)) := (a/[i))/[j)$.

In [Bonzio et al., 2024] it is first proved that:

Lemma 25. *$\mathbb{A}$ is a semilattice direct system of Boolean algebras, whence $\mathcal{P}_{\ell}(\mathbb{A})$ is an involutive bisemilattice.*

Hence, by theorems 22 and 23, $\mathcal{P}_{\ell}(\mathbb{A})$ can be expanded to a Bochvar algebra in a unique way.

Theorem 26. *$\mathcal{P}_{\ell}(\mathbb{A})$ is the involutive bisemilattice reduct of a unique Bochvar algebra $\mathbf{A}_{\mathbb{B}}$.*

In the other direction, we start with a Bochvar algebra:

$$\mathbf{A} = \langle A, \vee, \wedge, \neg, J_2, 0, 1\rangle$$

whose involutive bisemilattice reduct decomposes as $\mathcal{P}_{\ell}(\mathbb{A})$. We define $\mathbb{B}_{\mathbf{A}} := \langle \mathbf{A}_0, \mathbf{K}\rangle$, where:

- $K = \{J_2^{\mathbf{A}}(1^{A_i}) : i \in S\}$;
- for $i, j \in S$, $J_2^{\mathbf{A}}(1^{A_i}) \leq_{\mathbf{K}} J_2^{\mathbf{A}}(1^{A_j})$ iff $j \leq_{\mathbf{I}} i$.

We obtain that:

Theorem 27. *$\mathbb{B}_{\mathbf{A}}$ is a Bochvar system.*

The above maps, from Bochvar systems to Bochvar algebras and back, are mutually inverse. As implied by the definition of homomorphism to be found below, two Bochvar systems $\mathbb{B}_1 = \langle \mathbf{B}_1, \mathbf{S}_1\rangle$ and $\mathbb{B}_2 = \langle \mathbf{B}_2, \mathbf{S}_2\rangle$ are isomorphic when there exists an isomorphism g from $\mathbf{B}_1$ to $\mathbf{B}_2$ such that $g(i) \in S_2$ whenever $i \in S_1$.

Theorem 28 ([Bonzio et al., 2024], theorems 12-13)**.**

1. *If $\mathbf{A}$ is a Bochvar algebra, then $\mathbf{A}_{\mathbb{B}_{\mathbf{A}}}$ is isomorphic to $\mathbf{A}$;*
2. *If $\mathbb{B} = \langle \mathbf{B}, \mathbf{S}\rangle$ is a Bochvar system, then $\mathbb{B}$ is isomorphic to $\mathbb{B}_{\mathbf{A}_{\mathbb{B}}}$.*

This bijection can be lifted to a full-fledged categorical equivalence. Indeed, let:

- $\mathscr{B}$ denote the algebraic category of Bochvar algebras;
- $\mathscr{S}$ denote the category whose objects are Bochvar systems, and such that, if $\mathbb{B}_1 = \langle \mathbf{B}_1, \mathbf{S}_1 \rangle$ and $\mathbb{B}_2 = \langle \mathbf{B}_2, \mathbf{S}_2 \rangle$ are objects in $\mathfrak{S}$, a morphism from $\mathbb{B}_1$ to $\mathbb{B}_2$ is a homomorphism g from $\mathbf{B}_1$ to $\mathbf{B}_2$ such that $g(i) \in S_2$ for every $i \in S_1$.

We define the map Γ as follows:

- If $\mathbf{A}$ is an object in $\mathscr{B}$, $\Gamma(\mathbf{A}) := \mathbb{B}_{\mathbf{A}}$.
- If $f : \mathbf{A}_1 \to \mathbf{A}_2$ is a morphism in $\mathscr{B}$, $\Gamma(f)$ is the restriction of f to $\mathbf{A}_{1_0}$.

Similarly, we define the map Ξ as follows:

- If $\mathbb{B}$ is an object in $\mathscr{S}$, $\Xi(\mathbb{B}) := \mathbf{A}_{\mathbb{B}}$.
- If $g : \mathbb{B}_1 \to \mathbb{B}_2$ is a morphism in $\mathscr{S}$, $\Xi(g)$ is defined as follows: $\Xi(g)(a/[i]) := g(a)/[g(i))$.

Theorem 29 ([Bonzio et al., 2024], theorem 16)**.** *Γ and Ξ are functors that induce a categorical equivalence between the categories $\mathscr{B}$ and $\mathscr{S}$.*

1.4 Appendix: Algebraizability of $\mathbf{PWK}_\mathrm{e}$

Recall from definition 10 that $\mathbf{PWK}_\mathrm{e}$ is the logic induced by the matrix $\langle \mathbf{WK}^e, \{1, 1/2\} \rangle$. We are going show that the quasivariety of Bochvar algebras BCA plays the role of equivalent algebraic semantics for $\mathbf{PWK}_\mathrm{e}$, that is it plays the same role as for $\mathbf{B}_\mathrm{e}$ ([Bonzio et al., 2023, Theorem 35]).

As notational convention, in the following proof we will indicate by $Hom(\mathbf{Fm}, \mathbf{WK}^e)$ the set of homomorphisms from the formula algebra (in the language of BCA) in $\mathbf{WK}^e$.

Theorem 30. $\mathbf{PWK}_\mathrm{e}$ *is algebraizable w.r.t.* BCA *with transformers* $\tau(\varphi) := \{\neg J_0 \varphi \approx 1\}$ *and* $\rho(\epsilon \approx \delta) := \{\epsilon \equiv \delta\}$.

Proof. In our case (ALG1) and (ALG4) translate to:

$$\begin{aligned} &\text{(ALG1)} \quad \Gamma \vdash_{\mathsf{BCA}} \varphi \Leftrightarrow \tau[\Gamma] \vDash_{\mathsf{BCA}} \tau(\varphi), \\ &\text{(ALG4)} \quad \varepsilon \approx \delta \dashv\vDash_{\mathsf{BCA}} \tau(\rho(\varepsilon \approx \delta)). \end{aligned}$$

Moreover, since BCA is the quasivariety generated by $\mathbf{WK}^e$, the above claims amount to the following:

$$\begin{aligned} &\text{(ALG1)} \quad \Gamma \vdash_{\mathbf{PWK}_e} \varphi \Leftrightarrow \tau[\Gamma] \vDash_{\mathsf{BCA}} \tau(\varphi), \\ &\text{(ALG4)} \quad \varepsilon \approx \delta \dashv\vDash_{\mathbf{PWK}_e} \tau(\rho(\varepsilon \approx \delta)). \end{aligned}$$

That the chosen transformers are structural is guaranteed by [Font, 2016, proposition 3.3 and pp. 115-116], which states that τ and ρ are structural iff they are defined, respectively, by a set of equations in at most one variable and by a set of formulae in at most two variables. In the current case these are $\tau(x) := \{\neg J_0 x \approx 1\}$ for and $\rho(x \approx y) := \{x \equiv y\}$.

(ALG1) ($\Rightarrow$) Suppose $\Gamma \vdash_{\mathsf{PWK}_e} \varphi$. Take $h \in Hom(\mathbf{Fm}, \mathbf{WK}^e)$ s.t. $h(\neg J_0 \gamma) = h(1) = 1$ for every $\gamma \in \Gamma$, which implies $\neg J_0 h(\gamma) = 1$, i.e. $h(\gamma) \neq 0$. Since $\Gamma \vdash_{\mathsf{PWK}_e} \varphi$, by definition 10 it holds $h(\varphi) \neq 0$, hence $h(\neg J_0 \varphi) = 1 = h(1)$. Thus, we have shown that $\tau[\Gamma] \vDash_{\mathbf{WK}^e} \tau(\varphi)$.
($\Leftarrow$) Suppose $\tau[\Gamma] \vDash_{\mathbf{WK}^e} \tau(\varphi)$ and let $h \in Hom(\mathbf{Fm}, \mathbf{WK}^e)$ be s.t. $h(\gamma) \neq 0$ for every $\gamma \in \Gamma$. Therefore, by the hypothesis, $h(\neg J_0 \varphi) = h(1) = 1$, which implies $h(\varphi) \neq 0$, giving the desired conclusion.

(ALG4) ($\Rightarrow$) Consider an arbitrary equation $\varepsilon \approx \delta$ in the language of BCA. Unraveling the definition we have that $\tau(\rho(\varepsilon \approx \delta))$ is $\neg J_0(\varepsilon \equiv \delta) \approx 1$. Let $h \in Hom(\mathbf{Fm}, \mathbf{WK}^e)$ be s.t. $h(\varepsilon) = h(\delta)$, therefore $h(\varepsilon \equiv \delta) = 1$. This implies $h(\neg J_0(\varepsilon \equiv \delta)) = 1 = h(1)$, so we conclude $\varepsilon \approx \delta \models_{\mathbf{WK}^e} \tau(\rho(\varepsilon \approx \delta))$.
($\Leftarrow$) Suppose $h(\neg J_0(\varepsilon \equiv \delta)) = \neg J_0(h(\varepsilon) \equiv h(\delta)) = 1$, for $h \in Hom(\mathbf{Fm}, \mathbf{WK}^e)$. Now $J_0(h(\varepsilon) \equiv h(\delta)) = 0$ implies $(J_2 h(\varepsilon) \leftrightarrow J_2 h(\delta)) \wedge (J_0 h(\varepsilon) \leftrightarrow J_0 h(\delta)) \in \{1, 1/2\}$, but since $(J_2 h(\varepsilon) \leftrightarrow J_2 h(\delta)) \wedge (J_0 h(\varepsilon) \leftrightarrow J_0 h(\delta))$ is an external formula it must be that $(J_2 h(\varepsilon) \leftrightarrow J_2 h(\delta)) \wedge (J_0 h(\varepsilon) \leftrightarrow J_0 h(\delta)) = 1$. The fact that $J_2 h(\varepsilon) \leftrightarrow J_2 h(\delta) = 1$ and $J_0 h(\varepsilon) \leftrightarrow J_0 h(\delta) = 1$ implies

$J_2h(\varepsilon) = J_2h(\delta)$ and $J_0h(\varepsilon) = J_0h(\delta)$. Applying the quasiequation (13), we conclude $h(\varepsilon) = h(\delta)$. $\square$

Chapter 2

Structure theory of Bochvar algebras

In chapter 1 we have explained the reason behind the algebraic focus of this work: with the introduction of external operators to the language of weak Kleene we obtain logics that are algebraizable. This means that we can obtain useful information about the logics we are considering by investigating their algebraic counterpart, the quasivariety of Bochvar algebras BCA. Hence the interest in their properties and their strcuture theory.

Of utmost utility in the understanding of BCA is their Płonka-style representation provided by [Bonzio and Pra Baldi, 2024] and recalled in theorem 21: while only their IBSL-reduct can be strictly representedin this way, there is a precise interaction between that reduct seen as a Płonka sum of Boolean algebras and the external operator J_2 that expands the language, by producing isomorphic copies of the fibres inside the lowest one in the form of interval Boolean algebras. By observing the endpoints of the interval we can gain a further insight about the truly significant elements of a Bochvar algebra, namely the Boolean one - those belonging to the bottom fibre, which is the largest Boolean subalgebra of any BCA - and the images of the local tops of the fibres through J_2. We will emphasize the role of these points by adapting the notions of sharp, dense and dually dense elements to the Bochvar setting.

These elements are crucially important since they carry all the information needed to individuate any other point in the algebra. This idea is at the base of the notion of Bochvar systems (definition 24) and it will be further developed with the construction of variant twist product that we are going to introduce in this chapter. This is another technique for representing BCA which makes use precisely of the feature of the bottom fibre of a Bochvar algebra of encoding information, and more in general it seems to be a viable tool for analyzing algebras which can be represented in terms of some of their substructures (see [Paoli et al., 202x]). While the essential notions of Płonka sums, like those of fibres and their linking homomorphisms, can be recovered in this new construction, the twist product highlights different feature, above all the direct way in which all computations can be performed only by operating over elements of the bottom fibre. In the next section we enter in technical the details.

2.1 Twist products of Bochvar systems

In the paper [Paoli et al., 202x], the categorical equivalence from the previous chapter is presented using an expanded *twist product* construction. Twist products are a technique introduced by [Kalman, 1958] (see [Kracht, 1998]). Applied to algebras with a lattice reduct, it yields expansions of such which are still lattice-ordered. This construction resulted very helpful in the study of several classes of algebras, like subvarieties of residuated lattices ([Tsinakis and Wille, 2006], [Busaniche and Cignoli, 2009], [Busaniche et al., 2022]).

Differently from [Paoli et al., 202x], in the present chapter we will resort to an equivalent construction first introduced by [Katriňák, 1973] in the context of regular double Stone algebras. Applied to algebras with a lattice reduct, it produces expansions of their *direct* product (as opposed to the twist product).

We first explain the intuitive insight behind the construction. The categorical equivalence between Bochvar algebras and Bochvar systems implies that, in the representation of a Bochvar algebra as an expanded Płonka sum of its IBSL-reduct, all the relevant information about the entire algebra is encoded: (a) in its bottom fibre; (b) in the meet-semilattice of generators of the principal filters of such whose corresponding quotients yield the remaining fibres, and which are isomorphic to the interval Boolean algebras ranging from 0 to the image via J_2 of 1_i, the local top of $\mathbf{A}_i$.

From a different perspective, we can also say that to express the information needed to identify a point b in a Bochvar algebra we need two coordinates:

- a first element of the bottom algebra $\mathbf{A}_0$, which denotes the image of b through J_2;
- a second element of $\mathbf{A}_0$, denoting the image through J_2 of the local top 1_i of the fibre to which b belongs.

Definition 31. Let $\mathbb{B} = \langle \mathbf{B}, \mathbf{S} \rangle$ be a Bochvar system. The *variant twist product* of $\mathbb{B}$ is the algebra $\mathcal{P}_v(\mathbb{B}) = \langle T(B,S), \vee, \wedge, \neg, J_2, 0, 1 \rangle$ of type $\langle 2,2,1,1,0,0 \rangle$. The support[1] is defined as:

$$T(B,S) = \{(x,y) \in B^2 \mid x \leq y \text{ and } x \vee \neg y \in S\}.$$

On $\mathcal{P}_v(\mathbb{B})$ we define the operations $\wedge$ and $\vee$ by setting, for all $(x,y),(z,w) \in T(B,S)$:

$$(x,y) \wedge^{\mathcal{P}_v(\mathbb{B})} (z,w) = (x \wedge z, (y \vee w) \wedge (y \vee \neg z) \wedge (\neg x \vee w)); \tag{2.1}$$

$$(x,y) \vee^{\mathcal{P}_v(\mathbb{B})} (z,w) = ((x \wedge z) \vee (x \wedge \neg w) \vee (\neg y \wedge z), y \vee w). \tag{2.2}$$

We further define the unary operations:

$$\neg^{\mathcal{P}_v(\mathbb{B})}(x,y) = (\neg y, \neg x); \quad J_2^{\mathcal{P}_v(\mathbb{B})}(x,y) = (x,x), \tag{2.3}$$

and the constants:

$$0^{\mathcal{P}_v(\mathbb{B})} = (0,0); \quad 1^{\mathcal{P}_v(\mathbb{B})} = (1,1). \tag{2.4}$$

Unless otherwise specified, in the following by "twist product" (or simply "product") we always mean the above defined variant twist product, since we are not going to study other notions of more standard twist products from the literature.

Observe that in this construction, which we will see is able to represent every Bochvar algebra (theorem 48), the absorbing element (if there is any

[1]We employ the notation of ordered pairs of elements as (x,y) with round brackets instead of the acute ones to ease reading.

such) is represented by:

$$(0, 1).$$

In fact an absorbing element $(x, y) \in T(B, S)$ is a fixpoint for negation, i.e. $(x, y) = \neg(x, y) = (\neg y, \neg x)$. Hence $x = \neg y$, and since $x \leq y$ (because $(x, y) \in T(B, S)$), then $x = 0$ and $y = \neg x = 1$.

Before proceeding with the proof of the general properties of the structure just defined, we begin with an introductory lemma that will simplify some of the forthcoming proofs. Indeed, we observe that for particular pairs the definitions of $\wedge$ and $\vee$ simplify considerably.

Lemma 32. *Let $\mathbb{B} = \langle \mathbf{B}, \mathbf{S} \rangle$ be a Bochvar system and consider the product $\mathcal{P}_v(\mathbb{B})$. For all $(x, y), (z, w) \in T(B, S)$ such that $x = y$ and $z = w$, the following equalities hold:*

$$\begin{aligned}
(x, x) \wedge (z, z) &= (x \wedge z, x \wedge z), \\
(x, x) \vee (z, z) &= (x \vee z, x \vee z).
\end{aligned}$$

Proof. Let $(x, y), (z, w) \in T(B, S)$ be such that $x = y$ and $z = w$. We compute the meet as defined in equations 2.1:

$$\begin{aligned}
(x, x) \wedge (z, z) &= (x \wedge z, (x \vee z) \wedge (x \vee \neg z) \wedge (\neg x \vee z)) \\
&= (x \wedge z, (x \vee (z \wedge \neg z)) \wedge (\neg x \vee z)) \\
&= (x \wedge z, (x \vee 0) \wedge (\neg x \vee z)) \\
&= (x \wedge z, x \wedge (\neg x \vee z)) \\
&= (x \wedge z, (x \wedge \neg x) \vee (x \wedge z)) \\
&= (x \wedge z, x \wedge z).
\end{aligned}$$

The computation for the join is analogous and thus omitted. □

Theorem 33. *Let $\mathbb{B} = \langle \mathbf{B}, \mathbf{S} \rangle$ be a Bochvar system. The structure $\mathcal{P}_v(\mathbb{B})$ forms a Bochvar algebra.*

Proof. We begin by showing that the operations defined in equations 2.1 are well-defined. Let $(x, y), (z, w) \in T(B, S)$. Concerning the join operation, we need to show that:

$$(x \wedge z) \vee (x \wedge \neg w) \vee (\neg y \wedge z) \leq y \vee w \tag{2.5}$$

$$((x \wedge z) \vee (x \wedge \neg w) \vee (\neg y \wedge z) \vee \neg(y \vee w)) \in S \tag{2.6}$$

The inequality in 2.5 is straightforward, it follows from the fact that each of the three terms is bounded above by $y \vee w$:

$$x \wedge z \leq z \leq w \leq y \vee w, \quad x \wedge \neg w \leq x \leq y \leq y \vee w, \quad \neg y \wedge z \leq z \leq w \leq y \vee w.$$

To establish the inclusion in (2.6), note that since $(x, y), (z, w) \in T(B, S)$, we have:

$$x \leq y, \quad z \leq w, \quad x \vee \neg y \in S, \quad z \vee \neg w \in S.$$

Since S is closed under finite meets, it follows that:

$$(x \vee \neg y) \wedge (z \vee \neg w) \in S.$$

We now compute:

$$\begin{aligned}
&(x \wedge z) \vee (x \wedge \neg w) \vee (\neg y \wedge z) \vee \neg(y \vee w) = \\
&= (x \wedge z) \vee (x \wedge \neg w) \vee (\neg y \wedge z) \vee (\neg y \wedge \neg w) \\
&= ((x \vee \neg y) \wedge z) \vee ((x \vee \neg y) \wedge \neg w) \\
&= (x \vee \neg y) \wedge (z \vee \neg w).
\end{aligned}$$

Hence the inclusion in (2.6) is established. The proof for the meet operation is analogous.

Next we need to check the quasiequational theory of Bochvar algebras (definition 18).

1. $\varphi \vee \varphi \approx \varphi$

$$\begin{aligned}
(x, y) \vee (x, y) &= ((x \wedge x) \vee (x \wedge \neg y) \vee (\neg y \wedge x), y \vee y) \\
&= (x \vee (x \wedge \neg y), y) \\
&= (x, y).
\end{aligned}$$

2. $\varphi \vee \psi \approx \psi \vee \varphi$

$$\begin{aligned}(x,y) \vee (z,w) &= ((x \wedge z) \vee (x \wedge \neg w) \vee (\neg y \wedge z), y \vee w) \\ &= ((z \wedge x) \vee (z \wedge \neg y) \vee (\neg w \wedge x), w \vee y) \\ &= (z,w) \vee (x,y).\end{aligned}$$

3. $(\varphi \vee \psi) \vee \delta \approx \varphi \vee (\psi \vee \delta)$

$$\begin{aligned}&((x,y) \vee (z,w)) \vee (p,q)) = ((x \wedge z) \vee (x \wedge \neg w) \vee (\neg y \wedge z), y \vee w) \vee (p,q) = \\ &= ((((x \wedge z) \vee (x \wedge \neg w) \vee (\neg y \wedge z)) \wedge p) \vee (((x \wedge z) \vee (x \wedge \neg w) \vee (\neg y \wedge z)) \wedge \neg q)) \vee \\ &\vee (\neg(y \vee w) \wedge p, y \vee w \vee q) \\ &= ((x \wedge z \wedge p) \vee (x \wedge \neg w \wedge p) \vee (\neg y \wedge z \wedge p) \vee ((x \wedge z \wedge \neg q) \vee (x \wedge \neg w \wedge \neg q) \vee \\ &\vee (\neg y \wedge z \wedge \neg q)) \vee (\neg y \wedge \neg w \wedge p), y \vee w \vee q) \\ &= ((x \wedge ((z \wedge p) \vee (z \wedge \neg q) \vee (\neg w \wedge p))) \vee (x \wedge \neg(w \vee q)) \vee (\neg y \wedge ((z \wedge p) \vee \\ &\vee (z \wedge \neg q) \vee (\neg w \wedge p))), y \vee w \vee q) \\ &= (x,y) \vee ((z \wedge p) \vee (z \wedge \neg q) \vee (\neg w \wedge q), w \vee q) \\ &= (x,y) \vee ((z,w) \vee (p,q)).\end{aligned}$$

4. $\varphi \vee (\psi \wedge \delta) \approx (\varphi \vee \psi) \wedge (\varphi \vee \delta)$

We begin by computing the left-hand side of the identity:

$$(x,y) \vee ((z,w) \wedge (p,q)) = (x,y) \vee (z \wedge p,\ (w \vee q) \wedge (w \vee \neg p) \wedge (\neg z \vee q)).$$

Left-hand side, first component:

$$\begin{aligned}&(x \wedge z \wedge p) \vee (x \wedge \neg((w \vee q) \wedge (w \vee \neg p) \wedge (\neg z \vee q))) \vee (\neg y \wedge z \wedge p) = \\ &= (x \wedge z \wedge p) \vee (x \wedge ((\neg w \wedge \neg q) \vee (\neg w \wedge p) \vee (z \wedge \neg q))) \vee (\neg y \wedge z \wedge p) \\ &= (x \wedge z \wedge p) \vee (x \wedge \neg w \wedge \neg q) \vee (x \wedge \neg w \wedge p) \vee (x \wedge z \wedge \neg q) \vee (\neg y \wedge z \wedge p).\end{aligned}$$

Left-hand side, second component:

$$y \vee ((w \vee q) \wedge (w \vee \neg p) \wedge (\neg z \vee q)) =$$
$$= (y \vee w \vee q) \wedge (y \vee w \vee \neg p) \wedge (y \vee \neg z \vee q).$$

Now we compute the right-hand side:

$$((x, y) \vee (z, w)) \wedge ((x, y) \vee (p, q)) =$$
$$= ((x \wedge z) \vee (x \wedge \neg w) \vee (\neg y \wedge z),\ y \vee w) \wedge ((x \wedge p) \vee (x \wedge \neg q) \vee (\neg y \wedge p),\ y \vee q).$$

Right-hand side, first component, we apply distributivity of $\wedge$ over $\vee$:

$$((x \wedge z) \vee (x \wedge \neg w) \vee (\neg y \wedge z)) \wedge ((x \wedge p) \vee (x \wedge \neg q) \vee (\neg y \wedge p))$$
$$= (x \wedge z \wedge p) \vee (x \wedge \neg w \wedge p) \vee (\neg y \wedge z \wedge x \wedge p) \vee$$
$$\vee (x \wedge z \wedge \neg q) \vee (x \wedge \neg w \wedge \neg q) \vee (\neg y \wedge z \wedge x \wedge \neg q) \vee$$
$$\vee (\neg y \wedge z \wedge x \wedge p) \vee (\neg y \wedge z \wedge x \wedge \neg q) \vee (\neg y \wedge z \wedge p)$$
$$= (x \wedge z \wedge p) \vee (x \wedge \neg w \wedge p) \vee (x \wedge z \wedge \neg q) \vee (x \wedge \neg w \wedge \neg q) \vee (\neg y \wedge z \wedge p).$$

Observe that in the second line all the members of the join containing four elements amount to 0 due to $x \leq y$.

Right-hand side, second component:

$$(y \vee w \vee q) \wedge (y \vee w \vee \neg((x \wedge p) \vee (x \wedge \neg q) \vee (\neg y \wedge p)) \wedge$$
$$\wedge (\neg((x \wedge z) \vee (x \wedge \neg w) \vee (\neg y \wedge z))) \vee y \vee q))$$
$$= (y \vee w \vee q) \wedge (y \vee w \vee \neg p) \wedge (y \vee \neg z \vee q)$$

Hence, both the first and second components of the left-hand and right-hand sides coincide, and the distributive law is verified.

5. $\neg\neg\varphi \approx \varphi$

$$\neg\neg(x, y) = \neg(\neg y, \neg x) = (\neg\neg x, \neg\neg y) = (x, y).$$

6. $\neg 1 \approx 0$

$$\neg 1 = \neg(1,1) = (\neg 1, \neg 1) = (0,0) = 0.$$

7. $\neg(\varphi \vee \psi) \approx \neg\varphi \wedge \neg\psi$

$$\begin{aligned}\neg((x,y) \vee (z,w)) &= \neg((x \wedge z) \vee (x \wedge \neg w) \vee (\neg y \wedge z), y \vee w) \\ &= (\neg(y \vee w), \neg((x \wedge z) \vee (x \wedge \neg w) \vee (\neg y \wedge z))) \\ &= (\neg y \wedge \neg w, (\neg x \vee \neg z) \wedge (\neg x \vee w) \wedge (y \vee \neg z)) \\ &= (\neg y, \neg x) \wedge (\neg w, \neg z) \\ &= \neg(x,y) \wedge \neg(z,w).\end{aligned}$$

8. $0 \vee \varphi \approx \varphi$

$$(0,0) \vee (z,w) = ((0 \wedge z) \vee (0 \wedge \neg w) \vee (\neg 0 \wedge z), 0 \vee w) = (z,w).$$

9. $J_0 J_2 \varphi \approx \neg J_2 \varphi$

$$J_0 J_2(x,y) = J_2 \neg J_2(x,y) = J_2 \neg(x,x) = J_2(\neg x, \neg x) = (\neg x, \neg x) = \neg J_2(x,y).$$

10. $J_2 \varphi \approx \neg(J_0 \varphi \vee J_1 \varphi)$

$$\begin{aligned}\neg(J_0(x,y) \vee J_1(x,y)) &= \neg(J_2 \neg(x,y) \vee \neg(J_2(x,y) \vee J_2 \neg(x,y))) \\ &= \neg(J_2(\neg y, \neg x) \vee \neg((x,x) \vee J_2(\neg y, \neg x))) \\ &= \neg((\neg y, \neg y) \vee \neg((x,x) \vee (\neg y, \neg y))) \\ &= \neg((\neg y, \neg y) \vee ((\neg x, \neg x) \wedge (y,y))) \\ &= \neg((\neg y, \neg y) \vee (\neg x, \neg x)) \wedge ((\neg y, \neg y) \vee (y,y)) \\ &= \neg((\neg x, \neg x) \wedge (1,1)) \\ &= \neg(\neg x, \neg x) \\ &= (x,x) \\ &= J_2(x,y).\end{aligned}$$

11. $J_2 \varphi \vee \neg J_2 \varphi \approx 1$

$$J_2(x,y) \vee \neg J_2(x,y) = (x,x) \vee \neg(x,x) = (1,1).$$

12. $J_2(\varphi \vee \psi) \approx (J_2\varphi \wedge J_2\psi) \vee (J_2\varphi \wedge J_2\neg\psi) \vee (J_2\neg\varphi \wedge J_2\psi)$

$$\begin{aligned} J_2((x,y) \vee (z,w)) &= J_2((x \wedge z) \vee (x \wedge \neg w) \vee (\neg y \wedge z), y \vee w) \\ &= ((x \wedge z) \vee (x \wedge \neg w) \vee (\neg y \wedge z), (x \wedge z) \vee (x \wedge \neg w) \vee (\neg y \wedge z)). \end{aligned}$$

$$\begin{aligned} &(J_2(x,y) \wedge J_2(z,w)) \vee (J_2(x,y) \wedge J_2\neg(z,w)) \vee (J_2\neg(x,y) \wedge J_2(z,w)) \\ &= ((x,x) \wedge (z,z)) \vee ((x,x) \wedge (\neg w, \neg w)) \vee ((\neg y, \neg y) \wedge (z,z)) \\ &= (x \wedge z, x \wedge z) \vee (x \wedge \neg w, x \wedge \neg w) \vee (\neg y \wedge z, \neg y \wedge z) \\ &= ((x \wedge z) \vee (x \wedge \neg w), (x \wedge z) \vee (x \wedge \neg w)) \vee (\neg y \wedge z, \neg y \wedge z) \\ &= ((x \wedge z) \vee (x \wedge \neg w) \vee (\neg y \wedge z), (x \wedge z) \vee (x \wedge \neg w) \vee (\neg y \wedge z)). \end{aligned}$$

Hence our claim is obtained.

13. $J_0\varphi \approx J_0\psi$ & $J_2\varphi \approx J_2\psi \Rightarrow \varphi \approx \psi$
Suppose $J_0(x,y) = J_0(z,w)$, and $J_2(x,y) = J_2(z,w)$, then $(\neg y, \neg y) = J_0(x,y) = J_0(z,w) = (\neg w, \neg w)$ and $(x,x,) = J_2(x,y) = J_2(z,w) = (z,z)$, namely $(x,y) = (z,w)$.

□

From the results contained in the paper [Paoli et al., 202x], it can be deduced that the operator $\mathcal{P}_v$ can be turned into a full-fledged functor from the category of Bochvar systems to the category of Bochvar algebras, and moreover, that it factors through the *standard* Katrinak-style product construction used in [Katriňák, 1973] to represent regular double Stone algebras, and a translation from the language of Bochvar algebras to the language of regular double Stone algebras. We will not consider these aspects here.

We have shown that the product of a Bochvar system $\mathcal{P}_v(\mathbb{B})$ forms a Bochvar algebra. Our next goal is to prove every Bochvar algebra can be represented with the product construction of definition 31.

Let us now characterize the sets of its sharp, dense and dually dense elements in terms of the product construction (for some of the following results, compare also [Bonzio et al., 2024]).

Definition 34. Let $\mathbf{A}$ be a Bochvar algebra, and let $a \in A$. We say that:

- a is *sharp* if $a \vee \neg a = 1$, or equivalently $a \wedge \neg a = 0$;
- a is *dense* if $J_0 a = 0$;
- a is *dually dense* if $J_2 a = 0$.

This nomenclature is taken from the theory of quantum logics and orthomodular lattices ([Dalla Chiara et al., 2004], [Kalmbach, 1983]), where sharp elements play an essential role. In an algebraic setting whose aim is to generalize classical structures (e.g., as a model for quantum physics where uncertainty is fundamental and some sets of observables cannot be simultaneously be determined in a classical - sharp - way), the sharp elements are those corresponding to Boolean substructures, or, intuitively, they stand for classical values, in contrast with the unsharp ones.

In the current discussion, sharp elements are the properly classical ones in a non-classical setting. In a Bochvar algebra they correspond to the support of the lowest fibre, i.e. all the fixpoints for the J_2 operator, and the lowest fibre is the largest Boolean subalgebra of any Bochvar algebra. External operators, which we have called classical recapture operators, can be considered sharpness operators as well, since they turn arbitrary elements into sharp ones, by mapping any fibre of the IBSL-reduct of a Bochvar algebra inside its bottom fibre.

Lemma 35. *Let $\mathbb{B} = \langle \mathbf{B}, \mathbf{S} \rangle$ be a Bochvar system. Then the set of sharp elements of the Bochvar algebra $\mathcal{P}_v(\mathbb{B})$, denoted as $J(B,S)$, is the universe of a Boolean subalgebra of $\mathcal{P}_v(\mathbb{B})$ which is isomorphic to $\mathbf{B}$, via the map:*

$$g\colon B \to J(B,S), \qquad g(x) = (x,x).$$

Proof. Let us define the map $g\colon B \to J(B,S)$ as above. We first show that the image of $g(B)$ lies entirely within $J(B,S)$. Indeed, for each $x \in B$ we have

$$(x,x) \vee \neg(x,x) = (x,x) \vee (\neg x, \neg x) = (x \vee \neg x, x \vee \neg x) = (1,1).$$

Hence (x, x) is sharp for all $x \in B$, so g is well-defined.

It is evident that g is an injective lattice homorphism that also preserves $\neg$. To show that g is surjective, suppose $(x, y) \in J(B, S)$. By definition of sharp elements

$$\begin{aligned}(x, y) \wedge \neg(x, y) &= (x, y) \wedge (\neg y, \neg x) \\ &= (x \wedge \neg y, (y \vee \neg x) \wedge (y \vee y) \wedge (\neg x \vee \neg x)) \\ &= (x \wedge \neg y, y \wedge \neg x) \\ &= (0, 0).\end{aligned}$$

This implies that $x \wedge \neg y = 0$ and $y \wedge \neg x = 0$, i.e., $x \leq y$ and $y \leq x$. Hence, $x = y$ and any sharp element is necessarily of the form (x, x). Moreover g is an isomorphism. □

Remark 36. In view of lemma 35, the set of sharp elements of $\mathcal{P}_v(\langle \mathbf{B}, \mathbf{S} \rangle)$ can be described as

$$J(B, S) = \{(x, x) \mid x \in B\} = \{y \in T(B, S) \mid J_2 y = y\}.$$

Switching back to the perspective of Płonka sums, the set of sharp elements of a Bochvar algebra is precisely the universe of its lowest fibre $\mathbf{A}_0$. These are the elements for which $J_2 x = x$ (a consequence of theorem 21 for the fact that J_2 is the inverse of the identity homomorphism $p_{00} : A_0 \to A_0$). This fact supports the intuitive reading of the sharp elements as the properly Boolean ones, as they compose the only fibre which is a Boolean subalgebra of a Bochvar algebra (in fact the other fibres are Boolean algebras, but not subalgebras, since they lack the constants).

Similarly to what happens with the set of sharp elements, the set of dense and dually dense elements[2] also admits a neat representation in the product

[2]These notions, dense and dually dense, arise naturally in the theory of Stone algebras ([Chen and Grätzer, 1969a], [Chen and Grätzer, 1969b]) and their expansions, like regular double Stone algebras. Like it happened with orthomodular lattices, this is another context related to uncertainty and partial information. In the particular case of regular double Stone algebras, from a logical perspective they generalize classical logic to a substructural context where the principles of non-contradiction and excluded middle are not simultaneously valid. Classical negation splits into two unary operations: $^{\sim}$, which enforces non-contradiction, and $^{+}$, which enforces excluded middle.
Returning to the notion of sharp element, recent results have highlighted that reg-

construction.

Lemma 37. *Let* $\mathbb{B} = \langle \mathbf{B}, \mathbf{S} \rangle$ *be a Bochvar system and consider the product* $\mathcal{P}_v(\mathbb{B})$*. Its dense elements are denoted by* $D(B, S)$ *and can be described as:*

$$D(B, S) = \{(x, 1) : x \in S\};$$

The dually dense elements are denoted by $D^{\partial}(B, S)$ *and can be described as:*

$$D^{\partial}(B, S) = \{(0, y) : \neg y \in S\};$$

Proof. For the claim concerning dense elements, suppose $(x, y) \in D(B, S)$. By definition, $J_0(x, y) = (\neg y, \neg y) = (0, 0)$, which implies $\neg y = 0$, i.e. $y = 1$. Furthermore, if $(x, 1) \in (B, S)$ then $x \vee \neg 1 = x \vee 0 = x \in S$.
The proof of the second claim is dual.

□

As a consequence of lemma 37, one can observe a remarkably tight connection between the sets of dense and dually dense elements and the meet-semilattice S used in the construction. This observation will play a crucial role in establishing the categorical equivalence discussed in section 2.4.

It is an immediate consequence of the above lemma that these sets are not just collections of points but previse subreducts of the underlying BOchvar algebra.

Corollary 38. *Let* $\mathbb{B} = \langle \mathbf{B}, \mathbf{S} \rangle$ *be a Bochvar system and consider the product* $\mathcal{P}_v(\mathbb{B})$*. Then:*

1. $D(B, S)$ *is the universe of a meet-subsemilattice with unit;*
2. $D^{\partial}(B, F)$ *is the universe of a join-subsemilattice with zero.*

ular double Stone algebras are also relevant from the perspective of quantum logics ([Giuntini et al., 2024], [Ledda and Vergottini, 2025]), insofar as they represent sharp contexts in generalizations of orthomodular lattices, in much the same way as Boolean algebras represent classical contexts within orthomodular theory ([Ledda and Vergottini, 2024]).

Actually, the correspondence between the set of dense elements and the Boolean meet-semilattice used in the construction can be shown to be a lattice isomorphism, as the following lemma is going to prove. Intuitively, this means that the dense elements represent, in the product construction, what in the Płonka representation are the local tops of the fibres of the direct system (we will see this precisely in the section). An analogous result holds dually for the set of dually dense elements.

Lemma 39. *Let* $\mathbb{B} = \langle \mathbf{B}, \mathbf{S} \rangle$ *be a Bochvar system and consider the product* $\mathcal{P}_v(\mathbb{B})$*. Then* $D((B, S))$ *is the universe of a meet-semilattice isomorphic to* $\mathbf{S}$*.*

Proof. Define the mapping $h : D(B, S) \to S$ by

$$h(x, 1) = x.$$

Since every element of $D(B, S)$ is of the form $(x, 1)$ with $x \in S$, h is clearly well-defined. It is immediate to verify that h is bijective and that it preserves the meet operation and the top element. Hence, h is a meet-semilattice isomorphism. □

2.2 Representing Bochvar algebras with twist products

Having shown that the twist product construction introduced in the previous section yields a Bochvar algebra for any Bochvar system, we now turn to the converse direction.

In this subsection, we demonstrate that every Bochvar algebra can be represented, up to isomorphism, as the twist product of a certain Bochvar system. This product will be obtained from a Bochvar system consisting of sharp and dense elements of the starting algebra, showing how these two sets of elements are all the information needed to individuate the totality of points of a Bochvar algebra. This result establishes the completeness of the representation and ensures that the class of Bochvar algebras is captured entirely by the product construction of definition 31.

Recall that for any Bochvar algebra $\mathbf{A}$, $J_2[\mathbf{A}]$ is the lowest fibre of its

Płonka sum representation and a Boolean subalgebra (actually the largest) of **A**. Then the following characterization, which states that the sharp elements of a Bochvar algebra coincides with the support of its lowest fibre in the Płonka representation, is straightforward:

Lemma 40. *Let* $\mathbf{A} \in \mathsf{BCA}$. *The set of sharp elements of* $\mathbf{A}$ *is :*

$$J(A) = \{a \in A \mid J_2 a = a\}.$$

Proof. For $\mathbf{A} \in \mathsf{BCA}$, a sharp element $x \in A$ is s.t. $x \vee \neg x = 1$, therefore for the Płonka sum representation of **A**, since constants belong to the bottom fibre A_0, we have that $x \in A_0$, but $J_2 \restriction_{A_0} = id_{A_0}$, so $J_2 x = x$. For the other direction, if $J_2 x = x$ then $x \in A_0$, since J_2 is a map to the bottom fibre, hence $x \vee \neg x = 1$ since $\mathbf{A}_0$ is a Boolean subalgebra. □

We now provide a characterization of the sets of dense and dually dense elements, and investigate their operational properties.

Lemma 41. *Let* $\mathbf{A} \in \mathsf{BCA}$. *Then, for any* $x \in A$, *the following holds:*

$$J_0(x \vee \neg x) = 0,$$

that is, the element $x \vee \neg x$ *belongs to the set of dense elements of* $\mathbf{A}$.

Proof. Let $x \in A$. By definition of J_0 on join operation, we have:

$$J_0(x \vee \neg x) = J_0 x \wedge J_0 \neg x.$$

Using the identity $J_0 \neg x = J_2 x$ (which is valid in BCA), we obtain:

$$J_0 x \wedge J_0 \neg x = J_0 x \wedge J_2 x.$$

Now, since $J_0 x \leq \neg J_2 x$ (again, by the axioms of the algebra), we have:

$$J_0 x \wedge J_2 x \leq \neg J_2 x \wedge J_2 x = 0.$$

Therefore,

$$J_0(x \vee \neg x) = 0,$$

as required. $\square$

The lemma justifies when we were identifying the dense elements with the local tops of the fibres of the Płonka representation of a Bochvar algebra.

Corollary 42. *Let* $\mathbf{A} \in \mathsf{BCA}$. *The set of dense elements of* $\mathbf{A}$ *is :*

$$D(A) = \{a \vee \neg a \mid a \in A\}.$$

Proof. Suppose $x \in A$ is not of the form $a \vee \neg a$ for any $a \in A$. Consider the Płonka decomposition of $\mathbf{A}$. x cannot belong to a trivial fibre, so assume $a \in A_i$ for a certain non-trivial fibre. By hypothesis, $x \neq x \vee \neg x$, so it is not the local top. Hence $x < x \vee \neg x$, therefore $\neg(x \vee \neg x) < \neg x$ (since A_i is a Boolean algebra). J_2 is a Boolean isomorphism (theorem 21), so $0 = J_0(x \vee \neg x) = J_2 \neg(x \vee \neg x) < J_2 \neg x = J_0 x$. Hence x is not dense.

The other inclusion is a consequence of lemma 41. $\square$

Lemma 43. *Let* $\mathbf{A} \in \mathsf{BCA}$. *Then, for any* $x \in L$*, the following holds:*

$$\neg J_2(x \wedge \neg x) = 1,$$

that is, the element $x \wedge \neg x$ *belongs to the set of dually dense elements of* L*.*

Proof. The prove is dual to the argument in lemma 41. $\square$

We now turn our attention to some structural equations that hold in every Bochvar algebra and concern the interaction between arbitrary elements and the sharp projections defined by J_0 and J_2. In particular, we show that the expressions $x \vee \neg x$ and $x \wedge \neg x$ can be characterized in terms of $J_0(x)$ and $J_2(x)$, respectively. This highlights the role of the sharp components in capturing tautologies and contradictions within the algebra.

Lemma 44. *Let* $\mathbf{A} \in \mathsf{BCA}$. *The following identities hold for all* $x \in A$*:*

$$x \vee \neg x = x \vee J_0 x \quad x \wedge \neg x = x \wedge \neg J_2 x.$$

Proof. Let $x \in A$. By lemma 41 and the axioms of the Bochvar algebra, we have:

$$J_0(x \vee \neg x) = 0 = J_0(x \vee J_0 x).$$

We now consider the operator J_2:

$$\begin{aligned}
J_2(x \vee \neg x) &= (J_2x \wedge J_2\neg x) \vee (J_2x \wedge J_2\neg\neg x) \vee (J_2\neg x \wedge J_2\neg x) \\
&= (J_2x \wedge J_2\neg x) \vee (J_2x \wedge J_2x) \vee (J_2\neg x \wedge J_2\neg x) \\
&= (J_2x \wedge J_2\neg x) \vee J_2x \vee J_2\neg x \\
&= J_2x \vee J_2\neg x.
\end{aligned}$$

Next, we compute:

$$\begin{aligned}
J_2(x \vee J_0x) &= (J_2x \wedge J_2J_0x) \vee (J_2x \wedge J_2\neg J_0x) \vee (J_2\neg x \wedge J_2J_0x) \\
&= (J_2x \wedge J_0x) \vee (J_2x \wedge J_2J_0J_0x) \vee (J_2\neg x \wedge J_0x) \\
&= (J_2x \wedge J_0x) \vee (J_2x \wedge \neg J_0x) \vee (J_2\neg x \wedge J_0x) \\
&= (J_2x \wedge J_0x) \vee J_2x \vee (J_2\neg x \wedge J_0x) \\
&= (J_2x \wedge J_0x) \vee J_2x \vee (J_0x \wedge J_0x) \\
&= (J_2x \wedge J_0x) \vee J_2x \vee J_0x \\
&= J_2x \vee J_2\neg x.
\end{aligned}$$

Therefore, both $J_0(x \vee \neg x) = J_0(x \vee J_0x)$ and $J_2(x \vee \neg x) = J_2(x \vee J_0x)$. By quasiequation (13) of the definition 18 of Bochvar algebras, this implies:

$$x \vee \neg x = x \vee J_0x.$$

The proof is similar for $x \wedge \neg x = x \wedge \neg J_2x$. □

Lemma 45. *Let* $\mathbf{A} \in \mathsf{BCA}$*. The set of dense elements is a meet-subsemilattice with unit of* $\mathbf{A}$*.*

Proof. Clearly, $1 \in D(A)$, since $J_0(1) = 0$.

Let $x, y \in D(A)$, so that $J_0x = J_0y = 0$. We verify that $x \wedge y \in D(A)$:

$$\begin{aligned}
J_0(x \wedge y) &= (J_2x \wedge J_0y) \vee (J_0x \wedge \neg J_1y) \\
&= (J_2x \wedge 0) \vee (0 \wedge \neg J_1y) \\
&= 0.
\end{aligned}$$

□

The next theorem shows that there exists an embedding of the set of dense elements $D(A)$ into the lowest fibre $J_2[\mathbf{A}]$ of $\mathbf{A}$, thus establishing a structural connection between the dense and the sharp components of the algebra.

Lemma 46. *Let* $\mathbf{A} \in \mathsf{BCA}$. *The map:*

$$f : D(A) \to J_2[A], \quad f(x) = J_2 x,$$

is an injective homorphism on meet-semilattices (with unit).

Proof. First, observe that $J_2(1) = 1$, and since $J_0(1) = 0$, we have $1 \in D(A)$ and hence $1 \in f[D(A)]$, so $f[D(A)]$ contains the unit.

Let $x, y \in D(A)$. By properties of J_2 it follows:

$$J_2(x \wedge y) = J_2 x \wedge J_2 y.$$

Therefore, $f(x \wedge y) = f(x) \wedge f(y)$.

To prove injectivity, suppose $x, y \in D$ and $f(x) = f(y)$, i.e., $J_2 x = J_2 y$. Since $J_0 x = J_0 y = 0$, it follows from by quasiequation (13) of the definition 18 that $x = y$. Hence f is injective. □

Before proceeding with the main result of this section, we wish to highlight how the sets $J(A)$ and $D(A)$ intrinsically carry information about the entire structure. Indeed, lemma 47 states that every element of a Bochvar algebra can be expressed as a composition of a sharp element and a dense one, or as a sharp element and a dually dense one.

Lemma 47. *Let* $\mathbf{A} \in \mathsf{BCA}$. *Then, for every* $x \in A$, *the following representations hold:*

$$x = J_2 x \vee (x \wedge \neg J_2 x),$$
$$x = J_2 x \vee (x \wedge \neg x),$$
$$x = \neg J_0 x \wedge (x \vee J_0 x),$$
$$x = \neg J_0 x \wedge (x \vee \neg x).$$

Proof. Straightforward. □

Observe that:

$$J_2x \vee (x \wedge \neg J_2x)$$

is a decomposition of x into its sharp part J_2x and the residual part $(x \wedge \neg J_2x)$, which is disjoint from the sharp component due to the Boolean behavior of J_2. Hence, all the above expressions provide equivalent normal forms for x in terms of the sharp and dense components of $\mathbf{A}$.

Theorem 48. *Every Bochvar algebra is isomorphic to a variant twist product of the form $\mathcal{P}_v(\mathbb{B})$, for some Bochvar system $\mathbb{B}$. Moreover, $\mathbb{B}$ is uniquely determined up to isomorphism.*

Proof. Let $\mathbf{A}$ be a Bochvar algebra, and consider the set of sharp elements $J(A)$ and that of dense elements $D(A)$. Recall that the $J(A)$ is the universe of the Boolean subalgebra $J_2[\mathbf{A}]$ of $\mathbf{A}$, the lowest fibre of the IBSL-reduct of $\mathbf{A}$. Let us fix $\mathbf{B} := J_2[\mathbf{A}]$ as the first component of our desired Bochvar system.

By theorem 46, the map J_2 embeds $D(A)$, as a semilattice, into $J_2[A]$, and $J_2[D(A)]$ is the universe of a meet-subsemilattice with unit of $J_2[\mathbf{A}]$. Let us set $\mathbf{S} := \langle J_2[D(A)], \wedge, 1\rangle$ as the second component of the Bochvar system we are building.

We can now perform $\mathcal{P}_v(\langle \mathbf{B}, \mathbf{S}\rangle)$ and define the mapping $\varphi : A \to T(B, S)$ by:

$$\varphi(x) = (J_2x, \neg J_0x).$$

It is evident that φ is well-defined.

We check that φ is a homomorphism. Regarding the meet operation:

$$\begin{aligned}
\varphi(x \wedge y) &= (J_2(x \wedge y), \neg J_0(x \wedge y)) \\
&= (J_2x \wedge J_2y, \neg J_0(x \wedge y)) \\
&= (J_2x \wedge J_2y, \neg J_2\neg(x \wedge y)) \\
&= (J_2x \wedge J_2y, \neg J_2(\neg x \vee \neg y)) \\
&= (J_2x \wedge J_2y, \neg((J_2\neg x \wedge J_2\neg y) \vee (J_2x \wedge J_2\neg y) \vee (J_2\neg x \wedge J_2y))) \\
&= (J_2x \wedge J_2y, (\neg J_2\neg x \vee \neg J_2\neg y) \wedge (\neg J_2x \vee \neg J_2\neg y) \wedge (\neg J_2\neg x \vee \neg J_2y))) \\
&= (J_2x \wedge J_2y, (\neg J_0x \vee \neg J_0y) \wedge (\neg J_2x \vee \neg J_0y) \wedge (\neg J_0x \vee \neg J_2y))) \\
&= (J_2x, \neg J_0x) \wedge (J_2y, \neg J_0y) \\
&= \varphi(x) \wedge \varphi(y).
\end{aligned}$$

For J_2:

$$\begin{aligned}
\varphi(J_2x) =& (J_2J_2x, \neg J_0J_2x) \\
=& (J_2x, \neg\neg J_2x) \\
=& (J_2x, J_2x) \\
=& J_2\varphi(x).
\end{aligned}$$

About negation:

$$\begin{aligned}
\varphi(\neg x) =& (J_2\neg x, \neg J_0\neg x) \\
=& (J_0x, \neg J_2x) \\
=& \neg(J_2(x), J_0x) \\
=& \neg\varphi(x).
\end{aligned}$$

By quasiequation (13) of definition 18, if:

$$\varphi(x) = (J_2x, \neg J_0x) = (J_2y, \neg J_0y) = \varphi(y)$$

then $x = y$. Therefore, φ is injective.

We now prove that φ is surjective. Let $(x, y) \in T(B, S)$. By construction $x, y \in B = J_2[A]$, which means that they are sharp elements, $x \leq y$ and

$x \vee \neg y \in S = J_2[D(A)]$. The last fact implies that there exists $c \in D(A)$ such that $J_2 c = x \vee \neg y$; notice that such c is unique, since the application of J_2 to $D(A)$ is injective (theorem 46). c is dense, that means $J_0 c = 0$. Then, consider the element of A:

$$z = y \wedge c.$$

On the one hand:

$$\begin{aligned} J_2 z &= J_2(y \wedge c) \\ &= J_2 y \wedge J_2 c \\ &= y \wedge (x \vee \neg y) \\ &= y \wedge (x \vee \neg y) \\ &= y \wedge x \\ &= x. \end{aligned}$$

On the third line we have used the fact that y is sharp and that $J_2 c = x \vee \neg y$; on the last line that $x \leq y$.

On the other hand:

$$\begin{aligned} \neg J_0 z &= \neg J_0(y \wedge c) \\ &= \neg((J_0 y \wedge \neg J_1 c) \vee (\neg J_1 y \wedge J_0 c)) \\ &= \neg((J_2 \neg y \wedge \neg\neg(J_2 c \vee J_0 c)) \vee (\neg J_1 y \wedge 0)) \\ &= \neg(\neg y \wedge (J_2 c \vee J_0 c)) \\ &= \neg(\neg y \wedge ((x \vee \neg y) \vee 0)) \\ &= \neg\neg y \\ &= y. \end{aligned}$$

On the second line we have employed the equation $J_0(x \wedge y) = (J_0 x \wedge \neg J_1 y) \vee (\neg J_1 x \wedge J_0 y)$ valid in BCA; moreover we used the facts that $\neg y$ is sharp and c is dense ($J_0 c = 0$).

Thus, $\varphi(z) = (J_2 z, \neg J_0 z) = (x, y)$, which establishes that φ is surjective.

Finally we have to prove that given $\mathbf{A} \in$ BCA and one of its product representations $\mathcal{P}_v(\mathbb{B})$, $\mathbb{B}$ is unique up to isomorphism. Suppose $\mathbf{A} \cong \mathcal{P}_v(\mathbb{B})$ for some Bochvar system $\mathbb{B} = \langle \mathbf{B}, \mathbf{S} \rangle$. By what proved before, $\mathbf{A} \cong$

$\mathcal{P}_v(\langle J_2[\mathbf{A}], J_2[D(\mathbf{A})]\rangle)$, with $J_2[D(\mathbf{A})] := \langle J_2[D(A)], \wedge, 1\rangle$, hence there exists isomorphism $\psi : \mathcal{P}_v(\langle J_2[\mathbf{A}], J_2[D(\mathbf{A})]\rangle) \cong \mathcal{P}_v(\langle \mathbf{B}, \mathbf{S}\rangle)$. The isomorphism ψ induces a map $\psi_1 : J_2[A] \to B$ defined as $\psi_1 := \pi_1 \circ \psi \restriction_{J_2[A]}$ (with π_1 the projection over the first component). We show that ψ_1 is an isomorphim of Bochvar systems between $\langle J_2[\mathbf{A}], J_2[D(\mathbf{A})]\rangle$ and $\langle \mathbf{B}, \mathbf{S}\rangle$.

ψ_1 is a total function over the domain, since for all $x \in J_2[A], (x, x) \in \mathcal{P}_v(\langle J_2[\mathbf{A}], J_2[D(\mathbf{A})]\rangle)$, and it is well-defined because ψ is. It is moreover a Boolean homomorphism, since ψ is a homomorphism of Bochvar algebras. We need to show that ψ_1 preserves the semilattice structure. Let $x \in J_2[D(A)]$, so $x = J_2 y$ for some dense $y \in A$, i.e. $J_0 y = 0$. Therefore $\psi_1(x) = \psi_1(J_2 y) = J_2\psi_1(y)$. Now $J_0\psi_1(y) = \psi_1(J_0 y) = \psi_1(0) = 0$, so $\psi_1(y)$ is dense, hence $\psi_1(x) \in S$, which proves that ψ_1 is a homomorphism of Bochvar systems. The injectivity of ψ_1 is directly inherited from that of ψ. For surjectivity, let $y \in B$. By definition of twist product, we know that at least $(y, y) \in T(\langle B, S\rangle)$. By surjectivity of ψ, there exists $(x, x') \in T(\langle J_2[A], J_2[D(A)]\rangle)$ s.t. $\psi((x, x')) = (y, y)$, for $x, x' \in J_2[A]$. Hence $y = \pi_1 \circ \psi(x, x') = \psi_1(x)$. We conclude that ψ_1 is indeed an isomorphism of Bochvar systems. □

As a corollary, we obtain a characterization of the non-trivial subquasivarieties of BCA. These are the quasivariety NBCA of non-paraconsistent Bochvar algebras, which are those Bochvar algebras whose IBSL-reduct contains no trivial fibre (i.e. those Bochvar algebras without the fix-point for negation), and the quasivariety JBA of Bochvar algebras whose IBSL-reduct are Boolean algebras, which is term-equivalent to the variety of Boolean algebras ([Bonzio and Pra Baldi, 2024, theorem 4.9]).

Corollary 49. *Every* $\mathbf{A} \in \mathsf{NBCA}$ *is isomorphic to a product of the form* $\mathcal{P}_v(\mathbb{B})$*, where* $\mathbb{B} = \langle \mathbf{B}, \mathbf{S}\rangle$ *is a Bochvar system s.t.* $0 \notin S$*.*

Every $\mathbf{A} \in \mathsf{JBA}$ *is isomorphic to a product of the form* $\mathcal{P}_v(\mathbb{B})$*, where* $\mathbb{B} = \langle \mathbf{B}, \mathbf{S}_1\rangle$*, with* $\mathbf{S}_1 = \langle \{1\}, \wedge, 1\rangle$ *the trivial semilattice (with unit).*

In both cases, the pair $\mathbb{B}$ *is uniquely determined up to isomorphism.*

2.3 Relating twist products with the Płonka-style representation

The main result of the previous section is that variant twist products are an appropriate tool for representing Bochvar algebras as Bochvar systems: as stated in theorem 48, every Bochvar algebra is the variant twist product of the Bochvar system composed by, repsectively, its sharp and its dense elements.

In the current chapter we investigate how the product construction relates with the other representation method of BCA, the Płonka sum decomposition in an IBSL expanded with the external operator J_2.

Lemma 50. *Let $\mathbf{A} \in$ BCA, and consider its dense elements $D(A) = \{x \in A \mid J_0(x) = 0\}$. Then the bisemilattice operations $\wedge$ and $\vee$ restricted to $D(A)$ coincide and define a semilattice structure on $D(A)$. The same holds when the operations are restricted to the set of dually dense elements $D^{\partial}(A) = \{x \in L \mid J_2(x) = 0\}$.*

Proof. By theorem 48, we can assume without loss of generality that $\mathbf{A}$ is of the form $\mathcal{P}_v(\mathbb{B})$ for some Bochvar system $\mathbb{B}$.

Let $(x_1, x_2), (y_1, y_2) \in D(A)$. By definition of $D(A)$ we have $J_0(x_1, x_2) = (\neg x_2, \neg x_2) = (0, 0)$ and $J_0(y_1, y_2) = (\neg y_2, \neg y_2) = (0, 0)$, hence the pairs are of the form $(x_1, x_2) = (x_1, 1)$ and $(y_1, y_2) = (y_1, 1)$. Applying the operations presented in definition (31), we compute:

Join:

$$\begin{aligned}(x_1, 1) \vee (y_1, 1) &= ((x_1 \wedge y_1) \vee (x_1 \wedge \neg 1) \vee (\neg 1 \wedge y_1),\ 1 \vee 1) \\ &= (x_1 \wedge y_1, 1).\end{aligned}$$

Meet:

$$\begin{aligned}(x_1, 1) \wedge (y_1, 1) &= (x_1 \wedge y_1,\ (1 \vee 1) \wedge (1 \vee \neg y_1) \wedge (\neg x_1 \vee 1)) \\ &= (x_1 \wedge y_1, 1).\end{aligned}$$

Thus, both operations coincide and reduce to $(x_1 \wedge y_1, 1)$. Hence, the induced operation defines a meet-semilattice structure on $D(A)$.

The proof of the claim for $D^\partial(A)$ is dual. $\square$

Looking at a Bochvar algebra in the Płonka representation, the above lemma states that the set of local tops of the fibres composing the sum form a sub-semilattice (or a sub-bisemilattice, given the coincidence of the two semilattice operations) of the Bochvar algebra, under both semilattice operations[3].

In the theory of Płonka sums the notion of fibre plays a pivotal role. We rephrase it in order to obtain a new notion which is easier to adapt to the product representation.

Definition 51. Let $\mathbf{A} \in \mathsf{BCA}$. For any $x \in A$, we define the *block* generated by x as the set:

$$B_x := \{y \in A \,|\, x \wedge \neg x = y \wedge \neg y\}.$$

or, equivalently, $x \vee \neg x = y \vee \neg y$.

In other words, a block of a Bochvar algebra $\mathbf{A}$ is a set of all the elements which, in the Płonka representation, share a certain local bottom (or, equivalently, a local top). A fibre of the Płonka sum is therefore a block. We are going to show how the two notions actually coincide, but before we do that, we shift the perspective to the product construction and rephrase the definition of block for that setting.

In order to do that, observe that for any Bochvar algebra in the product representation $\mathcal{P}_v(\langle \mathbf{B}, \mathbf{S} \rangle)$, if $(x_1, x_2) \in T(B, S)$ we have:

$$\begin{aligned}(x_1, x_2) \wedge \neg(x_1, x_2) &= (x_1 \wedge \neg x_2, \neg x_1 \wedge x_2) \\ &= (0, \neg x_1 \wedge x_2),\end{aligned}$$

since $x_1 \leq x_2$. Therefore the only relevant component when comparing meets of the above form is the second one. This explains the following definition.

Definition 52. For $\mathbf{A} \in \mathsf{BCA}$, let $\mathcal{P}_v(\langle \mathbf{B}, \mathbf{S} \rangle)$ be its product representation as in theorem 48. For any element $(x_1, x_2) \in T(B, S)$, the *block* generated by

[3]The algebra $\langle D(A), \wedge, 1 \rangle$ is term-equivalent to $\langle D(A), \wedge, \vee, id_{D(A)}, 1 \rangle$, which is an example of a $\langle \wedge, \vee, \neg \rangle$-semilattice (a case of τ-semilattice, see [Bonzio et al., 2022, def. 2.3.6]), a subvariety of IBSL composed by bisemilattices whose Płonka decomposition is made only of trivial fibres.

(x_1, x_2) is defined as the set:

$$B_{(x_1,x_2)} := \{(y_1, y_2) \in T(B, S) \mid \neg x_1 \wedge x_2 = \neg y_1 \wedge y_2\}.$$

or, equivalently, $x_1 \vee \neg x_2 = y_1 \vee \neg y_2$.

Lemma 53. *Let* $\mathbf{A} \in \mathsf{BCA}$ *and let* B_x *be a block of* $\mathbf{A}$*. Then* B_x *contains exactly one dense element and exactly one dually dense element.*

Proof. Let y be a dense element of $\mathbf{A}$. Then, by representation theorem 48 and lemma 37, we can write $y = (y_1, 1)$. Consider the block $B_{(x_1,x_2)}$. By the reasoning above, $(x_1, x_2) \vee \neg(x_1, x_2)$ is a dense element of the block. We prove it is unique. Suppose $(y_1, 1) \in B_{(x_1,x_2)}$. By definition 51, we then have:

$$\begin{aligned}(x_1, x_2) \vee \neg(x_1, x_2) &= (y_1, 1) \vee \neg(y_1, 1) \\ &= (y_1, 1).\end{aligned}$$

The argument for dense elements is dual.

Therefore B_x contains exactly one dense and one dually dense element, which are, respectively, its upper and lower bounds. □

In light of lemma 53, every Bochvar algebra admits a one-to-one correspondence between its blocks and its dense (respectively, dually dense) elements. This correspondence extends to the fibres of the Płonka sum representation of a Bochvar algebras, in fact a dense element is just the local top of a fibre. Therefore blocks and fibres are in bijective correspondence. We can state more, they coincide.

Lemma 54. *Let* $\mathbf{A} \in \mathsf{BCA}$ *and* $x \in A$*. The block* B_x *is the support of the fibre* $\mathbf{A}_x := \langle [0_x, 1_x], \vee, \wedge, \neg, 0_x, 1_x \rangle$*, where* $0_x := x \wedge \neg x, 1_x := x \vee \neg x$*.*

Proof. Considering the Płonka sum decomposition of a the IBSL-reduct of any $\mathbf{A} \in \mathsf{BCA}$, we know that its fibres are Boolean algebras, therefore the interval nomenclature $[0_x, 1_x]$ is well-defined. Furthermore, $\mathbf{A}_x$ is closed under the Boolean operations by the way these are defined in a Płonka sum, therefore as a Boolean algebra it can be rephrased as $A_x = \{y \in A \mid x \wedge \neg x = y \wedge \neg y\} = B_x$ by definition 51. □

Therefore the notion of block is really a rephrasing of that of fibre, allowing us to recapture that notion within the product construction. As a straightforward consequences, the facts about the fibre structure in the Płonka decomposition can be translated in the product construction, such as:

- every block B_x when equipped when the bisemilattice operations and negation are restricted to it becomes a Boolean algebra, with bounds 0_x and 1_x;
- the bottom fibre of the Płonka sum representation corresponds to the block B_0 and it contains isomorphic images of all the blocks;
- for any two blocks B_x, B_y, with $x \neq y$, s.t. $1_x \leq_\wedge 1_y$ (or, equivalently, $0_x \leq_\vee 0_y$), the map $f\colon B_x \to B_y$ defined by $f(z) = z \vee 0_y$ is a surjective and non-injective Boolean homomorphism;
- for $\mathbf{A} \in \mathsf{BCA}$, consider the semilattice direct system of Boolean algebras:

$$\mathbb{A} = \left\langle \{\mathbf{B}_x\}_{x \in D^\partial(A)},\ D^\partial(\mathbf{A}),\ \{f_{xy} \mid x, y \in D^\partial(A), x \leq y\} \right\rangle,$$

  where $D^\partial(\mathbf{A}) = \left\langle D^\partial(A), \vee, 0 \right\rangle$ is the join-semilattice with zero built over the set of dually dense elements of $\mathbf{A}$, and for all $z \in A$, $f_{xy}(z) = z \vee 0_y$. Then the Płonka sum $\mathcal{P}_l(\mathbb{A})$ is isomorphic to the IBSL-reduct of the Bochvar algebra $\mathcal{P}_v(\langle \mathbf{B}_0, J_2[D(\mathbf{A})] \rangle)$, with $J_2[D(\mathbf{A})] = \langle J_2[D(A)], \wedge, 1 \rangle$ a meet-subsemilattice with unit of $\mathbf{B}_0$ and $D(A)$ the set of dense elements of $\mathbf{A}$.

2.4 Categorical Equivalence

Here we examine how the correspondence drawn in the previous section between Bochvar algebras and Bochvar systems is actually a categorical equivalence[4]. In order to do that we introduce the categories that will be under investigation:

[4]The same result is achieved by [Bonzio et al., 2024, theorem 16]. Here we prove the equivalence passing through the twist product construction, which will be helpful for expanding the result to the modal setting in section 3.5.

- $\mathscr{B}$ is the category of Bochvar algebras, whose objects are Bochvar algebras and whose morphisms are homomorphisms of Bochvar algebras;
- $\mathscr{S}$ is the category of Bochvar systems, whose objects are Bochvar systems and whose morphisms are Boolean homomorphisms that preserve the meet-semilattice structure, i.e. if $h : \langle \mathbf{B}_1, \mathbf{S}_1 \rangle \to \langle \mathbf{B}_2, \mathbf{S}_2 \rangle$ then $f[S_1] \subseteq S_2$.

Let us now introduce two functors:

- $E : \mathscr{B} \to \mathscr{S}$ is defined as:

  $E(\mathbf{A}) := \langle J_2[\mathbf{A}], J_2[D(\mathbf{A})] \rangle$, where $J_2[D(\mathbf{A})] := \langle J_2[D(A)], \wedge, 1 \rangle$;
  if $f : \mathbf{A} \to \mathbf{B}$ is a morphism in $\mathscr{B}$, then $E(h) := f \restriction_{A_0}$, with $\mathbf{A}_0$ the bottom fibre of the IBSL-reduct of $\mathbf{A}$.

- $P : \mathscr{S} \to \mathscr{B}$ is defined as:

  $P(\mathbb{B}) := \mathcal{P}_v(\mathbb{B})$;
  if $g : \mathbb{B}_1 \to \mathbb{B}_2$ is a morphism in $\mathscr{S}$, then $P(g)(x, y) = (g(x), g(y))$.

Observe that in the definition of the action of P over morphisms we have used the fact that every Bochvar algebra is representable as a twist product.

We need to check that those defined above are actually functors.

Lemma 55. *E and P are functors between the respective categories.*

Proof. Let us start with $E : \mathscr{B} \to \mathscr{S}$. It is well-defined on objects by theorem 48, where it is shown that for $\mathbf{A} \in \mathsf{BCA}$, $\langle J_2[\mathbf{A}], J_2[D(\mathbf{A})] \rangle$ is indeed a Bochvar system. Concerning morphisms, let $f : \mathbf{A} \to \mathbf{B}$ be a homomorphism of BCA. We have defined $E(f) = f \restriction_{A_0}$, so the domain is well-defined. To check that the codomain is as well, notice that $a \in A_0$ amounts to state that a is sharp, we need to check that $f \restriction_{A_0} (a)$ is sharp in $\mathbf{B}$. The sharpness of a is equivalent to $J_2 a = a$, so $f \restriction_{A_0} (a) = f \restriction_{A_0} (J_2 a) = J_2 f \restriction_{A_0} (a)$ (since f is a homomorphism of BCA, so its restriction is as well), as desired.

Next we check that $f \restriction_{A_0}$ is a homomorphism of Bochvar systems. That it is a Boolean homomorphism follows immediately by its definition. We need to show that it preserves the semilattice structure. Let $a \in J_2[D(A)]$, so $a = J_2 b$ for some dense $b \in A$, i.e. $J_0 b = 0$. Therefore $f(a) = f(J_2 b) = J_2 f(b)$. Now $J_0 f(b) = f(J_0 b) = f(0) = 0$, so $f(b)$ is dense, hence $f(a) \in J_2[D(B)]$.

Let us move to the case of $P : \mathscr{S} \to \mathscr{B}$. It is well-defined on objects by theorem 33. For morphisms, take $g : \langle \mathbf{B}_1, \mathbf{S}_1 \rangle \to \langle \mathbf{B}_2, \mathbf{S}_2 \rangle$ and $(x, y) \in T(B_1, S_1)$. $(g(x), g(y)) \in T(B_2, S_2)$, since $x \leq y$ implies $g(x) \leq g(y)$, being g a Boolean homomorphism, furthermore $x \vee \neg y \in S_1$ implies $g(x \vee \neg y) = g(x) \vee \neg g(y) \in g[S_1] \subseteq S_2$, because g preserve the semilattice structure. Then the image of a morphism is well-defined as well.

Finally we need to prove that P preserves the operations of a Bochvar algebra:

$$\begin{aligned}
P(g)((x_1, x_2) \vee (y_1, y_2)) &= P(g)(((x_1 \wedge y_1) \wedge (x_1 \wedge \neg y_2) \wedge (\neg x_2 \wedge y_1), x_2 \vee y_2)) \\
&= (g((x_1 \wedge y_1) \wedge (x_1 \wedge \neg y_2) \wedge (\neg x_2 \wedge y_1)), g(x_2 \vee y_2)) \\
&= (g(x_1) \wedge g(y_1) \wedge g(x_1) \wedge \neg g(y_2) \wedge g(\neg x_2) \wedge g(y_1), g(x_2) \vee g(y_2)) \\
&= (P(g)(x_1, x_2)) \vee (P(g)(y_1, y_2));
\end{aligned}$$

$$\begin{aligned}
P(g)(\neg(x_1, x_2)) &= P(g)((\neg x_2, \neg x_1)) \\
&= (g(\neg x_2), g(\neg x_1)) \\
&= (\neg g(x_2), \neg g(x_1)) \\
&= \neg(g(x_1), g(x_2)) \\
&= \neg(P(g)((x_1, x_2)));
\end{aligned}$$

$$\begin{aligned}
P(g)(J_2(x_1, x_2)) &= P(g)((x_1, x_1)) \\
&= (g(x_1), g(x_1)) \\
&= J_2(P(g)((x_1, x_2))).
\end{aligned}$$

□

We can finally prove that E and P determine a categorical equivalence.

Theorem 56. *$\mathscr{B}$ and $\mathscr{S}$ are equivalent categories.*

Proof. We are going to prove that E and P are the functors establishing the equivalence. First, we show that the two concatenations of E and P produces isomorphic objects:

- take $\mathbb{B} \in \mathscr{S}$, the application of P yields $P(\mathbb{B}) = \mathcal{P}_v(\mathbb{B}) =: \mathbf{A}$. By theorem 48, $\mathbf{A} \cong \mathcal{P}_v(\langle J_2[\mathbf{A}], J_2[D(\mathbf{A})]\rangle)$, and by the same theorem all the Bochvar systems underlying a product representation of $\mathbf{A}$ are isomorphic, so $\mathbb{B} \cong \langle J_2[\mathbf{A}], J_2[D(\mathbf{A})]\rangle$, but $\langle J_2[\mathbf{A}], J_2[D(\mathbf{A})]\rangle = E(\mathbf{A})$. We conclude that $\mathbb{B} \cong EP(\mathbb{B})$;

- once again, from the proof of theorem 48 we know that for every $\mathbf{A} \in \mathscr{B}$, $\mathbf{A} \cong \mathcal{P}_v(\langle J_2[\mathbf{A}], J_2[D(\mathbf{A})]\rangle) = PE(\mathbf{A})$.

Then we need to show the result of concatenations on morphisms. For PE we need to check the commutativity of the diagram:

$$\begin{array}{ccc}
\mathbf{A}_1 & \xrightarrow{\quad f \quad} & \mathbf{A}_2 \\
\downarrow{\scriptstyle PE} & & \downarrow{\scriptstyle PE} \\
\mathcal{P}_v(J_2[\mathbf{A}_1], J_2[D(\mathbf{A}_1)]) & \xrightarrow{PE(f)} & \mathcal{P}_v(J_2[\mathbf{A}_2], J_2[D(\mathbf{A}_2)])
\end{array}$$

Recall that the isomorphism between $\mathbf{A}$ and $\mathcal{P}_v(J_2[\mathbf{A}], J_2[D(\mathbf{A})])$ is witnessed by $\varphi(x) = (J_2 f(x), \neg J_0 f(x))$ (theorem 48). Take $\mathbf{A}_1, \mathbf{A}_2 \in \mathscr{B}$:

$$\begin{aligned}
PE(f(a)) &= (J_2 f(a), \neg J_0 f(a)) \\
&= (f(J_2 a), f(\neg J_0 a)) \\
&= (f \restriction_{J_2[A_1]} (J_2 a), f \restriction_{J_2[A_1]} (\neg J_0 a)) \\
&= (E(f)(J_2 a), E(f)(\neg J_0 a)) \\
&= PE(f)(J_2 a, \neg J_0 a) \\
&= PE(f)(PE(a)).
\end{aligned}$$

The case for EP requires the following diagram to be commutative:

$$
\begin{array}{ccc}
\mathbb{B}_1 & \xrightarrow{\quad g \quad} & \mathbb{B}_2 \\
\downarrow{\scriptstyle EP} & & \downarrow{\scriptstyle EP} \\
\langle J_2[\mathcal{P}_v(\mathbb{B}_1)], J_2[D(\mathcal{P}_v(\mathbb{B}_1))]\rangle & \xrightarrow{EP(g)} & \langle J_2[\mathcal{P}_v(\mathbb{B}_2)], J_2[D(\mathcal{P}_v(\mathbb{B}_2))]\rangle
\end{array}
$$

From the final part of the proof of theorem 48 we can easily deduce that the isomorphism between $\mathbb{B}$ and $\langle J_2[\mathcal{P}_v(\mathbb{B})], J_2[D(\mathcal{P}_v(\mathbb{B}))]\rangle$ is witnessed by $\alpha(x) = (x, x)$. Let $\mathbb{B}_1, \mathbb{B}_2 \in \mathscr{S}$:

$$
\begin{aligned}
EP(g(a)) &= (g(a), g(a)) \\
&= P(g)(a, a) \\
&= P(g) \restriction_{J_2[\mathcal{P}_v(\mathbb{B}_1)]} (a, a) \\
&= EP(g)(a, a) \\
&= EP(g)(EP(a)).
\end{aligned}
$$

□

Chapter 3

Modal external weak Kleene logics

We have explained the reason behind the interest in weak Kleene logics enriched with external operators of classical recapture. The virtue of this move is to recover a meaningful connection with an algebraic counterpart. Now we expand the language further, introducing a primitive unary modal operator.

Modal logics are formalisms devised to talk about necessity and possibility of formulae belonging to classical propositional, or first-order, logic, originally with the objective of finding a satisfying conditional free from the paradoxes of material implication. The impetuous development of standard modal logic is a very well-known history, but the broad realm of non-classical systems have been explored within modal logic. Modalities have been successfully extended to non-classical logics, including (but not limited to) intuitionistic logic ([Fischer Servi, 1977],[Fischer Servi, 1981],[Font, 1986]), strong Kleene logic ([Fitting, 1991],[Fitting, 1992]), Belnap-Dunn ([Rivieccio et al., 2015]), various fuzzy logics ([Hájek et al., 1996], [Hansoul and Teheux, 2013], [Blondeel et al., 2015], [Caicedo and Rodriguez, 2010], [Vidal et al., 2015]), and, more in general, the realm of substructural logics [Bou et al., 2011], just to provide a very concise list.

To the best of my knowledge, the only existing proposals of modal logics based on (some) weak Kleene logics are [Segerberg, 1967] and [Correia, 2002]. Both papers focus on the modalized extension of paraconsistent weak Kleene.[1] While Correia's work introduces an axiomatization and a relational semantics

[1] Actually, Correia [Correia, 2002] introduces also the idea of a modalized version of Bochvar logic, but gives no axiomatization nor semantical analysis for that.

for the modal version of **PWK**, [Segerberg, 1967]'s is based on the external version of paraconsistent weak Kleene $\mathbf{PWK_e}$. Segerberg takes into consideration the existing difference between truth and non-falsity within a three-valued settung and he incorporates this difference by allowing two different modal necessity operators in the primitive language, a box expressing truth at every successor (a modality reflecting the expected reading of truth in **B**) and one expressing non-falsity (a modality reflecting the expected reading of truth in **PWK**). By retaining at the same time a logical consequence which is the one of paraconsistent weak Kleene, Segerberg's study produces a rather confusing duplication where the two weak Kleene logics are both simulataneously present as the intended interpretation of the two boxes, yet the logic itself choose the **PWK** approach by taking both non-false truth-values as designated.

In the present work, inspired by Segerberg's intuitions but guided by the above explained distinction, we will introduce modal logics with a unique necessity operator, based on both (the external versions of) Bochvar and paraconsistent weak Kleene. Remarkably, the work of Segerberg relies on the external version of **PWK**. We first approach the object of our investigation by following Segerberg's steps, that is providing Kripke-style semantics for the two modal external weak Kleene logics. Interestingly, we obtain two notions of three-valued Kripke model that deliver two semantically different necessity operators. After that, we study the algebraic counterparts of the global version of the modal external weak Kleene logics, which are algebraizable and therefore allow us to employ all the algebraic observations that emerged in the previous chapters. We will see how this modal algebraic semantics relates with the algebraic counterparts of non-modal external weak Kleene logics, that is the class of Bochvar algebras, which now splits into two different (partially overlapping) classes of modal Bochvar algebras. We both examine these algebraic structures via a Płonka-style representation and a twist product one, observing in both the peculiar interaction of the new box operator with the BCA-reduct of these new modal algebras.

3.1 Modal Bochvar logic

For the remainder of the chapter, by **Fm** we will denote the formula algebra in the language $\mathscr{L}_J^\Box : \langle \neg, \vee, J_2, \Box, 0, 1\rangle$ of type $\langle 1, 2, 1, 1, 0, 0\rangle$, obtained from $\mathscr{L}_J$ by adding a unary operator $\Box$. The connectives $\wedge, \rightarrow$ are defined as usual, while recall that $J_0\varphi$ and $J_1\varphi$ are abbreviations for $J_2\neg\varphi$ and $\neg(J_2\varphi \vee J_2\neg\varphi)$, respectively. Let, moreover, $\diamond\varphi$ be an abbreviation for $\neg\Box\neg\varphi$. Our aim with the above introduced language is to define a (local) modal logic whose propositional basis is Bochvar external logic and whose interpretation of the formula $\Box\varphi$, in a relational semantics, is that $\Box\varphi$ holds in a state when φ holds (is equal to 1) in all related states and it is locally meaningful (in a sense that we will precisely define soon).

We introduce the logic $\mathbf{B}_{\mathrm{e}}^\Box = \langle \mathscr{L}_J^\Box, \vdash_{\mathbf{B}_{\mathrm{e}}^\Box}\rangle$ as induced by the following Hilbert-style axiomatization.

Axioms

- the axioms of **B** in definition 12;

(B1) $\Box(J_2\varphi \rightarrow J_2\psi) \rightarrow (\Box J_2\varphi \rightarrow \Box J_2\psi)$;

(B2) $+\varphi \leftrightarrow +\Box\varphi$;

(B3) $J_2\Box\varphi \rightarrow \Box J_2\varphi$;

(B4) $J_0\Box\varphi \rightarrow \neg\Box J_0\neg\varphi$.

Rules

- (ρ-B13) $J_2\varphi \leftrightarrow J_2\psi, J_0\varphi \leftrightarrow J_0\psi \vdash \varphi \equiv \psi$;
- (B-Alg3) $\varphi \dashv\vdash J_2\varphi \leftrightarrow 1$;
- (N): if $\vdash \varphi$ then $\vdash \Box\varphi$.

Throughout this section, for ease of notation, we will write $\vdash$ instead of $\vdash_{\mathbf{B}_{\mathrm{e}}^\Box}$. The axiom (B1) can be generalized to all external formulae, as follows.

Lemma 57. *For α, β external formulae, the following is a theorem of $\mathbf{B}_{\mathrm{e}}^\Box$:*

(BK) $\Box(\alpha \to \beta) \to (\Box\alpha \to \Box\beta)$

Proof. Consider arbitrary external formulae α, β. Let us start by instantiating (B1) as $\vdash \Box(J_2\alpha \to J_2\beta) \to (\Box J_2\alpha \to \Box J_2\beta)$. Recalling lemma 13, for γ external it holds $\vdash \gamma \leftrightarrow J_2\gamma$, therefore we can substitute equivalent formulae and obtain $\vdash \Box(\alpha \to \beta) \to (\Box\alpha \to \Box\beta)$. $\Box$

Remark 58. The rule of *modus ponens* (MP) obviously holds for $\mathbf{B}_{\mathrm{e}}^{\Box}$ (see lemma 13).

Lemma 59. *In the logic* $\mathbf{B}_{\mathrm{e}}^{\Box}$ *the following facts hold:*

1. $J_2\Box\varphi \vdash J_2\Box J_2\varphi$;

2. $\Box(J_i\varphi \wedge J_k\psi) \leftrightarrow \Box J_i\varphi \wedge \Box J_k\psi$ *for every* $i, k \in \{0, 1, 2\}$;

3. $\diamond(J_i\varphi \vee J_k\psi) \leftrightarrow \diamond J_i\varphi \vee \diamond J_k\psi$ *for* $i, k \in \{0, 1, 2\}$;

4. $\vdash J_2 \diamond \varphi \to \diamond J_2\varphi \vee \diamond J_1\varphi$.

Proof. Since $\mathbf{B}_{\mathrm{e}}^{\Box}$ contains all axioms from $\mathbf{B}$, we will freely make use of theorems and rules holding in the latter logic. In particular, notice that classical logic can always be employed on external formulae (by lemma 13 and the fact that *modus ponens* is a rule of $\mathbf{B}$).

1. By (B3) and (MP), $J_2\Box\varphi \vdash \Box J_2\varphi$; by lemma 13 (and the transitivity of $\vdash$) we get $\Box J_2\varphi \vdash J_2\Box J_2\varphi$.

2. By classical logic, $\vdash J_i\varphi \wedge J_k\psi \to J_i\varphi$, by (N) $\vdash \Box(J_i\varphi \wedge J_k\psi \to J_i\varphi)$. Applying (BK) and (MP), $\vdash \Box(J_i\varphi \wedge J_k\psi) \to \Box J_i\varphi$. By the same reasoning, $\vdash \Box(J_i\varphi \wedge J_k\psi) \to \Box J_k\psi$ as well. We conclude $\vdash \Box(J_i\varphi \wedge J_k\psi) \to \Box J_i\varphi \wedge \Box J_k\psi$. For the other direction: $\vdash J_i\varphi \to (J_i\psi \to J_i\varphi \wedge J_k\psi)$ by classical logic. By (N), $\vdash \Box(J_i\varphi \to (J_k\psi \to J_i\varphi \wedge J_k\psi))$, and by (BK) and (MP) twice, we have $\vdash \Box J_i\varphi \to (\Box J_k\psi \to \Box(J_i\varphi \wedge J_k\psi))$. Again by classical logic $\vdash \Box J_i\varphi \wedge \Box J_k\psi \to \Box(J_i\varphi \wedge J_k\psi)$.

3. By classical logic, $\vdash \neg J_i\varphi \wedge \neg J_k\psi \to \neg J_i\varphi$, by (N) $\vdash \Box(\neg J_i\varphi \wedge \neg J_k\psi \to \neg J_i\varphi)$. Applying (BK) and (MP), $\vdash \Box(\neg J_i\varphi \wedge \neg J_k\psi) \to \Box\neg J_i\varphi$. By contraposition and de Morgan, $\vdash \neg\Box\neg J_i\varphi \to \neg\Box\neg(J_i\varphi \vee J_k\psi)$, which is the definition of $\vdash \diamond J_i\varphi \to \diamond(J_i\varphi \vee J_k\psi)$. By the same reasoning, $\vdash$

$\diamond J_k\psi \to \diamond(J_i\varphi \vee J_k\psi)$ as well. We conclude $\vdash \diamond J_i\varphi \vee \diamond J_k\psi \to \diamond(J_i\varphi \vee J_k\psi)$. For the other direction, $\vdash \neg J_i\varphi \wedge \neg J_k\psi \to \neg(J_i\varphi \vee J_k\psi)$ by classical logic. By (N), $\vdash \Box(\neg J_i\varphi \wedge \neg J_k\psi \to \neg(J_i\varphi \vee J_k\psi))$, and by (BK) and (MP) twice we have $\vdash \Box(\neg J_i\varphi \wedge \neg J_k\psi) \to \Box\neg(J_i\varphi \vee J_k\psi)$. By point (2) we can distribute box, $\vdash (\Box\neg J_i\varphi \wedge \Box\neg J_k\psi) \to \Box\neg(J_i\varphi \vee J_k\psi)$. By classical logic, $\vdash \neg(\neg\Box\neg J_i\varphi \vee \neg\Box\neg J_k\psi) \to \Box\neg(J_i\varphi \vee J_k\psi)$. By contraposition we conclude $\vdash \diamond(J_i\varphi \vee J_k\psi) \to \diamond J_i\varphi \vee \diamond J_k\psi$.

4. By (B4) $\vdash J_0\Box\neg\varphi \to \neg\Box J_0\varphi$. For the linguistic abbreviations introduced, we have that the antecedent $J_0\Box\neg\varphi = J_2\neg\Box\neg\varphi = J_2 \diamond \varphi$; while the consequent $\neg\Box J_0\varphi = \neg\Box\neg(J_2\varphi \vee J_1\varphi) = \diamond(J_2\varphi \vee J_1\varphi) = \diamond J_2\varphi \vee \diamond J_1\varphi$.

□

Theorem 60 (Deduction theorem). *For the logic* $\mathbf{B}_e^{\Box}$*, it holds that* $\Gamma \vdash \varphi$ *iff there exist some formulae* $\gamma_1, \ldots, \gamma_n \in \Gamma$ *such that* $\vdash J_2\gamma_1 \wedge \cdots \wedge J_2\gamma_n \to J_2\varphi$.

Proof. The right to left direction is obvious. The other direction is proved by induction on the length of the derivation of φ from Γ. We just show the inductive case of the rule (N). Let $\varphi = \Box\psi$, for some $\psi \in Fm$, and $\Gamma \vdash \Box\psi$, and the last deduction rule applied is (N), hence it holds $\vdash \psi$. By the latter fact, we have $\vdash \Box\psi$, hence $\vdash J_2\Box\psi$ (by lemma 13). By induction hypothesis, there exists some formulae $\gamma_1, \ldots, \gamma_n \in \Gamma$ such that $\vdash J_2\gamma_1 \wedge \cdots \wedge J_2\gamma_n \to J_2\Box\psi$. Since the axioms (and rule) of classical logic hold for external formulae (Lemma 13), we have $\vdash J_2\Box\psi \to (J_2\gamma_1 \wedge \cdots \wedge J_2\gamma_n \to J_2\Box\psi)$, hence, by (MP), $\vdash J_2\gamma_1 \wedge \cdots \wedge J_2\gamma_n \to J_2\Box\psi$. □

The intended semantics of this modal logic consists of a relational (Kripke-style) structure where formulae, in each world, are evaluated into $\mathbf{WK}^e$ (this has been already implemented, for instance, in [Bonzio et al., 2023]). We introduce these structures according to the current terminology adopted in many-valued modal logics.

Definition 61. A *weak three-valued Kripke model* $\mathfrak{M}$ is a structure $\langle W, R, v\rangle$ such that W is a non-empty set (of possible worlds); R is a binary relation over W; v is a map assigning to each world and each variable, an element in $\mathbf{WK}^e$ ($v\colon W \times \mathbf{Fm} \to \mathbf{WK}^e$).

Non-modal formulae will be interpreted as in **B**, i.e. we assume that v is a homomorphism, in its second component, with respect to $\neg, \vee, J_2, 1, 0$. The reduct $\mathfrak{F} = \langle W, R \rangle$ of a model $\mathfrak{M}$ is as usual called frame.

The semantic interpretation of the modality $\Box$ is what characterize a special family of weak three-valued Kripke models:

Definition 62. A *Bochvar-Kripke model* is a weak three-valued Kripke model $\langle W, R, v \rangle$ such that v evaluates formulae of the form $\Box\varphi$ according to:

(1) $v(w, \Box\varphi) = 1$ iff $v(w, \varphi) \neq 1/2$ and $v(s, \varphi) = 1$ for every $s \in W$ such that wRs.

(0) $v(w, \Box\varphi) = 0$ iff $v(w, \varphi) \neq 1/2$ and there exists $s \in W$ such that wRs and $v(s, \varphi) \neq 1$.

(1/2) $v(w, \Box\varphi) = 1/2$ iff $v(w, \varphi) = 1/2$.

To simplify, within this section by *model* we intend *Bochvar-Kripke model*.

The above definition explains what we meant at the beginning of this section when we said that the truth of a formula $\Box\varphi$ requires φ to be *locally meaningful*, i.e. meaningful (classical) at the world of evaluation. In other words, the infectious feature of the non-classical value 1/2 propagates only locally: $\Box\varphi$ is meaningless if its argument is currently meaningless, the presence of accessible worlds (other than, potentially, the world of evaluation) where φ is meaningless doesn't alter the status of meaningfulness of φ at the world of evaluation.

Remark 63. Observe that, in every model $\langle W, R, v \rangle$ with $w \in W$, it holds that $v(w, \Diamond\varphi) = 1$ iff $v(w, \varphi) \neq 1/2$ and there exists $s \in W$ such that wRs and $v(s, \varphi) \neq 0$, while $v(w, \Diamond\varphi) = 0$ iff $v(w, \varphi) \neq 1/2$ and $v(s, \varphi) = 0$ for every $s \in W$ such that wRs.

Definition 64. A formula φ is satisfied in a model $\mathfrak{M} = \langle W, R, v \rangle$ if $v(w, \varphi) = 1$ for some $w \in W$, while it is valid in $\mathfrak{M}$ if $v(w, \varphi) = 1$ for all $w \in W$ (notation $\mathfrak{M} \vDash \varphi$). A formula φ is valid in a frame $\mathfrak{F}$ (notation $\mathfrak{F} \vDash \varphi$) if it is valid in $\langle \mathfrak{F}, v \rangle$, for all valuations v. A formula φ is valid in a class of frames $\mathcal{K}$ (notation $\mathcal{K} \vDash \varphi$) if it is valid in every frame $\mathfrak{F} \in \mathcal{K}$.

As usual in modal logic, one can opt to study the local or the global consequence relation related to a class of frame. Accordingly, we denote by $\models^{l}_{\mathbf{B}_{e}^{\Box}}$ the *local* modal Bochvar external logic on the class of all frames obtained by taking $\{1\}$ as designated value, while $\models^{g}_{\mathbf{B}_{e}^{\Box}}$ will be the *global* modal Bochvar external logic on the same class of frames.

Definition 65. $\Gamma \models^{l}_{\mathbf{B}_{e}^{\Box}} \varphi$ iff for all models $\langle W, R, v\rangle$ and all $w \in W$, if $v(w,\gamma) = 1, \forall\gamma \in \Gamma$, then $v(w,\varphi) = 1$. $\Gamma \models^{g}_{\mathbf{B}_{e}^{\Box}} \varphi$ iff for all models $\mathfrak{M}$, if $\mathfrak{M} \models \gamma, \forall\gamma \in \Gamma$, then $\mathfrak{M} \models \varphi$.

In the following we will focus on local logics and write simply $\models$ when we intend the local consequence $\models^{l}_{\mathbf{B}_{e}^{\Box}}$, unless the context requires an explicit distinction. The following semantic notions are standard. We now start the completeness proof with some preliminary definitions.

Definition 66. A set $\Gamma \subset Fm$ is *consistent* if $\Gamma \nvdash \varphi$, for some $\varphi \in Fm$. It is inconsistent if it is not consistent.

Remark 67. Equivalently, a set $\Gamma \subset Fm$ is consistent if there is no formula $\varphi \in Fm$, such that $\Gamma \vdash J_2\varphi$ and $\Gamma \vdash \neg J_2\varphi$. Observe that this is equivalent to say that $\Gamma \nvdash 0$.

Definition 68. A consistent set Γ is *maximally consistent* (or *complete*) whenever $\Gamma \subset \Gamma'$ implies that Γ' is inconsistent. Equivalently, Γ is maximally consistent iff, for every $\varphi \in Fm$ exactly one of the following holds:

(i) $\varphi \in \Gamma$;

(ii) $\neg\varphi \in \Gamma$;

(iii) $J_1\varphi \in \Gamma$;

Definition 69. A formula φ is *meaningful* in a maximally consistent set w if $x \in w$ or $\neg x \in w$, for every open variable $x \in \varphi$.

Observe that the definition of meaningful formulae implies, semantically, that such formulae are those evaluated, in a state, into the 2-element Boolean algebra $\mathbf{B}_2$ only. The definition of meaningful formula obviously applies to variables as well.

Lemma 70. *Let w be a maximally consistent set of formulae, then:*

1. *if $\neg\varphi \notin w$ and all the variables occurring in φ are meaningful in w, then $\varphi \in w$;*

2. *if all the variables of φ are covered, then φ is meaningful in w;*

3. *if $\vdash \varphi$ then $\varphi \in w$.*

Proof. We just show (3) (as the other claims can be found also in [Segerberg, 1967, Lemma 4.6]). Suppose that $\vdash \varphi$ and, by contradiction, that either $\neg\varphi \in w$ or $J_1\varphi \in w$. Let us assume that $\neg\varphi \in w$. From $\vdash \varphi$ it follows $\varphi \in w$. By lemma 13 we have $w \vdash J_2\varphi$ and $w \vdash J_0\varphi$. Applying the same lemma, the latter yields $w \vdash \neg J_2\varphi$, in contradiction with the assumption that w is (maximally) consistent (see remark 67). One can reason similarly for the case $J_1\varphi \in w$. $\square$

Lemma 71 ([Segerberg, 1967], lemma 4.7)**.** *Let w be a maximally consistent set of formulae, t.f.a.e.*

1. *φ is meaningful in w;*

2. *either $\varphi \in w$ or $\neg\varphi \in w$;*

3. *$+\varphi \in w$;*

4. *$\Box\varphi$ is meaningful in w;*

5. *$\Diamond\varphi$ is meaningful in w.*

Lemma 72 ([Segerberg, 1967], lemma 4.8)**.** *For every maximally consistent set w the following hold:*

1. *If $\varphi \to \psi \in w$ and $\varphi \in w$ then $\psi \in w$;*

2. *$\varphi \wedge \psi \in w$ if and only if $\varphi, \psi \in w$;*

3. *$\varphi \vee \psi \in w$ if and only if $\varphi \in w$ or $\psi \in w$;*

4. *$\varphi \in w$ if and only if $J_2\varphi \in w$;*

5. *$J_2\varphi \in w$ if and only if $\neg J_2\varphi \notin w$.*

Lemma 73. *Let Γ be a consistent set of formulae. If $\Gamma \cup \{\varphi\}$ is inconsistent then $\Gamma \vdash J_1\varphi \vee J_0\varphi$.*

Proof. Let Γ be a consistent set of formulae.Let $\Gamma \cup \{\varphi\}$ be inconsistent. By assumption, $\Gamma \cup \{\varphi\} \vdash 0$. By theorem 60, there exist formulae $\gamma_1, \dots, \gamma_n \in \Sigma$ such that $\vdash J_2\gamma_1 \wedge \dots \wedge J_2\gamma_n \wedge J_2\varphi \to J_2 0$, hence $\vdash J_2\gamma_1 \wedge \dots \wedge J_2\gamma_n \to \neg J_2\varphi$, thus $\Gamma \vdash \neg J_2\varphi$. Since $\vdash \neg J_2\varphi \to J_1\varphi \vee J_0\varphi$ by lemma 13, $\Gamma \vdash J_1\varphi \vee J_0\varphi$. $\square$

Lemma 74 (Lindenbaum's lemma)**.** *Let Γ be a consistent set of formulae such that $\Gamma \not\vdash \varphi$, for some $\varphi \in Fm$, then there exists a maximally consistent set of formulae w such that $\Gamma \subseteq w$ and such that $\varphi \notin w$.*

Proof. Consider an enumeration $\psi_1, \psi_2, \psi_3, \dots$ of the formulae in Fm. Define:

$$\Gamma_0 = \begin{cases} \Gamma \cup \{\neg\varphi\} \text{ if consistent,} \\ \Gamma \cup \{J_1\varphi\} \text{ otherwise.} \end{cases}$$

$$\Gamma_{i+1} = \begin{cases} \Gamma_i \cup \{\psi_i\} \text{ if consistent, else} \\ \Gamma_i \cup \{\neg\psi_i\} \text{ if consistent, else} \\ \Gamma_i \cup \{J_1\psi_i\}. \end{cases}$$

$$w = \bigcup_{i \in \mathbb{N}} \Gamma_i.$$

Observe that, by construction, w is maximal. We want to show that w is also consistent. We first claim that Γ_0 is consistent. If $\Gamma_0 = \Gamma \cup \{\neg\varphi\}$ it is consistent by construction. Differently, $\Gamma_0 = \Gamma \cup \{J_1\varphi\}$ means that $\Gamma \cup \{\neg\varphi\}$ is inconsistent, which by lemma 73 implies $\Gamma \vdash J_1\neg\varphi \vee J_0\neg\varphi$, that is equivalent to $\Gamma \vdash J_1\varphi \vee J_2\varphi$. Suppose towards a contradiction that $\Gamma \vdash J_1J_1\varphi \vee J_0J_1\varphi$. Using $\vdash_{\mathbf{B}} J_1J_1\varphi \leftrightarrow 0$ we obtain $\Gamma \vdash J_0J_1\varphi$, which amounts to $\Gamma \vdash \neg J_1\varphi$. Since $\vdash_{\mathbf{B}} \neg J_1\varphi \to J_2\varphi \vee J_0\varphi$, we have $\Gamma \vdash J_2\varphi \vee J_0\varphi$. This last fact together with $\Gamma \vdash J_1\varphi \vee J_2\varphi$ implies, by lemma 13, $\Gamma \vdash J_2\varphi$. Therefore $\Gamma \vdash \varphi$, which contradicts the assumption. We conclude that $\Gamma \nvdash J_1J_1\varphi \vee J_0J_1\varphi$, which, using lemma 73 by contraposition, yields that $\Gamma \cup \{J_1\varphi\}$ is consistent. This shows that Γ_0 is consistent.

We claim that Γ_{i+1} is consistent, given that Γ_i is. So suppose that $\Gamma_i \cup \{\psi_i\}$ and $\Gamma_i \cup \{\neg\psi_i\}$ are inconsistent. Then, by lemma 73, $\Gamma_i \vdash J_1\psi_i \vee J_0\psi_i$ and

$\Gamma_i \vdash J_1\neg\psi_i \vee J_0\neg\psi_i$, where the latter is equivalent to $\Gamma_i \vdash J_1\psi_i \vee J_2\psi_i$. By lemma 13 these facts imply $\Gamma_i \vdash J_1\psi_i$. Since Γ_i was assumed to be consistent, this means that $\Gamma_i \cup \{J_1\psi_i\}$ is consistent, therefore Γ_{i+1} is as well. This shows that w is maximal and consistent and, by construction, $\neg\varphi \in w$ or $J_1\varphi \in w$, therefore $\varphi \notin w$.

□

As a first step to introduce canonical models, let us define the *canonical relation.*

Definition 75. Let $\mathcal{W}$ be the set of all maximally consistent set of formulae. Then the *canonical relation* $\mathcal{R} \subseteq \mathcal{W} \times \mathcal{W}$ for $\mathbf{B}_{\mathrm{e}}^{\Box}$ is defined, for every $w, s \in \mathcal{W}$ as:

$$w\mathcal{R}s \text{ iff } \forall\varphi \in Fm \text{ s.t. } \Box\varphi \in w \text{ then } \varphi \in s.$$

Lemma 76 (Existence lemma). *For every maximally consistent set of formulae $w \in \mathcal{W}$ such that $\Diamond\varphi \in w$ (for some $\varphi \in Fm$) then φ is meaningful in w and there exists a maximally consistent set $s \in \mathcal{W}$ such that $w\mathcal{R}s$ and either $\varphi \in s$ or $J_1\varphi \in s$.*

Proof. Suppose that $\Diamond\varphi \in w$, for some w maximally consistent set of formulae. Consider the set

$$s^- = \{J_2\psi | \Box J_2\psi \in w\}.$$

Observe that $s^- \neq \emptyset$, as for every formula ψ such that $\vdash \psi$, then $\vdash J_2\psi$, hence $\vdash \Box J_2\psi$, which implies $\Box J_2\psi \in w$, by lemma 70-(3). Let us show that s^- is consistent. Suppose, by contradiction, that s^- is inconsistent, then $s^- \vdash \gamma$, for every $\gamma \in Fm$, thus, in particular, $s^- \vdash \neg\varphi$. By deduction theorem there are formulae $J_2\psi_1, \ldots, J_2\psi_n \in s^-$ such that $\vdash J_2J_2\psi_1 \wedge \cdots \wedge J_2J_2\psi_n \to J_2\neg\varphi$. Recall that $\vdash_{\mathbf{B}} J_2J_2\gamma \leftrightarrow J_2\gamma$, for every $\gamma \in Fm$, thus $\vdash J_2\psi_1 \wedge \cdots \wedge J_2\psi_n \to J_2\neg\varphi$. By applying (N), we get $\vdash \Box(J_2\psi_1 \wedge \cdots \wedge J_2\psi_n \to J_2\neg\varphi)$ and by distributing box ((BK) and lemma 59), $\vdash \Box J_2\psi_1 \wedge \cdots \wedge \Box J_2\psi_n \to \Box J_2\neg\varphi)$. Observe that, by construction of s^-, $\Box J_2\psi_i \in w$, for every $i \in \{1, \ldots, n\}$, hence $\Box J_2\neg\varphi \in w$, i.e. $\neg\Diamond\neg J_2\neg\varphi \in w$. By lemma 13, $\vdash \neg J_2\neg\varphi \leftrightarrow J_2\varphi \vee J_1\varphi$, thus $\neg\Diamond(J_2\varphi \vee J_1\varphi) \in w$, which implies (by distributivity of diamond, lemma 59), $\neg(\Diamond J_2\varphi \vee \Diamond J_1\varphi) \in w$. On the other hand, $\Diamond\varphi \in w$, hence $J_2 \Diamond \varphi \in w$, which implies $\Diamond J_2\varphi \vee \Diamond J_1\varphi \in w$, by lemma 59, giving raise to a contradiction with the fact that w is consistent.

Observe that we have also proved that $s^- \nvdash \neg\varphi$, hence by Lindenbaum's lemma there exists a maximally consistent set s such that $s^- \subseteq s$ and $\neg\varphi \notin s$. By maximality, we have that either $\varphi \in s$ or $J_1\varphi \in s$. To show that $w\mathcal{R}s$, suppose $\Box\gamma \in w$, for some $\gamma \in Fm$, then by lemma 72 $J_2\Box\gamma \in w$, hence, by (M3), $\Box J_2\gamma \in w$, and by construction $J_2\gamma \in s^- \subseteq s$, thus $\gamma \in s$ (by lemma 72), showing that $w\mathcal{R}s$. Finally, let us show that φ is meaningful in w. Since $\varphi \in w$, then $J_2\varphi \in w$ and $+\varphi \in w$ (as $\vdash_{\mathbf{B}} J_2\varphi \to +\varphi$), namely that $\diamond\varphi$ is meaningful in w and so is φ (Lemma 71). $\Box$

We are ready to define the concept of *canonical model*.

Definition 77. The *canonical model* for $\mathbf{B}_{\mathrm{e}}^{\Box}$ is a model $\mathfrak{M} = \langle \mathcal{W}, \mathcal{R}, v \rangle$ where $\mathcal{W}$ is the set of all maximally consistent sets of formulae, $\mathcal{R}$ is the canonical relation for $\mathbf{B}_{\mathrm{e}}^{\Box}$ and v is defined as follows:

- $v(w, x) = 1$ if and only if $x \in w$;
- $v(w, x) = 0$ if and only if $\neg x \in w$;
- $v(w, x) = 1/2$ if and only if $J_1 x \in w$,

for every $w \in \mathcal{W}$ and propositional variable x.

Lemma 78 (Truth lemma)**.** *Let $\mathfrak{M} = \langle \mathcal{W}, \mathcal{R}, v \rangle$ be the canonical model for $\mathbf{B}_{\mathrm{e}}^{\Box}$. Then, for every formula $\varphi \in Fm$ and every $w \in \mathcal{W}$, the following hold:*

1. $v(w, \varphi) = 1$ *if and only if* $\varphi \in w$;
2. $v(w, \varphi) = 0$ *if and only if* $\neg\varphi \in w$;
3. $v(w, \varphi) = 1/2$ *if and only if* $J_1\varphi \in w$.

Proof. By induction on the length of the formula φ. We just show (1) for the inductive step when $\varphi = \Box\psi$, for some $\psi \in Fm$.
Observe that $v(w, \Box\psi) = 1$ iff ψ is meaningful in w (i.e. $v(w, \psi) \neq 1/2$) and $\forall s$ s.t. $w\mathcal{R}s$, $v(s, \psi) = 1$, thus, by induction hypothesis, iff $J_1\psi \notin w$ and $\psi \in s$ $\forall s$ s.t. $w\mathcal{R}s$.
($\Rightarrow$) Suppose, by contradiction, that $v(w, \Box\psi) = 1$ but $\Box\psi \notin w$, hence, by maximality of w, $J_1\Box\psi \in w$ or $\neg\Box\psi \in w$. Since ψ is meaningful in w, so is $\Box\psi$

(Lemma 71), thus $J_1\Box\psi \notin w$. So, $\neg\Box\psi \in w$, i.e. $\Diamond\neg\psi \in w$, hence, by Existence lemma 76, there exists $s' \in \mathcal{W}$ such that $w\mathcal{R}s'$ such that $\neg\psi \in s'$ or $J_1\psi \in s'$, which implies, by induction hypothesis, that $v(s',\psi) = 0$ or $v(s',\psi) = 1/2$, a contradiction.
($\Leftarrow$) Let $\Box\psi \in w$. Then $J_2\Box\psi \in w$ (by lemma 72) and $+\Box\psi \in w$ (since $\vdash_{\mathbf{B}} J_2\gamma \to +\gamma$). This means that $\Box\psi$ is meaningful in w, hence so is ψ. Moreover, for every $s \in \mathcal{W}$ such that $w\mathcal{R}s$, we have that $\psi \in s$ (by definition of $\mathcal{R}$), hence, by induction hypothesis, $v(s,\psi) = 1$, from which $v(w,\Box\psi) = 1$. $\Box$

Theorem 79 (Completeness). *$\Gamma \vdash_{\mathbf{B}_e^\Box} \varphi$ if and only if $\Gamma \models^l_{\mathbf{B}_e^\Box} \varphi$.*

Proof. ($\Rightarrow$) It is easily checked that all the axioms are sound and the rules preserve soundness.
($\Leftarrow$) Suppose $\Gamma \nvdash \varphi$. Then Γ is a consistent set of formulae, therefore, by the Lindenbaum's lemma 74, there exist a maximally consistent set s such that $\Gamma \subseteq s$ and $\varphi \notin s$, hence, by Truth lemma 78, there exists a canonical countermodel, namely $(\mathcal{W},\mathcal{R},v)$, with $v(s,\gamma) = 1$ for all $\gamma \in \Gamma$ and $v(s,\varphi) \neq 1$. $\Box$

In order to prove decidability for $\mathbf{B}_e^\Box$, we employ the filtration technique (see [Blackburn et al., 2001, pp. 77-80]). First we need to provide an extended notion of closure under subformulae.

Definition 80. A set of formulae Σ is closed under subformulae if $\forall\varphi,\psi \in \Sigma$:

1. if $\varphi \circ \psi \in \Sigma$ for any binary connective $\circ$, then $\varphi,\psi \in \Sigma$;
2. if $\neg\varphi \in \Sigma$ or $J_2\varphi \in \Sigma$, then $\varphi \in \Sigma$;
3. if $\Box\varphi \in \Sigma$, then $\varphi \in \Sigma$ and $+\varphi \in \Sigma$;

Notice that if a set of formulae is finite, its closure under subformulae is still finite.

Definition 81. Let $\langle W,R,v\rangle$ be a model and Σ be a finite set of formulae closed under subformulae. This set induces an equivalence relation over W defined as follows: $w \equiv_\Sigma s$ iff $\forall\varphi \in \Sigma(v(w,\varphi) = 1$ iff $v(s,\varphi) = 1)$.

When the reference set Σ is clear from the context, we denote the equivalence class $[w]_{\equiv_\Sigma}$ simply by $[w]$.

Definition 82. Let $M = \langle W, R, v\rangle$ be a model and Σ be a finite set of formulae closed under subformulae. The filtration of M through Σ is the model $\langle W^f, R^f, v^f\rangle$ defined as:

1. $W^f = W/{\equiv_\Sigma}$;

2. $[w]R^f[s]$ iff $\exists w' \in [w], s' \in [s]$ s.t. $w'Rs'$;

3. $v^f([w], p) = 1$ iff $v(w, p) = 1$, for all variables $p \in \Sigma$.

Lemma 83. *Let $\langle W^f, R^f, v^f\rangle$ be a filtration of $M = \langle W, R, v\rangle$ through Σ. For all $\varphi \in \Sigma, w \in W$, it holds $v(w, \varphi) = 1$ iff $v^f([w], \varphi) = 1$.*

Proof. By induction on the complexity of $\varphi \in \Sigma$. The Boolean cases are straightforward. Let $\varphi = J_2\psi$, for some $\psi \in Fm$. $v(w, J_2\psi) = 1$ iff $v(w, \psi) = 1$ iff, by induction hypothesis, $v^f([w], \psi) = 1$ iff $v^f([w], J_2\psi) = 1$. Notice that by closure $\psi \in \Sigma$.

Let $\varphi = \Box\psi$, for some $\psi \in Fm$. Suppose $v(w, \Box\psi) = 1$, which means that $v(w, +\psi) = 1$ and for all $s \in W$ s.t. $wRs, v(s, \psi) = 1$. Now $+\psi := J_2\psi \vee J_2\neg\psi$, by the Boolean cases and the previous one we conclude $v^f([w], +\psi) = 1$. By definition of filtration, $[w]R^f[s]$, and by induction hypothesis $v^f([s], \psi) = 1$. Since this covers all the successors of $[w]$, then $v^f([w], \Box\psi) = 1$. Notice that by closure of ψ under subformula, $J_2\psi \vee J_2\neg\psi \in \Sigma$. The other direction follows similarly. □

Theorem 84. *If a formula φ is satisfiable in a model, it is satisfiable in a finite model.*

Proof. Let φ be satisfied by a model $M = \langle W, R, v\rangle$, and let Σ be the closure under subformulae of $\{\varphi\}$. Σ is finite. Now consider the filtration $M^f_\Sigma = \langle W^f, R^f, v^f\rangle$ of M through Σ. By theorem 83, M^f_Σ satisfies φ. Consider the mapping $g : W^f \to \mathscr{P}(\Sigma)$ s.t. $g([w]) = \{\psi | v(w, \psi) = 1\}$. By definition of $\equiv_\Sigma$, g is well-defined and injective. Denoting by $card(X)$ the cardinality of a set X, we have $card(W^f) \leq card(\mathscr{P}(\Sigma)) = 2^{card(\Sigma)}$. □

Corollary 85 (Decidability)**.** *The logic $\mathbf{B}^{\Box}_{\mathrm{e}}$ is decidable.*

3.2 Modal $\mathbf{PWK}_{\mathrm{e}}$ logic

The modal extension of the propositional logic $\mathbf{PWK}_{\mathrm{e}}$ is defined over the same formula algebra $\mathbf{Fm}$ of $\mathbf{B}_{\mathrm{e}}^{\Box}$. The substantial (semantical) difference between $\mathbf{PWK}_{\mathrm{e}}^{\Box}$ and $\mathbf{B}_{\mathrm{e}}^{\Box}$ concern the interpretation of the modal formulae $\Box\varphi$, which follows the choice of the different truth-set in $\mathbf{PWK}_{\mathrm{e}}$: namely a modal formula $\Box\varphi$ will hold in a state w iff it will also hold in all the related states s, namely in those the formula is not false.

The logic $\langle \mathscr{L}_J^{\Box}, \vdash_{\mathbf{PWK}_{\mathrm{e}}^{\Box}} \rangle$ is the consequence relation induced by the following Hilbert-style axiomatization.

Axioms

- the axioms for $\mathbf{PWK}_{\mathrm{e}}$ introduced in definition 15;

(P1) $\Box(J_2\varphi \to J_2\psi) \to (\Box J_2\varphi \to \Box J_2\psi)$;

(P2) $+\varphi \leftrightarrow +\Box\varphi$.

(P3) $\Box\varphi \leftrightarrow \Box\neg J_0\varphi$;

Rules

- (ρ-B13) $J_2\varphi \leftrightarrow J_2\psi, J_0\varphi \leftrightarrow J_0\psi \vdash \varphi \equiv \psi$;
- (PWK-Alg3) $\varphi \dashv\vdash \neg J_0\varphi \leftrightarrow 1$.
- (N): if $\vdash \varphi$ then $\vdash \Box\varphi$.

In this Section, by $\vdash$ we will mean $\vdash_{\mathbf{PWK}_{\mathrm{e}}^{\Box}}$.

Lemma 86. *For α, β external formulae, the following is a theorem of* $\mathbf{PWK}_{\mathrm{e}}^{\Box}$*:*

(BK) $\Box(\alpha \to \beta) \to (\Box\alpha \to \Box\beta)$

Proof. It is the same of lemma 57. $\Box$

The Deduction theorem holding for $\mathbf{PWK}_{\mathrm{e}}$ can be actually extented to its modal version.

Theorem 87 (Deduction theorem)**.** *For the logic* $\mathbf{PWK}_{\mathrm{e}}^{\Box}$*, it holds that* $\Gamma \vdash \varphi$ *iff there exist some formulae* $\gamma_1, \dots, \gamma_n \in \Gamma$ *such that* $\vdash \neg J_0\gamma_1 \wedge \cdots \wedge \neg J_0\gamma_n \rightarrow \neg J_0\varphi$*.*

Proof. ($\Rightarrow$). By induction on the length of the derivation of φ from Γ.
Basis. If φ is an axiom ($\vdash \varphi$), then by lemma 16 we have $\vdash \neg J_0\varphi$.
Inductive step. We just show the case for the rule (N). Let $\varphi = \Box\psi$, for some $\psi \in Fm$, and $\Gamma \vdash \Box\psi$, and the last deduction rule applied is (N), hence it holds $\vdash \psi$. Therefore we have $\vdash \Box\psi$, hence $\vdash \neg J_0\Box\psi$, by lemma 16. By induction hypothesis, there exist some formulae $\gamma_1, \dots, \gamma_n \in \Gamma$ such that $\vdash \neg J_0\gamma_1 \wedge \cdots \wedge \neg J_0\gamma_n \rightarrow \neg J_0\psi$. Observe that (by classical logic, lemma 16) $\vdash \neg J_0\Box\psi \rightarrow (\neg J_0\gamma_1 \wedge \cdots \wedge \neg J_0\gamma_n \rightarrow \neg J_0\Box\psi)$, hence, by *modus ponens* (on external formulae, see lemma 16), $\vdash \neg J_0\gamma_1 \wedge \cdots \wedge \neg J_0\gamma_n \rightarrow \neg J_0\Box\psi$.
($\Leftarrow$). Suppose $\vdash \neg J_0\gamma_1 \wedge \cdots \wedge \neg J_0\gamma_n \rightarrow \neg J_0\varphi$. By lemma 16, we have $\Gamma \vdash \neg J_0\gamma$, $\forall \gamma \in \Gamma$, therefore $\Gamma \vdash \neg J_0\gamma_1 \wedge \cdots \wedge \neg J_0\gamma_n$, thus applying *modus ponens* (on external formulae by lemma 16), we get $\Gamma \vdash \neg J_0\varphi$. Again, by lemma 16 we have $\Gamma \vdash \varphi$ (as $\neg J_0\varphi \vdash \varphi$). $\Box$

Lemma 88. *In the logic* $\mathbf{PWK}_{\mathrm{e}}^{\Box}$ *it holds:*

$$\Box(J_i\varphi \wedge J_k\psi) \leftrightarrow \Box J_i\varphi \wedge \Box J_k\psi \text{ for } i, k \in \{0, 1, 2\}.$$

Proof. The proof is identical to that of lemma 59. Observe that even though $\mathbf{PWK}_{\mathrm{e}}^{\Box}$ does not possess a full *modus ponens*, in this proof we are dealing exclusively with external formulae for which full *modus ponens* holds (see lemma 16).

$\Box$

The semantics for $\mathbf{PWK}_{\mathrm{e}}^{\Box}$ employs the weak three-valued Kripke models of definition 61. In particular we consider the following subclass:

Definition 89. A *Halldén-Kripke* model is a weak three-valued Kripke model $\langle W, R, v\rangle$ such that v evaluates formulae of the form $\Box\varphi$ according to:

(1) $v(w, \Box\varphi) = 1$ iff $v(w, \varphi) \neq 1/2$ and $v(s, \varphi) \neq 0$ for every $s \in W$ such that wRs.

(0) $v(w, \Box\varphi) = 0$ iff $v(w, \varphi) \neq 1/2$ and there exists $s \in W$ such that wRs and $v(s, \varphi) = 0$.

(1/2) $v(w, \Box\varphi) = 1/2$ iff $v(w, \varphi) = 1/2$.

Within this section by *model* we intend *Halldén-Kripke model.*

Remark 90. Observe that, in every model $\langle W, R, v\rangle$ with $w \in W$, it holds that $v(w, \diamond\varphi) = 1$ iff $v(w, \varphi) \neq 1/2$ and there exists $s \in W$ such that wRs and $v(s, \varphi) = 1$, $v(w, \diamond\varphi) = 0$ iff $v(w, \varphi) \neq 1/2$ and $v(s, \varphi) \neq 1$ for every $s \in W$ such that wRs.

We denote by $\models^l_{\mathbf{PWK}^\Box_e}$ the *local* modal **PWK** external logic on the class of all frames obtained by taking $\{1, 1/2\}$ as designated values, and similarly we denote the *global* modal **PWK** external logic by $\models^g_{\mathbf{PWK}^\Box_e}$, following definition 65:

Definition 91. $\Gamma \models^l_{\mathbf{PWK}^\Box_e} \varphi$ iff for all models $\langle W, R, v\rangle$ and all $w \in W$, if $v(w, \gamma) \neq 0, \forall\gamma \in \Gamma$, then $v(w, \varphi) \neq 0$. $\Gamma \models^g_{\mathbf{PWK}^\Box_e} \varphi$ iff for all models $\mathfrak{M}$, if $\mathfrak{M} \models \gamma, \forall\gamma \in \Gamma$, then $\mathfrak{M} \models \varphi$.

In the following we simply write $\models$ for $\models^l_{\mathbf{PWK}^\Box_e}$ (until the context require an explicit distionction), since we will focus on the local modal logic. Notice that the notion of satisfiability in $\mathbf{PWK}^\Box_e$ differs from $\mathbf{B}^\Box_e$, according to the difference at the propositional level between $\mathbf{PWK}_e$ and **B**.

Definition 92. A formula φ is satisfied (valid) in a model $\langle W, R, v\rangle$ if $v(w, \varphi) \neq 0$, for some (all) $w \in W$.

Validity in a frame and in a class of frames follow definition 64, with the exception that we consider only Halldén-Kripke models built on a frame.

The notion of consistency has to be changed for $\mathbf{PWK}^\Box_e$, because the sublogic **PWK** is paraconsistent. Therefore we substiture consistent (and maximally consistent) sets with non-trivial ones.

Remark 93. A set $\Gamma \subset Fm$ is *non-trivial* if there is no formula $\varphi \in Fm$, such that $\Gamma \vdash J_i\varphi$ and $\Gamma \vdash \neg J_i\varphi$, for any $i \in \{0, 1, 2\}$. It is called trivial otherwise.

Definition 94. A non-trivial set Γ is *maximally non-trivial* whenever $\Gamma \subset \Gamma'$ implies that Γ' is trivial.

A *meaningful* formula is the same of definition 69) for modal Bochvar logic, by just considering that $\vdash$ here refers to $\mathbf{PWK}_{\mathrm{e}}^{\Box}$ (and not to $\mathbf{B}_{\mathrm{e}}^{\Box}$).

The following is the analogous of lemma 70, for $\mathbf{PWK}_{\mathrm{e}}^{\Box}$ (indeed, the first claims coincide).

Lemma 95. *Let w be a maximally non-trivial set of formulae, then:*

1. *if $\neg\varphi \notin w$ and all the variables occurring in φ are meaningful in w, then $\varphi \in w$;*
2. *if all the variables of φ are covered, then φ is meaningful in w;*
3. *if $\vdash \varphi$ then $\neg\varphi \notin w$;*
4. *if $\vdash \varphi$ and every variable in φ is meaningful then $\varphi \in w$.*

Proof. (3). Suppose that $\vdash \varphi$ and, by contradiction, that $\neg\varphi \in w$. By lemma 16 $\vdash \neg J_0\varphi$, thus $w \vdash J_0\varphi$. On the other hand $w \vdash J_0\varphi$ (by lemma 16). But this implies that w is trivial (by Remark 93). (4) follows from the previous. $\Box$

Lemma 96. *Let Γ be a non-trivial set of formulae. If $\Gamma \cup \{\varphi\}$ is trivial then $\Gamma \vdash \neg\varphi$.*

Proof. Let Γ be a non-trivial set of formulae. Suppose $\Gamma \cup \{\varphi\}$ is trivial. Therefore $\Gamma \cup \{\varphi\} \vdash 0$; by theorem 87, there exist formulae $\gamma_1, \dots, \gamma_n \in \Gamma$ s.t. $\vdash \neg J_0\gamma_1 \wedge \cdots \wedge \neg J_0\gamma_n \wedge \neg J_0\varphi \to \neg J_0 0$, where the non-triviality of Γ assures that $\neg J_0\varphi$ actually appears in the antecedent. We have $\vdash_{\mathbf{PWK}_{\mathrm{e}}} \neg J_0 0 \leftrightarrow 0$ by lemma 16, hence $\vdash \neg J_0\gamma_1 \wedge \cdots \wedge \neg J_0\gamma_n \wedge \neg J_0\varphi \to 0$, therefore $\vdash \neg J_0\gamma_1 \wedge \cdots \wedge \neg J_0\gamma_n \to J_0\varphi$. Since $\vdash J_0\varphi \to \neg J_0\neg\varphi$, by classical logic $\vdash \neg J_0\gamma_1 \wedge \cdots \wedge \neg J_0\gamma_n \to \neg J_0\neg\varphi$. Thus, by theorem 87, $\gamma_1 \wedge \cdots \wedge \gamma_n \vdash \neg\varphi$, hence by monotonicity $\Gamma \vdash \neg\varphi$. $\Box$

Lindenbaum's lemma for $\mathbf{PWK}_{\mathrm{e}}^{\Box}$ has a slightly different form from that of $\mathbf{B}_{\mathrm{e}}^{\Box}$.

Lemma 97 (Lindenbaum's lemma)**.** *Let Γ be a non-trivial set of formulae such that $\Gamma \nvdash \varphi$, for some $\varphi \in Fm$, then there exists a maximally non-trivial set of formulae w such that $\Gamma \subseteq w$ and $\neg\varphi \in w$.*

Proof. Suppose $\Gamma \nvdash \varphi$. Let $\psi_1, \psi_2, \psi_3, \ldots$ be an enumeration of the formulae of $\mathbf{PWK}_e^{\Box}$. We define the sets inductively:

$$\Gamma_0 = \Gamma \cup \{\neg\varphi\}.$$

$$\Gamma_{i+1} = \begin{cases} \Gamma_i \cup \{\psi_i\}, & \text{if non-trivial} \\ \Gamma_i \cup \{\neg\psi_i\}, & \text{if non-trivial} \\ \Gamma_i \cup \{J_1\psi_i\}, & \text{otherwise} \end{cases}$$

$$w = \bigcup_{i \in \mathbb{N}} \Gamma_i.$$

By construction w is maximal, $\Gamma \subseteq w$ and $\neg\varphi \in w$. We prove the non-triviality of w by induction on $n \in \mathbb{N}$. For the base step, since $\Gamma \nvdash \varphi$, by lemma 96 $\Gamma_0 = \Gamma \cup \{\neg\varphi\}$ is non-trivial. For the inductive step, suppose Γ_i is non-trivial, while both $\Gamma_i \cup \{\psi_i\}$ and $\Gamma_i \cup \{\neg\psi_i\}$ are trivial. Therefore $\Gamma_{i+1} = \Gamma_i \cup \{J_1\psi_i\}$. By the same lemma the previous facts yield $\Gamma_i \vdash \neg\psi_i$ and $\Gamma_i \vdash \psi_i$. By lemma 16 we obtain $\Gamma_i \vdash J_2\neg\psi_i \vee J_1\neg\psi_i$ and $\Gamma_i \vdash J_2\psi_i \vee J_1\psi_i$, of which the former can be rewritten by the same lemma as $\Gamma_i \vdash J_0\psi_i \vee J_1\psi_i$. Since $J_0\psi_i, J_1\psi_i, J_2\psi_i$ are pairwise contradictory, by classical logic we conclude $\Gamma_i \vdash J_1\psi_i$. Since Γ_i is taken as non-trivial, this implies $\Gamma_i \nvdash \neg J_1\psi_i$. By lemma 96, we conclude that $\Gamma_i \cup \{J_1\psi_i\} = \Gamma_{i+1}$ is non-trivial. □

Definition 98. Let $\mathcal{W}$ be the set of all maximally non-trivial set of formulae. Then the *canonical relation* $\mathcal{R} \subseteq \mathcal{W} \times \mathcal{W}$ for $\mathbf{PWK}_e^{\Box}$ is defined, for every $w, s \in \mathcal{W}$ as:

$$w\mathcal{R}s \text{ iff } \forall\varphi \in Fm \text{ s.t. } \Box\varphi \in w \text{ then } \neg\varphi \notin s.$$

Lemma 99 (Existence lemma). *For every maximally non-trivial set of formulae w such that $\diamond\varphi \in w$ (for some $\varphi \in Fm$) then φ is meaningful in w and there exists a maximally non-trivial set $s \in \mathcal{W}$ such that $w\mathcal{R}s$ and $\varphi \in s$.*

Proof. Let $\diamond\varphi \in w$ and consider the set

$$s^- = \{\psi \mid \Box\neg J_0\psi \in w\}.$$

Observe that $s^- \neq \emptyset$, in fact by lemma 16, $\vdash \neg J_0 1$, therefore by (N)

$\vdash \Box\neg J_0 1$. Since w is maximally non-trivial, $\Box\neg J_0 1 \in w$, hence $1 \in s^-$. Suppose by contradiction that s^- is trivial. Therefore s^- derives every formula, in particular $s^- \vdash \neg\varphi$. By theorem 87, there are $\psi_1, \dots, \psi_n \in s^-$ s.t. $\vdash \neg J_0\psi_1 \wedge \dots \wedge \neg J_0\psi_n \to \neg J_0\neg\varphi$. By (N) $\vdash \Box(\neg J_0\psi_1 \wedge \dots \wedge \neg J_0\psi_n \to \neg J_0\neg\varphi)$, and using (BK) and *modus ponens* (since the formulae considered here are external) we get $\vdash \Box(\neg J_0\psi_1 \wedge \dots \wedge \neg J_0\psi_n) \to \Box\neg J_0\neg\varphi$. Now lemma 88 can be generalized to $\vdash \Box\neg J_0\psi_1 \wedge \dots \wedge \Box\neg J_0\psi_n \to \Box(\neg J_0\psi_1 \wedge \dots \wedge \neg J_0\psi_n)$, obtaining, by transitivity, $\vdash \Box\neg J_0\psi_1 \wedge \dots \wedge \Box\neg J_0\psi_n \to \Box\neg J_0\neg\varphi$. Observe that $\Box\neg J_0\psi_1, \dots, \Box\neg J_0\psi_n \in w$, therefore $\Box\neg J_0\neg\varphi \in w$ (as w is maximally non-trivial). By (P3) $\Box\neg\varphi \in w$, which can be rewritten as $\neg\Diamond\varphi \in w$, contradicting the non-triviality of w.

Notice that we have also proved that in particular $s^- \nvdash \neg\varphi$. We can now apply Lindembaum's lemma and extend s^- to a maximally non-trivial $s \supseteq s^-$ s.t. $\varphi \in s$, hence φ is meaningful in s. Finally we show that $w\mathcal{R}s$ according to the canonical relation: let for arbitrary $\chi \in Fm, \Box\chi \in w$, then by (P3) $\Box\neg J_0\chi \in w$, so $\chi \in s^- \subseteq s$ and by non-triviality $\neg\chi \notin s$. $\Box$

We now adapt the definition of canonical model to $\mathbf{PWK}_{\mathrm{e}}^{\Box}$:

Definition 100. The *canonical model* for $\mathbf{PWK}_{\mathrm{e}}^{\Box}$ is a weak three-valued Kripke model
$\mathfrak{M} = \langle \mathcal{W}, \mathcal{R}, v\rangle$ where $\mathcal{W}$ is the set of all maximally non-trivial sets of formulae, $\mathcal{R}$ is the canonical relation for $\mathbf{PWK}_{\mathrm{e}}^{\Box}$ and v is defined as follows:

- $v(w, x) = 1$ if and only if $x \in w$;
- $v(w, x) = 0$ if and only if $\neg x \in w$;
- $v(w, x) = 1/2$ if and only if $J_1 x \in w$,

for every $w \in \mathcal{W}$ and propositional variable x.

Lemma 101 (Truth lemma)**.** *Let* $\mathfrak{M} = \langle \mathcal{W}, \mathcal{R}, v\rangle$ *be the canonical model. Then, for every formula* $\varphi \in Fm$ *and every* $w \in \mathcal{W}$*, the following hold:*

1. $v(w, \varphi) = 1$ *if and only if* $\varphi \in w$*;*
2. $v(w, \varphi) = 0$ *if and only if* $\neg\varphi \in w$*;*

3. $v(w, \varphi) = 1/2$ *if and only if* $J_1\varphi \in w$.

Proof. By induction on the length of the formula φ. We just show (1) for the inductive step when $\varphi = \Box\psi$, for some $\psi \in Fm$.
Observe that $v(w, \Box\psi) = 1$ iff ψ is meaningful in w (i.e. $v(w, \psi) \neq 1/2$) and $\forall s$ s.t. $w\mathcal{R}s$, $v(s, \psi) \neq 0$, thus, by induction hypothesis, iff $J_1\psi \notin w$ and $\neg\psi \notin s$ $\forall s$ s.t. $w\mathcal{R}s$.
($\Rightarrow$) Suppose, by contradiction, that $v(w, \Box\psi) = 1$ but $\Box\psi \notin w$, hence, by maximality of w, $J_1\Box\psi \in w$ or $\neg\Box\psi \in w$. Since ψ is meaningful in w, so is $\Box\psi$ (Lemma 95), thus $J_1\Box\psi \notin w$. So, $\neg\Box\psi \in w$, i.e. $\diamond\neg\psi \in w$, hence, by Existence lemma 99, there exists $s' \in \mathcal{W}$ such that $w\mathcal{R}s'$ such that $\neg\psi \in s'$, which implies, by induction hypothesis, that $v(s', \psi) = 0$, a contradiction.
($\Leftarrow$) The proof is similar to one of the Truth lemma 78 for $\mathbf{B}_{\mathrm{e}}^{\Box}$. Observe that $\vdash_{\mathbf{PWK}_{\mathrm{e}}} \varphi \to J_2\varphi$ and $\vdash_{\mathbf{PWK}_{\mathrm{e}}} J_2\varphi \to +\varphi$, moreover maximally non-trivial sets are closed under unrestricted *modus ponens.* □

Theorem 102 (Completeness)**.** $\Gamma \vdash_{\mathbf{PWK}_{\mathrm{e}}^{\Box}} \varphi$ *if and only if* $\Gamma \models^{l}_{\mathbf{PWK}_{\mathrm{e}}^{\Box}} \varphi$.

Proof. ($\Rightarrow$) It is easily checked that all the axioms are sound and the rules preserve soundness.
($\Leftarrow$) Suppose $\Gamma \nvdash \varphi$. Then Γ is a non-trivial set of formulae, therefore, by the Lindenbaum's lemma 97, there exist a maximally non-trivial set s such that $\Gamma \subseteq s$ and $\neg\varphi \in s$, hence, by Truth lemma 78, there exists a canonical countermodel, namely $(\mathcal{W}, \mathcal{R}, v)$, with $v(s, \gamma) = 1$ for all $\gamma \in \Gamma$ and $v(s, \varphi) = 0$. □

The decidability of $\mathbf{PWK}_{\mathrm{e}}^{\Box}$ is proved similarly to the case for $\mathbf{B}_{\mathrm{e}}^{\Box}$. Once again we make use of filtrations. The notion of closure under subformulae is the same given in definition 80. The only essential change concerns the equivalence relation that induces the partition.

Definition 103. Let $\langle W, R, v\rangle$ be a model and Σ be a finite set of formulae closed under subformulae. This set induces an equivalence relation over W such that: $w \equiv_\Sigma s$ iff $\forall\varphi \in \Sigma(v(w, \varphi) = 1$ iff $v(s, \varphi) \neq 0)$.

Again, when there is no risk of confusion we omit the reference set Σ and denote the equivalence class $[w]_{\equiv_\Sigma}$ as $[w]$. We can adopt definition 82 for filtration.

Theorem 104. *Let* $\langle W^f, R^f, v^f \rangle$ *be a filtration of* $M = \langle W, R, v \rangle$ *through* Σ. *For all* $\varphi \in \Sigma, w \in W$, *it holds* $v(w, \varphi) \neq 0$ *iff* $v^f([w], \varphi) \neq 0$.

Proof. By induction on the complexity of $\varphi \in \Sigma$. The Boolean cases and the case for $\varphi = J_2\psi$ follows lemma 83.

Let $\varphi = \Box\psi$, for some $\psi \in Fm$. When $v(w, \Box\psi) = 1$ the proof is similar to the case covered in lemma 83. Suppose $v(w, \Box\psi) = 1/2$: this holds iff $v(w, \psi) = 1/2$ iff (by induction hypothesis) $v([w], \psi) = 1/2$ iff $v([w], \Box\psi) = 1/2$. □

In accordance with the notion of satisfiability in $\mathbf{PWK}_e^\Box$ we reobtain the main theorem:

Theorem 105. *If a formula* φ *is satisfiable in a model, it is satisfiable in a finite model.*

Proof. Identical to theorem 84. □

Corollary 106 (Decidability)**.** *The logic* $\mathbf{PWK}_e^\Box$ *is decidable.*

3.3 Extensions of modal weak Kleene logics

The aim of this Section is to axiomatize some extensions of both the modal logics $\mathbf{B}_e^\Box$ and $\mathbf{PWK}_e^\Box$. In particular, we focus on those extensions whose semantical counterpart is given by reflexive, transitive and/or Euclidean models (clearly, the properties refer all to the relational part of models). To this end, consider the following formulae:

$$(\mathrm{T}_e) \quad \Box J_2\varphi \to J_2\varphi,$$

$$(4_e) \quad \Box J_2\varphi \to \Box\Box J_2\varphi,$$

$$(5_e) \quad \Diamond J_2\varphi \to \Box \Diamond J_2\varphi,$$

which substantially consist of "external versions"[2] of the standard modal formulae (T), (4) and (5). The formulae introduced above allows to capture some frame properties within $\mathbf{B}_e^\Box$, as shown by the following.

[2]The problem with standard modal formulae is the same with Bochvar (non-external) propositional logic, which is well-known as a logic without theorems, since every formula can be evaluated into 1/2. Similarly, in the modal context using the internal formulae (T), (4) or (5) would provide an unsound axiomatization.

Proposition 107. *Let $\mathfrak{F} = \langle W, R\rangle$ be a frame. Then*

1. $\mathfrak{F} \models_{\mathbf{B}_e^\Box} \mathrm{T}_e$ *iff R is reflexive;*

2. $\mathfrak{F} \models_{\mathbf{B}_e^\Box} 4_e$ *iff R is transitive;*

3. $\mathfrak{F} \models_{\mathbf{B}_e^\Box} 5_e$ *iff R is euclidean.*

Proof. (1) ($\Rightarrow$) Suppose $\mathfrak{F} \models \mathrm{T}_e$ and, for arbitrary $w \in W$, let $X = \{s \in W \mid wRs\}$. Consider the valuation $v(s,p) = 1$ iff $s \in X$, for any propositional variable p. It follows that $v(s, J_2p) = 1$, and since X is the set of successors of w, we have $v(w, \Box J_2p) = 1$. By assumption $\mathfrak{F} \models \mathrm{T}_e$, therefore in particular $v(w, \Box J_2p \to J_2p) = 1$. It follows that $v(w, J_2p) = 1$, thus $v(w,p) = 1$, so, by definition, $w \in X$, that is wRw, showing that R is reflexive.
($\Leftarrow$) It is immediate to check that T_e is valid in every reflexive frame.

(2) ($\Rightarrow$) Assume $\mathfrak{F} \models 4_e$, let $w \in W$ such that wRs' and $s'\mathcal{R}t$, for some $s', t \in W$. Consider the valuation $v(s,p) = 1$ iff wRs, for every $s \in W$ and any propositional variable p. This implies that $v(s, J_2p) = 1$ for every wRs, thus $v(w, \Box J_2p) = 1$ (observing that $v(w, J_2p) \neq 1/2$. Since $\mathfrak{F} \models 4_e$ then $v(w, \Box\Box J_2p) = 1$. Since wRs', then $v(s', \Box J_2p) = 1$, therefore $v(t, J_2p) = 1$ (since $s'Rt$), thus $v(t,p) = 1$. This implies that wRt, i.e. R is transitive as desired.
($\Leftarrow$) It is immediate to check that 4_e is valid in every transitive frame.

(3) ($\Rightarrow$) Suppose $\mathfrak{F}$ to be non-euclidean, therefore for some $w, s', s'' \in W, wRs', wRs''$ but $\langle s', s''\rangle \notin R$. Define the valuation v such that for an arbitrary variable p, $v(w,p) = v(s'',p) = 1$, while $v(s',p) = 0$ and for all t s.t. $s'Rt, v(t,p) = 0$. It follows that $v(w, \Diamond J_2p) = 1$ but $v(s', \Diamond J_2p) = 0$, therefore $v(w, \Box \Diamond J_2p) = 0$, hence $v(w, \Diamond J_2p \to \Box \Diamond J_2p) = 0$. This countermodel proves $\mathfrak{F} \not\models 5_e$.
($\Leftarrow$) It is immediate to check that 5_e is valid in every euclidean frame. □

In the case of the logic $\mathbf{PWK}_e^\Box$, the same frame properties are expressed by the standard modal formulae:

$$\text{(T) } \Box\varphi \to \varphi$$

$$(4)\ \Box\varphi \to \Box\Box\varphi$$

$$(5)\ \diamond\varphi \to \Box\diamond\varphi$$

Proposition 108. *Let $\mathfrak{F} = \langle W, R\rangle$ be a frame. Then*

1. $\mathfrak{F} \models_{\mathbf{PWK}_e^\Box}$ (T) *iff R is reflexive;*

2. $\mathfrak{F} \models_{\mathbf{PWK}_e^\Box}$ (4) *iff R is transitive;*

3. $\mathfrak{F} \models_{\mathbf{PWK}_e^\Box}$(5) *iff R is euclidean.*

Proof. The proofs runs similarly as proposition 107. □

The correspondences established by Propositions 107 and 108 allow us to immediately prove completeness for some extensions of $\mathbf{B}_e^\Box$ and $\mathbf{PWK}_e^\Box$. For an axiomatic calculus L, let $\mathrm{LAx}_1 \dots \mathrm{Ax}_n$ be the logic obtained by adding $(\mathrm{Ax}_1), \dots, (\mathrm{Ax}_n)$ to L. Moreover we use the following abbreviations: S4 := T + 4, S5 := T + 5, $\mathrm{S4}_e := \mathrm{T}_e + 4_e$, $\mathrm{S5}_e := \mathrm{T}_e + 5_e$.

Theorem 109. *The relation $\mathcal{R}$ of the canonical models*[3] *for the following logics have the properties:*

- *In* $\mathbf{B}_e^\Box\mathrm{T}_e$ *and* $\mathbf{PWK}_e^\Box\mathrm{T}$ $\mathcal{R}$ *is reflexive;*
- *In* $\mathbf{B}_e^\Box 4_e$ *and* $\mathbf{PWK}_e^\Box 4$ $\mathcal{R}$ *is transitive;*
- *In* $\mathbf{B}_e^\Box 5_e$ *and* $\mathbf{PWK}_e^\Box 5$ $\mathcal{R}$ *is euclidean;*
- *In* $\mathbf{B}_e^\Box\mathrm{S4}_e$ *and* $\mathbf{PWK}_e^\Box\mathrm{S4}$ $\mathcal{R}$ *is reflexive and transitive;*
- *In* $\mathbf{B}_e^\Box\mathrm{S5}_e$ *and* $\mathbf{PWK}_e^\Box\mathrm{S5}$ $\mathcal{R}$ *is an equivalence relation.*

Proof. It follows from Propositions 107 and 108. □

That the accessibility relation of the canonical model has the desired properties is enough to obtain the following completeness results:

Corollary 110. *The following hold:*

[3]The canonical model for a certain extension $\mathbf{B}_e^\Box\mathrm{Ax}$ of $\mathbf{B}_e^\Box$ differs from definition 77 only for the set of worlds, which now consist not of all maximal consistent sets (w.r.t. $\mathbf{B}_e^\Box$), but only of the maximal consistent theories of $\mathbf{B}_e^\Box\mathrm{Ax}$. The canonical model for $\mathbf{PWK}_e^\Box\mathrm{Ax}$ is adapted from definition 100 in a similar fashion.

- $\mathbf{B}_{\mathrm{e}}^{\Box}\mathrm{T}_e$ *is complete with respect to the class of reflexive frames;*
- $\mathbf{B}_{\mathrm{e}}^{\Box}4_e$ *is complete with respect to the class of transitive frames;*
- $\mathbf{B}_{\mathrm{e}}^{\Box}5_e$ *is complete with respect to the class of euclidean frames;*
- $\mathbf{B}_{\mathrm{e}}^{\Box}\mathrm{S4}_e$ *is complete with respect to the class of reflexive and transitive frames;*
- $\mathbf{B}_{\mathrm{e}}^{\Box}\mathrm{S5}_e$ *is complete with respect to the class of frames whose relation is an equivalence.*

Corollary 111. *The following hold:*

- $\mathbf{PWK}_{\mathrm{e}}^{\Box}\mathrm{T}$ *is complete with respect to the class of reflexive frames;*
- $\mathbf{PWK}_{\mathrm{e}}^{\Box}4$ *is complete with respect to the class of transitive frames;*
- $\mathbf{PWK}_{\mathrm{e}}^{\Box}5$ *is complete with respect to the class of euclidean frames;*
- $\mathbf{PWK}_{\mathrm{e}}^{\Box}\mathrm{S4}$ *is complete with respect to the class of reflexive and transitive frames;*
- $\mathbf{PWK}_{\mathrm{e}}^{\Box}\mathrm{S5}$ *is complete with respect to the class of frames whose relation is an equivalence.*

Notice that depending on the choice of the basic logic we obtain a different notion of completeness: in the case of $\mathbf{B}_{\mathrm{e}}^{\Box}$ the completeness is w.r.t. the logical consequence relation $\vDash_{\mathbf{B}_{\mathrm{e}}^{\Box}}$, in the case of $\mathbf{PWK}_{\mathrm{e}}^{\Box}$ the relation is $\vDash_{\mathbf{PWK}_{\mathrm{e}}^{\Box}}$.

The decidability of $\mathbf{B}_{\mathrm{e}}^{\Box}$ and $\mathbf{PWK}_{\mathrm{e}}^{\Box}$ established by Corollaries 85 and 106 immediately follows for their (finitely axiomatizable) axiomatic extensions.

Theorem 112. *For* $\mathrm{E} \in \{\mathrm{T}_e, 4_e, 5_e, \mathrm{S4}_e, \mathrm{S5}_e\}$*, the logic* $\mathbf{B}_{\mathrm{e}}^{\Box}\mathrm{E}$ *is decidable. For* $\mathrm{E} \in \{\mathrm{T}, 4, 5, \mathrm{S4}, \mathrm{S5}\}$*, the logic* $\mathbf{PWK}_{\mathrm{e}}^{\Box}\mathrm{E}$ *is decidable.*

Proof. We give the proof for $\mathbf{B}_{\mathrm{e}}^{\Box}\mathrm{T}_e$, the others cases employs the same strategy. Using the same definitions of set closed under subformulae and filtration from definitions 80 and 82, we have that lemma 83 still holds. Suppose that φ is satisfiable in a reflexive model M, therefore by proposition 107 T_e is valid in M. Let $\Gamma = \{\mathrm{T}_e, \varphi\}$, Σ its closure under subformulae, and consider the filtration M^Σ of M through Σ. By lemma 83, T_e is valid in M^Σ, hence by proposition 107 M^Σ is reflexive. Now we repeat theorem 84 using the filtration M^Σ to prove that if φ is satisfiable in reflexive model, it is satisfiable in a finite reflexive model of cardinality at most $2^{card(\Sigma)}$. We have the desired finite model property, which, together with the completeness theorem stated in Corollary 110, gives the decisability of $\mathbf{B}_{\text{e}}^{\Box}\mathrm{T}_e$. $\Box$

3.4 Modal Bochvar and Halldén algebras

In this section we investigate the algebraic counterparts of the modal external weak Kleene logics. We will make use of the in section 1.3 about Bochvar algebras and their structure theory. In particular, recall that any Bochvar algebra has a SIBSL-reduct, which is a Płonka sum of Boolean algebras, and that its underlying semilattice direct system has surjective and non-injective homomorphisms. Furthermore the lowest fibre of the semilattice contains isomorphic copies of all the fibres through the mapping induced by the external operator J_2.

The modal logics we are going to examine are not precisely the ones studied in sections 3.1 and 3.2, instead we will put our focus on the *global* versions of modal external Bochvar ($\mathbf{B}_{\text{e}}^{g\Box}$, definition 65) and modal external **PWK** ($\mathbf{PWK}_{\text{e}}^{g\Box}$, definition 91), which are obtained by switching the rule of necessitation with the stronger global necessitation:

$$\text{(GN)}\ \varphi \vdash \Box\varphi$$

Like in standard modal logic, these global versions are extensions of their local counterparts, and the global logic coincide with the local on theorems. The motivation behind this move will be clear in a moment, first we need to extend the completeness theorems proved for the local counterparts to the new global modal logics.

Theorem 113 (Completeness)**.** *(1)* $\Gamma \vdash_{\mathbf{B}_{\text{e}}^{g\Box}} \varphi$ *if and only if* $\Gamma \models^{g}_{\mathbf{B}_{\text{e}}^{\Box}} \varphi$*; (2)* $\Gamma \vdash_{\mathbf{PWK}_{\text{e}}^{g\Box}} \varphi$ *if and only if* $\Gamma \models^{g}_{\mathbf{PWK}_{\text{e}}^{\Box}} \varphi$*.*

Proof. The proof is an adaptation of the completeness theorem for the minimal global modal logic $\mathbf{K}^g$. (1) We modify the proof of theorem 79 in the following way. Given a certain $\mathbf{B}_{\text{e}}^{g\Box}$-theory Γ, change definition 77 of canonical model

$\mathfrak{M} = \langle \mathcal{W}, \mathcal{R}, v\rangle$ by putting the set of worlds as $\mathcal{W} = \{\Delta \supseteq \Gamma \mid \Delta$ is a maximal $\mathbf{B}_{\mathrm{e}}^{\Box}$-consistent theory$\}$. Observe that we are considering all *local* theories that extend Γ. We prove that the truth lemma 78 holds also for $\mathfrak{M}$. Observe that the truth lemma requires the existence lemma 76 to work, but that lemma can be easily adapted to the case of $\mathbf{B}_{\mathrm{e}}^{g\Box}$, in particular notice that since we are working with *local* theories as points of our canonical model, we can employ the deduction theorem for $\mathbf{B}_{\mathrm{e}}^{\Box}$, which otherwise wouldn't hold for its global counterpart. In the case of the global logic though we need to prove that the set s of lemma 76 - the set that extends s^- and witnesses the instantiation of the argument of a diamond formula - belongs to the new canonical model, that is $\Gamma \subseteq s$. Recall that s is obtained from $s^- = \{J_2\psi | \Box J_2\psi \in w\}$, with $w \in \mathcal{W}$. By definition of the canonical model, $\Gamma \subseteq W$; since Γ is a $\mathbf{B}_{\mathrm{e}}^{g\Box}$-theory, for all $\gamma \in \Gamma, J_2\gamma \in \Gamma$ (by $\varphi \vdash_{\mathbf{B}_{\mathrm{e}}^{\Box}} \Box\varphi$) and $\Box J_2\varphi \in w$ (by (GN)). Therefore, for all $\gamma \in \Gamma, \Box J_2\gamma \in w$, hence $J_2\gamma \in s^-$. Now we extend s^- to a maximal $\mathbf{B}_{\mathrm{e}}^{\Box}$-consistent set by Lindenbaum's model 74, which we can do, since we are extending it to a theory of the local logic.

Now that we have all the needed lemma, completeness follows. Let $\Gamma \nvdash_{\mathbf{B}_{\mathrm{e}}^{g\Box}} \varphi$, expand it to the least $\mathbf{B}_{\mathrm{e}}^{g\Box}$-theory $\Gamma' \supseteq \Gamma$, for which still $\varphi \notin \Gamma'$. Since $\mathbf{B}_{\mathrm{e}}^{g\Box}$ extends $\mathbf{B}_{\mathrm{e}}^{\Box}$, Γ is also a $\mathbf{B}_{\mathrm{e}}^{\Box}$-theory, which we can further expand by Lindenbaum's model 74 to a maximal consistent $\mathbf{B}_{\mathrm{e}}^{\Box}$-theory Δ. Consider the canonical model $\mathfrak{M} = \langle \mathcal{W}, \mathcal{R}, v\rangle$ for Γ': $\Delta \in \mathcal{W}$, furthermore for all $\Sigma \in \mathcal{W}, \Gamma \subseteq \Gamma' \subseteq \Sigma$, so $\mathfrak{M} \vDash_{\mathbf{B}_{\mathrm{e}}^{\Box}} \gamma$, for all $\gamma \in \Gamma$, but $\varphi \notin \Delta$, hence $\mathfrak{M} \nvDash_{\mathbf{B}_{\mathrm{e}}^{\Box}} \varphi$. We conclude that $\Gamma \nvDash^g_{\mathbf{B}_{\mathrm{e}}^{\Box}} \varphi$.

For claim (2) it is straightforward to adapt the above strategy to $\mathbf{PWK}_{\mathrm{e}}^{g\Box}$, using theorems 97, 99 and 101. $\Box$

With the completeness result it is now legitimate to employ Kripke semantics to check validity.

The decision to move to global modal logics is due to their better algebraic behaviour compared to their local versions, which tend to have a weak connection with any algebraic counterpart[4]. On the contrary, the stronger inferential power of global modal logics allow for a meaningful connection between the

[4]It is a well-known result that the local normal modal logics based on **S5** and weaker systems are not algebraizable, while their global versions are not only algebraizable but even implicative (see e.g. [Font, 2016], example 3.61).

logic and a class of algebras, in our case it will be in the sense of algebraizability, which will amount to the possibility to translate the logical consequence and the relative equational consequence of to a certain class of algebras into each other and without any loss of information.

To prove that $\mathbf{B}_{\mathrm{e}}^{g\Box}$ and $\mathbf{PWK}_{\mathrm{e}}^{g\Box}$ are algebraizable we will employ a purely syntactic characterization of algebrazability ([Font, 2016, theorem 3.21]), which is equivalent to the standard definition 1. This allow us to check the algebraizability of expansions of algebraziable logics directly through their proof theory, before knowing their equivalent algebraic semantics:

Proposition 114 ([Font, 2016], proposition 3.31)**.** *Let* $\mathbf{\Lambda}$ *is an algebraizable logic and* $\Delta(x, y)$ *one of its sets of equivalence formulae. An expansion* $\mathbf{\Lambda}'$ *of* $\mathbf{\Lambda}$ *that satisfies for each n-ary symbol* λ *in the expanded type of* $\mathbf{\Lambda}'$*:*

$$\bigcup_{1 \leq i \leq n} \Delta(x_i, y_i) \vdash_{\mathbf{\Lambda}'} \Delta(\lambda x_1, ...x_n, \lambda y_1, ...y_n)$$

is algebraizable, and its algebraizability is witnessed by the same transformers of $\mathbf{\Lambda}$*.*

Since $\mathbf{B}_{\mathrm{e}}^{g\Box}$ and $\mathbf{PWK}_{\mathrm{e}}^{g\Box}$ are expansions of algebraizable logics, we use the above proposition to prove their algebraizability.

Theorem 115. $\mathbf{B}_{\mathrm{e}}^{g\Box}$ *and* $\mathbf{PWK}_{\mathrm{e}}^{g\Box}$ *are algebraizable with the same transformers of, respectively,* $\mathbf{B}$ *and* $\mathbf{PWK}_{\mathrm{e}}$*.*

Proof. Consider $\mathbf{B}_{\mathrm{e}}^{g\Box}$. By proposition 114, to prove our claim we simply need to check whether $x \equiv y \vdash_{\mathbf{B}_{\mathrm{e}}^{g\Box}} \Box x \equiv \Box y$. By completeness theorem 113, we can prove semantically that $x \equiv y \vDash^{g}_{\mathbf{B}_{\mathrm{e}}^{\Box}} \Box x \equiv \Box y$. Take an arbitrary Bochvar-Kripke model $\mathfrak{M}$ s.t. for all $x \equiv y$ is valid. For arbitrary $w \in W$, we have three cases:

(i) If $v(w, \Box x) = 1/2$ then $v(w, x) = 1/2$, which by $v(w, x \equiv y) = 1$ yields $v(w, y) = 1/2$, so also $v(w, \Box y) = 1/2$. We conclude $v(w, \Box x \equiv \Box y) = 1$.

(ii) If $v(w, \Box x) = 1$, $v(w, x) \neq 1/2$ and then $v(w, y) \neq 1/2$ as well, so $v(w, \Box y) \neq 1/2$. Since $v(w, \Box x) = 1$, in all successors $w' \in R[w], v(w', x) = 1$, moreover we assumed $v(w', x \equiv y) = 1$, therefore

$v(w', y) = 1$, which implies $v(w, \Box y) = 1$. We conclude $v(w, \Box x \equiv \Box y) = 1$.

(iii) $v(w, \Box x) = 0$ implies $v(w, \Box y) \neq 1/2$, as in the previous point. Since $v(w, \Box x) = 0$, there is a successors $w' \in R[w]$ s.t. $v(w', x) \neq 1$, hence by the assumption $v(w', x \equiv y) = 1$ we also have $v(w', y) \neq 1$, which gives $v(w, \Box y) = 0$. We conclude $v(w, \Box x \equiv \Box y) = 1$.

$\equiv$ is symmetric, so we don't need to prove anything for y on the right-hand side. The case for $\mathbf{PWK}_{\mathrm{e}}^{g\Box}$ is completely similar, employing Halldén-Kripke models instead. □

In the above proof the global reading of the logical consequence played a key role, this motivates our shift towards global modal logics.

From a finitary algebraizable logic with finitary transformers it is possible to extract a quasiequational theory for its equivalent algebraic semantics using the algorithm provided by [Font, 2016, proposition 3.44]. Since our global modal logics inherited the finitary transformers of their non-modal counterparts, we can apply the mentioned algorithm and obtain the following quasivarieties.

Definition 116. The class of *modal Bochvar algebras* $\mathsf{MBCA_B}$, in the language $\mathscr{L}_J^{\Box}$, is the quasivariety axiomatized by the following:

- all the equations and quasiequations of the basis of BCA (definition 18);
- $J_2(\Box(J_2\varphi \to J_2\psi) \to (\Box J_2\varphi \to \Box J_2\psi)) \approx 1$;
- $J_2(+\varphi \leftrightarrow +\Box\varphi) \approx 1$;
- $J_2(J_2\Box\varphi \to \Box J_2\varphi) \approx 1$;
- $J_2(J_0\Box\varphi \to \neg\Box J_0\neg\varphi) \approx 1$;
- $J_2(J_2\varphi \leftrightarrow J_2\psi) \approx 1 \ \& \ J_2(J_0\varphi \leftrightarrow J_0\psi) \approx 1 \Rightarrow J_2(\varphi \equiv \psi) \approx 1$;
- $J_2\varphi \approx 1 \Rightarrow J_2(\varphi \leftrightarrow 1) \approx 1$;
- $J_2(\varphi \leftrightarrow 1) \approx 1 \Rightarrow J_2\varphi \approx 1$;
- $J_2\varphi \approx 1 \Rightarrow J_2\Box\varphi \approx 1$;

- $J_2(\varphi \equiv \psi) \approx 1 \Rightarrow \varphi \approx \psi$.

Since $\varphi \approx 1 \dashv\vDash_{\mathsf{BCA}} J_2\varphi \approx 1$ and deleting what already derives from the quasiequational theory of BCA, this axiomatization can be equivalently rewritten as:

(B_θ) all the equations and quasiequations of the basis of BCA;

(M1) $\Box(J_2\varphi \to J_2\psi) \preccurlyeq (\Box J_2\varphi \to \Box J_2\psi)$;

(M2) $+\varphi \approx +\Box\varphi$;

(MB3) $J_2\Box\varphi \preccurlyeq \Box J_2\varphi$;

(MB4) $J_0\Box\varphi \preccurlyeq \neg\Box J_0\neg\varphi$;

(τ-GN) $\Box 1 \approx 1$.

$\mathsf{MBCA_B}$ is the equivalent algebraic semantics of $\mathbf{B}_{\mathrm{e}}^{g\Box}$.

We have made use of the symbol $\preccurlyeq$ despite the structures we are considering have a bisemilattice reduct (fact guaranteed by the presence of all the quasiequational theory of BCA in the above axiomatization). This notation is not ambiguous though, since each fibre of the Płonka sum representation of said bisemilattice is a Boolean algebra and, as recalled in proposition 20, the J_i operators map every point in the algebra to the lowest fibre. Therefore as long as we work with external formulae α, β, an equation $\alpha \to \beta \approx 1$ can be rewritten as $\alpha \preccurlyeq \beta$ by standard properties of Boolean algebras, and similarly $\alpha \leftrightarrow \beta \approx 1$ becomes $\alpha \approx \beta$. We still need to prove that the $\Box$ operator is computed inside its origin fibre, otherwise the above remarks would be illegitimate. We are going to do that after the next definition, to which the previous comments apply.

Definition 117. The class of *modal Halldén algebras* $\mathsf{MBCA_H}$, in the language $\mathscr{L}_J^{\Box}$, is the quasivariety axiomatized by the following:

- all the equations and quasiequations of the basis of BCA;
- $\neg J_0(\Box(J_2\varphi \to J_2\psi) \to (\Box J_2\varphi \to \Box J_2\psi)) \approx 1$;
- $\neg J_0(+\varphi \leftrightarrow +\Box\varphi) \approx 1$;

- $\neg J_0(\Box\varphi \leftrightarrow \Box\neg J_0\varphi) \approx 1$;
- $\neg J_0(J_2\varphi \leftrightarrow J_2\psi) \approx 1 \;\&\; \neg J_0(J_0\varphi \leftrightarrow J_0\psi) \approx 1 \Rightarrow \neg J_0(\varphi \equiv \psi) \approx 1$;
- $\neg J_0\varphi \approx 1 \Rightarrow \neg J_0(\neg J_0\varphi \leftrightarrow 1) \approx 1$;
- $\neg J_0(\neg J_0\varphi \leftrightarrow 1) \approx 1 \Rightarrow \neg J_0\varphi \approx 1$;
- $\neg J_0\varphi \approx 1 \Rightarrow \neg J_0\Box\varphi \approx 1$;
- $J_2(\varphi \equiv \psi) \approx 1 \Rightarrow \varphi \approx \psi$.

Since for α external, $\mathsf{BCA} \vDash J_0\alpha \approx \neg\alpha$ and deleting what already derives from the quasiequational theory of BCA, this axiomatization can be equivalently rewritten as:

(B_θ) all the equations and quasiequations of the basis of BCA;

(M1) $\Box(J_2\varphi \rightarrow J_2\psi) \preccurlyeq (\Box J_2\varphi \rightarrow \Box J_2\psi)$;

(M2) $+\varphi \approx +\Box\varphi$;

(MP3) $\neg J_0(\Box\varphi \leftrightarrow \Box\neg J_0\varphi) \approx 1$;

(τ-GN) $\Box 1 \approx 1$.

$\mathsf{MBCA_H}$ is the equivalent algebraic semantics of $\mathbf{PWK}_{\mathrm{e}}^{g\Box}$.

Lemma 118. *Let* $\mathbf{A} \in \mathsf{MBCA_B}$ *or* $\mathbf{A} \in \mathsf{MBCA_H}$*. Consider the Płonka sum representation of the* IBSL*-reduct of* $\mathbf{A}$ *and let* $\mathbf{A}_i$ *be a fibre of the representation. If* $a \in A_i$ *then* $\Box a \in A_i$*.*

Proof. Assume either $\mathbf{A} \in \mathsf{MBCA_B}$ or $\mathbf{A} \in \mathsf{MBCA_H}$, and $a \in A_i$. Suppose $\Box a \in A_k$. Since both classes share the equation (M2), we have $\mathbf{A} \vDash +x \approx +\Box x$, hence $+a = +\Box a$. Since $+\varphi := \neg J_1\varphi$ and $\mathsf{IBSL} \vdash x \approx \neg\neg x$, we have $J_1 a = J_1\Box a$. By equations of BCA (which can all easily be checked on the algebra $\mathbf{WK}^e$ that generates the quasivariety) we have $J_1 a = \neg(J_2 a \vee J_2\neg a) = \neg J_2(a \vee \neg a) = \neg J_2 1_i$, where the second to last identity follows from [Bonzio and Pra Baldi, 2024, lemma 3.9]: if $a, b \in A_i$ then $J_2(a \vee b) = J_2 a \vee J_2 b$; while 1_i is the top element of the fibre A_i. At the same time $J_1\Box a = \neg(J_2\Box a \vee J_2\neg\Box a) = \neg J_2(\Box a \vee \neg\Box a) = \neg J_2 1_k$, where we used again the just mentioned lemma to justify the second to last identity. So $\neg J_2 1_i = \neg J_2 1_k$, therefore $J_2 1_i = J_2 1_k$, which allow us to conlcude $i = k$ by [Bonzio and Pra Baldi, 2024, theorem 3.15]. $\square$

The lemma states that the modal operator is computed fibre by fibre, therefore it was correct to claim that once an element is mapped to the lowest fibre by a J_i it remains there even when prefixed by $\Box$.

On its $\{J_2, \Box\}$-free reduct a modal Bochvar or Halldén algebra is an involutive bisemilattice and as such it can be represented as a Płonka sum of Boolean algebras. At this point we might hope the if we add the box and consider only the J_2-free reduct this would provide fibres that are modal algebras. Unfortunately this is not the case, but first we remind their definition.

Definition 119. A *Boolean algebra with operators* (BAO), which for simplicity we will call *modal algebra*, is an algebra $\langle \mathbf{A}, \Box \rangle$ s.t. $\mathbf{A}$ is a Boolean algebra and $\Box$ is a unary operation satisfying:

($K_\vee$) $\Box(x \vee \neg x) \approx x \vee \neg x$;

($K_\wedge$) $\Box(x \wedge y) \approx \Box x \wedge \Box y$.

Lemma 120. *($K_\wedge$) is valid in* $\mathsf{MBCA_B}$ *but fails in* $\mathsf{MBCA_H}$*; ($K_\vee$) is valid in* $\mathsf{MBCA_H}$ *but fails in* $\mathsf{MBCA_B}$*.*

Proof. We prove all claims by checking the corresponding translations of the equations on Kripke frames, using algebraizability. By (ALG2), $\mathsf{MBCA_B} \vDash$ ($K_\wedge$) iff $\vdash_{\mathbf{B}_e^{g\Box}} \Box(x \wedge y) \equiv (\Box x \wedge \Box y)$ iff, by completeness theorem 113, $\vDash^g_{\mathbf{B}_e^\Box} \Box(x \wedge y) \equiv \Box x \wedge \Box y$; it is now a straightforward exercise on Bochvar-Kripke models to prove the last statement. Similarly, $\mathsf{MBCA_H} \vDash$ ($K_\vee$) iff $\vdash_{\mathbf{PWK}_e^{g\Box}} \Box(x \vee \neg x) \equiv (x \vee \neg x)$ iff $\vDash^g_{\mathbf{PWK}_e^\Box} \Box(x \vee \neg x) \equiv (x \vee \neg x)$, and we check the last statement on Halldén-Kripke models.

For the invalid equations, by the same reasoning above $\mathsf{MBCA_B} \nvDash$ ($K_\vee$) iff $\nvDash^g_{\mathbf{B}_e^\Box} \Box(x \vee \neg x) \equiv (x \vee \neg x)$, which is witnessed by the following Bochvar-Kripke model: $W = \{w_1, w_2\}, Rw_1w_2, v(w_1, x) = 1$ and $v(w_2, x) = 1/2$. Similarly, $\mathsf{MBCA_H} \nvDash$ ($K_\wedge$) iff $\nvDash^g_{\mathbf{PWK}_e^{g\Box}} \Box(x \wedge y) \approx \Box x \wedge \Box y$, witnessed by the following Halldén-Kripke model: $W = \{w_1, w_2\}, Rw_1w_2, v(w_1, x) = v(w_1, y) = 1$, $v(w_2, x) = 0$ and $v(w_2, y) = 1/2$. $\square$

We can weaken the definition of modal algebra and produce two larger classes.

Definition 121. Let $\mathbf{B} = \langle \mathbf{A}, \Box \rangle$ be a structure comprising a Boolean algebra and a unary operation $\Box$. $\mathbf{B}$ is a *meet-modal algebra* ($\mathsf{BAO}_\wedge$) if it satisfies ($\mathrm{K}_\wedge$); it is a *join-modal algebra* ($\mathsf{BAO}_\vee$) if it satisfies ($\mathrm{K}_\vee$).

Clearly the intersection of $\mathsf{BAO}_\wedge$ and $\mathsf{BAO}_\wedge$ returns precisely all the modal algebras.

Using the fact that box is computed fibre by fibre (lemma 118) and the valid equations of lemma 120, we conclude:

Corollary 122. *For $\mathbf{A} \in \mathsf{MBCA}_\mathsf{B}$, each fibre of the Płonka sum representation of the IBSL-reduct of $\mathbf{A}$ is, considered in the J_2-free reduct of $\mathbf{A}$, a meet-modal algebra. For $\mathbf{A} \in \mathsf{MBCA}_\mathsf{H}$, each fibre of the Płonka sum representation of the IBSL-reduct of $\mathbf{A}$ is, considered in the J_2-free reduct of $\mathbf{A}$, a join-modal algebra.*

The structure of the fibres of a MBCA_H is actually richer.

Lemma 123. *Let $\mathbf{A} \in \mathsf{MBCA}_\mathsf{H}$, and $\mathbf{A}_i$ a fibre of the Płonka sum representation of its IBSL-reduct. $\langle \mathbf{A}_i, \Box \rangle \in \mathsf{BAO}$.*

Proof. We need to prove that for $a, b \in A_i$ it holds $\Box(a \wedge b) = \Box a \wedge \Box b$. We can express syntactically the fact that two elements belong to the same fibre, using the property of BCA that for indexes j, k, if $J_2 1_j = J_2 1_k$ then $j = k$ (proposition 20). Since $a, b \in A_i$, $a \vee \neg a = b \vee \neg b = 1_i$. Hence we need to prove that $\langle \mathbf{A}_i, \Box \rangle$ satisfies the quasiequation $J_2(x \vee \neg x) \approx J_2(y \vee \neg y) \Rightarrow \Box(x \wedge y) \approx \Box x \wedge \Box y$, that is $J_2(x \vee \neg x) \approx J_2(y \vee \neg y) \vDash_{\langle \mathbf{A}_i, \Box \rangle} \Box(x \wedge y) \approx \Box x \wedge \Box y$. By (ALG2) this amounts to $J_2(x \vee \neg x) \equiv J_2(y \vee \neg y) \vdash_{\mathbf{PWK}_\mathrm{e}^{g\Box}} \Box(x \wedge y) \equiv \Box x \wedge \Box y$, which by completeness is the same as $J_2(x \vee \neg x) \equiv J_2(y \vee \neg y) \vDash^g_{\mathbf{PWK}_\mathrm{e}^\Box} \Box(x \wedge y) \equiv \Box x \wedge \Box y$.

Let $\mathfrak{M} = \langle W, R, v \rangle$ be a Halldén-Kripke model s.t. $\mathfrak{M} \vDash J_2(x \vee \neg x) \equiv J_2(y \vee \neg y)$. Observe that the last statement means that at each world $J_2(x \vee \neg x)$ and $J_2(y \vee \neg y)$ have the same value: if $v(w, J_2(x \vee \neg x)) = 1$ then $v(w, x \vee \neg x) = 1$ and $v(w, x) \neq 1/2$, otherwise $v(w, x) = 1/2$. Therefore the assumption forces at each world x and y to be either both classical or both non-classical.

Now to prove the main claim, suppose for $w \in W, v(w, \Box(x \wedge y)) = 1$, so $v(w, x) \neq 1/2 \neq v(w, y)$. For all $w' \in R[w], v(w', x \wedge y) \neq 0$, and since by hypothesis $v(w', J_2(x \vee \neg x) \equiv J_2(y \vee \neg y)) = 1$, either $v(w', x) \neq 1/2 \neq v(w', y)$

or $v(w', x) = v(w', y) = 1/2$. In the former case, since $v(w, x \wedge y) \neq 0$ it must be $v(w', x) = v(w', y) = 1$. Hence in both cases $v(w', x) \neq 0 \neq v(w', y)$, so $v(w, \Box x) = v(w, \Box y) = 1$. We conclude $v(w, \Box x \wedge \Box y) = 1$. The opposite direction that assumes $v(w, \Box x \wedge \Box y) = 1$ holds in general and doesn't make use of the hypothesis.

Suppose $v(w, \Box x \wedge \Box y) = 0$, which holds if $v(w, x) \neq 1/2 \neq v(w, y)$ and either $v(w, \Box x) = 0$ or $v(w, \Box y) = 0$. Therefore it exists $w' \in R[w]$ where $v(w, x) = 0$ or $v(w, y) = 0$. By hypothesis $v(w', J_2(x \vee \neg x) \equiv J_2(y \vee \neg y)) = 1$, hence $v(w', x) \neq 1/2$ iff $v(w', y) \neq 1/2$, therefore $v(w, x \wedge y) = 0$. We conclude $v(w, \Box(x \wedge y)) = 0$. As before, the opposite direction doesn't make use of the hypothesis and it holds in general.

Finally, $v(w, \Box(x \wedge y)) = 1/2$ iff $(v(w, x) = 1/2$ or $v(w, y) = 1/2)$ iff $(v(w, \Box x) = 1/2$ or $v(w, \Box y) = 1/2)$ iff $v(w, \Box x \wedge \Box y) = 1/2$. $\Box$

The result of corollary 122 unfortunately does not translate into the possibility to extend the Płonka homomorphisms to the box operator: considering the Płonka sum representation of the IBSL-reduct of a modal Bochvar of Halldén algebra, we are not guaranteed that the homomorphisms of their semilattice direct system are such w.r.t. $\Box$. The algebras in figure 3.1 witness this failure (full arrows represent $\Box$, dashed arrows represent J_2, which on the bottom fibre is always the identity). In the figure, if we denote by p_{01} the Płonka (Boolean) homomorphism from the bottom fibre $\mathbf{A}_0$ to the upper one $\mathbf{A}_1$, in the left $\mathsf{MBCA_B}$ observe that $p_{01}(\Box 1) = p_{01}(1) = 1' \neq 0' = \Box 1' = \Box p_{01}(1)$. Similarly in the right $\mathsf{MBCA_H}$ we have $p_{01}(\Box a) = p_{01}(0) = 0' \neq 1' = \Box 1' = \Box p_{01}(a)$.

We can actually prove more.

Lemma 124. *There is no possible Płonka sum decomposition of* $\mathsf{MBCA_B}$ *s.t. its* J_2*-free reduct belongs entirely to* $\mathcal{R}(\mathsf{BAO}_\wedge)$*. Similarly, There is no possible Płonka sum decomposition of* $\mathsf{MBCA_H}$ *s.t. its* J_2*-free reduct belongs entirely to* $\mathcal{R}(\mathsf{BAO}_\vee)$ *nor* $\mathcal{R}(\mathsf{BAO})$*.*

Proof. Let $\mathsf{MBCA_B}^-$ be the J_2-free reduct of $\mathsf{MBCA_B}$. Suppose by contradiciton that $\mathsf{MBCA_B}^- \subseteq \mathcal{R}(\mathsf{BAO}_\wedge)$. Notice that $\mathsf{BAO}_\wedge$ is a strongly irregular variety, since by its Boolean base it holds $\mathsf{BAO}_\wedge \vDash x \wedge (x \vee y) \approx x$. [Romanowska, 1986, theorem 7.6] describes how to obtain from the axiomatization of a strongly irregular variety that of its regularization, in particular $\mathcal{R}(\mathsf{BAO}_\wedge) \vDash \Box x \wedge (\Box x \vee$

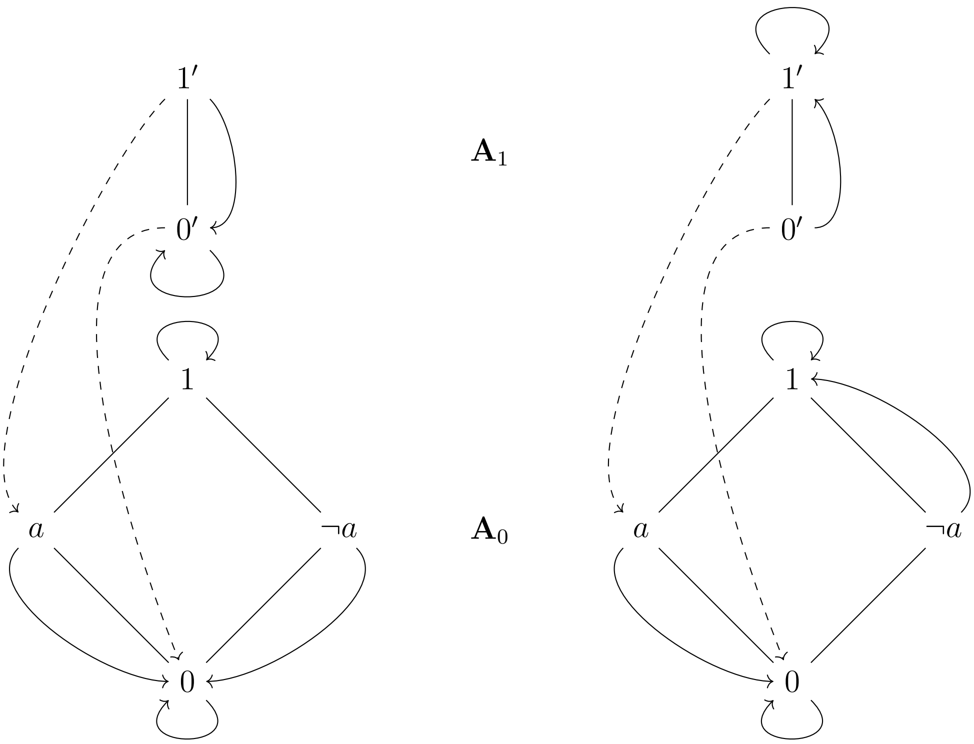


Figure 3.1: A modal Bochvar algebra (left) and a modal Halldén algebra (right).

$y) \approx \Box(x \wedge (x \vee y))$, so by our assumption also $\mathsf{MBCA_B} \vDash \Box x \wedge (\Box x \vee y) \approx \Box(x \wedge (x \vee y))$. By (ALG3) this amounts to $\vdash_{\mathbf{B}_e^{\Box}} (\Box x \wedge (\Box x \vee y)) \equiv \Box(x \wedge (x \vee y))$, which by completeness amounts to $\vDash_{\mathbf{B}_e^{\Box}} (\Box x \wedge (\Box x \vee y)) \equiv \Box(x \wedge (x \vee y))$. But this fails: consider the Bochvar-Kripke model with $W = w_1, w_2$ and $w_1 R w_2$ as only relation. Put $v(w_1, x) = v(w_1, y), v(w_2, x) = 1$, while $v(w_2, y) = 1/2$; it results $v(w_1, \Box x \wedge (\Box x \vee y)) = 1 \neq 0 = \Box(x \wedge (x \vee y))$.

The strategy for $\mathsf{MBCA_H}$ is the same. Suppose by contradiciton that the for the J_2-free reduct $\mathsf{MBCA_H}^- \subseteq \mathcal{R}(\mathsf{BAO_\vee})$. This amounts to $\vDash_{\mathbf{PWK}_e^{\Box}} (\Box x \wedge (\Box x \vee y)) \approx \Box(x \wedge (x \vee y))$. This fails: consider the Halldén-Kripke model with $W = w_1, w_2$ and $w_1 R w_2$ as only relation. Put $v(w_1, x) = v(w_1, y), v(w_2, y) = 1$, while $v(w_2, x) = 0$; it results $v(w_1, \Box x \wedge (\Box x \vee y)) = 0 \neq 1 = \Box(x \wedge (x \vee y))$. $\Box$

Corollary 125. *For* $\mathbf{A} \in \mathsf{MBCA_B}, \mathbf{A} \in \mathcal{R}(\mathsf{BAO_\wedge})$ *iff* $\mathbf{A} \vDash \Box x \wedge (\Box x \vee y) \approx \Box(x \wedge (x \vee y))$. *For* $\mathbf{A} \in \mathsf{MBCA_H}, \mathbf{A} \in \mathcal{R}(\mathsf{BAO_\vee})$ *iff* $\mathbf{A} \vDash \Box x \wedge (\Box x \vee y) \approx \Box(x \wedge (x \vee y))$.

Proof. By the proof of lemma 124, we know that if $\mathbf{A} \in \mathcal{R}(\mathsf{BAO_\wedge})$ then in particular $\mathbf{A} \models \Box x \wedge (\Box x \vee y) \approx \Box(x \wedge (x \vee y))$. For the other direction, let $\mathbf{A} \models \Box x \wedge (\Box x \vee y) \approx \Box(x \wedge (x \vee y))$. Let us abbreviate $x \cdot y := x \wedge (x \vee y)$. Observe that an axiomatization of $\mathsf{BAO}_\wedge$ with at most one irregular equation of the form $\varphi(x, y) \approx x$ is given by the equations of IBSL plus $x \cdot y \approx x$ (which delivers BA), plus the modal equation ($\mathrm{K}_\wedge$) $\Box(x \wedge y) \approx \Box x \wedge \Box y$, which is regular. This base meets the criteria to apply [Romanowska, 1986, theorem 7.6], which provides an axiomatization of $\mathcal{R}(\mathsf{BAO}_\wedge)$ composed by the base of IBSL and ($\mathrm{K}_\wedge$), plus:

(i) $x \cdot y \approx x$;

(ii) $(x \cdot y) \cdot z \approx x \cdot (y \cdot z)$;

(iii) $x \cdot (y \cdot z) \approx x \cdot (z \cdot y)$;

(iv) $\lambda(x_1, \dots x_n) \cdot y \approx \lambda(x_1 \cdot y, \dots x_n \cdot y)$;

(v) $y \cdot \lambda(x_1, \dots x_n) \approx y \cdot x_1 \cdot \dots \cdot x_n$;

with λ any n-ary operation of the type. In $\mathsf{MBCA_B}$ ($\mathrm{K}_\wedge$) holds and this class has a IBSL-reduct, therefore all equations are immediately satisfied except for (iv-v) in the case $\lambda = \Box$. (v) can be checked on Bochvar-Kripke models. (iv) is the critical equation, but we assumed it to be valid in the particular algebra $\mathbf{A}$. Therefore $\mathbf{A}$ satisfies all the equational base of $\mathcal{R}(\mathsf{BAO}_\wedge)$, hence it belongs to the class.

The identical reasoning applies for the second claim about $\mathsf{MBCA_H}$, with the exception that in the equational base for $\mathcal{R}(\mathsf{BAO}_\vee)$ we will have ($\mathrm{K}_\vee$) $\Box(x \vee \neg x) \approx x \vee \neg x$ instead of ($\mathrm{K}_\wedge$) (and both equations in the case of $\mathcal{R}(\mathsf{BAO})$, the claim about which follows immediately since $\mathsf{BAO} \subset \mathsf{BAO}_\vee$). $\Box$

Despite the fibres of the algebras described don't have the full structure of modal algebras, the bottom fibre is actually a BAO:

Lemma 126. *Let $\mathbf{A} \in \mathsf{MBCA_B}$ or $\mathbf{A} \in \mathsf{MBCA_H}$. If $\mathbf{A}_0$ is the bottom fibre of the Płonka sum representation of the IBSL-reduct of $\mathbf{A}$, $\langle \mathbf{A}_0, \Box \rangle \in \mathsf{BAO}$.*

Proof. Take $\mathbf{A} \in \mathsf{MBCA_B}$. By corollary 122, we already know that $\langle \mathbf{A}_0, \Box \rangle \in \mathsf{BAO}_\wedge$. To prove that ($\mathrm{K}_\vee$) holds, since we are also in a Boolean algebra,

$\langle \mathbf{A}_0, \Box\rangle \vDash x \vee \neg x \approx 1$. By (GN) $\langle \mathbf{A}_0, \Box\rangle \vDash \Box 1 \approx 1$, hence for all $a \in A_0, \Box(a \vee \neg a) = \Box(1) = 1 = a \vee \neg a$.

Now take $\mathbf{A} \in \mathsf{MBCA_H}$. Again by corollary 122, we have $\langle \mathbf{A}_0, \Box\rangle \in \mathsf{BAO}_\vee$. The validity of ($\mathrm{K}_\wedge$) amounts to prove that for $a, b \in A_0, \Box(a \wedge b) = \Box a \wedge \Box b$. In general in a BCA for an element a to belong to the bottom fibre is equivalent to $J_2 a = a$, therefore our claim can be rephrased as $J_2 x \approx x, J_2 y \approx y \vDash_{\mathsf{MBCA_H}} \Box(x \wedge y) \approx \Box x \wedge \Box y$. Since $\mathsf{MBCA_H}$ is algebraizable, by (ALG2) the previous holds iff $J_2 x \equiv x, J_2 y \equiv y \vdash_{\mathbf{PWK}_e^{g\Box}} \Box(x \wedge y) \equiv \Box x \wedge \Box y$ iff, by completeness theorem 113, $J_2 x \equiv x, J_2 y \equiv y \vDash^g_{\mathbf{PWK}_e^{\Box}} \Box(x \wedge y) \equiv \Box x \wedge \Box y$. Consider a Halldén-Kripke model $\langle W, R, v\rangle$ s.t. for all $w \in W, v(w, J_2 x \equiv x) = v(w, J_2 y \equiv y) = 1$ (the case $1/2$ is exluded since we are dealing with external formulae). Take $w \in W$ s.t. $v(w, \Box(x \wedge y)) = 1$, then $v(w, x \wedge y) \neq 1/2$ and for all $w' \in R[w], v(w', x \wedge y) \neq 0$, which holds iff either $v(w', x) = 1/2$ or $v(w', y) = 1/2$ or $v(w', x) = v(w', x) = 1$. The first two cases cannot be realized, since by assumption $v(w', J_2 x \equiv x) = v(w', J_2 y \equiv y) = 1$, that is x and y receive only classical values at w'. Therefore $v(w', x) = v(w', x) = 1$, which implies $v(w, \Box x) = v(w, \Box y) = 1$, hence $v(w, \Box x \wedge \Box y) = 1$. The opposite direction from the assumption $v(w, \Box x \wedge \Box y) = 1$ is similar and the cases for 0 are straightforward. $\Box$

Despite the lack of homorphism that respect $\Box$ between the fibres of the Płonka sum representation of a $\mathsf{MBCA_B}$, we can extend the isomorphism of proposition 20 to the modal operator. Remember that for a fibre $\mathbf{A}_i$, $1_i := 1^{\mathbf{A}_i}$.

Theorem 127. *Let $\mathbf{A} \in \mathsf{MBCA_B}$. If $\mathbf{A}_i$ is a fibre of the Płonka sum representation of the IBSL-reduct of $\mathbf{A}$, J_2 is a isomorphism of $\mathsf{BAO}_\wedge$ between $\langle \mathbf{A}_i, \Box\rangle$ and $\langle [\mathbf{0}, \mathbf{J_2 1_i}], \Box^*\rangle$, where $\Box^* x := \Box x \wedge J_2 1_i$.*

Proof. First we prove that that $\langle [0, J_2 1_i], \Box^*\rangle \in \mathsf{BAO}_\wedge$: $\Box^*(a \wedge b) = \Box(a \wedge b) \wedge J_2 1_i = \Box a \wedge \Box b \wedge J_2 1_i = \Box a \wedge J_2 1_i \wedge \Box b \wedge J_2 1_i = \Box^*(a) \wedge \Box^*(b)$, where all the identities hold because $\mathbf{A}_0$ is a BAO.

By proposition 20 we already know that J_2 is a Boolean isomorphism. For $\Box$ we need to prove $\mathsf{MBCA_B} \vDash J_2 \Box x \approx \Box^* J_2 x$, where $\Box^* J_2 x := \Box J_2 x \wedge J_2 1_i$ and 1_i can be expressed as $x \vee \neg x$, since any value we are going to assign to x belongs to $\mathbf{A}_i$ (we are working inside a Boolean algebra, we can express the local zero in the usual way). Therefore we need to prove $\mathsf{MBCA_B} \vDash J_2 \Box x \approx$

$\Box J_2 x \wedge J_2(x \vee \neg x)$, which by algebraizability plus Kripke completeness amounts to $\vDash_{\mathbf{B}_e^{\Box}} J_2 \Box x \equiv \Box J_2 x \wedge J_2(x \vee \neg x)$. Suppose for any Bochvar-Kripke model and $w \in W$, $v(w, J_2 \Box x) = 1$, so $v(w, \Box x) = 1$ and $v(w, x) \neq 1/2$; the latter yields $v(w, J_2(x \vee \neg x)) = 1$. For all $w' \in R[w], v(w', x) = 1$, then $v(w', J_2 x) = 1$, hence $v(w, \Box J_2 x) = 1$. The right-to-left reasoning is similar. Suppose now $v(w, J_2 \Box x) = 0$, then either $v(w, \Box x) = 1/2$ or $v(w, \Box x) = 0$. In the former case $v(w, x) = 1/2$, so $v(w, J_2(x \vee \neg x)) = 0$ and always $v(w, \Box J_2 x) \neq 1/2$, hence $v(w, \Box J_2 x \wedge J_2(x \vee \neg x)) = 0$. In the latter, $v(w, x) \neq 1/2$ and $v(w, J_2(x \vee \neg x)) = 1$, moreover there is $w' \in R[w]$ s.t. $v(w', x) \neq 1$, then $v(w', J_2 x) \neq 1$, hence $v(w, \Box J_2 x) = 0$. The right-to-left reasoning is similar. Finally, if $v(w, x) = 1/2$ the formula is satisfied, since both of its sides are external formulae. $\Box$

As in the case for BCA, the above theorem states that bottom fibre of a $\mathsf{MBCA_B}$ stores all the information of the upper fibres, in the form of isomoprhic copies. The isomorphism doesn't extend to the case for $\mathsf{MBCA_H}$, as the right algebra in figure 3.1 shows.

3.5 Modal Bochvar algebras as twist products

So far we have provided two complete semantics for the global modal logics $\mathbf{B}_e^{g\Box}$ and $\mathbf{PWK}_e^{g\Box}$: a Kripke-style one and an algebraic one. On the side of the algebraic semantics, we have seen how the case for the counterpart of $\mathbf{B}_e^{g\Box}$, the class of modal Bochvar algebras $\mathsf{MBCA_B}$, is amenable to a Płonka-style representation of its IBSL-reduct, which, although this representation doesn't properly include the $\Box$ operator because it violates the homomorphic condtion of a semilattice direct system (lemma 124), at the same time we can recover the isomorphism theorem of non-modal BCA in the form of theorem 127. Therefore in the $\mathsf{MBCA_B}$ we have once more that the bottom fibre encodes all the information of all the fibres of the algebras[5].

[5]We will not investigate the case of modal Halldén algebras, due to the failure of a suitable modal isomorphism between the upper fibres and the bottom one. As we have seen in the example provided by figure 3.1, there are cases in which the Boolean isomorphism induced by J_2 might not be extendable to $\Box$. While this is not a limitative result concerning the applicability of the twist product representation, it seems to point against such inquiry, since the product construction for BCA relied strongly on the part of their structure theory that is lost in the passage to $\mathsf{MBCA_H}$.

This feature preserved by the modal setting suggests that modal Bochvar algebras could be representable as special twist products, as we did for the non-modal case in chapter 2. We need to adapt the notion of Bochvar system.

Definition 128. A *modal Bochvar system* is a pair $\mathbb{B} = \langle \mathbf{B}, \mathbf{S} \rangle$ such that $\mathbf{B} = \langle B, \vee, \wedge, \vee, \neg, \Box, 0, 1 \rangle$ is a modal algebra and $\mathbf{S} = \langle S, \wedge, 1 \rangle$ is a meet-subsemilattice with unit of $\mathbf{B}$.

The first component $\mathbf{B}$ of the modal Bochvar system is the one encoding all the information of the algebra that we are aiming to represent, which in light of theorem 127 is expected to be a modal algebra. Concerning the second component $\mathbf{S}$, in chapter 2 it was a subreduct of the first element whose task was to contain the elements that could uniquely determine the fibres of the Płonka sum representation, namely the dense elements, i.e. the local tops of the fibres (represented in $\mathbf{S}$ as their images through J_2 in the bottom fibre). We want $\mathbf{S}$ to play the same role here.

Notice that $\mathbf{S}$ it is still a just semilattice, we didn't add $\Box$ to its type. The motivation is entirely explained by corollary 122: a fibre $\mathbf{A}_i$ of the IBSL-reduct of a $\mathsf{MBCA_B}$ when expanded with box is, in general, not a modal algebra, instead $\langle \mathbf{A}_i, \Box \rangle \in \mathsf{BAO}_\wedge$, because local ones are not required to be fixpoints for $\Box$ (lemma 120: $\mathsf{MBCA_B} \not\models \Box(x \vee \neg x) \approx x \vee \neg x$). Moreover box is computed inside the fibre of its argument (lemma 118), hence in general we are not guaranteed that the application of box to a dense element returns a dense element (figure 3.1 is an example of this failure), hence we should not require for S to be closed under $\Box$.

We proceed to define the twist product of a modal Bochvar system.

Definition 129. Let $\mathbb{B} = \langle \mathbf{B}, \mathbf{S} \rangle$ be a modal Bochvar system. The variant twist product of $\mathbb{B}$ is the algebra $\mathcal{P}_v(\mathbb{B}) = \langle T(B, S), \vee, \wedge, \neg, \Box, J_2, 0, 1 \rangle$ of type $\langle 2, 2, 1, 1, 1, 0, 0 \rangle$. The structure is defined as per definition 31, with the addition of the modal operator $\Box$:

$$\Box(x_1, x_2) = (\Box x_1 \wedge (x_1 \vee \neg x_2), \Box x_1 \vee (\neg x_1 \wedge x_2)). \tag{3.1}$$

The notions of sharp, dense and dually dense element are unchanged from definition 34. We extend lemma 32 and show that sharp elements are much easier than the others to manipulate.

Lemma 130. *Let $\mathbb{B} = \langle \mathbf{B}, \mathbf{S} \rangle$ be a modal Bochvar system. For all $(x, y) \in T(B, S)$ such that $x = y$ the following equality holds:*

$$\Box(x, x) = (\Box x, \Box x).$$

Proof. Let $(x, y) \in T(B, S)$ be such that $x = y$. We compute the box operator as defined in equation 3.1:

$$\begin{aligned} \Box(x, x) &= (\Box x \wedge (x \vee \neg x), \Box x_{\vee}(\neg x \wedge x)) \\ &= (\Box x \wedge 1, \Box x \vee 0) \\ &= (\Box x, \Box x). \end{aligned}$$

□

In the following we show the correspondence between modal Bochvar algebras and products of modal Bochvar systems.

Theorem 131. *Let $\mathbb{B} = \langle \mathbf{B}, \mathbf{S} \rangle$ be a modal Bochvar system. The structure $\mathcal{P}_v(\mathbb{B})$ forms a Modal Bochvar algebra.*

Proof. We first show that the operation defined in equation 3.1 is well-defined. Let $(x_1, x_2) \in T(B, S)$, and apply the operation $\Box$:

$$\Box(x_1, x_2) = (\Box x_1 \wedge (x_1 \vee \neg x_2),\ \Box x_1 \vee (\neg x_1 \wedge x_2)).$$

It is immediate to observe that:

$$\Box x_1 \wedge (x_1 \vee \neg x_2)\ \leq\ \Box x_1 \vee (\neg x_1 \wedge x_2).$$

Concerning the second condition of the elements of $T(B, S)$, consider:

$$(\Box x_1 \wedge (x_1 \vee \neg x_2)) \vee \neg(\Box x_1 \vee (\neg x_1 \wedge x_2)).$$

We compute:

$$
\begin{aligned}
(\Box x_1 \wedge (x_1 \vee \neg x_2)) \vee \neg(\Box x_1 \vee (\neg x_1 \wedge x_2)) &= (\Box x_1 \wedge (x_1 \vee \neg x_2)) \vee (\neg\Box x_1 \wedge \neg(\neg x_1 \wedge x_2)) \\
&= (\Box x_1 \wedge (x_1 \vee \neg x_2)) \vee (\neg\Box x_1 \wedge (x_1 \vee \neg x_2)) \\
&= (\Box x_1 \vee \neg\Box x_1) \wedge (x_1 \vee \neg x_2) \\
&= 1 \wedge (x_1 \vee \neg x_2) \\
&= x_1 \vee \neg x_2.
\end{aligned}
$$

Since $x_1 \vee \neg x_2 \in S$, the claim follows.

Next we check he quasiequational theory of $\mathsf{MBCA_B}$ (definition 116):

- $(M1)$ $\Box(J_2\varphi \to J_2\psi) \leq \Box J_2\varphi \to \Box J_2\psi$

$$
\begin{aligned}
\Box(J_2(x_1, x_2) \to J_2(y_1, y_2)) &= \Box(\neg J_2(x_1, x_2) \vee J_2(y_1, y_2)) \\
&= \Box(\neg(x_1, x_1) \vee (y_1, y_1)) \\
&= \Box((\neg x_1, \neg x_1) \vee (y_1, y_1)) \\
&= \Box(\neg x_1 \vee y_1, \neg x_1 \vee y_1) \\
&= (\Box(\neg x_1 \vee y_1), \Box(\neg x_1 \vee y_1)) \\
&\leq (\neg\Box x_1 \vee \Box y_1, \neg\Box x_1 \vee \Box y_1) \\
&= (\neg\Box x_1, \neg\Box x_1) \vee (\Box y_1, \Box y_1) \\
&= \neg(\Box x_1, \Box x_1) \vee (\Box y_1, \Box y_1) \\
&= \neg\Box(x_1, x_1) \vee \Box(y_1, y_1) \\
&= \neg\Box J_2(x_1, x_2) \vee \Box J_2(y_1, y_2) \\
&= \Box J_2(x_1, x_2) \to \Box J_2(y_1, y_2)
\end{aligned}
$$

The inequality line is justified by the fact that sharp elements belong to the bottom fibre of the Płonka sum representation (this still holds for the modal setting since fibres are closed under box), which is a modal algebra.

- $(M2)$ $+\varphi \approx +\Box\varphi$

$$
\begin{aligned}
+(x_1, x_2) &= J_2(x_1, x_2) \vee J_2\neg(x_1, x_2) \\
&= (x_1, x_1) \vee J_2(\neg x_2, \neg x_1) \\
&= (x_1, x_1) \vee (\neg x_2, \neg x_2) \\
&= (x_1 \vee \neg x_2, x_1 \vee \neg x_2)
\end{aligned}
$$

and

$$
\begin{aligned}
+\Box(x_1, x_2) &= +(\Box x_1 \wedge (x_1 \vee \neg x_2),\ \Box x_1 \vee (\neg x_1 \wedge x_2)) \\
&= J_2(\Box x_1 \wedge (x_1 \vee \neg x_2),\ \Box x_1 \vee (\neg x_1 \wedge x_2)) \\
&\vee J_2\neg(\Box x_1 \wedge (x_1 \vee \neg x_2),\ \Box x_1 \vee (\neg x_1 \wedge x_2)) \\
&= (\Box x_1 \wedge (x_1 \vee \neg x_2),\ \Box x_1 \wedge (x_1 \vee \neg x_2)) \\
&\vee J_2(\neg\Box x_1 \wedge \neg(\neg x_1 \wedge x_2),\ \neg\Box x_1 \vee \neg(x_1 \vee \neg x_2)) \\
&= (\Box x_1 \wedge (x_1 \vee \neg x_2),\ \Box x_1 \wedge (x_1 \vee \neg x_2)) \\
&\vee (\neg\Box x_1 \wedge \neg(\neg x_1 \wedge x_2),\ \neg\Box x_1 \wedge \neg(\neg x_1 \wedge x_2)) \\
&= ((\Box x_1 \wedge (x_1 \vee \neg x_2)) \vee (\neg\Box x_1 \wedge \neg(\neg x_1 \wedge x_2)), \\
&(\Box x_1 \wedge (x_1 \vee \neg x_2)) \vee (\neg\Box x_1 \wedge \neg(\neg x_1 \wedge x_2))) \\
&= ((\Box x_1 \wedge (x_1 \vee \neg x_2)) \vee (\neg\Box x_1 \wedge (x_1 \vee \neg x_2)), \\
&(\Box x_1 \wedge (x_1 \vee \neg x_2)) \vee (\neg\Box x_1 \wedge (x_1 \vee \neg x_2))) \\
&= ((\Box x_1 \vee \neg\Box x_1) \wedge (x_1 \vee \neg x_2), (\Box x_1 \vee \neg\Box x_1) \wedge (x_1 \vee \neg x_2)) \\
&= (x_1 \vee \neg x_2, x_1 \vee \neg x_2)
\end{aligned}
$$

- $(MB3)$ $J_2\Box\varphi \leq \Box J_2\varphi$

$$
\begin{aligned}
J_2\Box(x_1, x_2) &= (\Box x_1 \wedge (x_1 \vee \neg x_2), \Box x_1 \wedge (x_1 \vee \neg x_2)) \\
&\leq (\Box x_1, \Box x_1) \\
&= \Box(x_1, x_1) \\
&= \Box J_2(x_1, x_2)
\end{aligned}
$$

- $(MB4)$ $J_0\Box\varphi \leq \neg\Box J_0\neg\varphi$

$$
\begin{aligned}
J_0\Box(x_1,x_2) &= J_2\neg\Box(x_1,x_2)\\
&= J_2\neg(\Box x_1 \wedge (x_1 \vee \neg x_2), \Box x_1 \vee (\neg x_1 \wedge x_2))\\
&= J_2(\neg\Box x_1 \wedge (x_1 \vee \neg x_2), \neg\Box x_1 \vee (\neg x_1 \wedge x_2))\\
&= (\neg\Box x_1 \wedge (x_1 \vee \neg x_2), \neg\Box x_1 \wedge (x_1 \vee \neg x_2))\\
&\leq (\neg\Box x_1, \neg\Box x_1)\\
&= \neg(\Box x_1, \Box x_1)\\
&= \neg\Box(x_1, x_1)\\
&= \neg\Box J_2(x_1,x_2)\\
&= \neg\Box J_2\neg\neg(x_1,x_2)\\
&= \neg\Box J_0\neg(x_1,x_2)
\end{aligned}
$$

- $(\tau - GN)$ $\Box 1 \approx 1$

$$
\begin{aligned}
\Box(1,1) &= (\Box 1 \wedge (1 \vee \neg 1), \Box 1 \vee (\neg 1 \wedge 1))\\
&= (\Box 1, \Box 1)\\
&= (1,1)
\end{aligned}
$$

□

Theorem 132. *Every modal Bochvar algebra is isomorphic to a variant twist product of the form $\mathcal{P}_v(\mathbb{B})$, for some modal Bochvar system $\mathbb{B}$. Moreover, $\mathbb{B}$ is uniquely determined up to isomorphism.*

Proof. The strategy of the proof is an extension of that of theorem 48. Let $\mathbf{A} \in \mathsf{MBCA_B}$ and let us fix $\mathbf{B} := J_2[\mathbf{A}]$, which by lemma 126 is a BAO. Theorem 46 still holds, so put $\mathbf{S} := \langle J_2[D(A)], \wedge, 1\rangle$ is a meet-subsemilattice with unit of $\mathbf{B}$. $\mathbb{B} = \langle \mathbf{B}, \mathbf{S}\rangle$ is a modal Bochvar system.

We know by theorem 48 that the mapping $\varphi : A \to T(B,S)$ defined as $\varphi(x) = (J_2x, \neg J_0x)$ is an isomorphism of BCA. To prove the main claim it is enough to check that this is a homomorphism for $\Box$ as well:

$$\begin{aligned}
\varphi(\Box x) &= (J_2\Box x, \neg J_0\Box x) \\
&= (\Box J_2 x \wedge J_2(x \vee \neg x), \neg J_2\neg\Box x) \\
&= (\Box J_2 x \wedge J_2(x \vee \neg x), \neg(\neg(\Box J_2 x) \wedge J_2(x \vee \neg x))) \\
&= (\Box J_2 x \wedge J_2(x \vee \neg x), \Box J_2 x \vee \neg J_2(x \vee \neg x)) \\
&= (\Box J_2 x \wedge (J_2 x \vee J_2\neg x), \Box J_2 x \vee \neg(J_2 x \vee J_2\neg x)) \\
&= (\Box J_2 x \wedge (J_2 x \vee J_0 x), \Box J_2 x \vee (\neg J_2 x \wedge \neg J_2\neg x)) \\
&= (\Box J_2 x \wedge (J_2 x \vee J_0 x), \Box J_2 x \vee (\neg J_2 x \wedge \neg J_0 x)) \\
&= \Box(J_2 x, \neg J_0 x) \\
&= \Box\varphi(x).
\end{aligned}$$

To obtain the first member of the second line we have applied the fact that J_2 is a homomorphism from any fibre to an interval algebra, where $\Box$ is defined as truncated ($\Box^* x := \Box x \wedge J_2(x \vee \neg x)$, theorem 127). This is the same as the validity of the equation $J_2\Box x \approx \Box J_2 x \wedge J_2(x \vee \neg x)$, which by algebrizability and completeness can be easily checked on Bochvar-Kripke frames. The same happens on the second member of the third line, and for negation on the same member of the fourth line. $\Box$

As a concluding result of the chapter, we extend the catergorical equivalence of theorem 56 to the modal case. We introduce the categories:

- $\mathscr{B}^{\Box}$ is the category of modal Bochvar algebras, whose objects are $\mathsf{MBCA_B}$ and whose morphisms are the corresponding homomorphisms;
- $\mathscr{S}^{\Box}$ is the category of Bochvar systems, whose objects are Bochvar systems and whose morphisms are homomorphisms of BAO that preserve the meet-semilattice structure, i.e. if $h : \langle \mathbf{B}_1, \mathbf{S}_1 \rangle \to \langle \mathbf{B}_2, \mathbf{S}_2 \rangle$ then $f[S_1] \subseteq S_2$.

To not complicate reading any further, we employ a slight abuse of notation and name the functors between these categories the same name as for the non-modal case, since their actions are defined identically:

- $E : \mathscr{B}^\Box \to \mathscr{S}^\Box$ is defined as:

  $E(\mathbf{A}) := \langle J_2[\mathbf{A}], J_2[D(\mathbf{A})]\rangle$, where $J_2[D(\mathbf{A})] := \langle J_2[D(A)], \wedge, 1\rangle$;
  if $f : \mathbf{A} \to \mathbf{B}$ is a morphism in $\mathscr{B}$, then $E(h) := f \restriction_{A_0}$, with $\mathbf{A}_0$ the bottom fibre of the IBSL-reduct of $\mathbf{A}$.

- $P : \mathscr{S}^\Box \to \mathscr{B}^\Box$ is defined as:

  $P(\mathbb{B}) := \mathcal{P}_v(\mathbb{B})$;
  if $g : \mathbb{B}_1 \to \mathbb{B}_2$ is a morphism in $\mathscr{S}$, then $P(g)(x, y) = (g(x), g(y))$.

Lemma 133. *E and P are functors over the respective categories.*

Proof. The proof runs essentially as in lemma 55. To prove that $E : \mathscr{B}^\Box \to \mathscr{S}^\Box$ is a functor the only differences are that the well-definedness on objects is obtained from theorem 132; for morphisms, take f homorphism of $\mathsf{MBCA_B}$, the fact that $E(f)$ is a homorphism of BAO is a consequence of the fact that f is a homorphism of $\mathsf{MBCA_B}$, hence its restriction $E(f)$ is a Boolean homorphism that is compatible with $\Box$, hence of BAO. The preservation of the semilattice structure follows the same argument of the non-modal case.

For $P : \mathscr{S}^\Box \to \mathscr{B}^\Box$, that it is well-defined on objects comes from theorem 131. For morphisms, the only integration needed is to prove that P preserves $\Box$:

$$\begin{aligned} P(g)(\Box(x_1, x_2)) &= P(g)((\Box x_1 \wedge (x_1 \vee \neg x_2), \Box x_1 \vee (\neg x_1 \wedge x_2))) \\ &= (g(\Box x_1 \wedge (x_1 \vee \neg x_2)), g(\Box x_1 \vee (\neg x_1 \wedge x_2))) \\ &= (\Box g(x_1) \wedge (g(x_1) \vee \neg g(x_2))), \Box g(x_1) \vee (\neg g(x_1) \wedge g(x_2)) \\ &= \Box(g(x_1), g(x_2)) \\ &= \Box P(g)(x_1, x_2). \end{aligned}$$

□

The above lemma allow us to straightforwardly adapt the proof of theorem 56 to the following.

Theorem 134. *$\mathscr{B}^\Box$ and $\mathscr{S}^\Box$ are equivalent categories.*

3.6 Conclusions

Through these pages I have explored weak Kleene logics under many different aspects, working in a semantical approach. On the linguistic side, we have started from the basic propositional language in which weak Kleene is traditionally formulated, then considered the expansion with external operators, and later we introduced modalities. In particular the addition of external operators greatly enhances the expressivity of the language, thus allowing to consider a proper algebraic counterpart of these external weak Kleene logics through the notion of algebraziability.

On the technical side, we have considered Płonka sums and introduced variant twist products. While Płonka sums are at this point a consolidated technique in the study of variable inclusion logics, the motivation behind twist products is different and concerned the possibility to find a certain core of the algebraic structures in question that encodes the information of the entire algebra. In our case this was the set of sharp elements together with the subsemilattice of (images of) dense elements. This correspond to [Bonzio and Pra Baldi, 2024]'s isomorphism theorem 21 that states the presence of isomorphic copies of the fibres of the Płonka sum decomposition of a Bochvar algebra within its bottom fibre, or, from again another perspective, to the equivalence between BCA and Bochvar systems first noted by [Bonzio et al., 2024] and here proved working with the product construction.

When modalities entered the scene, we have furthermore considered Kripke-style semantics for modal weak Kleene logics, which returned semantically different box operators and consequently different classes of relational models. The corresponding modal logics offered many different options, not only due to the possibility to strengthen them with formulae corresponding to standard frame conditions, but most importantly for the consideration of a global reading of the logical consequence instead of the usual local one. Global modal logics tend to have more interesting algebraic properties, as it was the case here with the algebraizability of global modal external weak Kleene logics. The introduction of the modal operator $\Box$ has revealed to be a non-trivial addition in the light of its interactions, different according to the base non-modal logic, with the external operator J_2.

With the introduction of the two different algebraic semantics for these global modal weak Kleene logics, an open question is the possible direct relation between weak three-valued Kripke models and modal Bochvar and modal Halldén in the form of a Jónsson-Tarski-style duality, similar to that for classical normal modal logic which is able to extract a modal algebra from a Kripke frame and viceversa. Another question concerns specifically $\mathsf{MBCA_H}$ and it is whether it could be possible to find some sort of representation method of the upper fibres of the Płonka sum representation, when equipped with $\Box$ and considered as modal algebras, in terms of the information stored inside the lowest fibre. As we have seen the isomorphic copies that are still present in $\mathsf{MBCA_B}$ are no longer available in the case of the algebras for $\mathbf{PWK}_{\mathrm{e}}^{g\Box}$, yet maybe some other kind of substructures or subreducts of the bottom fibre might encode the information needed to rebuild the full algebra.

The conncetion between logic and algebra had been maintained by algebraizability, which justified in the final part our exclusive focus on global modal logics. The study of local modal systems though had been functional to provide a Kripke semantics for modal external weak Kleene, giving a helpful semantic tool that lead to the individuation of the algebraic counterparts of the global modal logics, moreover it is a framework that might shed more light on the new box operators than their behaviour at the algebraic level. It is likely that the local modal logics we have studied would result to be the logics of order of the classes of modal algebras we have introduced, as it happens with standard modal logics, but we haven't explored the algebraic counterpart of local modal Kleene logics so far.

A more promising line of inquiry is the extent to which the variant twist product construction can be applied to obtain interesting result. While here we employed twist products to give a representation of Bochvar algebras alternative to Płonka sums and with its own virtues, in [Paoli et al., 202x] we proved the relation between Bochvar algebras and regular double Stone algebras passing precisely through the product representation of them. The seemingly best candidates for a study via twist products are algebras which have a representation in some form of Bochvar-style systems, i.e. those algebras containing some substructures storing the entire information of the algebra. A general theory of algebras exhibiting this feature might be worth studying.

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