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Keywords: Boundary control; coupled partial differential equations (PDEs)–ordinary differential equations (ODEs); diffusion equation; distributed parameter systems; dynamic actuators; sliding mode (SM) control; unmatched disturbances.

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Sliding mode boundary control of an uncertain parabolic PDE-ODE cascade with dynamic actuation and unmatched disturbance

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Abstract—Dynamic boundary control design is addressed for an uncertain heat process in order to regulate its state to a constant set-point value while also rejecting a smooth unmatched boundary disturbance. The diffusivity parameter is unknown *a priori*, and a control signal is applied to the plant at its boundary through a first-order dynamic actuation with uncertain time constant, thereby yielding a coupled PDE/ODE representation of the overall system dynamics. The proposed output feedback design is sliding mode based, and it is made under collocated sensing. Its Lyapunov stability analysis is developed in the Sobolev state space $H^2(0, 1)$ to additionally conclude the uniform point-wise-in-space exponential stabilization of the closed-loop error dynamics. Numerical simulations are involved to corroborate the theoretical findings.

Index Terms—Distributed parameter systems; Diffusion equation; coupled PDE-ODE; unmatched disturbances; boundary control; sliding mode control; dynamic actuators.

I. INTRODUCTION

Boundary control design for processes modeled by parabolic partial differential equations (PDEs) is widely investigated because this class of PDEs describes large classes of physical processes of engineering relevance such as fluid, thermal, and chemical processes (see [1, Section I.A] and references therein).

A significant trend in the recent literature involves boundary control synthesis for distributed-parameters systems modeled by PDEs coupled with ordinary differential equations (ODEs)

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representing, e.g., actuator or sensor dynamics in a variety of different configurations (see, e.g., [1]–[10]).

Most of these references consider the backstepping boundary control problem of parabolic PDEs cascaded with an ODE or “sandwiched” between two ODEs with coupling at the boundaries of the 1-D spatial domain. In [3], backstepping design for a Burgers’ equation coupled with two integrator ODEs was studied. Reference [5] investigates a diffusion PDE coupled with an ODE system whose state is also wanted to be stabilized in addition to the state of the PDE. Reference [1] considers a more general ODE-diffusion PDE-ODE cascade system. In [7] it is proposed a constructive LMI-based method to provide exponential stability of a diffusion PDE coupled with an ODE. Recently, more complicated coupling configurations were also explored such as the chain of parabolic PDE-ODE cascades investigated in [10].

Although the generalization to the PDE setting of finite-dimensional control strategies dealing with external disturbances and/or parameter uncertainties has received some attention in recent years (see, e.g., [8], [9], [11]–[13] and references therein), only a few works have investigated this class of problems in the setting of uncertain parabolic PDEs coupled with ODEs. To the best of our knowledge, one of the first attempts was [12] where a dynamic input extension similar to that proposed in [3] was considered, thus yielding an uncertain diffusion PDE coupled to an integrator ODE fed by a discontinuous sliding-mode (SM) based control law. Such a scheme was capable of providing the asymptotic rejection of a possibly unbounded matched disturbance. In [8], a combined backstepping/SM controller is proposed to asymptotically stabilize a coupled diffusion PDE-ODE system affected by a bounded matched disturbance.

A. Main Contribution

None of the previously quoted references addressed the presence of unmatched disturbances in the combined PDE/ODE setting. It is also worth mentioning that among the reviewed studies, only [1] was concerned with ODE actuator dynamics, but neither uncertainties nor disturbances were considered there. Thus motivated, this paper aims to ensure robust boundary control of an uncertain diffusion PDE coupled with an uncertain linear first-order ODE representing the actuator dynamics. Specifically, the investigation is carried

out in the presence of unmatched unknown inputs acting at the coupling point between the actuator and the diffusion process.

From a methodological viewpoint several innovative steps are made beyond the existing literature for a PDE-ODE system affected by an unmatched boundary disturbance and whose internal dynamics are not asymptotically stable. In addition, the underlying PDE-ODE model is subject to some uncertainties in both its finite- and infinite-dimensional parts, namely the actuator time-constant and the spatially invariant diffusivity. Potential extension to spatially varying uncertain plant parameters is rather straightforward, however, it remains beyond the scope of the present work.

To this end, a discontinuous boundary control strategy is devised to rely on the SM control approach [14], with a sliding manifold with respect to which the system in question is minimum phase. The strategy is accompanied with some proportional and derivative (PD) control terms to exponentially stabilize the closed-loop system in the Sobolev $H^2(0,1)$ -norm, thereby yielding its pointwise stabilization in the spatial domain. This constitutes the main contribution of the paper. The main novelties compared to the related paper [12] are mainly due to the uncertain actuator's dynamics considered in the present work, and they are discussed in more detail in the dedicated Remark 1, after that the statement of the problem will be fully illustrated.

The stability proof is based on a strict Lyapunov functional, invented ad-hoc and unusually featuring an explicit dependence on the uncertain disturbance to be rejected. To the best of our knowledge, such a class of “disturbance-dependent” Lyapunov functionals was rarely used in the existing literature. Few examples include: [15], where a similar relay-control was suggested for stabilizing a perturbed second-order ODE system; [16], where a disturbance-dependent Lyapunov function addresses the rejection of unmatched disturbances in a more general n -th order linear ODE system equipped with a SM control law embodying multiple relay terms. Very recently, a disturbance-dependent Lyapunov functional was invoked in [17] to deal with the adaptive SM control of a diffusion process affected by a matched disturbance with unknown upper bound.

The Lyapunov analysis to be presented has led to a simple set of tuning rules for the controller gains represented as a function of the bounds, which were imposed on the uncertain parameters and on the magnitude of unmatched disturbances and its derivatives up to the second order.

It is worth mentioning that a conference version of the present work was recently presented in [18]. Beyond that work, unmatched disturbances are involved in the system's dynamics, well-posedness issues of the proposed relay-based design are addressed in terms of Filippov solutions, and the robust exponential stability of the resulting perturbed closed-loop system is established in the Sobolev state space $H^2(0,1)$.

B. Structure of the paper

Section II introduces the class of perturbed coupled PDE-ODE BVPs to deal with and formulates the control problem to be addressed. The robust boundary control design is then proposed in Section III, where an appropriate solution notion is

specified, the closed-loop well-posedness is analyzed, and the convergence proofs are given. Numerical simulations corroborating the theoretical analysis are presented in Section IV. Finally, Section V collects some concluding remarks and points out some perspectives for further related investigations.

C. Notation

Let $a \in \mathbb{R}$. $\text{sign}(a)$ denotes the multi-valued function

$$\text{sign}(a) \in \begin{cases} 1 & a > 0 \\ [-1, 1] & a = 0 \\ -1 & a < 0 \end{cases}. \quad (1)$$

$H^\ell(0,1)$, with $\ell = 0, 1, 2, \dots$, denotes the Sobolev space of absolutely continuous scalar functions $y(\zeta)$ with square integrable derivatives $y^{(i)}(\zeta)$ on $(0,1)$ up to the order ℓ , equipped with the H^ℓ -norm

$$\|y(\cdot)\|_\ell = \left(\int_0^1 \sum_{i=0}^{\ell} \left[y^{(i)}(\zeta) \right]^2 d\zeta \right)^{\frac{1}{2}}. \quad (2)$$

Notice that the H^0 -norm corresponds to the standard L_2 -norm.

II. PROBLEM FORMULATION

Consider the space- and time-varying scalar field $q(\xi, t)$ evolving in a Hilbert space $H^0(0,1)$, with the spatial variable $\xi \in [0,1]$ and the time variable $t \in [0, \infty)$. Let $q(\xi, t)$ be governed by the diffusion equation

$$q_t(\xi, t) = \theta q_{\xi\xi}(\xi, t), \quad (3)$$

where q_t and $q_{\xi\xi}$ denote temporal and second-order spatial derivatives of the scalar field $q(\xi, t)$, and $\theta > 0$ is referred to as the diffusivity parameter.

Throughout, we consider controlled and perturbed Neumann-type boundary conditions (BCs) of the form

$$q_\xi(1, t) = d(t) + u(t), \quad q_\xi(0, t) = 0, \quad (4)$$

where $d(t) \in \mathbb{R}$ is a sufficiently smooth unknown disturbance, and $u(t) \in \mathbb{R}$ is the boundary control input applied through a dynamic actuator governed by the first-order ODE

$$\tau \dot{u}(t) + u(t) = v(t), \quad (5)$$

where $\tau > 0$ is the unknown actuator's time constant, and $v(t)$ is the manipulable control variable to be designed. Such PDE-ODE systems and their well-posedness features have been addressed in the literature, e.g., in [19].

The initial condition (IC) is

$$q(\xi, 0) = q^0(\xi) \in H^2(0,1). \quad (6)$$

Let q^r be an *a priori* given time-independent and spatially-invariant constant reference.

The control objective is to design the actuator input $v(t)$ to exponentially steer to zero the H^2 -norm $\|x(\cdot, t)\|_2$ of the error variable

$$x(\xi, t) = q(\xi, t) - q^r \quad (7)$$

which, according to (2), admits the explicit representation

$$\|x(\cdot, t)\|_2^2 = \|x(\cdot, t)\|_0^2 + \|x_\xi(\cdot, t)\|_0^2 + \|x_{\xi\xi}(\cdot, t)\|_2^2. \quad (8)$$

Once the control objective is achieved, a complementary bonus is obtained of exponential decay of the L_∞ norms

$$\|x(\cdot, t)\|_\infty = \max_{\xi \in [0,1]} |x(\xi, t)|, \quad (9)$$

$$\|x_\xi(\cdot, t)\|_\infty = \max_{\xi \in [0,1]} |x_\xi(\xi, t)|, \quad (10)$$

of the error variable $x(\xi, t)$ and its spatial derivative $x_\xi(\xi, t)$. Hence, uniform pointwise-in-space convergence of the error and its spatial derivative is simultaneously addressed.

A. Operating assumptions and preliminary observations

The next assumptions specify the available information about the plant parameters and the admissible disturbances.

Assumption 1 *There exist a-priori known positive constants $\theta_m, \theta_M, \tau_m, \tau_M$ such that*

$$0 < \theta_m \leq \theta \leq \theta_M, \quad 0 < \tau_m \leq \tau \leq \tau_M. \quad (11)$$

Assumption 2 *There exist a-priori known nonnegative constants $D_0, D_1, D_2 \geq 0$ such that*

$$|d(t)| \leq D_0, \quad |\dot{d}(t)| \leq D_1, \quad |\ddot{d}(t)| \leq D_2, \quad \forall t \geq 0. \quad (12)$$

Let us make some preliminary observations regarding the stated control objective. Our main goal is to ensure the exponential decay of the H^2 -norm of the solution $x(\xi, t)$ of the combined PDE-ODE boundary value problem

$$x_t(\xi, t) = \theta x_{\xi\xi}(\xi, t), \quad (13)$$

$$x_\xi(0, t) = 0, \quad (14)$$

$$x_\xi(1, t) = d(t) + u(t), \quad (15)$$

$$\tau \dot{u}(t) = v(t) - u(t), \quad (16)$$

$$x(\xi, 0) = x^0(\xi), \quad (17)$$

where $x^0(\xi) = q^0(\xi) - q^r \in H^2(0, 1)$.

Provided that the input $u(t)$ is of class C^2 , and the initial and boundary conditions (14)-(17) are compatible at the end points $\xi = 0$ and $\xi = 1$, the PDE (13)-(17) possesses a unique classical solution [20, Theorem 3.3.3]. Since such a compatibility condition $x_\xi(1, 0) = d(0) + u(0)$ would unrealistically require to properly synchronize initial state and control input values with that of uncertain disturbance, the meaning of (13)-(17) is further adopted in the mild sense [20].

A mild solution of (13)-(17) is defined by means of the Green function

$$G(\xi, \zeta, t) = 1 + 2 \sum_{n=1}^{\infty} \cos(\pi n \xi) \cos(\pi n \zeta) e^{-\theta(\pi n)^2 t} \quad (18)$$

of the PDE (13) under the homogeneous BCs

$$x_\xi(1, t) = 0, \quad x_\xi(0, t) = 0. \quad (19)$$

By inspection, the Green function (18) is straightforwardly verified to read both the PDE (13) and BCs (19). An interested reader may also refer to, e.g., [21, Chapter 3] for more details on the corresponding Sturm–Liouville problem, which generates the Green function (18) of the BVP (13), (19).

Definition 1 *Consider the non-homogeneous BVP (13)-(17). An absolutely continuous function*

$$x(\xi, t) = \int_0^1 G(\xi, \zeta, t) x^0(\zeta) d\zeta + \int_0^t G(\xi, \zeta, t-s) [d(s) + u(s)] ds \quad (20)$$

is said to be a mild solution of (13)-(17). ■

Note that (20) becomes an integral equation when a feedback $u = u(x(\cdot, t))$ is involved. For later use, the BVP below

$$z_t(\xi, t) = \theta z_{\xi\xi}(\xi, t), \quad (21)$$

$$z_\xi(0, t) = 0, \quad (22)$$

$$z_\xi(1, t) = \dot{u}(t) + \dot{d}(t), \quad (23)$$

$$z(\xi, 0) = z^0(\xi) = \theta x_{\xi\xi}^0(\xi) \in H^0(0, 1) \quad (24)$$

which governs the evolution of $z(\xi, t) = x_t(\xi, t)$ is obtained by differentiating (13)-(17) in the time variable where the IC (24) is straightforwardly derived from (13) and (17). Being driven by the dynamic actuator (16) with $u(t) = x_\xi(1, t) - d(t)$ extracted from (15), the closed-loop BC (23) are

$$z_\xi(1, t) = \frac{1}{\tau} v(t) + \frac{d(t) - x_\xi(1, t)}{\tau} + \dot{d}(t). \quad (25)$$

In the closed-loop, the state variable $z(\xi, t) = x_t(\xi, t)$ can therefore be viewed as a mild solution

$$z(\xi, t) = \int_0^1 G(\xi, \zeta, t) z^0(\zeta) d\zeta + \int_0^t G(\xi, \zeta, t-s) \times \left[\frac{1}{\tau} v(s) + \frac{d(s) - x_\xi(1, s)}{\tau} + \dot{d}(s) \right] ds \quad (26)$$

of the auxiliary system (21)-(25). The closed-loop systems (13)-(17) and (21)-(25) are connected in cascade through the actuator input v and the output $x_\xi(1, t)$, which affects the BC (25) of the latter system. The subsequent controller synthesis is therefore carried out for the resulting cascade systems.

Remark 1 *The paper shares some similarities with [12]. Both consider the same BVP (13)-(15). However, the PDE input $u(t)$ was assumed to be directly manipulable in [12], while here it plays the role of an additional state variable associated to the uncertain actuator dynamics (16) driven by the manipulable input $v(t)$. Thus, the disturbance $d(t)$ can be considered unmatched in the present setting.* ■

III. ROBUST SM BOUNDARY CONTROL SYNTHESIS

To achieve the control goal, the following discontinuous boundary control law is proposed

$$v(t) = -\beta_1 x(1, t) - \beta_2 x_t(1, t) - W \text{sign}(x(1, t)), \quad (27)$$

where β_1, β_2 , and W are positive gains to be properly tuned. The proposed control law (27) consists of a linear PD controller plus a relay term, and it makes explicit use of the boundary measurement $x(1, t)$ and its time derivative $x_t(1, t)$.

Notice that explicit state differentiation is flawed in control synthesis with induced algebraic loops since the resulting closed-loop system would be extremely sensitive to measurement noise. However, the use of the state derivative $x_t(1, t)$ is acceptable in the control input (27) because it then passes through the integrator in the dynamic actuation (16), and the variable $x_t(\xi, t)$ constitutes a state component $z(\xi, t)$ of the overall closed-loop system (13)-(17), (21)-(25), (27).

A. On well-posedness of closed-loop Filippov solutions

Let us further note that since $v(t)$ in (27) is discontinuous on the manifold $x(1, t) = 0$, the right-hand side of (16) is differentiable almost everywhere. Thus, the meaning of (16), driven by (27), is understood according to the Filippov theory [22] and its generalization to parabolic PDEs [23]. The closed-loop (16), (27) is thus associated with a differential inclusion with the corresponding multi-valued sign function given by (1).

As in the finite-dimensional case, a motion along the discontinuity manifold $x(1, t) = 0$, if any, is referred to as a sliding mode. In the sliding mode that potentially may occur at the right boundary $\xi = 1$, the system motion is reduced to the underlying PDE (13), coupled to the same BC (14) at the left boundary $\xi = 0$ and to the homogeneous Dirichlet BC

$$x(1, t) = 0 \quad (28)$$

at the right boundary.

The SM dynamics (13), (14), (28) are established to be exponentially stable in the L_2 -norm by verifying that the quadratic Lyapunov function candidate $V(x) = |x(\cdot, t)|_0^2 = \int_0^1 x^2(\xi, t) d\xi$ possesses a negative definite time derivative along the sliding modes. Moreover, by differentiating (13), (14), (28) in the time variable, one arrives at the same BVP for successive state derivatives, thereby concluding that the SMs are of class C^∞ and their temporal derivatives of arbitrary order exponentially decay in the L_2 -norm as well. Certainly, this means that the SM BVP (13), (14), (28) is exponentially stable in the Sobolev space H^ℓ of arbitrary order $\ell = 1, 2, \dots$

Throughout, the closed-loop system in question is viewed in the mild sense of Definition 1 beyond the discontinuity manifold $x(1, t) = 0$, where it becomes a linear PDE–ODE system with the constant relay $W \text{sign}(x(1, t)) = 1$ or $W \text{sign}(x(1, t)) = -1$. The solutions of the resulting parabolic PDE–ODE system are well-recognized to locally exist [19, Theorem 1]. Furthermore, along the discontinuity manifold, the closed-loop is viewed in the Filippov sense [22] by relying on its extension, made for sliding-mode-controlled parabolic PDEs [23].

It is noteworthy that for such an ODE-PDE system, both the mild and Filippov concepts of solutions are covered by the unifying concept of generalized solutions [21]. The closed-loop generalized solutions are differentiable in time almost everywhere except for a countable set of switching instants when the trajectory crosses the discontinuity manifold, whereas the infinite switching frequency in sliding mode generates smooth SM solutions. The well-posedness of the closed-loop system will also be formally established by Theorem 1.

Since the SM dynamics are exponentially stable in the Sobolev space of interest, the next stability analysis focuses on the closed-loop dynamics beyond the discontinuity manifold that has become typical in both finite- and infinite-dimensional settings (see, e.g., [14] and respectively [21] for details).

B. Main results

For later use, define the constant scalar and matrix terms

$$T = \tau + \frac{8}{3\theta}, \quad \Gamma = \begin{bmatrix} \frac{1}{\tau} & 1 \\ 1 & T \end{bmatrix}, \quad (29)$$

and notice that, by Assumption 1, the following estimation

$$0 < T_m \leq T \leq T_M \quad (30)$$

is true, where

$$T_m = \tau_m + \frac{8}{3\theta_m}, \quad T_M = \tau_M + \frac{8}{3\theta_m}. \quad (31)$$

To achieve the desired closed-loop stability properties, the tuning parameters β_1 , β_2 , and W of the control law (27) are required to satisfy certain inequalities involving the bounding constants defined in the Assumption 1 and Assumption 2. These inequalities are specified by the following relations

$$\beta_1 > 0, \quad (32)$$

$$\beta_2 > \frac{2}{\tau_m \theta_m^2} + \frac{\tau_M}{4} + \frac{2\tau_M}{\theta_m T_m}, \quad (33)$$

$$W > D_0 + D_1 \tau_M + T_M (D_1 + D_2 \tau_M). \quad (34)$$

We can now state and demonstrate our main result.

Theorem 1 *Under Assumption 1 and Assumption 2, consider the BVP (13)–(17), driven by the control (27). Let the controller gains be tuned as in (32)–(34). Then, the solutions $x(\cdot, t)$ of the resulting closed-loop BVP are well-posed and globally exponentially stable in the space $H^2(0, 1)$. ■*

Remark 2 *Although inequality (32) establishes that β_1 can be selected arbitrarily close to zero, the proportional control component in (27) is instrumental for guaranteeing the exponential stability of the closed-loop system. Indeed, by setting β_1 to zero only asymptotic stability would be in force. □*

The proof of Theorem 1 is preceded by an instrumental lemma given below. To this end, let us introduce the functional

$$\begin{aligned} \mathcal{V}(\tilde{x}, \tilde{z}, \tilde{d}) &= \frac{1}{2} \int_0^1 [\tilde{x}(\zeta) \quad \tilde{z}(\zeta)] \Gamma \begin{bmatrix} \tilde{x}(\zeta) \\ \tilde{z}(\zeta) \end{bmatrix} d\zeta \\ &+ \frac{\theta}{2} \int_0^1 \tilde{x}_\zeta^2(\zeta) d\zeta + W \frac{\theta T}{\tau} |\tilde{x}(1)| \\ &- \frac{\theta T}{\tau} \tilde{x}(1) \tilde{d} + \frac{\theta}{2\tau} (T\beta_1 + \beta_2) \tilde{x}^2(1), \end{aligned} \quad (35)$$

mapping $H^0(0, 1) \times H^0(0, 1) \times \mathbb{R} \rightarrow \mathbb{R}$.

Lemma 1 *Let conditions of Theorem 1 be satisfied. The value*

$$V(t) = \mathcal{V}(x(\zeta, t), x_t(\zeta, t), d(t) + \tau \dot{d}(t)) \quad (36)$$

of the functional (35), computed along arbitrarily initialized solutions and admissible disturbances of the closed-loop system (13)–(17), (27), admits lower and upper bounds expressed in terms of the H^2 -norm of $x(\zeta, t)$ such that there exist positive constants $\rho_1 < \rho_2$ and ρ_3 for which the following estimation

$$\rho_1 \|x(\cdot, t)\|_2^2 \leq V(t) \leq \rho_2 \|x(\cdot, t)\|_2^2 + \rho_3 \|x(\cdot, t)\|_2 \quad (37)$$

holds true for all $t \geq 0$. ■

Proof of Lemma 1: It follows from the Assumption 1 and definitions (29) that the symmetric matrix Γ is positive definite. Thus it admits two strictly positive real eigenvalues, says λ_m and λ_M , with $\lambda_m \leq \lambda_M$. Therefore, exploiting the properties of positive definite quadratic forms, one has that

$$\begin{bmatrix} x(\zeta, t) & x_t(\zeta, t) \end{bmatrix} \Gamma \begin{bmatrix} x(\zeta, t) \\ x_t(\zeta, t) \end{bmatrix} \geq \lambda_m (x^2(\zeta, t) + x_t^2(\zeta, t)), \quad (38)$$

$$\begin{bmatrix} x(\zeta, t) & x_t(\zeta, t) \end{bmatrix} \Gamma \begin{bmatrix} x(\zeta, t) \\ x_t(\zeta, t) \end{bmatrix} \leq \lambda_M (x^2(\zeta, t) + x_t^2(\zeta, t)). \quad (39)$$

Considering the inequality (38) into the first integral term on the right-hand side of (36), one derives the next estimation

$$\begin{aligned} \frac{1}{2} \int_0^1 \begin{bmatrix} x(\zeta, t) & x_t(\zeta, t) \end{bmatrix} \Gamma \begin{bmatrix} x(\zeta, t) \\ x_t(\zeta, t) \end{bmatrix} d\zeta \geq \\ \frac{1}{2} \lambda_m (\|x(\cdot, t)\|_0^2 + \|x_t(\cdot, t)\|_0^2). \end{aligned} \quad (40)$$

By virtue of Assumption 1 and Assumption 2, one also derives the next straightforward estimation

$$-\frac{\theta T}{\tau} x(1, t) (d(t) + \tau \dot{d}(t)) \geq -\frac{\theta T}{\tau} |x(1, t)| (D_0 + D_1 \tau_M).$$

Finally, by (40) and (41), the lower estimate of (36) is

$$\begin{aligned} V(t) \geq \frac{1}{2} \lambda_m (\|x(\cdot, t)\|_0^2 + \|x_t(\cdot, t)\|_0^2) + \frac{\theta}{2} \|x_\xi(\cdot, t)\|_0^2 \\ + \frac{\theta T}{\tau} [W - (D_0 + D_1 \tau_M)] |x(1, t)|. \end{aligned} \quad (42)$$

The equality

$$\|x_t(\cdot, t)\|_0 = \theta \|x_{\xi\xi}(\cdot, t)\|_0 \quad (43)$$

readily follows from (13). Considering (43) into (42) and noticing that (34) implies that $W - (D_0 + D_1 \tau_M) > 0$, one straightforwardly obtains the left inequality of (37) with

$$\rho_1 = \min \left\{ \frac{1}{2} \lambda_m, \frac{\theta}{2}, \frac{1}{2} \lambda_m \theta^2 \right\}. \quad (44)$$

To reproduce the right inequality of (37), one first concludes from (39) and (43) that

$$\begin{aligned} \frac{1}{2} \int_0^1 \begin{bmatrix} x(\zeta, t) & x_t(\zeta, t) \end{bmatrix} \Gamma \begin{bmatrix} x(\zeta, t) \\ x_t(\zeta, t) \end{bmatrix} d\zeta \leq \\ \frac{1}{2} \lambda_M (\|x(\cdot, t)\|_0^2 + \|x_t(\cdot, t)\|_0^2) = \\ \frac{1}{2} \lambda_M (\|x(\cdot, t)\|_0^2 + \theta^2 \|x_{\xi\xi}(\cdot, t)\|_0^2). \end{aligned} \quad (45)$$

Then notice that by virtue of (14) and (2), and applying the Cauchy-Schwarz inequality, the next chain of inequalities

$$\begin{aligned} x_\xi^2(\xi, t) &= \left(\int_0^\xi x_{\zeta\zeta}(\zeta, t) d\zeta \right)^2 \leq \int_0^\xi x_{\zeta\zeta}^2(\zeta, t) d\zeta \\ &\leq \|x_{\xi\xi}(\cdot, t)\|_0^2 \end{aligned} \quad (46)$$

holds $\forall \xi \in [0, 1]$ and $\forall t \geq 0$. Thus, by (2) and (46), one easily derives the next estimation

$$x^2(1, t) = \int_0^1 [x(1, t) - x(\zeta, t) + x(\zeta, t)]^2 d\zeta$$

$$\begin{aligned} &\leq 2 \left\{ \int_0^1 [x(1, t) - x(\zeta, t)]^2 d\zeta + \int_0^1 x^2(\zeta, t) d\zeta \right\} \\ &\leq 2 \left\{ \int_0^1 \max_{\zeta \in [0, 1]} x_\zeta^2(\zeta, t) d\zeta + \|x(\cdot, t)\|_0^2 \right\} \\ &\leq 2 \left\{ \|x_{\xi\xi}(\cdot, t)\|_0^2 + \|x(\cdot, t)\|_0^2 \right\} \leq 2 \|x(\cdot, t)\|_2^2, \end{aligned} \quad (47)$$

which readily implies that

$$|x(1, t)| \leq \sqrt{2} \|x(\cdot, t)\|_2. \quad (48)$$

By virtue of Assumptions 1 and 2, one also concludes that

$$\left| \frac{\theta T}{\tau} x(1, t) \right| \left| d(t) + \tau \dot{d}(t) \right| \leq \frac{\theta T}{\tau} |x(1, t)| (D_0 + D_1 \tau_M). \quad (49)$$

Considering the inequalities (45) and (47)-(49), the right-hand side of (36) can be upper-estimated as

$$\begin{aligned} V(t) \leq \frac{1}{2} \lambda_M \|x(\cdot, t)\|_0^2 + \frac{1}{2} \lambda_M \theta^2 \|x_{\xi\xi}(\cdot, t)\|_0^2 + \frac{\theta}{2} \|x_\xi(\cdot, t)\|_0^2 \\ + \frac{\theta T}{\tau} [W + (D_0 + D_1 \tau_M)] \sqrt{2} \|x(\cdot, t)\|_2 \\ + \frac{\theta}{2\tau} (T\beta_1 + \beta_2) 2 \|x(\cdot, t)\|_2^2 \end{aligned} \quad (50)$$

To complete the proof, straightforward manipulations to the (41) right-hand side of (50) yield the right inequality of (37) with

$$\rho_2 = \max \left\{ \frac{1}{2} \lambda_M, \frac{\theta}{2}, \frac{1}{2} \lambda_M \theta^2, \frac{\theta}{\tau} (T\beta_1 + \beta_2) \right\}, \quad (51)$$

$$\rho_3 = \frac{\sqrt{2}\theta T}{\tau} [W + (D_0 + D_1 \tau_M)]. \quad (52)$$

□

C. Proof of Theorem 1

First, let us note that once the closed-loop system (13)-(17), (27) is initialized beyond the discontinuity manifold $x(1, t) = 0$, it possesses a local solution (cf. [19, Theorem 1]) to the parabolic BVP, specified with the linear boundary input (27) subject to the constant relay term $W \text{sign}(x(1, t)) = 1$ or $W \text{sign}(x(1, t)) = -1$. Provided that such a generic solution cannot escape to infinity in finite time, it is continuously extendable to the right by [24, Theorem 23.5], at least until sliding mode occurs on the discontinuity manifold.

Moreover, potential sliding modes are established in Section III-A to be governed by the mixed Neumann-Dirichlet BVP (13), (14), (28) and be exponentially stable in L_2 -norm. The closed-loop system is thus concluded to possess a local solution regardless of whether it evolves in sliding mode.

Next, let us compute the time derivative of the Lyapunov functional (36), (35) along the solutions of the closed-loop BVPs (13)-(17) and (21)-(25). The functional $V(t)$ is first rewritten in expanded form

$$\begin{aligned} V(t) &= \frac{1}{2\tau} \int_0^1 x^2(\zeta, t) d\zeta + \int_0^1 x(\zeta, t) z(\zeta, t) d\zeta \\ &+ \frac{T}{2} \int_0^1 z^2(\zeta, t) d\zeta + \frac{\theta}{2} \int_0^1 x_\zeta^2(\zeta, t) d\zeta + W \frac{\theta T}{\tau} |x(1, t)| \\ &- \frac{\theta T}{\tau} x(1, t) (d(t) + \tau \dot{d}(t)) + \frac{\theta}{2\tau} (T\beta_1 + \beta_2) x^2(1, t) \end{aligned} \quad (53)$$

to facilitate subsequent derivations.

Differentiating (53) outside the discontinuity manifold, where $x(1, t) = 0$, and taking into account that $z(\xi, t) = x_t(\xi, t)$ by definition yield the next expression

$$\begin{aligned} \dot{V}(t) = & \frac{1}{\tau} \int_0^1 x(\zeta, t) x_t(\zeta, t) d\zeta + \int_0^1 x(\zeta, t) z_t(\zeta, t) d\zeta \\ & + \int_0^1 x_t(\zeta, t) z(\zeta, t) d\zeta + T \int_0^1 z(\zeta, t) z_t(\zeta, t) d\zeta \\ & + \theta \int_0^1 x_\xi(\zeta, t) z_\xi(\zeta, t) d\zeta + W \frac{\theta T}{\tau} z(1, t) \operatorname{sign}(x(1, t)) \\ & - \frac{\theta T}{\tau} z(1, t) \left(\dot{d}(t) + \tau \dot{d}(t) \right) - \frac{\theta T}{\tau} x(1, t) \left(\dot{d}(t) + \tau \ddot{d}(t) \right) \\ & + \frac{\theta}{\tau} (T\beta_1 + \beta_2) x(1, t) z(1, t). \end{aligned} \quad (54)$$

It is worth noting that the above relation (54) remains in force along the manifold where $x(1, t) = 0$, provided that the directional Dini derivative of the functional $V(t)$ stands for $\dot{V}(t)$ and the multi-valued function (1) is accepted for $\operatorname{sign}(x(1, t))$. The interested reader may refer to [21, Section 3.5] for Dini differentials. Since in (54), the term $\operatorname{sign}(x(1, t))$ comes with the multiplier $z(1, t) = x_t(1, t)$, their product is nullified in the manifold $x(1, t) = 0$ when sliding occurs.

The integral terms in the right-hand side of (54) are then manipulated and expanded. Exploiting PDE (13) and BC (14), and applying integration by parts yield

$$\begin{aligned} \frac{1}{\tau} \int_0^1 x(\zeta, t) x_t(\zeta, t) d\zeta &= \frac{\theta}{\tau} \int_0^1 x(\zeta, t) x_{\xi\xi}(\zeta, t) d\zeta \\ &= \frac{\theta}{\tau} x(1, t) x_\xi(1, t) - \frac{\theta}{\tau} \|x_\xi(\cdot, t)\|_0^2. \end{aligned} \quad (55)$$

Similarly, by virtue of (21) and by applying integration by parts and using the BCs (22) and (25), one obtains

$$\begin{aligned} \int_0^1 x(\zeta, t) z_t(\zeta, t) d\zeta &= \theta \int_0^1 x(\zeta, t) z_{\xi\xi}(\zeta, t) d\zeta \\ &= \theta x(1, t) z_\xi(1, t) - \theta \int_0^1 x_\xi(\nu, t) z_\xi(\nu, t) d\nu \\ &= \frac{\theta}{\tau} x(1, t) \left[v(t) - x_\xi(1, t) + d(t) + \tau \dot{d}(t) \right] \\ &\quad - \theta \int_0^1 x_\xi(\nu, t) z_\xi(\nu, t) d\nu. \end{aligned} \quad (56)$$

The next relations hold due to definition $z(\xi, t) = x_t(\xi, t)$

$$\begin{aligned} \int_0^1 x_t(\zeta, t) z(\zeta, t) d\zeta &= \int_0^1 z^2(\zeta, t) d\zeta = \|z(\cdot, t)\|_0^2, \\ \theta \int_0^1 x_\xi(\zeta, t) x_{\xi t}(\zeta, t) d\zeta &= \theta \int_0^1 x_\xi(\zeta, t) z_\xi(\zeta, t) d\zeta. \end{aligned} \quad (57)$$

Finally, by (21), and then applying integration by parts by considering the BCs (22), (25) one derives

$$\begin{aligned} T \int_0^1 z(\nu, t) z_t(\nu, t) d\nu &= \theta T \int_0^1 z(\nu, t) z_{\xi\xi}(\nu, t) d\nu \\ &= -\theta T \|z_\xi(\cdot, t)\|_0^2 + \theta T z(1, t) z_\xi(1, t) = -\theta T \|z_\xi(\cdot, t)\|_0^2 \\ &\quad + \frac{\theta T}{\tau} z(1, t) \left[v(t) - x_\xi(1, t) + d(t) + \tau \dot{d}(t) \right]. \end{aligned} \quad (58)$$

Substituting (55)-(58) into (54), replacing the control $v(t)$ by its explicit expression (27) (which is multi-valued following Filippov's meaning of (1)) and performing lengthy but straightforward simplifications and reordering, one obtains

$$\begin{aligned} \dot{V}(t) = & -\frac{\theta}{\tau} \|x_\xi(\cdot, t)\|_0^2 - T\theta \|z_\xi(\cdot, t)\|_0^2 \\ & - \beta_1 \frac{\theta}{\tau} x^2(1, t) - W \frac{\theta}{\tau} |x(1, t)| - \beta_2 \frac{T\theta}{\tau} z^2(1, t) \\ & + \frac{\theta}{\tau} x(1, t) \left[\dot{d}(t) + \tau \dot{d}(t) - T \left(\dot{d}(t) + \tau \ddot{d}(t) \right) \right] \\ & + \|z(\cdot, t)\|_0^2 - \frac{T\theta}{\tau} z(1, t) x_\xi(1, t). \end{aligned} \quad (59)$$

In the above simplifications, the multi-valued sign-terms, coming from both differentiation of the absolute-value function $|x(1, t)|$ and control law (27), kill each other once $x(1, t) \neq 0$ and they are negligible when sliding $x(1, t) = 0$ occurs which nullifies their common multipliers $z(1, t) = \dot{x}(1, t)$. Thus, (59) is possibly violated only at a zero-measure discrete set of isolated time instants where the solutions $x(\xi, t)$ cross (rather than slide along) the discontinuity manifold $x(1, t) = 0$. Hence, (59) holds true almost always thereby validating all subsequent derivations and stability analysis.

The last three terms in the right-hand side of (59) are sign-indefinite and are now estimated to compensate them through the remaining "stabilizing" negative terms.

By Assumption 1, Assumption 2, and (30), one concludes

$$\begin{aligned} & \frac{\theta}{\tau} x(1, t) \left[\dot{d}(t) + \tau \dot{d}(t) - T \left(\dot{d}(t) + \tau \ddot{d}(t) \right) \right] \\ & \leq \frac{\theta}{\tau} |x(1, t)| \left[\left| \dot{d}(t) + \tau \dot{d}(t) \right| + T \left| \dot{d}(t) + \tau \ddot{d}(t) \right| \right] \\ & \leq \frac{\theta}{\tau} |x(1, t)| [D_0 + D_1 \tau_M + T_M (D_1 + \tau_M D_2)], \end{aligned} \quad (60)$$

whereas straight application of Poincare' inequality yields

$$\|z(\cdot, t)\|_0^2 \leq 2z^2(1, t) + 2\|z_\xi(\cdot, t)\|_0^2 \quad (61)$$

The generalized triangle inequality is employed to estimate the last term in the right-hand side of (59)

$$\begin{aligned} -\frac{T\theta}{\tau} z(1, t) x_\xi(1, t) &\leq \frac{T\theta}{\tau} |z(1, t)| |x_\xi(1, t)| \\ &\leq \frac{T\theta}{\tau} \left(\frac{1}{2\gamma} z^2(1, t) + \frac{\gamma}{2} x_\xi^2(1, t) \right), \end{aligned} \quad (62)$$

where γ is an arbitrary positive constant that will be specified later on. Notice that by (14), and applying the Cauchy-Schwarz and Poincare inequalities, the next chain of inequalities

$$\begin{aligned} x_\xi^2(1, t) &= (x_\xi(1, t) - x_\xi(0, t))^2 = \left(\int_0^1 x_{\xi\xi}(\zeta, t) d\zeta \right)^2 \\ &\leq \int_0^1 x_{\xi\xi}^2(\zeta, t) d\zeta = \|x_{\xi\xi}(\cdot, t)\|_0^2 \\ &= \frac{1}{\theta^2} \|z(\cdot, t)\|_0^2 \leq \frac{2}{\theta^2} (z^2(1, t) + \|z_\xi(\cdot, t)\|_0^2), \end{aligned} \quad (63)$$

which also takes advantage of the equality $\|x_{\xi\xi}(\cdot, t)\|_0^2 = \frac{1}{\theta^2} \|z(\cdot, t)\|_0^2$ that readily follows from (13) and from definition $z(\xi, t) = x_t(\xi, t)$, is derived. Substituting (63) into (62) and

specifying the chosen value $\gamma = \tau\theta^2/4 > 0$ of the arbitrary positive constant γ it follows that

$$\begin{aligned} & -\frac{T\theta}{\tau} z(1, t) x_\xi(1, t) \\ & \leq \frac{T\theta}{\tau} \left(\frac{2}{\tau\theta^2} + \frac{\tau}{4} \right) z^2(1, t) + \frac{T\theta}{4} \|z_\xi(\cdot, t)\|_0^2. \end{aligned} \quad (64)$$

Now, replacing the estimations (60), (61) and (64) into the right-hand side of (59) it yields

$$\begin{aligned} \dot{V}(t) & \leq -\frac{\theta}{\tau} \|x_\xi(\cdot, t)\|_0^2 - \lambda_1 \|z_\xi(\cdot, t)\|_0^2 \\ & - \beta_1 \frac{\theta}{\tau} x^2(1, t) - \frac{T\theta}{\tau} \lambda_2 z^2(1, t) - \frac{\theta}{\tau} \lambda_3 |x(1, t)|. \end{aligned} \quad (65)$$

where

$$\lambda_1 = \frac{3}{4}T\theta - 2, \quad (66)$$

$$\lambda_2 = \beta_2 - \left(\frac{2}{\tau\theta^2} + \frac{\tau}{4} + \frac{2\tau}{T\theta} \right), \quad (67)$$

$$\lambda_3 = W - [D_0 + D_1\tau_M + T_M(D_1 + \tau_M D_2)]. \quad (68)$$

By (29) and (11) one further manipulates (66) as follows

$$\lambda_1 = \frac{3}{4}T\theta - 2 = \frac{3}{4}\tau\theta > 0. \quad (69)$$

Due to the next estimation

$$\frac{2}{\tau\theta^2} + \frac{\tau}{4} + \frac{2\tau}{T\theta} \leq \frac{2}{\tau_m\theta_m^2} + \frac{\tau_M}{4} + \frac{2\tau_M}{T_m\theta_m}, \quad (70)$$

that readily derives from (11) and (30), it is clear that the tuning inequalities (33) and (34) are such that $\lambda_2 > 0$ and $\lambda_3 > 0$ hold, whence all the terms in the right-hand side of (65) are nonpositive. Let us further estimate the right-hand side of (65) to show that there exists $\lambda_4 > 0$ such that

$$\dot{V}(t) \leq -\lambda_4 \|x(\cdot, t)\|_2^2. \quad (71)$$

To this end, we rely on the next estimations

$$-(x^2(1, t) + \|x_\xi(\cdot, t)\|_0^2) \leq -\frac{1}{2} \|x(\cdot, t)\|_0^2, \quad (72)$$

$$-(z^2(1, t) + \|z_\xi(\cdot, t)\|_0^2) \leq -\frac{\theta^2}{2} \|x_{\xi\xi}(\cdot, t)\|_0^2, \quad (73)$$

which are consequences of the Poincare inequality and the identities $\|z(\xi, t)\|_0^2 = \|x_t(\cdot, t)\|_0^2 = \theta^2 \|x_{\xi\xi}(\xi, t)\|_0^2$. One can further manipulate (65) as follows

$$\begin{aligned} \dot{V}(t) & \leq -\frac{\theta}{2\tau} \|x_\xi(\cdot, t)\|_0^2 - \lambda_5 (x^2(1, t) + \|x_\xi(\cdot, t)\|_0^2) \\ & - \lambda_6 (z^2(1, t) + \|z_\xi(\cdot, t)\|_0^2), \end{aligned} \quad (74)$$

where $\lambda_5 = \min\{\frac{\theta}{2\tau}, \beta_1 \frac{\theta}{\tau}\}$, $\lambda_6 = \min\{\frac{T\theta}{\tau} \lambda_2, \lambda_1\}$. By considering (72) and (73) into (74) one easily concludes that relation (71) holds with $\lambda_4 = \min\{\frac{\theta}{2\tau}, \frac{\lambda_5}{2}, \frac{\lambda_6\theta^2}{2}\}$. If one considers together the relations (71) and (37), it yields

$$\dot{V}(t) \leq -\frac{\lambda_4}{\rho_1} V(t) \quad (75)$$

where the explicit expression of ρ_1 was given in (44).

Clearly, (75) ensures that the closed-loop solutions of (13)-(17), (27) cannot escape to infinity in finite time. Then

by applying [24, Th. 23.5], it follows that any solution is continuously extendable on the entire semi-infinite time axis $[0, \infty)$, thereby establishing the closed-loop well-posedness. Moreover, $V(t)$ exponentially decays by virtue of (75), which, due to (37), ensures the exponential decay of $\|x(\cdot, t)\|_2^2$. Thus, BVP (13)-(17) in closed-loop with the control (27), (32)-(34), is globally exponentially stable in the space $H^2(0, 1)$. $\square$

D. Uniform pointwise-in-space convergence analysis

Based on Theorem 1, the error variable $x(\xi, t)$ and its spatial derivative $x_\xi(\xi, t)$ are additionally concluded to pointwise exponentially decay uniformly in $\xi \in [0, 1]$.

Theorem 2 Under the conditions of Theorem 1, the L_∞ norms (9) and (10) escape to zero exponentially. $\blacksquare$

Proof of Theorem 2: The Agmon's inequality will be employed in this proof. It states that an arbitrary $w \in H^1(0, 1)$ fulfills the next estimation

$$\max_{\xi \in [0, 1]} |w(\xi, t)|^2 \leq w^2(i, t) + \|w(\cdot, t)\|_0 \|w_\xi(\cdot, t)\|_0, i \in \{0, 1\}. \quad (76)$$

Specifying (76) with $w = x$ and $i = 1$ yields

$$\max_{\xi \in [0, 1]} |x(\xi, t)|^2 \leq x^2(1, t) + \|x(\cdot, t)\|_0 \|x_\xi(\cdot, t)\|_0. \quad (77)$$

The boundary state $x^2(1, t)$ meets the estimation

$$x^2(1, t) \leq 2 \|x(\cdot, t)\|_2^2, \quad (78)$$

previously derived in (47). Since $\|x(\cdot, t)\|_2$ escapes to zero exponentially as proven in Theorem 1, then also $x^2(1, t)$ escapes to zero exponentially. The norms $\|x(\cdot, t)\|_0$ and $\|x_\xi(\cdot, t)\|_0$ both decay exponentially as proven in Theorem 1. Thus, by (77), the L_∞ norm (9) also escapes to zero exponentially. Finally, specifying (76) with $w = x_\xi$ and $i = 0$ yields

$$\max_{\xi \in [0, 1]} |x_\xi(\xi, t)|^2 \leq x_\xi^2(0, t) + \|x_\xi(\cdot, t)\|_0 \|x_{\xi\xi}(\cdot, t)\|_0. \quad (79)$$

Notice that $x_\xi^2(0, t)$ is identically zero according to (14), whereas the norms $\|x_\xi(\cdot, t)\|_0$ and $\|x_{\xi\xi}(\cdot, t)\|_0$ both escape to zero exponentially as for Theorem 1. Thus, by (79), the L_∞ norm (10) also escapes to zero exponentially. $\square$

Remark 3 A consequence of Theorem 2 is that $x_\xi(1, t) = u(t) + d(t)$ escapes exponentially to zero. This implies that the actuator output $u(t)$ exponentially approaches the sign-reversed disturbance $-d(t)$ as time grows. $\square$

IV. SIMULATIONS

Consider the BVP (3)-(6). Let θ and τ be taken as $\tau = 1$ and $\theta = 10$ within the domains $\theta \in [\theta_m, \theta_M] = [8, 12]$, and $\tau \in [\tau_m, \tau_M] = [0.5, 1]$ as specified in the Assumption 1. Let the unmatched disturbance $d(t)$ be chosen as $d(t) = \sin(t)$ such that Assumption 2 is in force with $D_0 = 1$, $D_1 = 1$ and $D_2 = 1$. The ICs are $u(0) = 0$ and $q^0(\xi) = 10 + 5\cos(4\pi\xi)$. The controller in (27) is tuned according to (32)-(34), as $\beta_1 = 1$, $\beta_2 = 3.1$, $W = 4.8$, and the constant reference is $q^r = 20$.

For solving numerically the closed-loop combined PDE-ODE BVP (13)-(17) the finite-difference approximation

method was used. The spatial domain $\xi \in [0, 1]$ has been discretized into $N = 30$ uniformly spaced nodes $\xi_i = ih$, with $h = 1/(N - 1)$, and $i = 0, 1, 2, \dots, N - 1$. Then, the resulting N -th order system is solved by using the Euler-Forward method with step 10^{-5} .

Fig. 1-left shows the spatiotemporal profile of $q(\xi, t)$ in the open loop, i.e. with $v(t) = 0$, whereas Fig. 1-right shows the same signal under the proposed control (27), (32)-(34). The point-wise convergence of the solution profile to the prescribed reference confirms the expected closed-loop system features.

The exponential decay of the norm $\|x(\cdot, t)\|_2$ displayed in Fig. 2 confirms the conclusions of Theorem 1. Finally, Fig. 3 shows the time evolutions of the L_∞ norms (9) and (10), which were proven in Theorem 2 to be exponentially decaying.

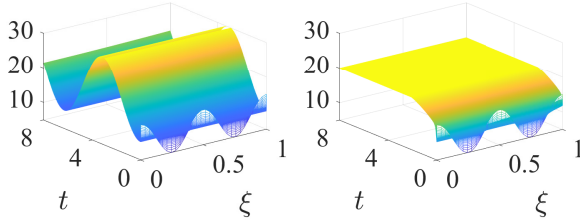


Fig. 1: Spatiotemporal profiles of $q(\xi, t)$ in the open-loop (left) and with the designed feedback control (right).

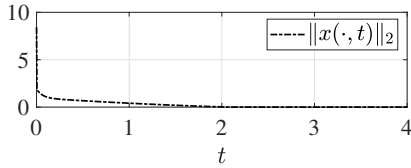


Fig. 2: Time evolution of the H^2 -norm $\|x(\cdot, t)\|_2$.

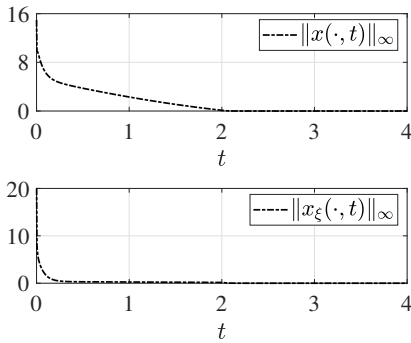


Fig. 3: Time evolutions of the L_∞ norms $\|x(\cdot, t)\|_\infty$ (top) and $\|x_\xi(\cdot, t)\|_\infty$ (bottom).

V. CONCLUSIONS

The problem of achieving global exponential stabilization of an uncertain heat process by means of Newmann-type boundary control is solved in the presence of smooth unmatched disturbances and uncertain actuator dynamics. The proposed discontinuous control law is of PD-plus-relay type and requires

boundary sensing only. The utilization of non-conventional disturbance-dependent Lyapunov functions has been crucial to demonstrate the asymptotic rejection of the unmatched disturbance. More general problems of unmatched disturbance rejection are worth addressing to exploit the features of this class of Lyapunov functions. The investigation of more sophisticated interconnections between (possibly multiple) PDEs and ODEs, e.g., “sandwiched” ODE-PDE-ODE systems, also calls for further research within the same framework of systems affected by unmatched disturbances to be asymptotically rejected.

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