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A NONLINEAR ATTRACTION-REPULSION KELLER–SEGEL MODEL WITH DOUBLE SUBLINEAR ABSORPTIONS: CRITERIA TOWARD BOUNDEDNESS

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Abstract. This paper deals with the zero-flux attraction-repulsion chemotaxis model

$$\begin{cases} u_t = \nabla \cdot \big((u+1)^{m_1-1}\nabla u - \chi u(u+1)^{m_2-1}\nabla v & \text{in } \Omega \times (0, T_{\max}), \\ \qquad\quad +\xi u(u+1)^{m_3-1}\nabla w\big) + h(u) & \\ v_t = \Delta v - f(u)v & \text{in } \Omega \times (0, T_{\max}), \\ w_t = \Delta w - g(u)w & \text{in } \Omega \times (0, T_{\max}), \end{cases} \tag{$\diamond$}$$

in the unknown $(u, v, w) = (u(x, t), v(x, t), w(x, t))$. Here, $x \in \Omega$, a bounded and smooth domain of $\mathbb{R}^n$ $(n \geq 1)$, $t, \chi, \xi > 0$, $m_1, m_2, m_3 \in \mathbb{R}$, and $f(u), g(u)$ and $h(u)$ sufficiently regular functions generalizing the prototypes $f(u) = K_1 u^\alpha$, $g(u) = K_2 u^\gamma$ and $h(u) = ku - \mu u^\beta$, with $K_1, K_2, \mu > 0$, $k \in \mathbb{R}$, $\beta > 1$ and suitable $\alpha, \gamma > 0$. Besides, further regular initial data $u(x, 0) = u_0(x), v(x, 0) = v_0(x), w(x, 0) = w_0(x) \geq 0$ are given, whereas $T_{\max} \in (0, \infty]$ stands for the maximal instant of time up to which solutions to the system exist. We will derive relations between the parameters involved in $(\diamond)$ capable to warrant that u, v, w are global and uniformly bounded in time. The article generalizes and extends to the case of nonlinear effects and logistic perturbations some results recently developed in [3] where, for the linear counterpart and in the absence of logistics, criteria towards boundedness are established.

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1. **Introduction and motivations.**

1.1. **General overview and state of the art.**

The pioneering papers by Keller and Segel ([9, 10, 11]), proposed in the 70's to model the biological phenomena concerning chemotaxis mechanisms, continue to inspire many researchers in the field towards the consideration of several related variants.

In this regard, in the article we dedicate ourselves to a specific problem intimately connected to the forthcoming coupled system of partial differential equations:

$$\begin{cases} u_t = \nabla \cdot (D(u)\nabla u - S(u)\nabla v + T(u)\nabla w) + h(u) & \text{in } \Omega \times (0, T_{\max}),\\ \tau v_t = \Delta v - av + k(u,v) & \text{in } \Omega \times (0, T_{\max}),\\ \tau w_t = \Delta w - bw + l(u,w) & \text{in } \Omega \times (0, T_{\max}),\\ u_\nu = v_\nu = w_\nu = 0 & \text{on } \partial\Omega \times (0, T_{\max}),\\ u(x,0) = u_0(x),\ \tau v(x,0) = \tau v_0(x),\ \tau w(x,0) = \tau w_0(x) & x \in \bar{\Omega}. \end{cases} \tag{1.1}$$

Herein Ω is a bounded open domain of $\mathbb{R}^n$ with $n \in \mathbb{N}$ and smooth boundary $\partial\Omega$, $\tau \in \{0,1\}$, $a, b > 0$, $D = D(\zeta), S = S(\zeta), T = T(\zeta), h = h(\zeta), k = k(\zeta, \eta)$ and $l = l(\zeta, \rho)$ are functions of their arguments $\zeta, \eta \geq 0$ and $\rho \geq 0$ with a certain regularity and proper behavior. Moreover, further regular initial data $u_0(x) \geq 0$, $\tau v_0(x) \geq 0$ and $\tau w_0(x) \geq 0$ are given as well, the subscript ν in $(\cdot)_\nu$ indicates the outward normal derivative on $\partial\Omega$, whereas $T_{\max} \in (0, \infty]$ the maximal instant of time up to which solutions to the system exist.

In the framework of real biological phenomena, it is commonly used to indicate with $u = u(x,t)$, $v = v(x,t)$ and $w = w(x,t)$, respectively, the nonnegative value of a certain cell distribution (populations, organisms), of the concentration of a chemoattractant (i.e. a chemical signal whose effect is attracting the cells to each other) and of a chemorepellent density (i.e. a chemical signal with exactly the opposite effect of the chemoattractant); naturally the couple (x,t) specifies the position and the instant of time where such values are considered. In this way, it is quite natural to interpret system (1.1) as a model describing the motion of the cells, inside an insulated domain (zero-flux on the border: homogeneous Neumann boundary conditions), under the flux effect $D(u)\nabla u - S(u)\nabla v + T(u)\nabla w$, a combination of diffusive ($D(u)$), attractive ($S(u)$) and repulsive ($T(u)$) impacts, and the external action of a source $h(u)$. As expected, the effect of such impacts is intimately connected to the expression of the diffusion, the attraction and the repulsion, while the source may increment, decrement or both the cells' density. Further, the attractive and repulsive signals growths evolve conforming the rates $k(u,v)$ and $l(u,w)$, respectively (second and third equation in (1.1)).

As expected, the cellular motility is extremely sensitive to the actions of the above factors governing model (1.1): in particular, in general no stable behavior is conceivable but, conversely, even weak changes in the related values may strongly influence the dynamics. Specifically, the evolution might relax towards global stabilization and convergence to equilibrium of the cell distribution u, but it could even degenerate into the so-called *chemotactic collapse*, the mechanism resulting in uncontrolled aggregation processes for u, eventually blowing up/exploding in finite time. From the mathematical point of view, in the first case solutions (u,v,w) are defined and bounded for all (x,t) in $\Omega \times (0, \infty)$, in the other case for a certain finite time $T_{\max}$, the solution (u,v,w) becomes unbounded when approaching $T_{\max}$.

Let us now mention some known results in the literature dealing with the so called signal-production models coming from (1.1); in this situation, k and l are positive functions only of u, and as u itself increases so v and w do:

1. For $\tau = 0$ in the equations for v and w, $h(u) \equiv 0$ and for the linear case (i.e., diffusion $D(u) = u$, chemosensitivities $S(u) = \chi u$, with $\chi > 0$, $T(u) = \xi u$ with $\xi > 0$, and chemoattractant and chemorepellent $k(u, v) = \alpha u$, $\alpha > 0$, and $l(u, w) = \gamma u$, $\gamma > 0$, respectively), the value $\xi\gamma - \chi\alpha$ is critical for $n = 2$: in particular, if $\xi\gamma - \chi\alpha > 0$ (repulsion dominates on attraction), in any dimension all solutions to the model are globally bounded, whereas for $\xi\gamma - \chi\alpha < 0$ (attraction dominates repulsion) unbounded solutions can be detected (see [7, 13, 17, 19]). Indeed, when $\xi\gamma - \chi\alpha > 0$ and for $h(u) = ku - \mu u^\beta$, $k \in \mathbb{R}, \mu > 0$ and some $\beta > 1$, in [2] some questions on the blow-up scenario are discussed.
2. For more general production laws, respectively $k(u, v) \approx \alpha u^s$, $s > 0$, and $l(u, w) \approx \gamma u^r$, $r \geq 1$ (which D, S, T and τ as before) the following is available in the literature ([20]): for $n \geq 2$, for every $\alpha, \beta, \gamma, \delta, \chi > 0$, and for $r > s \geq 1$ (resp. $s > r \geq 1$), there exists $\xi^* > 0$ (resp. $\xi_* > 0$) such that if $\xi > \xi^*$ (resp. $\xi \geq \xi_*$), any sufficiently regular initial distribution $u_0(x) \geq 0$ (resp. $u_0(x) \geq 0$ enjoying some smallness assumptions) leads to a unique classical and bounded solution. In addition, the same conclusion holds true for every $\alpha, \beta, \gamma, \delta, \chi, \xi > 0$, $0 < s < 1$, $r = 1$ and any sufficiently regular $u_0(x) \geq 0$.

The chemotactic collapse appearing in the signal-production situations, apparently (but this is an open question in the field yet) cannot occur for alike signal-absorption models, where to high values of u correspond small ones (eventually vanishing) of v and w. As a matter of fact, even for the so called Keller–Segel system with consumption, which is the simplified two-unknowns version of problem (1.1) reading

$$\begin{cases} u_t = \Delta u - \chi \nabla \cdot (u \nabla v) & \text{in } \Omega \times (0, T_{\max}), \\ v_t = \Delta v - uv & \text{in } \Omega \times (0, T_{\max}), \\ u_\nu = v_\nu = 0 & \text{on } \partial\Omega \times (0, T_{\max}), \\ u(x, 0) = u_0(x),\ v(x, 0) = v_0(x) & x \in \bar{\Omega}, \end{cases} \tag{1.2}$$

the occurrence of blow-up has not been found. In particular, all classical solutions (u, v) to (1.2) are uniformly bounded in one and two-dimensional settings, independently of some size of (u_0, v_0): the case $n = 1$ can be justified by standard procedures, while for $n = 2$ the result is a consequence of more general analyses discussed in [21, 22]. Conversely, for $n \geq 3$, boundedness requires the smallness assumption $\chi \|v_0\|_{L^\infty(\Omega)} \leq \frac{1}{6(n+1)}$, as proved in [16]. (This condition is improved in $\chi \|v_0\|_{L^\infty(\Omega)} < \frac{\pi}{\sqrt{2(n+1)}}$; see [1].) On the other hand, if in models like (1.2) some smoothing effects on the dynamics of problem are introduced, boundedness of related solutions remains valid even when $\chi \|v_0\|_{L^\infty(\Omega)}$ is larger than the values mentioned above. More specifically,

3. In [12], where the motion of the cells is affected by a logistic source reading

$$u_t = \Delta u - \chi \nabla \cdot (u \nabla v) + ku - \mu u^2, \quad k, \mu > 0,$$

the authors establish that the resulting Cauchy problem admits classical bounded solutions for arbitrarily large $\chi \|v_0\|_{L^\infty(\Omega)}$ provided μ is also larger than a certain expression increasing with $\chi \|v_0\|_{L^\infty(\Omega)}$;

4. In [15], together with the dampening logistic term the equation for the particles' density is also perturbed by nonlinear diffusion and sensitivity:
$$u_t = \nabla \cdot \big((u+1)^{m_1-1}\nabla u - u(u+1)^{m_2-1}\nabla v\big) + ku - \mu u^2, \quad m_1, m_2 \in \mathbb{R}.$$
Similarly to the previous case, globality and boundedness are derived whenever the smoothness parameter μ of the logistic is large enough and the diffusion dominates the attraction, in the sense that $m_1 > 2m_2 - 1$.

Naturally, in the frame of chemotaxis models with two signals, beyond the double-production cases aforementioned, double-saturation or consumption-production mechanisms can be considered. In this sense, and always with reference to (1.1):

5. When $k(u,v) \approx av - u^\alpha v$ in the second equation (with $\tau = 1$) and $l(u,w) \approx u^l$ in the third (with $\tau = 0$) are fixed, the cells' density produces chemorepellent and absorbs chemoattractant; in [6] boundedness is established (i) for $l = 1$, $n \in \{1,2\}$, $\alpha \in (0, \frac{1}{2} + \frac{1}{n}) \cap (0,1)$ and any $\xi > 0$, (ii) for $l = 1$, $n \geq 3$, $\alpha \in (0, \frac{1}{2} + \frac{1}{n})$ and ξ larger than a quantity depending on $\chi \|v_0\|_{L^\infty(\Omega)}$, (iii) for $l > 1$ any $\xi > 0$, and in any dimension;
6. For $\tau = 1$ in the equations for v and w, $h(u) \equiv 0$ and for the linear case (i.e., diffusion $D(u) = u$, chemosensitivities $S(u) = \chi u$ and $T(u) = \xi u$, $\chi, \xi > 0$) and chemoattractant and chemorepellent $k(u,v) = av - uv$, $l(u,w) = bw - uw$ (double-signal saturation), for $n \geq 3$ all solutions emanating from sufficiently regular data such that $0 < \chi \|v_0\|_{L^\infty(\Omega)} < \frac{1}{5n}$ and $0 < \xi \|w_0\|_{L^\infty(\Omega)} < \frac{1}{5n}$ are globally bounded. This is proved in [3] (where also two- and three-dimensional numerical simulations are presented) and through the lines of this paper we will give some more hints on this research since the present article represents a generalization of what derived in [3].
7. For $\tau = 1$ in the equation for v and $k(u,v) = av - uv$ (chemoattractant consumed), for $\tau = 0$ in the third and $l(u,w) \approx u^l$, $l \geq 1$ (chemorepellent produced), for the diffusion $D(u) \approx u^{m_1}$, chemosensitivities $S(u) \approx u^{m_2}$ and $T(u) \approx u^{m_3}$ ($m_1, m_2, m_3 \in \mathbb{R}$), for $h(u) \approx ku - \mu u^\beta$ ($k \in \mathbb{R}$, $\mu > 0$ and $\beta > 1$), for $n \geq 3$ all solutions emanating from sufficiently regular data u_0, v_0 are globally bounded provided $m_1 > \varphi(m_2, m_3, n, \alpha, \beta, l)$, for specific expressions of φ.

Remark 1.1. Even though the analysis of this paper is merely theoretical, it is worthwhile emphasizing that model (1.1) has some applications in the inflammation observed in Alzheimer's disease when $h(u) \equiv 0$, $k(u,v)$ and $l(u,w)$ are linear functions only of the cell density (the already mentioned signal-production models). More precisely, [14] deals with the description of the gathering mechanisms of microglia and dimensional, numerical and experimental analyses, in bounded intervals are proposed. But, without dwelling too much on aspects not belonging to our expertise and out of the scope of the paper, there is more: also the counterpart considering the absorption of both signals may have some biological interpretation. Indeed, *phagocytes* are cells that protect the body by ingesting harmful foreign particles, bacteria, and dead or dying cells. For instance, during the phagocytosis process, the hepatic cells filter toxic substances (toxins), engulf them and convert them into harmless substances (nutrients) or make sure they are released into the surrounding environment. (See [8, §1 and §2].)

2. Main claims and organization of the paper.

2.1. The model: presentation and some notations.

Let $\Omega \subset \mathbb{R}^n$, $n \geq 2$, be a bounded open domain with smooth boundary, $\chi, \xi > 0$, $m_1, m_2, m_3 \in \mathbb{R}$, $f(u), g(u)$ and $h(u)$ be reasonably regular functions generalizing the prototypes $f(u) = K_1 u^\alpha$, $g(u) = K_2 u^\gamma$, and $h(u) = ku - \mu u^\beta$ with $K_1, K_2, \mu > 0$, $k \in \mathbb{R}$ and suitable $\alpha, \gamma, \beta > 0$. Once nonnegative initial configurations u_0, v_0 and w_0 are fixed, herein we are interested in the following nonlinear attraction-repulsion chemotaxis model, naturally obtainable as a particular case of problem (1.1):

$$
\begin{cases}
u_t = \nabla \cdot \big((u+1)^{m_1-1}\nabla u - \chi u(u+1)^{m_2-1}\nabla v & \text{in } \Omega \times (0, T_{\max}), \\
\qquad\qquad + \xi u(u+1)^{m_3-1}\nabla w\big) + h(u) & \\
v_t = \Delta v - f(u)v & \text{in } \Omega \times (0, T_{\max}), \\
w_t = \Delta w - g(u)w & \text{in } \Omega \times (0, T_{\max}), \\
u_\nu = v_\nu = w_\nu = 0 & \text{on } \partial\Omega \times (0, T_{\max}), \\
u(x,0) = u_0(x),\ v(x,0) = v_0(x),\ w(x,0) = w_0(x) & x \in \bar{\Omega}.
\end{cases}
\tag{2.1}
$$

In the light of what previously said, it should be easy to convince ourselves that this model brings together many of the characteristics above discussed: nonlinear diffusion, sensitivities and growth rates, as well as general logistic terms.

To our knowledge, the literature provides partial results connected to model (2.1) only for the case $m_1 = m_2 = m_3 = 1$, $f(u) = g(u) = u$ and $h(u) = 0$ (see [3]); the attained results were summarized in item 6. In particular, system (2.1) appears as a natural continuation of the model described in item 6 itself, and it is worthwhile developing a general n-dimensional analysis in order to extend the mathematical comprehension. Specifically, we aim at deriving sufficient conditions involving the parameters used in problem (2.1) according to which it admits classical solutions which are global and uniformly bounded in time. Specifically, we look for nonnegative functions $u = u(x,t), v = v(x,t), w = w(x,t)$ defined for $(x,t) \in \bar{\Omega} \times [0, T_{\max})$, and $T_{\max} = \infty$, such that

$$
\begin{cases}
u \in C^0(\bar{\Omega} \times [0,\infty)) \cap C^{2,1}(\bar{\Omega} \times (0,\infty)) \cap L^\infty((0,\infty); L^\infty(\Omega)) \\
v, w \in C^0(\bar{\Omega} \times [0,\infty)) \cap C^{2,1}(\bar{\Omega} \times (0,\infty)) \cap L^\infty((0,\infty); W^{1,\infty}(\Omega))
\end{cases}
\tag{2.2}
$$

and satisfying for all $(x,t) \in \bar{\Omega} \times [0,\infty)$ all the relations in (2.1).

To this aim, we require that f, g and h comply with

$$
\begin{aligned}
&f, g \in C^1(\mathbb{R}) \quad \text{with} \quad 0 \leq f(\zeta) \leq K_1 \zeta^\alpha \text{ and } 0 \leq g(\zeta) \leq K_2 \zeta^\gamma, \\
&\text{for some} \quad K_1, K_2, \alpha, \gamma > 0 \quad \text{and all } \zeta \geq 0,
\end{aligned}
\tag{2.3}
$$

and

$$
\begin{aligned}
&h \in C^1(\mathbb{R}) \quad \text{with} \quad h(0) \geq 0 \text{ and } h(\zeta) \leq k\zeta - \mu\zeta^\beta, \\
&\text{for some} \quad k \in \mathbb{R},\ \mu > 0,\ \beta > 1 \quad \text{and all } \zeta \geq 0.
\end{aligned}
\tag{2.4}
$$

Remark 2.1. As the reader can expect, the results below will depend on the parameters $\alpha, \gamma, m_1, m_2, m_3$ and n. In particular, in view of the formulations of the second and third equation (which are somehow "exchangeable"), by permuting some of those constants a number of assumptions connecting their values can be seen as the symmetric case of the other. In this sense, even though the presentation of the forthcoming Theorems 2.2 and 2.3 may appear hard to read, we want to underline that we made an important effort to include all the aforementioned permutations in the clearest way. This is the reason why we dedicate a part of the manuscript to define crucial constants in the computation; this is precisely what §2.1.1 includes. (For the readers' convenience, in Example 2.4 below we include a case where the form of these assumptions is of easier readability.)

2.1.1. *Notations.* We will make reference to the following quantities in the absence of the logistic term:

$$\mathcal{A} := \min\left\{\begin{array}{l}\max\Big\{2m_2-1, 2m_3-1, \frac{n-2}{n}\Big\},\ \max\Big\{m_2-\frac{1}{n}, m_3-\frac{1}{n}, \frac{n-2}{n}\Big\}\\ \max\Big\{2m_2-1, m_3-\frac{1}{n}, \frac{n-2}{n}\Big\},\ \max\Big\{m_2-\frac{1}{n}, 2m_3-1, \frac{n-2}{n}\Big\}\\ \max\Big\{m_2-\frac{1}{n}, m_3-\frac{1}{n}\Big\}\end{array}\right\},$$

$$\mathcal{B} := \min\left\{\begin{array}{l}\max\Big\{m_2-\frac{2}{n}+\alpha, m_3-\frac{2}{n}+\gamma\Big\},\ \max\Big\{2m_2, 2m_3, \frac{n-2}{n}\Big\}\\ \max\Big\{m_2-\frac{2}{n}+\alpha, 2m_3, \frac{n-2}{n}\Big\},\ \max\Big\{2m_2, m_3-\frac{2}{n}+\gamma, \frac{n-2}{n}\Big\}\end{array}\right\},$$

$$\mathcal{C} := \min\left\{\begin{array}{l}\max\Big\{m_2+\frac{n\alpha-2}{n\alpha-1}, m_3+\frac{n\gamma-2}{n\gamma-1}\Big\},\ \max\Big\{2m_2, 2m_3, \frac{n-2}{n}\Big\}\\ \max\Big\{m_2+\frac{n\alpha-2}{n\alpha-1}, 2m_3, \frac{n-2}{n}\Big\},\ \max\Big\{2m_2, m_3+\frac{n\gamma-2}{n\gamma-1}, \frac{n-2}{n}\Big\}\end{array}\right\},$$

$$\mathcal{D} := \min\left\{\max\Big\{m_2+\frac{n\alpha-2}{n\alpha-1}, m_3+\frac{n\gamma-2}{n\gamma-1}\Big\},\ \max\Big\{m_2+\frac{n\alpha-2}{n\alpha-1}, 2m_3, \frac{n-2}{n}\Big\}\right\},$$

$$\mathcal{E} := \min\left\{\max\Big\{m_2+\frac{n\alpha-2}{n\alpha-1}, m_3+\frac{n\gamma-2}{n\gamma-1}\Big\},\ \max\Big\{2m_2, \frac{n-2}{n}, m_3+\frac{n\gamma-2}{n\gamma-1}\Big\}\right\},$$

$$\mathcal{F} := \max\Big\{m_2+\frac{n\alpha-2}{n\alpha-1}, m_3+\frac{n\gamma-2}{n\gamma-1}\Big\}.$$

$$\mathcal{G} := \min\left\{\begin{array}{l}\max\Big\{m_2-\frac{1}{n}, m_3-\frac{2}{n}+\gamma\Big\},\ \max\Big\{2m_2-1, 2m_3, \frac{n-2}{n}\Big\}\\ \max\Big\{m_2-\frac{1}{n}, 2m_3, \frac{n-2}{n}\Big\},\ \max\Big\{2m_2-1, m_3-\frac{2}{n}+\gamma, \frac{n-2}{n}\Big\}\\ \max\Big\{m_2-\frac{1}{n}, m_3-\frac{2}{n}+\gamma, \frac{n-2}{n}\Big\}\end{array}\right\},$$

$$\mathcal{H} := \min\left\{\begin{array}{l}\max\Big\{m_2-\frac{1}{n}, m_3+\frac{n\gamma-2}{n\gamma-1}\Big\},\ \max\Big\{2m_2-1, 2m_3, \frac{n-2}{n}\Big\}\\ \max\Big\{m_2-\frac{1}{n}, 2m_3, \frac{n-2}{n}\Big\},\ \max\Big\{2m_2-1, m_3+\frac{n\gamma-2}{n\gamma-1}, \frac{n-2}{n}\Big\}\\ \max\Big\{m_2-\frac{1}{n}, m_3+\frac{n\gamma-2}{n\gamma-1}, \frac{n-2}{n}\Big\}\end{array}\right\},$$

$$\mathcal{I} := \min\left\{\begin{array}{l}\max\Big\{m_2-\frac{1}{n}, m_3+\frac{n\gamma-2}{n\gamma-1}\Big\},\ \max\Big\{2m_2-1, m_3+\frac{n\gamma-2}{n\gamma-1}, \frac{n-2}{n}\Big\}\\ \max\Big\{m_2-\frac{1}{n}, m_3+\frac{n\gamma-2}{n\gamma-1}, \frac{n-2}{n}\Big\}\end{array}\right\},$$

$$\mathcal{J} := \min\left\{\begin{array}{l} \max\Big\{m_2 - \frac{2}{n} + \alpha, m_3 + \frac{n\gamma-2}{n\gamma-1}\Big\}, \max\Big\{2m_2, 2m_3, \frac{n-2}{n}\Big\} \\ \max\Big\{m_2 - \frac{2}{n} + \alpha, 2m_3, \frac{n-2}{n}\Big\}, \max\Big\{2m_2, m_3 + \frac{n\gamma-2}{n\gamma-1}, \frac{n-2}{n}\Big\} \end{array}\right\},$$

$$\mathcal{K} := \min\left\{\max\Big\{m_2 - \frac{2}{n} + \alpha, m_3 + \frac{n\gamma-2}{n\gamma-1}\Big\}, \max\Big\{2m_2, \frac{n-2}{n}, m_3 + \frac{n\gamma-2}{n\gamma-1}\Big\}\right\}.$$

For the logistic case we will refer to these quantities:

$$\mathcal{A}' := \min\left\{\begin{array}{l} \max\Big\{2m_2 - 1, 2m_3 - 1, \frac{n-2}{n}\Big\}, \max\Big\{m_2 - \frac{1}{n}, m_3 - \frac{1}{n}, \frac{n-2}{n}\Big\} \\ \max\Big\{2m_2 - 1, m_3 - \frac{1}{n}, \frac{n-2}{n}\Big\}, \max\Big\{m_2 - \frac{1}{n}, 2m_3 - 1, \frac{n-2}{n}\Big\} \\ \max\Big\{2m_2 - \beta, 2m_3 - \beta, \frac{n-2}{n}\Big\}, \max\Big\{2m_2 - \beta, 2m_3 - \beta\Big\} \\ \max\Big\{m_2 - \frac{1}{n}, 2m_3 - \beta, \frac{n-2}{n}\Big\}, \max\Big\{2m_2 - 1, 2m_3 - \beta, \frac{n-2}{n}\Big\} \\ \max\Big\{2m_2 - \beta, 2m_3 - 1, \frac{n-2}{n}\Big\}, \max\Big\{2m_2 - \beta, m_3 - \frac{1}{n}, \frac{n-2}{n}\Big\} \end{array}\right\},$$

$$\mathcal{B}' := \min\left\{\begin{array}{l} \max\Big\{2m_2, 2m_3, \frac{n-2}{n}\Big\}, \max\Big\{2m_2 + 1 - \beta, 2m_3 + 1 - \beta\Big\} \\ \max\Big\{2m_2, 2m_3 + 1 - \beta, \frac{n-2}{n}\Big\}, \max\Big\{2m_2 + 1 - \beta, 2m_3, \frac{n-2}{n}\Big\} \end{array}\right\},$$

$$\mathcal{C}' := \min\left\{\begin{array}{l} \max\Big\{2m_2 - 1, \frac{n-2}{n}, 2m_3\Big\}, \max\Big\{2m_2 - 1, \frac{n-2}{n}, 2m_3 + 1 - \beta\Big\} \\ \max\Big\{m_2 - \frac{1}{n}, 2m_3, \frac{n-2}{n}\Big\}, \max\Big\{m_2 - \frac{1}{n}, \frac{n-2}{n}, 2m_3 + 1 - \beta\Big\} \\ \max\Big\{2m_2 - \beta, \frac{n-2}{n}, 2m_3\Big\}, \max\Big\{2m_2 - \beta, \frac{n-2}{n}, 2m_3 + 1 - \beta\Big\} \\ \max\Big\{2m_2 - \beta, 2m_3 + 1 - \beta\Big\} \end{array}\right\}.$$

2.2. **Statements of the theorems and discussions.** With reference to the notations introduced in §2.1.1, let us now give the theorems proved in this paper.

We notice that the substantial indication behind the hypotheses below is rather natural; the diffusion parameter m_1 has to be large enough in order to ensure boundedness.

Theorem 2.2 (The non-logistic case)**.** *Let Ω be a bounded open domain of $\mathbb{R}^n$, with $n \geq 2$, and smooth boundary $\partial\Omega$, χ, ξ positive reals, and $h \equiv 0$. Moreover, for*

$\alpha, \gamma > 0$ and $m_1, m_2, m_3 \in \mathbb{R}$, let f and g fulfill (2.3) for each of the following cases:

A_1) $\alpha, \gamma \in \left(0, \frac{1}{n}\right]$, $m_1 > \mathcal{A}$;

A_2) $\alpha, \gamma \in \left(\frac{1}{n}, \frac{2}{n}\right)$, $m_1 > \mathcal{B}$;

A_3) $\alpha, \gamma \in \left[\frac{2}{n}, 1\right)$, $m_1 > \mathcal{C}$;

A_4) $\alpha \in \left[\frac{2}{n}, 1\right], \gamma \in \left[\frac{2}{n}, 1\right)$, $m_1 > \mathcal{D}$;

A_5) $\alpha \in \left[\frac{2}{n}, 1\right), \gamma \in \left[\frac{2}{n}, 1\right]$, $m_1 > \mathcal{E}$;

A_6) $\alpha, \gamma \in \left[\frac{2}{n}, 1\right]$, $m_1 > \mathcal{F}$;

A_7) $\alpha \in \left(0, \frac{1}{n}\right], \gamma \in \left(\frac{1}{n}, \frac{2}{n}\right)$, $m_1 > \mathcal{G}$;

A_8) $\alpha \in \left(0, \frac{1}{n}\right], \gamma \in \left[\frac{2}{n}, 1\right)$; $m_1 > \mathcal{H}$;

A_9) $\alpha \in \left(0, \frac{1}{n}\right], \gamma \in \left[\frac{2}{n}, 1\right]$, $m_1 > \mathcal{I}$;

A_{10}) $\alpha \in \left(\frac{1}{n}, \frac{2}{n}\right), \gamma \in \left(0, \frac{1}{n}\right]$, $m_1 > \mathcal{G}^t$;

A_{11}) $\alpha \in \left(\frac{1}{n}, \frac{2}{n}\right), \gamma \in \left[\frac{2}{n}, 1\right)$, $m_1 > \mathcal{J}$;

A_{12}) $\alpha \in \left(\frac{1}{n}, \frac{2}{n}\right), \gamma \in \left[\frac{2}{n}, 1\right]$, $m_1 > \mathcal{K}$;

A_{13}) $\alpha \in \left[\frac{2}{n}, 1\right),\ \gamma \in \left(0, \frac{1}{n}\right]$, $m_1 > \mathcal{H}^t$;

A_{14}) $\alpha \in \left[\frac{2}{n}, 1\right],\ \gamma \in \left(0, \frac{1}{n}\right]$, $m_1 > \mathcal{I}^t$;

A_{15}) $\alpha \in \left[\frac{2}{n}, 1\right),\ \gamma \in \left(\frac{1}{n}, \frac{2}{n}\right)$, $m_1 > \mathcal{J}^t$;

A_{16}) $\alpha \in \left[\frac{2}{n}, 1\right],\ \gamma \in \left(\frac{1}{n}, \frac{2}{n}\right)$, $m_1 > \mathcal{K}^t$,

the superscript t denoting the case where the roles of α and m_2 are taken by γ and m_3, respectively. Then for any initial data $(u_0, v_0, w_0) \in (W^{1,\infty}(\Omega))^3$, with $u_0, v_0, w_0 \geq 0$ on $\bar{\Omega}$, there exists a unique triplet (u, v, w) of nonnegative functions, uniformly bounded in time and classically solving problem (2.1).

Theorem 2.3 (The logistic case)**.** *Under the same hypotheses of Theorem 2.2 and $\beta > 1$, let h comply with (2.4). Then the same claim holds true whenever $\alpha, \gamma > 0$, $m_1, m_2, m_3 \in \mathbb{R}$, and f and g fulfill (2.3) for each of the following cases:*

A_{17}) $\alpha, \gamma \in \left(0, \frac{1}{n}\right]$, $m_1 > \mathcal{A}'$;

A_{18}) $\alpha, \gamma \in \left(\frac{1}{n}, 1\right)$, $m_1 > \mathcal{B}'$;

A_{19}) $\alpha \in \left(0, \frac{1}{n}\right], \gamma \in \left(\frac{1}{n}, 1\right)$, $m_1 > \mathcal{C}'$;

A_{20}) $\alpha \in \left(\frac{1}{n}, 1\right), \gamma \in \left(0, \frac{1}{n}\right]$, $m_1 > \mathcal{C}'^t$.

All these results are put together into Figure 1.

Example 2.4. Let $n = 2$, $m_1 = m_2 = m_3 = m \in \mathbb{R}$, $\alpha = \gamma \in \left(\frac{1}{2}, 1\right)$. We have

$$\mathcal{B} = \min\{m - 1 + \alpha, \max\{2m, 0\}, \max\{m - 1 + \alpha, 2m, 0\}\}.$$

Specifically, for $m < 0$

$$\mathcal{B} = m - 1 + \alpha,$$

and assumption A_2) of Theorem 2.2 are automatically fulfilled, in view of $\alpha < 1$.

Remark 2.5 (On the validity of Theorem 2.2)**.** For m_i, with $i = 1, 2, 3$, complying with its related assumptions, Theorem 2.2 provides a rather complete picture concerning boundedness of solutions to (2.1). Conversely, as far as the linear diffusion and sensitivities ($m_1 = m_2 = m_3 = 1$) model is concerned, it still holds except when α and/or γ belong to the intervals $\left[\frac{2}{n}, 1\right]$ and/or $\left[\frac{2}{n}, 1\right)$, namely under the assumptions A_3)—A_6), A_8), A_9) and A_{11})—A_{16}). Henceforth, what happens in these situations? In low dimensions or under further restrictions on the data, some cases can be saved. More precisely:

I) If $\alpha = \gamma = 1$ (included in A_6)), boundedness of solutions is ensured by requiring $0 < \chi < \frac{1}{5n\|v_0\|_{L^\infty(\Omega)}}$ and $0 < \xi < \frac{1}{5n\|w_0\|_{L^\infty(\Omega)}}$, with $n \geq 2$, as seen in [3, Theorem 1.1];

II) If $\alpha \in \left(0, \frac{1}{n}\right]$, $\gamma \in \left[\frac{2}{n}, 1\right]$ (A_9)) or $\alpha \in \left[\frac{2}{n}, 1\right]$, $\gamma \in \left(0, \frac{1}{n}\right]$ (A_{14})), boundedness of solutions is achieved for $\alpha \in (0, 1]$, $\gamma = 1$ (so for $n = 1$), for $\alpha \in \left(0, \frac{1}{2}\right]$, $\gamma = 1$ and $0 < \xi < K_2(n, \|w_0\|_{L^\infty(\Omega)})$ (so for $n = 2$), or by symmetry, for $\alpha = 1$, $\gamma \in (0, 1]$ (so for $n = 1$), for $\alpha = 1$, $\gamma \in \left(0, \frac{1}{2}\right]$ and $0 < \chi < K_1(n, \|v_0\|_{L^\infty(\Omega)})$ (so for $n = 2$), respectively;

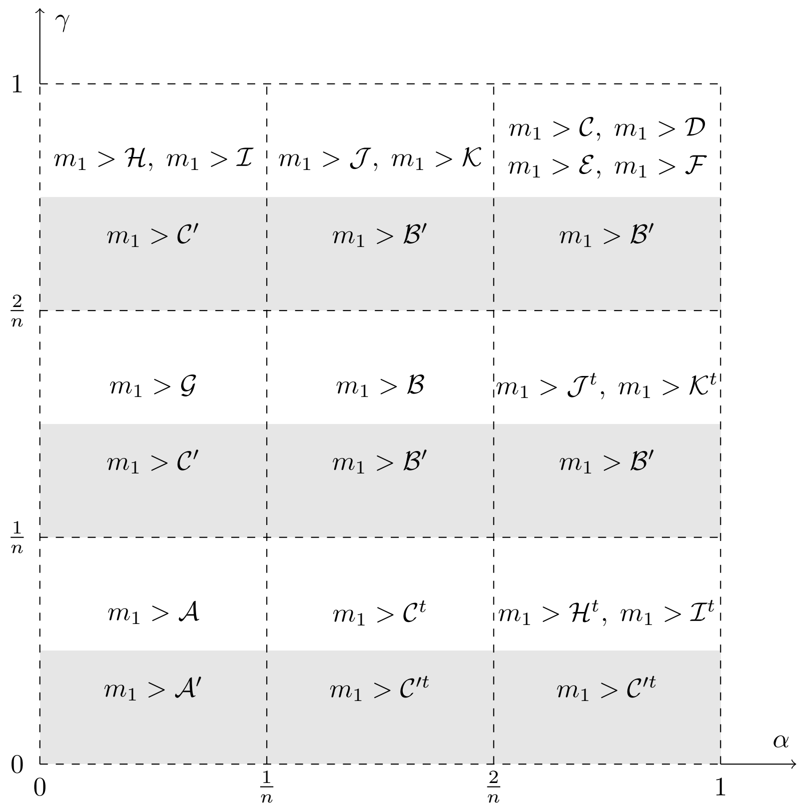

FIGURE 1. Schematization of the non-logistic case (Theorem 2.2, non-shadowed zones) and the logistic case (Theorem 2.3, shadowed zones). With reference to §2.1.1, herein we collect the ranges of the parameters involved in model (2.1) for which boundedness of its solutions is established for any fixed initial distribution u_0, v_0 and w_0. (Recall $m_2, m_3, k \in \mathbb{R}$ and $\chi, \xi \in \mathbb{R}^+$. For the analysis of the limit values of α and γ in the corresponding intervals, we refer to the mentioned theorems.)

III) If $\alpha \in \left(\frac{1}{n}, \frac{2}{n}\right)$, $\gamma \in \left[\frac{2}{n}, 1\right]$ (A_{12})) or $\alpha \in \left[\frac{2}{n}, 1\right]$, $\gamma \in \left(\frac{1}{n}, \frac{2}{n}\right)$ (A_{16})), Theorem 2.2 still holds for $n = 1$, $\alpha \in (1, 2)$, $\gamma = 1$, for $n = 2$, $\alpha \in \left(\frac{1}{2}, 1\right)$, $\gamma = 1$ and $0 < \xi < \tilde{K}_2(n, \|w_0\|_{L^\infty(\Omega)})$, or for $n = 1$, $\alpha = 1$, $\gamma \in (1, 2)$, for $n = 2$, $\alpha = 1$, $\gamma \in \left(\frac{1}{2}, 1\right)$ and $0 < \chi < \tilde{K}_1(n, \|v_0\|_{L^\infty(\Omega)})$, respectively.

Apparently, the remaining scenarios cannot be managed; some details on this discussion will be given in Remark 4.5.

Remark 2.6 (On the validity of Theorem 2.3)**.** If for the non logistic case Table 1 provides an exhaustive scenario, for the logistic one we discuss separately the linear and nonlinear situations.

In particular, for $m_1 = m_2 = m_3 = 1$, it is seen that Theorem 2.3 still applies for $\beta > 2$ and $\alpha, \gamma \in (0, 1)$. Which is the situation in those cases where the limit values of β and/or α, γ are considered? Theorem 2.3 applies (hints are given in Remark 4.6 at the end of the article) whenever

i) $\beta > 2$ and $\alpha, \gamma \in (0, 1]$;

ii) $\beta = 2, \alpha, \gamma \in (\frac{1}{n}, 1]$ and $\mu > K(n)(\chi^2 \|\chi v_0\|^{\frac{4}{n}}_{L^\infty(\Omega)} + \xi^2 \|\xi w_0\|^{\frac{4}{n}}_{L^\infty(\Omega)})$, with $K(n) > 0$;

iii) $\beta = 2, \alpha \in \left(\frac{1}{n}, 1\right], \gamma \in \left(0, \frac{1}{n}\right]$ and $\mu > K_1(n)\chi^2 \|\chi v_0\|^{\frac{4}{n}}_{L^\infty(\Omega)}$, with $K_1(n) > 0$;

iv) $\beta = 2, \alpha \in \left(0, \frac{1}{n}\right], \gamma \in \left(\frac{1}{n}, 1\right]$ and $\mu > K_2(n)\xi^2 \|\xi w_0\|^{\frac{4}{n}}_{L^\infty(\Omega)}$, with $K_2(n) > 0$.

On the other hand, for the nonlinear diffusion and sensitivities case, it continues to be valid when either $\beta > 2$ and

v) $\alpha, \gamma \in \left(\frac{1}{n}, 1\right]$ and $m_1 > \mathcal{B}'$;

vi) $\alpha \in \left(0, \frac{1}{n}\right], \gamma \in \left(\frac{1}{n}, 1\right]$ and $m_1 > \mathcal{C}'$;

vii) $\alpha \in \left(\frac{1}{n}, 1\right], \gamma \in \left(0, \frac{1}{n}\right]$ and $m_1 > \mathcal{C}'^t$;

or $\beta = 2$ and

viii) $\alpha, \gamma \in \left(\frac{1}{n}, 1\right]$, $m_1 > \mathcal{B}'$ and $\mu > \tilde{K}$, with some constant $\tilde{K} > 0$ depending on $n, m_1, m_2, m_3, \chi, \|v_0\|_{L^\infty(\Omega)}, \xi, \|w_0\|_{L^\infty(\Omega)}$;

ix) $\alpha \in \left(0, \frac{1}{n}\right], \gamma \in \left(\frac{1}{n}, 1\right]$, $m_1 > \mathcal{C}'$ and $\mu > \tilde{K}_1$, with some constant $\tilde{K}_1 > 0$ depending on $n, m_1, m_2, m_3, \xi, \|w_0\|_{L^\infty(\Omega)}$;

x) $\alpha \in \left(\frac{1}{n}, 1\right], \gamma \in \left(0, \frac{1}{n}\right]$, $m_1 > \mathcal{C}'^t$ and $\mu > \tilde{K}_2$, with some constant $\tilde{K}_2 > 0$ depending on $n, m_1, m_2, m_3, \chi, \|v_0\|_{L^\infty(\Omega)}$.

2.3. **Technical strategy and structure of the article.** As it will be formally made precise below, the mathematical requirements toward boundedness are connected to some *a priori* estimates of $\int_\Omega u^p = \int_\Omega u^p(x,t)dx$, for some $p > 1$ and some $t > 0$, with (u, v, w) being any given solution to problem (2.1). Whereas in [3], dealing with the situation where $m_1 = m_2 = m_3 = 1$, this issue is addressed by the employment of a functional of the form $\int_\Omega u^p \varphi$, being φ a suitable function of (v, w), apparently for the nonlinear case under investigation the same attempt does not work. Henceforth our idea is to consider, for some $p, q, r > 1$ properly large and some $t > 0$, the functional

$$y(t) := \int_\Omega (u+1)^p + \int_\Omega |\nabla v|^{2q} + \int_\Omega |\nabla w|^{2r}, \tag{2.5}$$

a natural adjustment of that used in many fully-parabolic Keller–Segel systems with two unknowns. An evolutive analysis of such a functional leads to a crucial absorption differential inequality in time for the functional itself, and in turn to the desired uniform-in-time bound for $\int_\Omega u^p$. In particular, our strategy towards the achievement of this differential inequality requires some *ad hoc* exploitation of well-known functional inequalities (see Remark 3.4 below for details).

The rest of the paper is structured as follows. First, in §3 we give some hints concerning the local existence and uniqueness of a classical solution to model (2.1) and some of its main properties. In this same section we give a *boundedness criterion*, establishing globality and boundedness of local solutions from proper *a priori* L^p-boundedness. In turn, in §4 we focus on the derivation of these bounds, by means of which we can deduce the claims of Theorems 2.2 and 2.3.

3. **Existence of local-in-time solutions and basic properties.** Once Ω, χ, ξ, m_1, m_2, m_3 and f, g, h are picked as above, with (u, v, w) we will indicate the classical and nonnegative solution of problem (2.1) defined for all $(x, t) \in \bar{\Omega} \times [0, T_{\max})$, for some finite $T_{\max}$, and emanating from the nonnegative initial data $(u_0, v_0, w_0) \in$

$(W^{1,\infty}(\Omega))^3$. In particular, u, v and w are such that

$$\int_\Omega u(x,t)dx \le m_0 := \min\left\{m,\ \left(\frac{k_+}{\mu}\right)^{\frac{1}{\beta-1}}|\Omega|\right\} \text{ on } (0,T_{\max}) \text{ and } m=\int_\Omega u_0, \quad (3.1)$$

and

$$0 \le v \le \|v_0\|_{L^\infty(\Omega)} \quad \text{and} \quad 0 \le w \le \|w_0\|_{L^\infty(\Omega)} \quad \text{in} \quad \Omega\times(0,T_{\max}). \qquad (3.2)$$

Further, globality and boundedness of (u,v,w) (in the sense of (2.2)) are ensured whenever (*boundedness criterion*, below) $u \in L^\infty((0,T_{\max});L^p(\Omega))$, with $p>1$ arbitrarily large: formally,

$$\begin{aligned}&\text{If for some } C>0 \text{ and } p=p(n,m_1,m_2,m_3)>1 \text{ arbitrarily large,}\\ &\text{we have } \int_\Omega u^p \le C \text{ on } (0,T_{\max}), \text{ then } (u,v,w)\in (L^\infty((0,\infty);L^\infty(\Omega)))^3.\end{aligned} \qquad (3.3)$$

We do not prove these basic statements, nor dedicate any lemma; we understand that the details in [3, §2] and [18, Appendix A], which take into consideration also relations (3.1) and (3.2), are sufficient in this regard. Conversely, we spend some words regarding estimate (3.1). When $h\equiv 0$, it immediately follows by integrating over Ω the first equation of (2.1) and it is the well-known mass conservation property. In the presence of the logistic terms h as in (2.4), oppositely, also an application of the Hölder inequality has to be invoked: precisely for $k_+=\max\{k,0\}$ and for all $t\in(0,T_{\max})$

$$\frac{d}{dt}\int_\Omega u = \int_\Omega h(u) = k\int_\Omega u - \mu\int_\Omega u^\beta \le k_+\int_\Omega u - \frac{\mu}{|\Omega|^{\beta-1}}\left(\int_\Omega u\right)^\beta,$$

thus concluding the proof by virtue of an ODI-comparison argument.

Crucial in our computations, beyond the above derivations, are as well some uniform bounds for $\|v(\cdot,t)\|_{W^{1,s}(\Omega)}$ and $\|w(\cdot,t)\|_{W^{1,s}(\Omega)}$, with $s\ge 1$. In this sense, the following lemma is a cornerstone.

Lemma 3.1. *For some $c_0, c_1>0$, we have that v and w comply with*

$$\int_\Omega |\nabla v(\cdot,t)|^s \le c_0 \quad \text{on } (0,T_{\max}) \begin{cases} \text{for all } s\in[1,\infty) & \text{if } \alpha\in\left(0,\frac{1}{n}\right],\\ \text{for all } s\in\left[1,\frac{n}{(n\alpha-1)}\right) & \text{if } \alpha\in\left(\frac{1}{n},1\right],\end{cases} \qquad (3.4)$$

and

$$\int_\Omega |\nabla w(\cdot,t)|^s \le c_1 \quad \text{on } (0,T_{\max}) \begin{cases} \text{for all } s\in[1,\infty) & \text{if } \gamma\in\left(0,\frac{1}{n}\right],\\ \text{for all } s\in\left[1,\frac{n}{(n\gamma-1)}\right) & \text{if } \gamma\in\left(\frac{1}{n},1\right].\end{cases} \qquad (3.5)$$

Proof. Fixing $\alpha,\gamma\in(0,1]$, it is possible to find $\rho,\rho_1>\frac{1}{2}$ such that for all $s\in[\frac{1}{\alpha},\frac{n}{(n\alpha-1)_+})$ and $s\in[\frac{1}{\gamma},\frac{n}{(n\gamma-1)_+})$, we have $\frac{1}{2}<\rho<1-\frac{n}{2}(\alpha-\frac{1}{s})$ and $\frac{1}{2}<\rho_1<1-\frac{n}{2}(\gamma-\frac{1}{s})$, respectively. From $1-\rho-\frac{n}{2}(\alpha-\frac{1}{s})>0$ and $1-\rho_1-\frac{n}{2}(\gamma-\frac{1}{s})>0$, the claims follow invoking properties related to the Neumann heat semigroup; details can be found in [4, Lemma 5.1]. □

We will also make use of these technical results.

Lemma 3.2. *Let $n\in\mathbb{N}$, with $n\ge 2$, $m_1>\frac{n-2}{n}$, $m_2,m_3\in\mathbb{R}$ and $\alpha,\gamma\in(0,1]$. Then there exists $s\in[1,\infty)$ such that for proper $p,q,r\in[1,\infty)$, θ and θ', $\tilde\theta$ and $\tilde\theta'$,*

μ and μ', $\tilde{\mu}$ and $\tilde{\mu}'$ conjugate exponents, we have that

$$a_1 = \frac{\frac{m_1+p-1}{2}\left(1-\frac{1}{(p+2m_2-m_1-1)\theta}\right)}{\frac{m_1+p-1}{2}+\frac{1}{n}-\frac{1}{2}}, \qquad a_2 = \frac{q\left(\frac{1}{s}-\frac{1}{2\theta'}\right)}{\frac{q}{s}+\frac{1}{n}-\frac{1}{2}},$$
$$a_3 = \frac{\frac{m_1+p-1}{2}\left(1-\frac{1}{2\alpha\mu}\right)}{\frac{m_1+p-1}{2}+\frac{1}{n}-\frac{1}{2}}, \qquad a_4 = \frac{q\left(\frac{1}{s}-\frac{1}{2(q-1)\mu'}\right)}{\frac{q}{s}+\frac{1}{n}-\frac{1}{2}},$$
$$\kappa_1 = \frac{\frac{p}{2}\left(1-\frac{1}{p}\right)}{\frac{m_1+p-1}{2}+\frac{1}{n}-\frac{1}{2}}, \qquad \kappa_2 = \frac{q-\frac{1}{2}}{q+\frac{1}{n}-\frac{1}{2}},$$

and

$$\tilde{a}_1 = \frac{\frac{m_1+p-1}{2}\left(1-\frac{1}{(p+2m_3-m_1-1)\tilde{\theta}}\right)}{\frac{m_1+p-1}{2}+\frac{1}{n}-\frac{1}{2}}, \qquad \tilde{a}_2 = \frac{r\left(\frac{1}{s}-\frac{1}{2\tilde{\theta}'}\right)}{\frac{r}{s}+\frac{1}{n}-\frac{1}{2}},$$
$$\tilde{a}_3 = \frac{\frac{m_1+p-1}{2}\left(1-\frac{1}{2\gamma\tilde{\mu}}\right)}{\frac{m_1+p-1}{2}+\frac{1}{n}-\frac{1}{2}}, \qquad \tilde{a}_4 = \frac{r\left(\frac{1}{s}-\frac{1}{2(r-1)\tilde{\mu}'}\right)}{\frac{r}{s}+\frac{1}{n}-\frac{1}{2}},$$
$$\kappa_3 = \frac{r-\frac{1}{2}}{r+\frac{1}{n}-\frac{1}{2}},$$

belong to the interval $(0,1)$. If, additionally,

$$\alpha \in \left(0, \frac{1}{n}\right] \text{ and } \quad m_1 > m_2 - \frac{1}{n}, \quad \gamma \in \left(0, \frac{1}{n}\right] \text{ and } \quad m_1 > m_3 - \frac{1}{n}, \tag{3.6}$$

$$\alpha \in \left(\frac{1}{n}, \frac{2}{n}\right) \text{ and } \quad m_1 > m_2 - \frac{2}{n} + \alpha, \quad \gamma \in \left(\frac{1}{n}, \frac{2}{n}\right) \text{ and } \quad m_1 > m_3 - \frac{2}{n} + \gamma, \tag{3.7}$$

or

$$\alpha \in \left[\frac{2}{n}, 1\right] \text{ and } \quad m_1 > m_2 + \frac{n\alpha-2}{n\alpha-1}, \quad \gamma \in \left[\frac{2}{n}, 1\right] \text{ and } \quad m_1 > m_3 + \frac{n\gamma-2}{n\gamma-1}, \tag{3.8}$$

the following further relations hold true:

$$\beta_1 + \gamma_1 = \frac{p+2m_2-m_1-1}{m_1+p-1} a_1 + \frac{1}{q} a_2 \in (0,1),$$
$$\beta_2 + \gamma_2 = \frac{2\alpha}{m_1+p-1} a_3 + \frac{q-1}{q} a_4 \in (0,1),$$

and

$$\tilde{\beta}_1 + \tilde{\gamma}_1 = \frac{p+2m_3-m_1-1}{m_1+p-1} \tilde{a}_1 + \frac{1}{r} \tilde{a}_2 \in (0,1),$$
$$\tilde{\beta}_2 + \tilde{\gamma}_2 = \frac{2\gamma}{m_1+p-1} \tilde{a}_3 + \frac{r-1}{r} \tilde{a}_4 \in (0,1).$$

Finally, the relations involving the sum of β_1 and γ_1, β_2 and γ_2, $\tilde{\beta}_1$ and $\tilde{\gamma}_1$, $\tilde{\beta}_2$ and $\tilde{\gamma}_2$ still hold true in each one of the following cases:

- $\triangleright$ $\alpha \in \left(0, \frac{1}{n}\right]$ *and* $m_1 > m_2 - \frac{1}{n}$, $\gamma \in \left(\frac{1}{n}, \frac{2}{n}\right)$ *and* $m_1 > m_3 - \frac{2}{n} + \gamma$,
- $\triangleright$ $\alpha \in \left(0, \frac{1}{n}\right]$ *and* $m_1 > m_2 - \frac{1}{n}$, $\gamma \in \left[\frac{2}{n}, 1\right]$ *and* $m_1 > m_3 + \frac{n\gamma-2}{n\gamma-1}$,
- $\triangleright$ $\alpha \in \left(\frac{1}{n}, \frac{2}{n}\right)$ *and* $m_1 > m_2 - \frac{2}{n} + \alpha$, $\gamma \in \left(0, \frac{1}{n}\right]$ *and* $m_1 > m_3 - \frac{1}{n}$,
- $\triangleright$ $\alpha \in \left(\frac{1}{n}, \frac{2}{n}\right)$ *and* $m_1 > m_2 - \frac{2}{n} + \alpha$, $\gamma \in \left[\frac{2}{n}, 1\right]$ *and* $m_1 > m_3 + \frac{n\gamma-2}{n\gamma-1}$,
- $\triangleright$ $\alpha \in \left[\frac{2}{n}, 1\right]$ *and* $m_1 > m_2 + \frac{n\alpha-2}{n\alpha-1}$, $\gamma \in \left(0, \frac{1}{n}\right]$ *and* $m_1 > m_3 - \frac{1}{n}$,

$\triangleright$ $\alpha \in \left[\frac{2}{n}, 1\right]$ *and* $m_1 > m_2 + \frac{n\alpha - 2}{n\alpha - 1}$, $\quad \gamma \in \left(\frac{1}{n}, \frac{2}{n}\right)$ *and* $m_1 > m_3 - \frac{2}{n} + \gamma$.

Proof. For any $s \geq 1$, we put $\theta', \tilde{\theta}' > \max\{\frac{n}{2}, \frac{s}{2}\}$, $\mu > \max\{\frac{1}{2\alpha}, \frac{n}{2}\}$ and $\tilde{\mu} > \max\{\frac{1}{2\gamma}, \frac{n}{2}\}$. Thereafter, for

$$\begin{cases} q > \max\left\{\frac{n-2}{n}\theta', \frac{s}{2\mu'} + 1\right\}, \quad r > \max\left\{\frac{n-2}{n}\tilde{\theta}', \frac{s}{2\tilde{\mu}'} + 1\right\} \\ p > \max\Big\{2 - \frac{2}{n} - m_1, \frac{1}{\theta} - 2m_2 + m_1 + 1, \frac{(2m_2 - m_1 - 1)(n-2)\theta - nm_1 + n}{n - (n-2)\theta}, \\ \qquad \frac{2\alpha\mu(n-2)}{n} - m_1 + 1, \frac{1}{\tilde{\theta}} - 2m_3 + m_1 + 1, \\ \qquad \frac{(2m_3 - m_1 - 1)(n-2)\tilde{\theta} - nm_1 + n}{n - (n-2)\tilde{\theta}}, \frac{2\gamma\tilde{\mu}(n-2)}{n} - m_1 + 1\Big\}, \end{cases} \tag{3.9}$$

it is possible to check that $a_i, \tilde{a}_i, \kappa_2, \kappa_3 \in (0,1)$, for any $i = 1,2,3,4$. On the other hand, $\kappa_1 \in (0,1)$ also thanks to the assumption $m_1 > \frac{n-2}{n}$.

As to the second part, we consider three cases: $\alpha \in (0, \frac{1}{n}]$, $\alpha \in (\frac{1}{n}, \frac{2}{n})$ and $\alpha \in [\frac{2}{n}, 1]$.

- $\alpha \in (0, \frac{1}{n}]$. For $s > \frac{2\mu'}{2\mu' - 1}$ arbitrarily large, consistently with (3.9), we take $p = q = s$ and $\theta' = s\omega$, for some $\omega > \frac{1}{2}$. Some standard computations entail

$$0 < \beta_1 + \gamma_1 = \frac{s + 2m_2 - m_1 - 1 - \frac{1}{\theta}}{m_1 + s - 2 + \frac{2}{n}} + \frac{2 - \frac{1}{\omega}}{s + \frac{2s}{n}},$$

  and

$$0 < \beta_2 + \gamma_2 = \frac{2\alpha - \frac{1}{\mu}}{m_1 + s - 2 + \frac{2}{n}} + \frac{2s - 2 - \frac{s}{\mu'}}{s + \frac{2s}{n}}.$$

  In light of the above statements, the largeness of s infers θ arbitrarily close to 1, in accordance with θ' large. Further, by choosing ω approaching $\frac{1}{2}$, continuity arguments imply that $\beta_1 + \gamma_1 < 1$ whenever restriction (3.6) is satisfied, whereas $\beta_2 + \gamma_2 < 1$ comes from $\mu > \frac{n}{2}$.
- $\alpha \in (\frac{1}{n}, \frac{2}{n})$. First let s be arbitrarily close to $\frac{n}{n\alpha - 1}$ and let $q = \frac{p}{2}$ such that (3.9) is accomplished. Then, it holds that $\max\{\frac{s}{2}, \frac{n}{2}\} = \frac{s}{2}$, so that restriction on θ' (see above) reads $\theta' > \frac{s}{2}$. Subsequently,

$$0 < \beta_1 + \gamma_1 = \frac{p + 2m_2 - m_1 - 1 - \frac{1}{\theta}}{m_1 + p - 2 + \frac{2}{n}} + \frac{2 - \frac{s}{\theta'}}{p + \frac{2s}{n} - s},$$

  and

$$0 < \beta_2 + \gamma_2 = \frac{2\alpha - \frac{1}{\mu}}{m_1 + p - 2 + \frac{2}{n}} + \frac{p - 2 - \frac{s}{\mu'}}{p + \frac{2s}{n} - s}.$$

  Since from $\theta' > \frac{s}{2}$ we have that θ' approaches $\frac{n}{2(n\alpha - 1)}$, an already used reasoning implies that upon enlarging p condition (3.7) yields $\beta_1 + \gamma_1 < 1$. On the other hand, in order to have $\beta_2 + \gamma_2 < 1$ we have to invoke the above constraint on μ, i.e., $\mu > \frac{1}{2\alpha}$.
- $\alpha \in [\frac{2}{n}, 1]$. By considering in the previous case $\theta' > \frac{n}{2}$, we conclude by means of (3.8).

By reasoning similarly to what we have done before for the range of α and exchanging μ' with $\tilde{\mu}'$, q with r, θ' with $\tilde{\theta}'$ and α with γ, we have the claim for the cases $\gamma \in (0, \frac{1}{n}]$, $\gamma \in (\frac{1}{n}, \frac{2}{n})$ and $\gamma \in [\frac{2}{n}, 1]$.

The final part is simply obtained by considering permutations of the ranges of α and γ. $\square$

In the concluding part of the paper we will also invoke the next result, by means of which products of powers will be estimated by suitable sums involving their bases and powers of sums controlled by sums of powers.

Lemma 3.3. *Let $a, b, c \geq 0$ and $d_1, d_2 > 0$ such that $d_1 + d_2 < 1$. Then for all $\epsilon > 0$ there exists $d > 0$ such that*

$$a^{d_1} b^{d_2} \leq \epsilon(a+b) + d.$$

Moreover, for further $d_3, d_4, d_5 > 0$, it is possible to find positive $d_6, \hat{d}$ and $\tilde{d}$ such that

$$a^{d_3} + b^{d_4} + c^{d_5} \geq \hat{d}(a+b+c)^{d_6} - \tilde{d}.$$

Proof. The proof is based on manipulations of Young's inequality and some details are available in [5, Lemma 4.3] and [15, Lemma 3.3]. □

Remark 3.4. In view of its importance in the computations, we have to point out that from the above Lemma 3.2, the parameter s can be chosen arbitrarily large only when $\alpha, \gamma \in \big(0, \frac{1}{n}\big]$ (this is connected to Lemma 3.1). In particular, as it will be clear later, in this interval the terms $\int_\Omega (u+1)^{p+2m_2-m_1-1}|\nabla v|^2$, $\int_\Omega (u+1)^{2\alpha}|\nabla v|^{2(q-1)}$, $\int_\Omega (u+1)^{p+2m_3-m_1-1}|\nabla w|^2$ and $\int_\Omega (u+1)^{2\gamma}|\nabla w|^{2(r-1)}$, appearing in our reasoning when dealing with the control of the functional defined in (2.5), can be controlled by invoking either the Young or the Gagliardo–Nirenberg inequality.

4. A priori estimates and proof of the theorems.

4.1. The non-logistic case.

In order to exploit the boundedness criterion (3.3), let us analyze the behavior of functional defined in (2.5), with $p, q, r > 1$ properly large.

In the spirit of Remark 3.4, the first steps toward the uniform bound of $\int_\Omega u^p$ will focus on controlling the evolution in time of the functional $y(t)$ by employing the Young inequality.

Lemma 4.1. *If $m_1, m_2, m_3 \in \mathbb{R}$ comply with $m_1 > \max\{2m_2 - 1, 2m_3 - 1, \frac{n-2}{n}\}$ or $m_1 > \max\{m_2 - \frac{1}{n}, m_3 - \frac{1}{n}, \frac{n-2}{n}\}$ or $m_1 > \max\{2m_2 - 1, m_3 - \frac{1}{n}, \frac{n-2}{n}\}$ or $m_1 > \max\{m_2 - \frac{1}{n}, 2m_3 - 1, \frac{n-2}{n}\}$ whenever $\alpha, \gamma \in (0, \frac{1}{n}]$, or $m_1 > \max\{2m_2, 2m_3, \frac{n-2}{n}\}$ whenever $\alpha, \gamma \in (\frac{1}{n}, 1)$, then there exist $p, q, r > 1$ such that (u, v, w) satisfies for some $c_{33}, c_{34}, c_{35}, c_{36} > 0$ and for all $t \in (0, T_{\max})$*

$$\frac{d}{dt}\left(\int_\Omega (u+1)^p + \int_\Omega |\nabla v|^{2q} + \int_\Omega |\nabla w|^{2r}\right) + c_{33}\int_\Omega |\nabla|\nabla v|^q|^2 + c_{34}\int_\Omega |\nabla|\nabla w|^r|^2 + c_{35}\int_\Omega |\nabla(u+1)^{\frac{m_1+p-1}{2}}|^2 \leq c_{36}. \quad (4.1)$$

Proof. Let $p = q = r > 1$ be sufficiently large; moreover, in view of Remark 3.4, if necessary we are allowed to arbitrarily enlarge these parameters.

For estimates of the term $\frac{d}{dt}\int_\Omega (u+1)^p$, standard testing procedures provide for all $t \in (0, T_{\max})$

$$\frac{d}{dt}\int_\Omega (u+1)^p = \int_\Omega p(u+1)^{p-1}u_t$$
$$= -p(p-1)\int_\Omega (u+1)^{p+m_1-3}|\nabla u|^2 + p(p-1)\chi\int_\Omega u(u+1)^{m_2+p-3}\nabla u \cdot \nabla v$$

$$-p(p-1)\xi\int_\Omega u(u+1)^{m_3+p-3}\nabla u\cdot\nabla w. \tag{4.2}$$

An application of the Young inequality to the second and the third integral in (4.2) give on $(0,T_{\max})$ for $\epsilon_1,\epsilon_2>0$ and some positive c_2,c_3

$$\begin{aligned}&p(p-1)\chi\int_\Omega u(u+1)^{m_2+p-3}\nabla u\cdot\nabla v\\&\leq \epsilon_1\int_\Omega (u+1)^{p+m_1-3}|\nabla u|^2+c_2\int_\Omega (u+1)^{p+2m_2-m_1-1}|\nabla v|^2,\end{aligned} \tag{4.3}$$

and

$$\begin{aligned}&-p(p-1)\xi\int_\Omega u(u+1)^{m_3+p-3}\nabla u\cdot\nabla w\\&\leq \epsilon_2\int_\Omega (u+1)^{p+m_1-3}|\nabla u|^2+c_3\int_\Omega (u+1)^{p+2m_3-m_1-1}|\nabla w|^2.\end{aligned} \tag{4.4}$$

Case 1: $\alpha,\gamma\in(0,\frac{1}{n}]$ and $m_1>\max\{2m_2-1,2m_3-1,\frac{n-2}{n}\}$ or $m_1>\max\{m_2-\frac{1}{n},m_3-\frac{1}{n},\frac{n-2}{n}\}$ or $m_1>\max\{2m_2-1,m_3-\frac{1}{n},\frac{n-2}{n}\}$ or $m_1>\max\{m_2-\frac{1}{n},2m_3-1,\frac{n-2}{n}\}$. The Young inequality and bound (3.4) yield for all $t\in(0,T_{\max})$

$$\begin{aligned}c_2\int_\Omega (u+1)^{p+2m_2-m_1-1}|\nabla v|^2&\leq \epsilon_3\int_\Omega|\nabla v|^s+c_4\int_\Omega (u+1)^{\frac{(p+2m_2-m_1-1)s}{s-2}}\\&\leq c_4\int_\Omega (u+1)^{\frac{(p+2m_2-m_1-1)s}{s-2}}+c_5,\end{aligned} \tag{4.5}$$

with $\epsilon_3>0$ and some positive c_4,c_5.

Let us now dedicate ourselves to the cases $m_1>\max\{2m_2-1,\frac{n-2}{n}\}$ and $m_1>m_2-\frac{1}{n}$, respectively. From $m_1>2m_2-1$, we have $\frac{(p+2m_2-m_1-1)s}{s-2}<p$, and for every $\epsilon_4>0$, Young's inequality yields some $c_6>0$ entailing

$$c_4\int_\Omega (u+1)^{\frac{(p+2m_2-m_1-1)s}{s-2}}\leq \epsilon_4\int_\Omega (u+1)^p+c_6\quad\text{on }(0,T_{\max}), \tag{4.6}$$

with $\epsilon_4>0$ and positive c_6. Further, an application of the Gagliardo–Nirenberg inequality and property (3.1) yield

$$\theta=\frac{\frac{n(m_1+p-1)}{2}\left(1-\frac{1}{p}\right)}{1-\frac{n}{2}+\frac{n(m_1+p-1)}{2}}\in(0,1),$$

so giving for $c_7,c_8>0$

$$\begin{aligned}\int_\Omega (u+1)^p&=\|(u+1)^{\frac{m_1+p-1}{2}}\|^{\frac{2p}{m_1+p-1}}_{L^{\frac{2p}{m_1+p-1}}(\Omega)}\\&\leq c_7\|\nabla(u+1)^{\frac{m_1+p-1}{2}}\|^{\frac{2p}{m_1+p-1}\theta}_{L^2(\Omega)}\|(u+1)^{\frac{m_1+p-1}{2}}\|^{\frac{2p}{m_1+p-1}(1-\theta)}_{L^{\frac{2}{m_1+p-1}}(\Omega)}\\&\quad+c_7\|(u+1)^{\frac{m_1+p-1}{2}}\|^{\frac{2p}{m_1+p-1}}_{L^{\frac{2}{m_1+p-1}}(\Omega)}\\&\leq c_8\Big(\int_\Omega|\nabla(u+1)^{\frac{m_1+p-1}{2}}|^2\Big)^{\kappa_1}+c_8\qquad\text{for all }t\in(0,T_{\max}).\end{aligned}$$

Since $\kappa_1\in(0,1)$ (see Lemma 3.2), for any positive ϵ_5 thanks to the Young inequality we arrive for some positive $c_9>0$ at

$$\epsilon_4\int_\Omega (u+1)^p\leq \epsilon_5\int_\Omega|\nabla(u+1)^{\frac{m_1+p-1}{2}}|^2+c_9\quad\text{on }(0,T_{\max}). \tag{4.7}$$

Alternatively, we can treat the integral in the left hand side of (4.6) by applying the Gagliardo–Nirenberg inequality and again bound (3.1), so having

$$\begin{aligned}
&\int_\Omega (u+1)^{\frac{(p+2m_2-m_1-1)s}{s-2}}\\
&= \|(u+1)^{\frac{m_1+p-1}{2}}\|_{L^{\frac{2s(p+2m_2-m_1-1)}{(s-2)(m_1+p-1)}}(\Omega)}^{\frac{2s(p+2m_2-m_1-1)}{(s-2)(m_1+p-1)}}\\
&\le c_{10}\|\nabla(u+1)^{\frac{m_1+p-1}{2}}\|_{L^2(\Omega)}^{\frac{2s(p+2m_2-m_1-1)}{(s-2)(m_1+p-1)}\theta_1}\|(u+1)^{\frac{m_1+p-1}{2}}\|_{L^{\frac{2}{m_1+p-1}}(\Omega)}^{\frac{2s(p+2m_2-m_1-1)}{(s-2)(m_1+p-1)}(1-\theta_1)}\\
&\quad + c_{10}\|(u+1)^{\frac{m_1+p-1}{2}}\|_{L^{\frac{2}{m_1+p-1}}(\Omega)}^{\frac{2s(p+2m_2-m_1-1)}{(s-2)(m_1+p-1)}}\\
&\le c_{11}\Big(\int_\Omega |\nabla(u+1)^{\frac{m_1+p-1}{2}}|^2\Big)^{\kappa} + c_{11} \quad \text{for all } t\in(0,T_{\max}),
\end{aligned}$$

with

$$\theta_1 = \frac{\frac{m_1+p-1}{2}\Big(1-\frac{s-2}{(p+2m_2-m_1-1)s}\Big)}{\frac{m_1+p-1}{2}+\frac{1}{n}-\frac{1}{2}} \text{ and } \kappa = \frac{s(p+2m_2-m_1-1)-(s-2)}{(s-2)(m_1+p-2+\frac{2}{n})},$$

for some $c_{10}, c_{11} > 0$. In particular $\theta_1 \in (0,1)$, and from $m_1 > m_2 - \frac{1}{n}$, since s can be arbitrarily enlarged, continuity arguments imply $\kappa \in (0,1)$; in addition, the Young inequality allows us to rephrase (4.6) as

$$c_4\int_\Omega (u+1)^{\frac{(p+2m_2-m_1-1)s}{s-2}} \le \epsilon_6\int_\Omega |\nabla(u+1)^{\frac{m_1+p-1}{2}}|^2 + c_{12} \quad \text{on } (0,T_{\max}), \tag{4.8}$$

with $\epsilon_6 > 0$ and some positive c_{12}.

Treating in a similar way the second integral on the right-hand side of (4.4) and exploiting bound (3.5) yield on $(0,T_{\max})$

$$c_3\int_\Omega (u+1)^{p+2m_3-m_1-1}|\nabla w|^2 \le c_{13}\int_\Omega (u+1)^{\frac{(p+2m_3-m_1-1)s}{s-2}} + c_{14}, \tag{4.9}$$

with positive c_{13}, c_{14}.

Let us now turn our attention to the situation where $m_1 > \max\{2m_3-1, \frac{n-2}{n}\}$ and $m_1 > m_3 - \frac{1}{n}$, respectively. In the same flavor as before, from $m_1 > 2m_3-1$, we have $\frac{(p+2m_3-m_1-1)s}{s-2} < p$, and for every $\epsilon_7, \epsilon_8 > 0$, the Young and the Gagliardo–Nirenberg inequalities yield some $c_{15}, c_{16} > 0$ entailing

$$\begin{aligned}
c_3\int_\Omega (u+1)^{\frac{(p+2m_3-m_1-1)s}{s-2}} &\le \epsilon_7\int_\Omega (u+1)^p + c_{15}\\
&\le \epsilon_8\int_\Omega |\nabla(u+1)^{\frac{m_1+p-1}{2}}|^2 + c_{16} \quad \text{on } (0,T_{\max}).
\end{aligned} \tag{4.10}$$

On the other hand, by exploiting the condition $m_1 > m_3 - \frac{1}{n}$ again the Gagliardo–Nirenberg inequality yields for all $t \in (0,T_{\max})$

$$\begin{aligned}
&\int_\Omega (u+1)^{\frac{(p+2m_3-m_1-1)s}{s-2}}\\
&= \|(u+1)^{\frac{m_1+p-1}{2}}\|_{L^{\frac{2s(p+2m_3-m_1-1)}{(s-2)(m_1+p-1)}}(\Omega)}^{\frac{2s(p+2m_3-m_1-1)}{(s-2)(m_1+p-1)}}\\
&\le c_{17}\|\nabla(u+1)^{\frac{m_1+p-1}{2}}\|_{L^2(\Omega)}^{\frac{2s(p+2m_3-m_1-1)}{(s-2)(m_1+p-1)}\theta_2}\|(u+1)^{\frac{m_1+p-1}{2}}\|_{L^{\frac{2}{m_1+p-1}}(\Omega)}^{\frac{2s(p+2m_3-m_1-1)}{(s-2)(m_1+p-1)}(1-\theta_2)}
\end{aligned}$$

$$+ c_{17}\|(u+1)^{\frac{m_1+p-1}{2}}\|^{\frac{2s(p+2m_3-m_1-1)}{(s-2)(m_1+p-1)}}_{L^{\frac{2}{m_1+p-1}}(\Omega)} \leq c_{18}\Big(\int_\Omega |\nabla(u+1)^{\frac{m_1+p-1}{2}}|^2\Big)^{\hat{\kappa}} + c_{18},$$

with

$$\theta_2 = \frac{\frac{m_1+p-1}{2}\left(1-\frac{s-2}{(p+2m_3-m_1-1)s}\right)}{\frac{m_1+p-1}{2}+\frac{1}{n}-\frac{1}{2}} \in (0,1) \text{ and } \hat{\kappa} = \frac{s(p+2m_3-m_1-1)-(s-2)}{(s-2)(m_1+p-2+\frac{2}{n})},$$

for some $c_{17}, c_{18} > 0$, and with $\theta_2, \hat{\kappa} \in (0,1)$. In this way, (4.10) becomes

$$c_3\int_\Omega (u+1)^{\frac{(p+2m_3-m_1-1)s}{s-2}} \leq \epsilon_9 \int_\Omega |\nabla(u+1)^{\frac{m_1+p-1}{2}}|^2 + c_{19} \quad \text{on } (0,T_{\max}), \tag{4.11}$$

with $\epsilon_9 > 0$ and some positive c_{19}. By plugging estimates (4.3) and (4.4) into relation (4.2), and by relying on bounds (4.5)-(4.7) and (4.9), (4.10) (or, alternatively to (4.6) and (4.10), relations (4.8) and (4.11)), infer for appropriate $\tilde{\epsilon}_1 > 0$ and proper $c_{20} > 0$ for all $t \in (0,T_{\max})$

$$\frac{d}{dt}\int_\Omega (u+1)^p \leq \left(-\frac{4p(p-1)}{(m_1+p-1)^2}+\tilde{\epsilon}_1\right)\int_\Omega |\nabla(u+1)^{\frac{m_1+p-1}{2}}|^2 + c_{20}, \tag{4.12}$$

where we also have taken into consideration

$$\int_\Omega (u+1)^{p+m_1-3}|\nabla u|^2 = \frac{4}{(m_1+p-1)^2}\int_\Omega |\nabla(u+1)^{\frac{m_1+p-1}{2}}|^2 \quad \text{on } (0,T_{\max}). \tag{4.13}$$

Now, we treat the terms $\frac{d}{dt}\int_\Omega |\nabla v|^{2q}$ and $\frac{d}{dt}\int_\Omega |\nabla w|^{2r}$ of the functional $y(t)$ under the assumption that $m_1 > \frac{n-2}{n}$. As to the term $\frac{d}{dt}\int_\Omega |\nabla v|^{2q}$, reasoning similarly as in [4, Lemma 5.3], we obtain for some $c_{21}, c_{22} > 0$ for all $t \in (0,T_{\max})$

$$\frac{d}{dt}\int_\Omega |\nabla v|^{2q} + q\int_\Omega |\nabla v|^{2q-2}|D^2 v|^2 \leq c_{21}\int_\Omega u^{2\alpha}|\nabla v|^{2q-2} + c_{22}. \tag{4.14}$$

Moreover, Young's inequality and bound (3.4) give for every arbitrary $\epsilon_{10}, \epsilon_{11}, \epsilon_{12} > 0$ and some $c_{23}, c_{24}, c_{25}, c_{26} > 0$

$$\begin{aligned} c_{21}\int_\Omega u^{2\alpha}|\nabla v|^{2q-2} &\leq \epsilon_{10}\int_\Omega u^p + c_{23}\int_\Omega |\nabla v|^{\frac{2(q-1)p}{p-2\alpha}} \\ &\leq \epsilon_{10}\int_\Omega (u+1)^p + \epsilon_{11}\int_\Omega |\nabla v|^s + c_{24} \leq \epsilon_{10}\int_\Omega (u+1)^p + c_{25} \\ &\leq \epsilon_{12}\int_\Omega |\nabla(u+1)^{\frac{m_1+p-1}{2}}|^2 + c_{26} \quad \text{on } (0,T_{\max}). \end{aligned} \tag{4.15}$$

As to the term $\frac{d}{dt}\int_\Omega |\nabla w|^{2r}$ of the functional $y(t)$, with bound (3.5) in our hands, through similar aforedescribed computations we obtain for all $t \in (0,T_{\max})$

$$\begin{aligned} &\frac{d}{dt}\int_\Omega |\nabla w|^{2r} + r\int_\Omega |\nabla w|^{2r-2}|D^2 w|^2 \leq c_{27}\int_\Omega u^{2\gamma}|\nabla w|^{2r-2} + c_{28} \\ &\leq \epsilon_{13}\int_\Omega u^p + c_{29}\int_\Omega |\nabla w|^{\frac{2(r-1)p}{p-2\gamma}} \leq \epsilon_{13}\int_\Omega (u+1)^p + \epsilon_{14}\int_\Omega |\nabla w|^s + c_{30} \\ &\leq \epsilon_{13}\int_\Omega (u+1)^p + c_{31} \leq \epsilon_{15}\int_\Omega |\nabla(u+1)^{\frac{m_1+p-1}{2}}|^2 + c_{32}, \end{aligned} \tag{4.16}$$

with $\epsilon_{13}, \epsilon_{14}, \epsilon_{15} > 0$ and some $c_{27}, c_{28}, c_{29}, c_{30}, c_{31}, c_{32} > 0$. Therefore, by inserting relation (4.15) into (4.14) and adding (4.12) and (4.16), we have the claim for a proper choice of $\tilde{\epsilon}_1$ and some positive $c_{33}, c_{34}, c_{35}, c_{36}$, once relations (see [4, page 17])

$$\begin{aligned}|\nabla|\nabla v|^q|^2 &= \frac{q^2}{4}|\nabla v|^{2q-4}|\nabla|\nabla v|^2|^2 = q^2|\nabla v|^{2q-4}|D^2 v\nabla v|^2\\ &\le q^2|\nabla v|^{2q-2}|D^2 v|^2 \quad \text{on } (0, T_{\max}),\end{aligned} \tag{4.17}$$

and

$$\begin{aligned}|\nabla|\nabla w|^r|^2 &= \frac{r^2}{4}|\nabla w|^{2r-4}|\nabla|\nabla w|^2|^2 = r^2|\nabla w|^{2r-4}|D^2 w\nabla w|^2\\ &\le r^2|\nabla w|^{2r-2}|D^2 w|^2 \quad \text{on } (0, T_{\max}),\end{aligned} \tag{4.18}$$

are considered too.

Case 2: $\alpha, \gamma \in (\frac{1}{n}, 1)$ and $m_1 > \max\{2m_2, 2m_3, \frac{n-2}{n}\}$. According to Remark 3.4, since in this case s has a finite upper bound, a different approach to deal with relations (4.5), (4.9), (4.15) and (4.16) has to be used. In particular, for $\bar{\epsilon}_1 > 0$ and some $\bar{c}_1 > 0$ we can estimate relation (4.5) on $(0, T_{\max})$ as follows:

$$c_2\int_\Omega (u+1)^{p+2m_2-m_1-1}|\nabla v|^2 \le \bar{\epsilon}_1\int_\Omega |\nabla v|^{2(p+1)} + \bar{c}_1\int_\Omega (u+1)^{\frac{(p+2m_2-m_1-1)(p+1)}{p}}.$$

Now, if $m_1 > 2m_2$, then some p sufficiently large infers $\frac{(p+2m_2-m_1-1)(p+1)}{p} < p$, so that for any positive $\bar{\epsilon}_2, \bar{\epsilon}_3$ and some $\bar{c}_2, \bar{c}_3 > 0$ we have

$$\begin{aligned}\bar{c}_1\int_\Omega (u+1)^{\frac{(p+2m_2-m_1-1)(p+1)}{p}} &\le \bar{\epsilon}_2\int_\Omega (u+1)^p + \bar{c}_2\\ &\le \bar{\epsilon}_3\int_\Omega |\nabla (u+1)^{\frac{m_1+p-1}{2}}|^2 + \bar{c}_3 \quad \text{for all } t \in (0, T_{\max}),\end{aligned}$$

where in the last implication we used $m_1 > \frac{n-2}{n}$ (as in the previous case). In a similar way, for $m_1 > 2m_3$ and $m_1 > \frac{n-2}{n}$ we have for any positive $\bar{\epsilon}_4, \bar{\epsilon}_5, \bar{\epsilon}_6$ and some $\bar{c}_4, \bar{c}_5, \bar{c}_6 > 0$

$$\begin{aligned}&c_3\int_\Omega (u+1)^{p+2m_3-m_1-1}|\nabla w|^2\\ &\le \bar{\epsilon}_4\int_\Omega |\nabla w|^{2(p+1)} + \bar{c}_4\int_\Omega (u+1)^{\frac{(p+2m_3-m_1-1)(p+1)}{p}}\\ &\le \bar{\epsilon}_4\int_\Omega |\nabla w|^{2(p+1)} + \bar{\epsilon}_5\int_\Omega (u+1)^p + \bar{c}_5\\ &\le \bar{\epsilon}_4\int_\Omega |\nabla w|^{2(p+1)} + \bar{\epsilon}_6\int_\Omega |\nabla (u+1)^{\frac{m_1+p-1}{2}}|^2 + \bar{c}_6 \quad \text{on } (0, T_{\max}).\end{aligned}$$

Let us focus on the integrals $\int_\Omega |\nabla v|^{\frac{2p(p-1)}{p-2\alpha}}$ and $\int_\Omega |\nabla w|^{\frac{2p(p-1)}{p-2\gamma}}$. Since $\alpha, \gamma < 1$, this implies that $\frac{2p(p-1)}{p-2\alpha} < 2(p+1)$ and $\frac{2p(p-1)}{p-2\gamma} < 2(p+1)$, and subsequently an application of the Young inequality leads to

$$c_{23}\int_\Omega |\nabla v|^{\frac{2p(p-1)}{p-2\alpha}} \le \bar{\epsilon}_7\int_\Omega |\nabla v|^{2(p+1)} + \bar{c}_7 \quad \text{for all } t \in (0, T_{\max}),$$

and

$$c_{29}\int_\Omega |\nabla w|^{\frac{2p(p-1)}{p-2\gamma}} \le \bar{\epsilon}_8\int_\Omega |\nabla w|^{2(p+1)} + \bar{c}_8 \quad \text{on } (0, T_{\max}),$$

with $\bar{\epsilon}_7, \bar{\epsilon}_8 > 0$ and some positive $\bar{c}_7, \bar{c}_8$. By taking into account [12, Lemma 2.2] and bounds (3.2), we get

$$\int_\Omega |\nabla v|^{2(p+1)} \le 2(4p^2+n)\|v_0\|^2_{L^\infty} \int_\Omega |\nabla v|^{2p-2}|D^2 v|^2 \quad \text{for all } t \in (0, T_{\max}),$$

and

$$\int_\Omega |\nabla w|^{2(p+1)} \le 2(4p^2+n)\|w_0\|^2_{L^\infty} \int_\Omega |\nabla w|^{2p-2}|D^2 w|^2 \quad \text{on } (0, T_{\max});$$

the rest of the proof is an evident adaptation of previous reasoning.

We conclude by observing that this lemma holds in each of the following cases:

▷ $\alpha \in (0, \frac{1}{n}]$ and $\gamma \in (\frac{1}{n}, 1)$, whenever $m_1 > \max\{2m_2 - 1, 2m_3, \frac{n-2}{n}\}$ or $m_1 > \max\{m_2 - \frac{1}{n}, 2m_3, \frac{n-2}{n}\}$

▷ $\alpha \in (\frac{1}{n}, 1)$ and $\gamma \in (0, \frac{1}{n}]$, whenever $m_1 > \max\{2m_2, 2m_3 - 1, \frac{n-2}{n}\}$ or $m_1 > \max\{2m_2, m_3 - \frac{1}{n}, \frac{n-2}{n}\}$. □

Let us now turn our attention to the case where, as mentioned before, the Gagliardo–Nirenberg inequality is employed. In this case, we can derive information also for $\alpha, \gamma = 1$; this is the reason why in Theorem 2.2 we distinguish the situations where the value 1 does or does not belong to the interval in question.

Lemma 4.2. *If $m_1, m_2, m_3 \in \mathbb{R}$ and $\alpha, \gamma > 0$ are taken accordingly to* (3.6), (3.7), (3.8)*, then there exist $p, q, r > 1$ such that (u, v, w) satisfies a similar inequality as in* (4.1).

Proof. For s, p, q and r taken according to Lemma 3.2 (in particular, $p = q = r$ for $\alpha, \gamma \in \left(0, \frac{1}{n}\right]$, and $q = r = \frac{p}{2}$ for $\alpha, \gamma \in \left(\frac{1}{n}, 1\right]$), let $\theta, \theta', \tilde{\theta}, \tilde{\theta}', \mu, \mu', \tilde{\mu}, \tilde{\mu}'$, $a_1, a_2, a_3, a_4, \tilde{a}_1, \tilde{a}_2, \tilde{a}_3, \tilde{a}_4$ and $\beta_1, \beta_2, \tilde{\beta}_1, \tilde{\beta}_2, \gamma_1, \tilde{\gamma}_1, \gamma_2, \tilde{\gamma}_2$ be therein defined.

With a view to Lemma 4.1, by manipulating relation (4.2) and focusing on the inequalities (4.3), (4.4), (4.14) and on the first inequality in (4.16), a proper $\tilde{\epsilon}_1$ leads for all $t \in (0, T_{\max})$ to

$$\begin{aligned} \frac{d}{dt}&\left(\int_\Omega (u+1)^p + \int_\Omega |\nabla v|^{2q} + \int_\Omega |\nabla w|^{2r}\right) + q\int_\Omega |\nabla v|^{2q-2}|D^2 v|^2 \\ &+ r\int_\Omega |\nabla w|^{2r-2}|D^2 w|^2 \\ \le &\left(-\frac{4p(p-1)}{(m_1+p-1)^2} + \tilde{\epsilon}_1\right)\int_\Omega |\nabla (u+1)^{\frac{m_1+p-1}{2}}|^2 \\ &+ c_2\int_\Omega (u+1)^{p+2m_2-m_1-1}|\nabla v|^2 + c_{21}\int_\Omega u^{2\alpha}|\nabla v|^{2q-2} \\ &+ c_3\int_\Omega (u+1)^{p+2m_3-m_1-1}|\nabla w|^2 + c_{27}\int_\Omega u^{2\gamma}|\nabla w|^{2r-2} + c_{37}, \end{aligned} \tag{4.19}$$

for some $c_{37} > 0$ (we also used relation (4.13)). In this way, we can estimate the last four integrals on the right-hand side of (4.19) by applying the Hölder inequality so to have for all $t \in (0, T_{\max})$

$$\int_\Omega (u+1)^{p+2m_2-m_1-1}|\nabla v|^2 \le \left(\int_\Omega (u+1)^{(p+2m_2-m_1-1)\theta}\right)^{\frac{1}{\theta}} \left(\int_\Omega |\nabla v|^{2\theta'}\right)^{\frac{1}{\theta'}}, \tag{4.20}$$

$$\int_\Omega (u+1)^{2\alpha}|\nabla v|^{2q-2} \le \left(\int_\Omega (u+1)^{2\alpha\mu}\right)^{\frac{1}{\mu}} \left(\int_\Omega |\nabla v|^{2(q-1)\mu'}\right)^{\frac{1}{\mu'}}, \tag{4.21}$$

and for all $t \in (0, T_{\max})$

$$\int_\Omega (u+1)^{p+2m_3-m_1-1}|\nabla w|^2 \le \left(\int_\Omega (u+1)^{(p+2m_3-m_1-1)\tilde{\theta}}\right)^{\frac{1}{\tilde{\theta}}}\left(\int_\Omega |\nabla w|^{2\tilde{\theta}'}\right)^{\frac{1}{\tilde{\theta}'}}, \quad (4.22)$$

$$\int_\Omega (u+1)^{2\gamma}|\nabla w|^{2r-2} \le \left(\int_\Omega (u+1)^{2\gamma\tilde{\mu}}\right)^{\frac{1}{\tilde{\mu}}}\left(\int_\Omega |\nabla w|^{2(r-1)\tilde{\mu}'}\right)^{\frac{1}{\tilde{\mu}'}}. \quad (4.23)$$

By invoking the Gagliardo–Nirenberg inequality and bound (3.1), we obtain for some $c_{38}, c_{39} > 0$

$$\begin{aligned}
&\left(\int_\Omega (u+1)^{(p+2m_2-m_1-1)\theta}\right)^{\frac{1}{\theta}}\\
&= \|(u+1)^{\frac{m_1+p-1}{2}}\|^{\frac{2(p+2m_2-m_1-1)}{m_1+p-1}}_{L^{\frac{2(p+2m_2-m_1-1)}{m_1+p-1}\theta}(\Omega)}\\
&\le c_{38}\|\nabla(u+1)^{\frac{m_1+p-1}{2}}\|^{\frac{2(p+2m_2-m_1-1)}{m_1+p-1}a_1}_{L^2(\Omega)}\|(u+1)^{\frac{m_1+p-1}{2}}\|^{\frac{2(p+2m_2-m_1-1)}{m_1+p-1}(1-a_1)}_{L^{\frac{2}{m_1+p-1}}(\Omega)}\\
&\quad + c_{38}\|(u+1)^{\frac{m_1+p-1}{2}}\|^{\frac{2(p+2m_2-m_1-1)}{m_1+p-1}}_{L^{\frac{2}{m_1+p-1}}(\Omega)}\\
&\le c_{39}\left(\int_\Omega |\nabla(u+1)^{\frac{m_1+p-1}{2}}|^2\right)^{\beta_1} + c_{39} \quad \text{on } (0, T_{\max}),
\end{aligned} \quad (4.24)$$

and for some $c_{40}, c_{41} > 0$

$$\begin{aligned}
&\left(\int_\Omega (u+1)^{(p+2m_3-m_1-1)\tilde{\theta}}\right)^{\frac{1}{\tilde{\theta}}}\\
&= \|(u+1)^{\frac{m_1+p-1}{2}}\|^{\frac{2(p+2m_3-m_1-1)}{m_1+p-1}}_{L^{\frac{2(p+2m_3-m_1-1)}{m_1+p-1}\tilde{\theta}}(\Omega)}\\
&\le c_{40}\|\nabla(u+1)^{\frac{m_1+p-1}{2}}\|^{\frac{2(p+2m_3-m_1-1)}{m_1+p-1}\tilde{a}_1}_{L^2(\Omega)}\|(u+1)^{\frac{m_1+p-1}{2}}\|^{\frac{2(p+2m_3-m_1-1)}{m_1+p-1}(1-\tilde{a}_1)}_{L^{\frac{2}{m_1+p-1}}(\Omega)}\\
&\quad + c_{40}\|(u+1)^{\frac{m_1+p-1}{2}}\|^{\frac{2(p+2m_3-m_1-1)}{m_1+p-1}}_{L^{\frac{2}{m_1+p-1}}(\Omega)}\\
&\le c_{41}\left(\int_\Omega |\nabla(u+1)^{\frac{m_1+p-1}{2}}|^2\right)^{\tilde{\beta}_1} + c_{41} \quad \text{for all } t \in (0, T_{\max}).
\end{aligned} \quad (4.25)$$

Moreover, we get for some $c_{42}, c_{43} > 0$

$$\begin{aligned}
\left(\int_\Omega (u+1)^{2\alpha\mu}\right)^{\frac{1}{\mu}} &= \|(u+1)^{\frac{m_1+p-1}{2}}\|^{\frac{4\alpha}{m_1+p-1}}_{L^{\frac{4\alpha\mu}{m_1+p-1}}(\Omega)}\\
&\le c_{42}\|\nabla(u+1)^{\frac{m_1+p-1}{2}}\|^{\frac{4\alpha}{m_1+p-1}a_3}_{L^2(\Omega)}\|(u+1)^{\frac{m_1+p-1}{2}}\|^{\frac{4\alpha}{m_1+p-1}(1-a_3)}_{L^{\frac{2}{m_1+p-1}}(\Omega)}\\
&\quad + c_{42}\|(u+1)^{\frac{m_1+p-1}{2}}\|^{\frac{4\alpha}{m_1+p-1}}_{L^{\frac{2}{m_1+p-1}}(\Omega)}\\
&\le c_{43}\left(\int_\Omega |\nabla(u+1)^{\frac{m_1+p-1}{2}}|^2\right)^{\beta_2} + c_{43} \quad \text{on } (0, T_{\max}),
\end{aligned} \quad (4.26)$$

and for some $c_{44}, c_{45} > 0$

$$\begin{aligned}\left(\int_\Omega (u+1)^{2\gamma\tilde{\mu}}\right)^{\frac{1}{\tilde{\mu}}} &= \|(u+1)^{\frac{m_1+p-1}{2}}\|_{L^{\frac{4\gamma\tilde{\mu}}{m_1+p-1}}(\Omega)}^{\frac{4\gamma}{m_1+p-1}}\\
&\le c_{44}\|\nabla(u+1)^{\frac{m_1+p-1}{2}}\|_{L^2(\Omega)}^{\frac{4\gamma}{m_1+p-1}\tilde{a}_3}\|(u+1)^{\frac{m_1+p-1}{2}}\|_{L^{\frac{2}{m_1+p-1}}(\Omega)}^{\frac{4\gamma}{m_1+p-1}(1-\tilde{a}_3)}\\
&\quad + c_{44}\|(u+1)^{\frac{m_1+p-1}{2}}\|_{L^{\frac{2}{m_1+p-1}}(\Omega)}^{\frac{4\gamma}{m_1+p-1}}\\
&\le c_{45}\left(\int_\Omega |\nabla(u+1)^{\frac{m_1+p-1}{2}}|^2\right)^{\tilde{\beta}_2} + c_{45} \text{ for all } t\in(0,T_{\max}). \quad (4.27)\end{aligned}$$

In a similar way, we can again apply the Gagliardo–Nirenberg inequality and bound (3.4) and get for some $c_{46}, c_{47} > 0$

$$\begin{aligned}\left(\int_\Omega |\nabla v|^{2\theta'}\right)^{\frac{1}{\theta'}} &= \||\nabla v|^q\|_{L^{\frac{2\theta'}{q}}(\Omega)}^{\frac{2}{q}}\\
&\le c_{46}\|\nabla|\nabla v|^q\|_{L^2(\Omega)}^{\frac{2}{q}a_2}\||\nabla v|^q\|_{L^{\frac{s}{q}}(\Omega)}^{\frac{2}{q}(1-a_2)} + c_{46}\||\nabla v|^q\|_{L^{\frac{s}{q}}(\Omega)}^{\frac{2}{q}}\\
&\le c_{47}\left(\int_\Omega |\nabla|\nabla v|^q|^2\right)^{\gamma_1} + c_{47} \quad \text{on } (0,T_{\max}), \quad (4.28)\end{aligned}$$

and for some $c_{48}, c_{49} > 0$

$$\begin{aligned}\left(\int_\Omega |\nabla v|^{2(q-1)\mu'}\right)^{\frac{1}{\mu'}} &= \||\nabla v|^q\|_{L^{\frac{2(q-1)}{q}\mu'}(\Omega)}^{\frac{2(q-1)}{q}}\\
&\le c_{48}\|\nabla|\nabla v|^q\|_{L^2(\Omega)}^{\frac{2(q-1)}{q}a_4}\||\nabla v|^q\|_{L^{\frac{s}{q}}(\Omega)}^{\frac{2(q-1)}{q}(1-a_4)} + c_{48}\||\nabla v|^q\|_{L^{\frac{s}{q}}(\Omega)}^{\frac{2(q-1)}{q}}\\
&\le c_{49}\left(\int_\Omega |\nabla|\nabla v|^q|^2\right)^{\gamma_2} + c_{49} \quad \text{for all } t\in(0,T_{\max}). \quad (4.29)\end{aligned}$$

Finally, an application of the Gagliardo–Nirenberg inequality and bound (3.5) imply for some $c_{50}, c_{51} > 0$

$$\begin{aligned}\left(\int_\Omega |\nabla w|^{2\tilde{\theta}'}\right)^{\frac{1}{\tilde{\theta}'}} &= \||\nabla w|^r\|_{L^{\frac{2\tilde{\theta}'}{r}}(\Omega)}^{\frac{2}{r}}\\
&\le c_{50}\|\nabla|\nabla w|^r\|_{L^2(\Omega)}^{\frac{2}{r}\tilde{a}_2}\||\nabla w|^r\|_{L^{\frac{s}{r}}(\Omega)}^{\frac{2}{r}(1-\tilde{a}_2)} + c_{50}\||\nabla w|^r\|_{L^{\frac{s}{r}}(\Omega)}^{\frac{2}{r}}\\
&\le c_{51}\left(\int_\Omega |\nabla|\nabla w|^r|^2\right)^{\tilde{\gamma}_1} + c_{51} \quad \text{on } (0,T_{\max}), \quad (4.30)\end{aligned}$$

and for some $c_{52}, c_{53} > 0$

$$\begin{aligned}\left(\int_\Omega |\nabla w|^{2(r-1)\tilde{\mu}'}\right)^{\frac{1}{\tilde{\mu}'}} &= \||\nabla w|^r\|_{L^{\frac{2(r-1)}{r}\tilde{\mu}'}(\Omega)}^{\frac{2(r-1)}{r}}\\
&\le c_{52}\|\nabla|\nabla w|^r\|_{L^2(\Omega)}^{\frac{2(r-1)}{r}\tilde{a}_4}\||\nabla w|^r\|_{L^{\frac{s}{r}}(\Omega)}^{\frac{2(r-1)}{r}(1-\tilde{a}_4)} + c_{52}\||\nabla w|^r\|_{L^{\frac{s}{r}}(\Omega)}^{\frac{2(r-1)}{r}}\\
&\le c_{53}\left(\int_\Omega |\nabla|\nabla w|^r|^2\right)^{\gamma_2} + c_{53} \quad \text{for all } t\in(0,T_{\max}). \quad (4.31)\end{aligned}$$

By plugging (4.20), (4.21), (4.22) and (4.23) into (4.19) and taking into account (4.24), (4.25), (4.28), (4.30), (4.26), (4.27), (4.29), (4.31), once inequalities (4.17) and (4.18) are considered we can derive for a positive suitable $\tilde{\epsilon}_1$ the following estimate, valid for certain $c_{54}, c_{55}, c_{56}, c_{57}, c_{58}, c_{59}, c_{60}, c_{61} > 0$ and all $t \in (0, T_{\max})$:

$$\begin{aligned}
&\frac{d}{dt}\left(\int_\Omega (u+1)^p + \int_\Omega |\nabla v|^{2q} + \int_\Omega |\nabla w|^{2r}\right) \\
&\quad + c_{54}\int_\Omega |\nabla|\nabla v|^q|^2 + c_{55}\int_\Omega |\nabla|\nabla w|^r|^2 + c_{56}\int_\Omega |\nabla(u+1)^{\frac{m_1+p-1}{2}}|^2 \\
&\le c_{57}\left(\int_\Omega |\nabla(u+1)^{\frac{m_1+p-1}{2}}|^2\right)^{\beta_1}\left(\int_\Omega |\nabla|\nabla v|^q|^2\right)^{\gamma_1} \\
&\quad + c_{57}\left(\int_\Omega |\nabla(u+1)^{\frac{m_1+p-1}{2}}|^2\right)^{\beta_1} + c_{57}\left(\int_\Omega |\nabla|\nabla v|^q|^2\right)^{\gamma_1} \\
&\quad + c_{58}\left(\int_\Omega |\nabla(u+1)^{\frac{m_1+p-1}{2}}|^2\right)^{\beta_2}\left(\int_\Omega |\nabla|\nabla v|^q|^2\right)^{\gamma_2} \\
&\quad + c_{58}\left(\int_\Omega |\nabla(u+1)^{\frac{m_1+p-1}{2}}|^2\right)^{\beta_2} + c_{58}\left(\int_\Omega |\nabla|\nabla v|^q|^2\right)^{\gamma_2} \\
&\quad + c_{59}\left(\int_\Omega |\nabla(u+1)^{\frac{m_1+p-1}{2}}|^2\right)^{\tilde{\beta}_1}\left(\int_\Omega |\nabla|\nabla w|^r|^2\right)^{\tilde{\gamma}_1} \\
&\quad + c_{59}\left(\int_\Omega |\nabla(u+1)^{\frac{m_1+p-1}{2}}|^2\right)^{\tilde{\beta}_1} + c_{59}\left(\int_\Omega |\nabla|\nabla w|^r|^2\right)^{\tilde{\gamma}_1} \\
&\quad + c_{60}\left(\int_\Omega |\nabla(u+1)^{\frac{m_1+p-1}{2}}|^2\right)^{\tilde{\beta}_2}\left(\int_\Omega |\nabla|\nabla w|^r|^2\right)^{\tilde{\gamma}_2} \\
&\quad + c_{60}\left(\int_\Omega |\nabla(u+1)^{\frac{m_1+p-1}{2}}|^2\right)^{\tilde{\beta}_2} + c_{60}\left(\int_\Omega |\nabla|\nabla w|^r|^2\right)^{\tilde{\gamma}_2} + c_{61}.
\end{aligned} \tag{4.32}$$

Since by Lemma 3.2 we have that $\beta_1+\gamma_1<1$, $\beta_2+\gamma_2<1$, $\tilde{\beta}_1+\tilde{\gamma}_1<1$ and $\tilde{\beta}_2+\tilde{\gamma}_2<1$ and in particular $\beta_1,\gamma_1,\beta_2,\gamma_2,\tilde{\beta}_1,\tilde{\gamma}_1,\tilde{\beta}_2,\tilde{\gamma}_2 \in (0,1)$, we can treat the four integral products and the remaining eight addenda of the right-hand side in a such way that eventually they are absorbed by the three integral terms involving the gradients in the left one. More exactly, to the products we apply the first inequality derived in Lemma 3.3, and to the other terms Young's inequality. In this way, the resulting linear combination of the terms $\int_\Omega |\nabla|\nabla v|^q|^2$, $\int_\Omega |\nabla|\nabla w|^r|^2$ and $\int_\Omega |\nabla(u+1)^{\frac{m_1+p-1}{2}}|^2$ can be written as $\frac{c_{54}}{2}\int_\Omega |\nabla|\nabla v|^q|^2 + \frac{c_{55}}{2}\int_\Omega |\nabla|\nabla w|^r|^2 + \frac{c_{56}}{2}\int_\Omega |\nabla(u+1)^{\frac{m_1+p-1}{2}}|^2$, which throughout relation (4.32) infers the claim. □

Remark 4.3. We observe that the argument of Lemma 4.2 can be applied to the linear case $m_1 = m_2 = m_3 = 1$ only for $\alpha \in \left(0, \frac{2}{n}\right)$ and/or $\gamma \in \left(0, \frac{2}{n}\right)$.

4.2. **The logistic case.** For the logistic case we retrace part of the computations above connected to the usage of the Young inequality only.

Lemma 4.4. *If $m_1, m_2, m_3 \in \mathbb{R}$ comply with $m_1 > \max\{2m_2-1, 2m_3-1, \frac{n-2}{n}\}$ or $m_1 > \max\{m_2-\frac{1}{n}, m_3-\frac{1}{n}, \frac{n-2}{n}\}$ or $m_1 > \max\{2m_2-1, m_3-\frac{1}{n}, \frac{n-2}{n}\}$ or $m_1 > \max\{m_2-\frac{1}{n}, 2m_3-1, \frac{n-2}{n}\}$ or $m_1 > \max\{2m_2-\beta, 2m_3-\beta, \frac{n-2}{n}\}$ or $m_1 > \max\{2m_2-\beta, 2m_3-\beta\}$ whenever $\alpha,\gamma \in (0,\frac{1}{n}]$, or $m_1 > \max\{2m_2, 2m_3, \frac{n-2}{n}\}$ or $m_1 > \max\{2m_2+1-\beta, 2m_3+1-\beta\}$ whenever $\alpha,\gamma \in (\frac{1}{n}, 1)$, then there exist $p,q,r>1$ such that (u,v,w) satisfies a similar inequality as in (4.1).*

Proof. As in Lemma 4.1, in view of inequalities (4.3) and (4.4) taking into account (4.5) and (4.9), and the properties of the logistic h in (2.4), relation (4.2) now becomes for some positive $\tilde{c}_3$ and for all $t \in (0, T_{\max})$

$$\begin{aligned}\frac{d}{dt}\int_\Omega (u+1)^p \le{}& (-p(p-1)+\delta_1)\int_\Omega (u+1)^{p+m_1-3}|\nabla u|^2\\ &+\tilde{c}_1\int_\Omega (u+1)^{\frac{(p+2m_2-m_1-1)s}{s-2}}+\tilde{c}_2\int_\Omega (u+1)^{\frac{(p+2m_3-m_1-1)s}{s-2}}\\ &+pk_+\int_\Omega (u+1)^p - p\mu\int_\Omega (u+1)^{p-1}u^\beta+\tilde{c}_3.\end{aligned}\tag{4.33}$$

Applying the inequality $(A+B)^p \le 2^{p-1}(A^p+B^p)$ with $A, B \ge 0$ and $p>1$ to the last integral in (4.33), implies that $-u^\beta \le -\frac{1}{2^{\beta-1}}(u+1)^\beta + 1$; therefore

$$-p\mu\int_\Omega (u+1)^{p-1}u^\beta \le -\frac{p\mu}{2^{\beta-1}}\int_\Omega (u+1)^{p-1+\beta}+p\mu\int_\Omega (u+1)^{p-1} \quad \text{on } (0,T_{\max}). \tag{4.34}$$

Henceforth, by taking into account the Young inequality, we have that for $t \in (0, T_{\max})$

$$\begin{aligned}&pk_+\int_\Omega (u+1)^p \le \delta_1\int_\Omega (u+1)^{p-1+\beta}+\tilde{c}_4 \quad \text{and}\\ &p\mu\int_\Omega (u+1)^{p-1} \le \delta_2\int_\Omega (u+1)^{p-1+\beta}+\tilde{c}_5,\end{aligned}\tag{4.35}$$

with $\delta_1, \delta_2 > 0$ and some $\tilde{c}_4, \tilde{c}_5 > 0$.

Case 1: $\alpha, \gamma \in (0, \frac{1}{n}]$ and $m_1 > \max\{2m_2-1, 2m_3-1, \frac{n-2}{n}\}$ or $m_1 > \max\{m_2-\frac{1}{n}, m_3-\frac{1}{n}, \frac{n-2}{n}\}$ or $m_1 > \max\{2m_2-1, m_3-\frac{1}{n}, \frac{n-2}{n}\}$ or $m_1 > \max\{m_2-\frac{1}{n}, 2m_3-1, \frac{n-2}{n}\}$ or $m_1 > \max\{2m_2-\beta, 2m_3-\beta, \frac{n-2}{n}\}$ or $m_1 > \max\{2m_2-\beta, 2m_3-\beta\}$. For $m_1 > \max\{2m_2-1, 2m_3-1, \frac{n-2}{n}\}$ or $m_1 > \max\{m_2-\frac{1}{n}, m_3-\frac{1}{n}, \frac{n-2}{n}\}$ or $m_1 > \max\{2m_2-1, m_3-\frac{1}{n}, \frac{n-2}{n}\}$ or $m_1 > \max\{m_2-\frac{1}{n}, 2m_3-1, \frac{n-2}{n}\}$, we refer to Lemma 4.1 and we take in mind inequalities (4.6), (4.7), (4.8), (4.9), (4.10) and (4.11). Conversely, when $m_1 > 2m_2-\beta$ and $m_1 > 2m_3-\beta$, we have that (recall s may be arbitrary large) $\frac{(p+2m_2-m_1-1)s}{s-2} < p-1+\beta$ and $\frac{(p+2m_3-m_1-1)s}{s-2} < p-1+\beta$, and by means of the Young inequality estimates (4.6) and (4.10) can alternatively read

$$\tilde{c}_1\int_\Omega (u+1)^{\frac{(p+2m_2-m_1-1)s}{s-2}} \le \delta_3\int_\Omega (u+1)^{p-1+\beta}+\tilde{c}_6 \quad \text{for all } t\in(0,T_{\max}), \tag{4.36}$$

and

$$\tilde{c}_2\int_\Omega (u+1)^{\frac{(p+2m_3-m_1-1)s}{s-2}} \le \delta_4\int_\Omega (u+1)^{p-1+\beta}+\tilde{c}_7 \quad \text{on } (0,T_{\max}), \tag{4.37}$$

with $\delta_3, \delta_4 > 0$ and positive $\tilde{c}_6, \tilde{c}_7$. By inserting estimates (4.34) and (4.35) into relation (4.33), as well as taking into account (4.6) and (4.10) (or, alternatively to (4.6) and (4.10), bound (4.36) and (4.37)), for suitable $\hat{\epsilon}, \tilde{\delta} > 0$ and some $\tilde{c}_7 > 0$ we arrive at

$$\begin{aligned}\frac{d}{dt}\int_\Omega (u+1)^p \le{}& \left(-\frac{4p(p-1)}{(m_1+p-1)^2}+\hat{\epsilon}\right)\int_\Omega |\nabla (u+1)^{\frac{m_1+p-1}{2}}|^2\\ &+\left(\tilde{\delta}-\frac{p\mu}{2^{\beta-1}}\right)\int_\Omega (u+1)^{p-1+\beta}+\tilde{c}_7 \quad \text{for all } t\in(0,T_{\max}),\end{aligned}$$

where we used again relation (4.13). We can conclude reasoning exactly as in the second part of the proof of Lemma 4.1, by exploiting $m_1 > \frac{n-2}{n}$ and by choosing suitable $\hat{\epsilon}, \tilde{\delta}$. On the other hand, by enlarging p, Young's inequality allows us to obtain the following alternative estimates

$$\begin{aligned} c_{21}\int_\Omega u^{2\alpha}|\nabla v|^{2q-2} &\leq \delta_5\int_\Omega u^{p-1+\beta}+\tilde{c}_8\int_\Omega |\nabla v|^{\frac{2(q-1)(p-1+\beta)}{p-1+\beta-2\alpha}}\\ &\leq \delta_5\int_\Omega u^{p-1+\beta}+\delta_6\int_\Omega|\nabla v|^s+\tilde{c}_9 \quad \text{on } (0,T_{\max}),\end{aligned}$$

and

$$\begin{aligned} c_{27}\int_\Omega u^{2\gamma}|\nabla w|^{2r-2} &\leq \delta_6\int_\Omega u^{p-1+\beta}+\tilde{c}_9\int_\Omega |\nabla w|^{\frac{2(r-1)(p-1+\beta)}{p-1+\beta-2\gamma}}\\ &\leq \delta_6\int_\Omega u^{p-1+\beta}+\delta_7\int_\Omega|\nabla w|^s+\tilde{c}_{10} \quad \text{for all } t\in(0,T_{\max}).\end{aligned}$$

Case 2: $\alpha,\gamma \in (\frac{1}{n},1)$ and $m_1>\max\{2m_2,2m_3,\frac{n-2}{n}\}$ or $m_1>\max\{2m_2+1-\beta,2m_3+1-\beta\}$.

For $m_1>\max\{2m_2,2m_3,\frac{n-2}{n}\}$ we will refer to the second case of Lemma 4.1. Now, if $m_1>2m_2+1-\beta$ and $m_1>2m_3+1-\beta$, then some p sufficiently large infers to $\frac{(p+2m_2-m_1-1)(p+1)}{p}<p-1+\beta$ and $\frac{(p+2m_3-m_1-1)(p+1)}{p}<p-1+\beta$, so that for any positive δ_8,δ_9 and some $\tilde{c}_{10},\tilde{c}_{11}>0$ we have

$$\bar{c}_1\int_\Omega (u+1)^{\frac{(p+2m_2-m_1-1)(p+1)}{p}} \leq \delta_8\int_\Omega (u+1)^{p-1+\beta}+\tilde{c}_{10} \quad \text{on } (0,T_{\max}),$$

and

$$\bar{c}_4\int_\Omega (u+1)^{\frac{(p+2m_3-m_1-1)(p+1)}{p}} \leq \delta_9\int_\Omega (u+1)^{p-1+\beta}+\tilde{c}_{11} \quad \text{for all } t\in(0,T_{\max}).$$

Now, the integrals $\int_\Omega u^{2\alpha}|\nabla v|^{2p-2}$ and $\int_\Omega u^{2\gamma}|\nabla w|^{2p-2}$ can be treated as in **Case 2** of Lemma 4.1 or alternatively, by exploiting $\alpha,\gamma<1$, a different application of Young's inequalities leads on $(0,T_{\max})$ to

$$\begin{aligned} c_{21}\int_\Omega u^{2\alpha}|\nabla v|^{2p-2} &\leq \delta_5\int_\Omega u^{p-1+\beta}+\tilde{c}_8\int_\Omega |\nabla v|^{\frac{2(p-1)(p-1+\beta)}{p-1+\beta-2\alpha}}\\ &\leq \delta_5\int_\Omega u^{p-1+\beta}+\delta_{10}\int_\Omega|\nabla v|^{2(p+1)}+\tilde{c}_{12}, \end{aligned} \tag{4.38}$$

and

$$\begin{aligned} c_{27}\int_\Omega u^{2\gamma}|\nabla w|^{2p-2} &\leq \delta_6\int_\Omega u^{p-1+\beta}+\tilde{c}_9\int_\Omega |\nabla w|^{\frac{2(p-1)(p-1+\beta)}{p-1+\beta-2\gamma}}\\ &\leq \delta_6\int_\Omega u^{p-1+\beta}+\delta_{11}\int_\Omega|\nabla w|^{2(p+1)}+\tilde{c}_{13}, \end{aligned} \tag{4.39}$$

with $\delta_{10},\delta_{11}>0$ and some positive $\tilde{c}_{12},\tilde{c}_{13}$. The remaining part of the proof follows as **Case 2** of Lemma 4.1 for the terms dealing with $\int_\Omega|\nabla v|^{2(p+1)}$ and $\int_\Omega|\nabla w|^{2(p+1)}$.

As before, this result applies also for

▷ $\alpha\in\left(0,\frac{1}{n}\right],\gamma\in\left(\frac{1}{n},1\right)$ and $m_1>\max\left\{2m_2-1,\frac{n-2}{n},2m_3\right\}$ or $m_1>\max\left\{2m_2-1,\frac{n-2}{n},2m_3+1-\beta\right\}$ or $m_1>\max\left\{m_2-\frac{1}{n},2m_3,\frac{n-2}{n}\right\}$ or $m_1>\max\left\{m_2-\frac{1}{n},2m_3+1-\beta\right\}$ or $m_1>\max\left\{2m_2-\beta,2m_3,\frac{n-2}{n}\right\}$ or $m_1>\max\left\{2m_2-\beta,2m_3+1-\beta,\frac{n-2}{n}\right\}$ or $m_1>\max\left\{2m_2-\beta,2m_3+1-\beta\right\}$,

▷ $\alpha\in\left(\frac{1}{n},1\right), \gamma\in\left(0,\frac{1}{n}\right]$ and $m_1>\max\Big\{2m_2,2m_3-1,\frac{n-2}{n}\Big\}$ or $m_1>\max\Big\{2m_2+1-\beta,2m_3-1,\frac{n-2}{n},\Big\}$ or $m_1>\max\Big\{2m_2,m_3-\frac{1}{n},\frac{n-2}{n}\Big\}$ or $m_1>\max\Big\{2m_2+1-\beta,m_3-\frac{1}{n}\Big\}$ or $m_1>\max\Big\{2m_2,2m_3-\beta,\frac{n-2}{n}\Big\}$ or $m_1>\max\Big\{2m_2+1-\beta,2m_3-\beta,\frac{n-2}{n}\Big\}$ or $m_1>\max\Big\{2m_2+1-\beta,2m_3-\beta\Big\}$. □

As a by-product of what has now been obtained we are in a position to conclude.

4.3. **Proof of Theorems 2.2 and 2.3.**

Proof. Let $(u_0,v_0,w_0)\in(W^{1,\infty}(\Omega))^3$ with $u_0,v_0,w_0\geq 0$ on $\bar{\Omega}$. For f and g as in (2.3) and, respectively, for f, g as in (2.3) and h as in (2.4), let $\alpha,\gamma>0$ and let $m_1,m_2,m_3\in\mathbb{R}$ comply with $A_1)$—$A_{16})$, respectively, $A_{17})$—$A_{20})$. Then, we refer to Lemmas 4.1 and 4.2, respectively, Lemma 4.4 and obtain for some $C_1,C_2,C_3,C_4>0$ and for all $t\in(0,T_{\max})$

$$y'(t)+C_1\int_\Omega|\nabla(u+1)^{\frac{m_1+p-1}{2}}|^2+C_2\int_\Omega|\nabla|\nabla v|^q|^2+C_3\int_\Omega|\nabla|\nabla w|^r|^2\leq C_4. \quad (4.40)$$

Successively, the Gagliardo–Nirenberg inequality again makes that for some positive constants c_{62},c_{63},c_{64} we have on the one hand

$$\int_\Omega(u+1)^p\leq c_{62}\Big(\int_\Omega|\nabla(u+1)^{\frac{m_1+p-1}{2}}|^2\Big)^{\kappa_1}+c_{62}\quad\text{for all } t\in(0,T_{\max}), \quad (4.41)$$

(as already done in the inequality immediately before (4.7)), and on the other hand on $(0,T_{\max})$

$$\begin{aligned}\int_\Omega|\nabla v|^{2q}=|||\nabla v|^q||^2_{L^2(\Omega)}&\leq c_{63}||\nabla|\nabla v|^q||^{2\kappa_2}_{L^2(\Omega)}|||\nabla v|^q||^{2(1-\kappa_2)}_{L^{\frac{1}{q}}(\Omega)}\\&\quad+c_{63}|||\nabla v|^q||^2_{L^{\frac{1}{q}}(\Omega)},\end{aligned} \quad (4.42)$$

and similarly for all $t\in(0,T_{\max})$

$$\begin{aligned}\int_\Omega|\nabla w|^{2r}=|||\nabla w|^r||^2_{L^2(\Omega)}&\leq c_{64}||\nabla|\nabla w|^r||^{2\kappa_3}_{L^2(\Omega)}|||\nabla w|^r||^{2(1-\kappa_3)}_{L^{\frac{1}{r}}(\Omega)}\\&\quad+c_{64}|||\nabla w|^r||^2_{L^{\frac{1}{r}}(\Omega)},\end{aligned} \quad (4.43)$$

with κ_2,κ_3 already defined in Lemma 3.2. Subsequently, the L^s-bound of ∇v in (3.4) and of ∇w in (3.5) infer some $c_{65},c_{66}>0$ such that

$$\int_\Omega|\nabla v|^{2q}\leq c_{65}\Big(\int_\Omega|\nabla|\nabla v|^q|^2\Big)^{\kappa_2}+c_{65}\quad\text{on } (0,T_{\max}),$$

and

$$\int_\Omega|\nabla w|^{2r}\leq c_{66}\Big(\int_\Omega|\nabla|\nabla w|^r|^2\Big)^{\kappa_3}+c_{66}\quad\text{for all } t\in(0,T_{\max}).$$

At this stage, by using estimates (4.41), (4.42) and (4.43), with the aid of the second inequality in Lemma 3.3, relation (4.40) entails positive constants c_{67} and c_{68}, and $\tilde{\kappa}=\min\{\frac{1}{\kappa_1},\frac{1}{\kappa_2},\frac{1}{\kappa_3}\}$ such that

$$\begin{cases}y'(t)\leq c_{67}-c_{68}y^{\tilde{\kappa}}(t)\quad\text{on } (0,T_{\max}),\\ y(0)=\int_\Omega(u_0+1)^p+\int_\Omega|\nabla v_0|^{2q}+\int_\Omega|\nabla w_0|^{2r}.\end{cases}$$

Finally, ODE comparison principles imply $u\in L^\infty((0,T_{\max});L^p(\Omega))$, and the conclusion is a consequence of the boundedness criterion in (3.3). □

Remark 4.5. First we note that the conclusions II) and III) in Remark 2.5 can be similarly justified by using reasonings given above and connected to the proof of Theorem 2.2. In particular, two issues are crucial: the regularity of $\nabla v, \nabla w$ (see Lemma 3.1) and the problematic integral term $\int_\Omega u^{p+1}$, which can be treated by applying the Gagliardo–Nirenberg and Young's inequalities for $n = 1$ without any condition, and for $n = 2$ provided some restrictions hold.

As to the other cases, we can discuss the following scenarios:

- If $\alpha, \gamma \in [\frac{2}{n}, 1)$ (A_3) we cannot apply Lemma 4.2 (see Remark 4.3). Moreover, even Lemma 4.1 does not work; in fact, from it we would obtain that $\alpha, \gamma \in [2, 1)$ for $n = 1$ and $\alpha, \gamma \in [1, 1)$ for $n = 2$, which is not possible. The same contradictions appear for the cases $\alpha \in [\frac{2}{n}, 1]$, $\gamma \in [\frac{2}{n}, 1)$ (A_4) or $\alpha \in [\frac{2}{n}, 1)$, $\gamma \in [\frac{2}{n}, 1]$ (A_5) and $\alpha, \gamma \in [\frac{2}{n}, 1]$ (A_6).
- If $\alpha \in (0, \frac{1}{n}]$, $\gamma \in [\frac{2}{n}, 1)$ (A_8) or $\alpha \in [\frac{2}{n}, 1)$, $\gamma \in (0, \frac{1}{n}]$ (A_{13}), even though both Lemmas 4.1 and 4.2 are applicable, a further contradiction appears.
- If $\alpha \in (\frac{1}{n}, \frac{2}{n})$, $\gamma \in [\frac{2}{n}, 1)$ (A_{11}) or $\alpha \in [\frac{2}{n}, 1)$, $\gamma \in (\frac{1}{n}, \frac{2}{n})$ (A_{15}), and similarly for $\alpha \in (\frac{1}{n}, \frac{2}{n})$, $\gamma \in [\frac{2}{n}, 1]$ (A_{12}) or $\alpha \in [\frac{2}{n}, 1]$, $\gamma \in (\frac{1}{n}, \frac{2}{n})$ (A_{16}), by reasoning as in the previous cases we obtain a contradiction, characterized by the fact that the intervals of α or γ are empty.

Remark 4.6. Let us spend some words on the hints mentioned in Remark 2.6.

- ▷ For the linear case $m_1 = m_2 = m_3 = 1$, a simple substitution ensures the validity of Theorem 2.3 for $\beta > 2$ and $\alpha, \gamma \in (0, 1)$; conversely, for $\alpha, \gamma \in (0, 1]$ relations (4.38) and (4.39) are still applicable for $\beta > 2$ (item i)); for the nonlinear case, the same reasoning can be carried out to show v), vi) and vii);
- ▷ For $\beta = 2$, in bound (4.33), the term associated to the logistic dampening effect takes the form $-p\mu \int_\Omega u^{p+1}$, so that for μ large, the other positive contributions proportional as well to $\int_\Omega u^{p+1}$ itself, can be absorbed. As to the expressions for μ, relations in ii), iii) and iv) come from the related range of α and γ; for $\beta = 2$ and the nonlinear case, a further largeness assumption on μ is required (viii), ix) and x)).

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