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Reduced Yang model and noncommutative geometry of curved spacetime

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Abstract

The Yang model describes a noncommutative geometry in a curved spacetime by means of an orthogonal algebra $o(1,5)$, whose 15 generators are identified with phase space variables and Lorentz generators, together with an additional scalar generator. In this paper we show that it is possible to define a nonlinear algebra with the same structure, but with only 14 generators, that better fits in phase space. The fifteenth generator of the Yang algebra can then be written as a function of the squares of the others.

As a simple application, we also consider the problem of the quantum harmonic oscillator in this theory, calculating the energy spectrum in the one- and three-dimensional nonrelativistic versions of the model.

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1. Introduction

It is well known that all theories of quantum gravity predict a breaking of the continuous spacetime structure at scales close to the Planck length [1]. Among the various proposals advanced to describe such regime, a relevant role is played by noncommutative geometry. The first idea of noncommutativity of spacetime arose already in 1947 when Snyder [2] proposed a model which managed to introduce a fundamental scale without breaking the Lorentz invariance. This was obtained by deforming the commutation relations between the position operators. Although this proposal was not successful at that time, noncommutative geometry became fashionable in the 90's when some variants were proposed [3], and some connections with string theory were found [4].

On the other hand, it is known that our universe is curved on large scale, so a theory of noncommutative geometry on a nonflat spacetime is of physical relevance. This is also interesting in view of phenomenological applications, like possible time delays in the propagation of photons from distant sources [5].

The first proposal for an extension of noncommutative geometry to curved spacetime was advanced by C.N. Yang [6] shortly after the original paper by Snyder [2]. The idea was to describe the phase space by means of its symmetry generators, namely the Lorentz generators, the de Sitter momenta and the position operators, that generate the symmetries of Snyder's curved momentum space. The original formulation was based on an orthogonal algebra $o(1,5)$ that included all the previous operators. However, a drawback of the formalism was that the closure of the algebra imposed the introduction of a further generator K, whose physical interpretation was not clear.

The mathematical interest of this framework lies in the combination of curved spacetime with curved momentum space and in the fact that it is the unique stable Lie algebraic deformation of the standard phase space symmetries [7]. It also displays a duality between position and momentum spaces [8]. It may also assume an important physical role as a possible low-energy limit of quantum gravity, since spacetime curvature seems to be an essential feature in this limit [9].

Also Yang's proposal was almost forgotten for fifty years, and it revived when Kowalski-Glikman and Smolin proposed a model based on a nonlinear quadratic algebra, that contained both de Sitter and Snyder algebras like the Yang model, but avoided the necessity of the introduction of the generator K [10]. This was called Triply Special Relativity (TSR) or alternatively Snyder-de Sitter model [11].

More recently, a series of papers has been devoted to the study of generalizations of the Yang algebra and to their realization in quantum phase space and in extended phase space (an extension of phase space that includes tensorial degrees of freedom) [12,13]. Also its general κ-deformations were investigated in depth in [14]. A review with further references is given in [15].

In this letter, we propose a generalization of the Yang model that, in the same spirit of ref. [10], exploiting nonlinearity avoids the inclusion of K as an independent generator, and can be considered a nonlinear realization of the Yang model. This permits to realize the model in ordinary phase space, without the necessity of introducing an extra independent generator, whose interpretation in terms of physical observables is problematic.

We also give examples of application of such model to the dynamics of the harmonic oscillator and compare our results with those obtained for the related TSR model [16]. Of course, it is possible to apply the formalism to more complex systems. For example, in [17] relations with generalized uncertainty relations are discussed for the Yang model. Although some experimental work has been made on possible deformations of the spectrum of harmonic oscillators [18], at present experimental observations of effects of the order of magnitude predicted by the Yang model do not seem feasible, so we refrain from discussing more realistic examples.

As mentioned above, the Yang algebra is generated by noncommuting position and momentum operators, Lorentz symmetry generators and one additional Lorentz-invariant operator K, and is isomorphic to an orthogonal algebra. In this paper, we show that K can be expressed in terms of a function of the squares of the other generators, thus obtaining a new algebra, whose dimension (in four-dimensional spacetime) is 14, instead of 15 as the original Yang algebra, and can be interpreted as a deformation of the Heisenberg-Lorentz algebra. We call this new nonlinear algebra reduced Yang algebra. It differs from other nonlinear algebras in phase space related to the Yang model, as for example TSR, since the deformed Heisenberg algebra has a different form in those cases.

More explicitly, the Yang algebra is defined by the commutation relations between the phase space coordinates $\hat{x}_\mu$ and $\hat{p}_\mu$, the Lorentz generators $\hat{M}_{\mu\nu}$ and K given by[1]

$$\begin{aligned}
&[\hat{x}_\mu, \hat{x}_\nu] = i\beta^2 \hat{M}_{\mu\nu}, \qquad [\hat{p}_\mu, \hat{p}_\nu] = i\alpha^2 \hat{M}_{\mu\nu}, \qquad [\hat{x}_\mu, \hat{p}_\nu] = i\eta_{\mu\nu} K, \\
&[K, \hat{x}_\mu] = i\beta^2 \hat{p}_\mu, \qquad [K, \hat{p}_\mu] = -i\alpha^2 \hat{x}_\mu, \qquad [\hat{M}_{\mu\nu}, K] = 0, \\
&[\hat{M}_{\mu\nu}, \hat{x}_\lambda] = i(\eta_{\mu\lambda}\hat{x}_\nu - \eta_{\nu\lambda}\hat{x}_\mu), \qquad [\hat{M}_{\mu\nu}, \hat{p}_\lambda] = i(\eta_{\mu\lambda}\hat{p}_\nu - \eta_{\nu\lambda}\hat{p}_\mu), \\
&[\hat{M}_{\mu\nu}, \hat{M}_{\rho\sigma}] = i\big(\eta_{\mu\rho}\hat{M}_{\nu\sigma} - \eta_{\mu\sigma}\hat{M}_{\nu\rho} - \eta_{\nu\rho}\hat{M}_{\mu\sigma} + \eta_{\nu\sigma}\hat{M}_{\mu\rho}\big),
\end{aligned} \tag{1.1}$$

where $\eta_{\mu\nu}$ is the flat metric and α and β are two real deformation parameters, that can take also negative values. Their physical dimensions are inverse length and inverse mass respectively, and are usually identified with the square root of the cosmological constant and the inverse Planck mass. Clearly, the properties of the algebra depend on the signs of α^2 and β^2. In fact the Yang algebra is isomorphic either to $o(1,5)$, $o(2,4)$ or $o(3,3)$, depending on these signs. It follows that it admits a quadratic Casimir operator

$$\mathcal{C} = \alpha^2 \hat{x}_\mu^2 + \beta^2 \hat{p}_\mu^2 + \frac{\alpha^2\beta^2}{2} \hat{M}_{\mu\nu}^2 + K^2. \tag{1.2}$$

From (1.1) it is clear that the position operators do not commute, as in Snyder space, and the same happens to the momenta, as is expected for de Sitter spacetime. In fact, neglecting K, in the limit $\alpha \to 0$ one recovers the Snyder algebra [2], while for $\beta \to 0$ one obtains the de Sitter algebra acting on a commutative de Sitter spacetime endowed with Beltrami coordinates.

2. Reduced Yang model

The main result of this paper is the proof that the original Yang algebra (1.1) can be reduced to a nonlinear algebra with the same form, where however the generator K is not independent, but is a given function of the other generators. We are not able to show that this representation is unique, but its postulated form appears natural in view of the expression of the quadratic Casimir operator (1.2).

In fact, the following theorem holds:

Theorem:

The Lorentz-invariant generator K of the algebra (1) can be expressed in terms of the Lorentz invariants $\hat{x}^2 = \hat{x}_\mu \hat{x}^\mu$, $\hat{p}^2 = \hat{p}_\mu \hat{p}^\mu$ and $\hat{M}_{\mu\nu}^2 = \hat{M}_{\mu\nu} M^{\mu\nu}$ as

$$K = \sqrt{1 - \alpha^2 \hat{x}_\mu^2 - \beta^2 \hat{p}_\mu^2 - \frac{\alpha^2\beta^2}{2} \hat{M}_{\mu\nu}^2}, \tag{2.1}$$

so that all the relations (1.1) and the Jacobi identities of the algebra are satisfied.

Proof:

It is useful to introduce the Lorentz-invariant operator $z = \alpha^2 \hat{x}_\mu^2 + \beta^2 \hat{p}_\mu^2 + \frac{\alpha^2\beta^2}{2} \hat{M}_{\mu\nu}^2$, so that $K = \sqrt{1-z}$. The operator z satisfies

$$[z, \hat{x}_\mu] = -i\beta^2(\hat{p}_\mu K + K\hat{p}_\mu), \qquad [z, \hat{p}_\mu] = i\alpha^2(\hat{x}_\mu K + K\hat{x}_\mu). \tag{2.2}$$

Now,

$$\begin{aligned}
[K, \hat{x}_\mu] &= [\sqrt{1-z}, \hat{x}_\mu] \\
&= i\beta^2 \sum_{n=1}^{\infty} (-1)^{n-1} \binom{\frac{1}{2}}{n} \Big(z^{n-1}(\hat{p}_\mu K + K\hat{p}_\mu) + z^{n-2}(\hat{p}_\mu K + K\hat{p}_\mu) z + \ldots + (\hat{p}_\mu K + K\hat{p}_\mu) z^{n-1} \Big) \\
&= i\beta^2 \sum_{n=1}^{\infty} (-1)^{n-1} \binom{\frac{1}{2}}{n} \sum_{k=0}^{n-1} z^k (\hat{p}_\mu K + K\hat{p}_\mu) z^{n-k-1} = i\beta^2 \sum_{k,l\geq 0} C_{kl} z^k \hat{p}_\mu z^l,
\end{aligned} \tag{2.3}$$

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[1] We adopt the following conventions: Greek indices run from 0 to 3, Latin indices from 1 to 3, and the flat metric is given by $(-1,1,1,1)$. We use natural units $c = \hbar = 1$.

where

$$C_{00} = 2\binom{\frac{1}{2}}{1}\binom{\frac{1}{2}}{0} = 1,$$

$$C_{10} = C_{01} = -2\binom{\frac{1}{2}}{2}\binom{\frac{1}{2}}{0} - \binom{\frac{1}{2}}{1}\binom{\frac{1}{2}}{1} = 0,$$

$$C_{20} = C_{11} = C_{02} = 2\binom{\frac{1}{2}}{3}\binom{\frac{1}{2}}{0} + \binom{\frac{1}{2}}{2}\binom{\frac{1}{2}}{1} + \binom{\frac{1}{2}}{1}\binom{\frac{1}{2}}{2} = 0.$$

By induction it follows in general

$$C_{kl} = (-1)^{k+l} \sum_{m=0}^{k+l+1} \binom{\frac{1}{2}}{k+l+1-m}\binom{\frac{1}{2}}{m}. \tag{2.4}$$

Then, by the Vandermonde identity follows

$$C_{kl} = (-1)^{k+l} \binom{1}{k+l+1}, \tag{2.5}$$

which implies $C_{00} = 1$ and $C_{kl} = 0$ if $k + l \geq 1$. Hence,

$$[\sqrt{1-z}, \hat{x}_\mu] = i\beta^2 \hat{p}_\mu. \tag{2.6}$$

Analogously,

$$[\sqrt{1-z}, \hat{p}_\mu] = -i\alpha^2 \hat{x}_\mu. \tag{2.7}$$

In one space dimension there is no Lorentz generator and

$$K = \sqrt{1 - \alpha^2 \hat{x}^2 - \beta^2 \hat{p}^2}. \tag{2.8}$$

In the limit $\alpha = 0$, $\beta = 0$, the reduced Yang algebra becomes the ordinary Heisenberg algebra together with the Lorentz algebra. In the limit $\alpha = 0$, it reduces to the Snyder algebra with

$$K = \sqrt{1 - \beta^2 \hat{p}^2}. \tag{2.9}$$

In the limit $\beta = 0$, it reduces to the dual Snyder algebra with

$$K = \sqrt{1 - \alpha^2 \hat{x}^2}. \tag{2.10}$$

In general, the reduced Yang algebra can be interpreted as a two-parameter deformed Heisenberg algebra with exact Lorentz algebra. The $o(1,5)$ Casimir operator $\mathcal{C}$ takes the value 1.

Clearly, the realizations of this algebra on canonical phase space with coordinates x_μ and p_μ coincide with those of the original Yang model, since the commutation relations are the same. For example, at first order in α^2 and β^2, a possible Hermitean realization with $\hat{M}_{\mu\nu} = x_\mu p_\nu - x_\nu p_\mu$ is

$$\hat{x}_\mu = x_\mu - \frac{\beta^2}{4}\left(x_\mu p^2 + p^2 x_\mu\right) + \dots, \qquad \hat{p}_\mu = p_\mu - \frac{\alpha^2}{4}\left(p_\mu x^2 + x^2 p_\mu\right) + \dots, \tag{2.11}$$

implying $K = 1 - \frac{1}{2}(\alpha^2 x^2 + \beta^2 p^2) + o(\alpha^4, \beta^4)$.

3. One-dimensional oscillator

In the following, we describe the applications of the reduced Yang model to simple quantum mechanical systems. To avoid the intricacies of the relativistic theory, we discuss the Euclidean case, and for definiteness we consider positive α^2 and β^2.

We start by considering the simplest nontrivial example of dynamics, namely the one-dimensional harmonic oscillator. In one dimensional space, the Yang algebra reduces to the $o(3)$ algebra

$$[\hat{x}, \hat{p}] = K, \qquad [K, \hat{x}] = i\beta^2\hat{p}, \qquad [K, \hat{p}] = -i\alpha^2\hat{x}, \tag{3.1}$$

and we assume the relation (2.8).

We recall that in standard quantum mechanics, the Hamiltonian of the harmonic oscillator of mass m is [19]

$$H_0 = \frac{1}{2}\left(\frac{p^2}{m} + m\omega^2 q^2\right) = \frac{\omega}{2}(a^\dagger a + aa^\dagger) = \omega\left(a^\dagger a + \frac{1}{2}\right), \tag{3.2}$$

where we have defined

$$a = \sqrt{\frac{m\omega}{2}}\left(x + i\frac{p}{m\omega}\right), \qquad a^\dagger = \sqrt{\frac{m\omega}{2}}\left(x - i\frac{p}{m\omega}\right), \tag{3.3}$$

with

$$[a, a^\dagger] = 1. \tag{3.4}$$

Defining $N = a^\dagger a$, with eigenvectors $|n{>}$, one then gets $N|n{>} = n|n{>}$ and

$$H_0|n{>} = \omega\left(n + \frac{1}{2}\right)|n{>}, \tag{3.5}$$

which gives the energy spectrum of the oscillator.

We proceed in a similar way in the reduced Yang case, by using a perturbative approach. We start from the Hamiltonian

$$H = \frac{1}{2}\left(\frac{\hat{p}^2}{m} + m\omega^2\hat{x}^2\right), \tag{3.6}$$

and define the ladder operators

$$\hat{a} = \sqrt{\frac{m\omega}{2}}\left(\hat{x} + i\frac{\hat{p}}{m\omega}\right), \qquad \hat{a}^\dagger = \sqrt{\frac{m\omega}{2}}\left(\hat{x} - i\frac{\hat{p}}{m\omega}\right), \tag{3.7}$$

which satisfy

$$[\hat{a}, \hat{a}^\dagger] = K. \tag{3.8}$$

We assume that $\hat{x} = x + o(\alpha^2, \beta^2)$ and $\hat{p} = p + o(\alpha^2, \beta^2)$, and therefore at order 0, $\hat{a} = a$ and $\hat{a}^\dagger = a^\dagger$.

On the other hand, from (2.8), at first order in α^2 and β^2,

$$K \approx 1 - \frac{1}{2}(\alpha^2\hat{x}^2 + \beta^2\hat{p}^2) \approx 1 - \frac{A}{2\omega}(\hat{a}\hat{a}^\dagger + \hat{a}^\dagger\hat{a}) - \frac{B}{2\omega}(\hat{a}\hat{a} + \hat{a}^\dagger\hat{a}^\dagger), \tag{3.9}$$

where

$$A = \frac{\alpha^2 + \beta^2 m^2\omega^2}{2m}, \qquad B = \frac{\alpha^2 - \beta^2 m^2\omega^2}{2m}. \tag{3.10}$$

At first order, eq. (3.8) is satisfied by

$$\hat{a} = a - \frac{A}{4\omega}aa^\dagger a - \frac{B}{8\omega}\big(aa^\dagger a^\dagger + a^\dagger a^\dagger a\big), \qquad \hat{a}^\dagger = a^\dagger - \frac{A}{4\omega}a^\dagger aa^\dagger - \frac{B}{8\omega}\big(aaa^\dagger + a^\dagger aa\big), \tag{3.11}$$

and then

$$\begin{aligned} H &= \frac{\omega}{2}\big(\hat{a}^\dagger\hat{a} + \hat{a}\hat{a}^\dagger\big) \approx \frac{\omega}{2}(a^\dagger a + aa^\dagger) - \frac{A}{8\omega}\big(a^\dagger aa^\dagger a + aa^\dagger aa^\dagger\big) - \frac{B}{16\omega}\big(a^3a^\dagger + aa^\dagger aa + a^\dagger aa^\dagger a^\dagger + (a^\dagger)^3 a\big) \\ &= \omega\left(N + \frac{1}{2}\right) - \frac{A}{2}\left(N^2 + 2N + \frac{1}{2}\right) - \frac{B}{4}\left(a^2N + (a^\dagger)^2\,(N+1)\right) = H_0 + H_1, \end{aligned} \tag{3.12}$$

where H_0 is the standard quantum mechanical Hamiltonian of eq. (3.2). At first order, the corrections to the spectrum are given by

$$<n|H_1|n> = -\frac{A}{2}\left(n^2 + n + \frac{1}{2}\right), \tag{3.13}$$

while the terms proportional to B do not contribute. Then,

$$E_n = \omega\left(n + \frac{1}{2}\right) - \frac{\alpha^2 + \beta^2\omega^2}{4m}\left(n^2 + n + \frac{1}{2}\right) + o(\alpha^2, \beta^2). \tag{3.14}$$

This is the same result obtained in [17] for the Yang model, using a different method based on the solution of the Schrödinger equation. It is also analogous to the results obtained for TSR [16], where however the corrections have opposite sign.

An interesting special limit is obtained for $\omega = \frac{\alpha}{\beta m}$. This value of ω is out of the physical range, but seems to be the most natural choice from an algebraic point of view, and permits to solve exactly the Hamilton equations. From a purely mathematical point of view, it may be interpreted as the Hamiltonian of a free particle in the reduced Yang model.

In this case, one has

$$H = \frac{1 - K^2}{2m\beta^2} = \frac{1}{2m}\left(\hat{p}^2 + \frac{\alpha^2}{\beta^2}\hat{x}^2\right). \tag{3.15}$$

Thus H and K commute and have common eigenvectors. Defining the ladder operators as in (3.3), with $\omega = \frac{\alpha}{\beta m}$, it follows that

$$[K, a] = \alpha\beta\, a, \qquad [K, a^\dagger] = -\alpha\beta\, a^\dagger. \tag{3.16}$$

Hence, the role of the number operator is taken in this case by

$$\hat{N} = -\frac{K}{\alpha\beta}. \tag{3.17}$$

In fact, given an eigenvector $|\eta>$ of $\hat{N}$, such that $\hat{N}|\eta>= \eta|\eta>$, one has

$$\hat{N}a^\dagger|\eta>= (\eta + 1)a^\dagger|\eta> . \tag{3.18}$$

Now, assuming the existence of a ground state $|0>$ such that $a|0> = 0$, one has

$$\hat{N}|0> = n_0|0>, \tag{3.19}$$

and defining recursively $|n+1> = a^\dagger|n>$, it is easy to show that

$$\hat{N}|n> = (n + n_0)|n> . \tag{3.20}$$

To calculate n_0, let us consider the equality

$$K^2 = 1 - 2\beta^2 H = 1 - 2\alpha\beta\, a^\dagger a - \alpha\beta K, \tag{3.21}$$

which, applied to the ground state, gives $\alpha^2\beta^2 n_0^2 = 1 + \alpha^2\beta^2 n_0$ and then

$$n_0 = \frac{1}{2} + \sqrt{\frac{1}{4} + \frac{1}{\alpha^2\beta^2}}. \tag{3.22}$$

Hence,

$$E_n =<n|\frac{1 - \alpha^2\beta^2 N^2}{2\beta^2 m}|n> = \frac{\alpha}{\beta m}\sqrt{1 + \frac{\alpha^2\beta^2}{4}}\left(n + \frac{1}{2}\right) - \frac{\alpha^2}{2m}\left(n^2 + n + \frac{1}{2}\right). \tag{3.23}$$

Therefore, at first order, $E_n \approx \frac{\alpha}{\beta m}\left(n+\frac{1}{2}\right) - \frac{\alpha^2}{2m}\left(n^2+n+\frac{1}{2}\right)$, which coincides with (3.14) for $\omega = \frac{a}{\beta m}$.

4. Three-dimensional oscillator

Let us now consider the Hamiltonian of a generic harmonic oscillator of mass m in three dimensions. The relevant relations are simply obtained by replacing Greek indices by Latin ones in the equations of sect. (2), and the symmetry algebra reduces to $o(5)$. Unfortunately, to discuss it in a perturbative approach, it is necessary to find a realization of the reduced Yang model on standard or extended phase space, which at leading order reduces to the same as the standard Yang model.

Here, we shall consider a simple Hermitian realization on standard phase space, given by [12]

$$\hat{x}_i = x_i - \frac{\beta^2}{4}(p^2 x_i + x_i p^2), \qquad \hat{p}_i = p_i - \frac{\alpha^2}{4}(x^2 p_i + p_i x^2), \qquad M_{ij} = x_i p_j - x_j p_i, \tag{4.1}$$

that of course at leading order in α and β satisfies the commutation relations (1.1).

We choose a Hamiltonian

$$H = \frac{1}{2}\left(\frac{\hat{p}_i^2}{m} + m\omega^2 \hat{x}_i^2 + \frac{\gamma}{2}\hat{M}_{ij}^2\right). \tag{4.2}$$

The M_{ij}^2 term is due to the curvature of position and momentum spaces, and γ is assumed to be linear in α^2 and β^2. In particular, it is known that the correct Hamiltonian in de Sitter space corresponds to $\gamma = \alpha^2/m$. However, in principle one may add to γ terms proportional to β^2 to take into account the curvature of momentum space as well [16]. We perturb the Hamiltonian around commutative flat space with $\gamma = 0$. We have at first order

$$H = H_0 - \frac{\alpha^2 + \beta^2 m^2 \omega^2}{4m}(x_i^2 p_j^2 + p_i^2 x_j^2 + 3) + \frac{\gamma}{4}M_{ij}^2, \tag{4.3}$$

with $H_0 = \frac{1}{2}(\frac{p_i^2}{m} + m\omega^2 x_i^2)$ and $M_{ij}^2 = 2x_i p_j(x_i p_j - x_j p_i)$.

In this case, it is more convenient to study the Schrödinger equation. In the unperturbed case, it separates in spherical coordinates, with

$$\psi(r,\theta,\phi) = \sum_{l\mathrm{m}} \frac{u_{nl\mathrm{m}}(r)}{r} Y_{l\mathrm{m}}(\theta,\phi), \tag{4.4}$$

where $Y_{l\mathrm{m}}(\theta,\phi)$ are spherical harmonics and u_{nlm} satisfies the equation

$$\frac{d^2 u_{nl\mathrm{m}}}{dr^2} - \left(m^2\omega^2 r^2 + \frac{l(l+1)}{r^2}\right) u_{nl\mathrm{m}} = 2mE_{nl} u_{nl\mathrm{m}}, \tag{4.5}$$

with eigenvalues $E_{nl} = \omega(2n + l + \frac{3}{2})$ and solutions [20]

$$u_{nl\mathrm{m}} = C_{nl}\, e^{-m\omega r^2/2}\, r^{l+1}\, L_n^{l+1/2}(m\omega r^2), \qquad C_{nl}^2 = \frac{2(m\omega)^{l+1/2} n!}{(n+l+\frac{1}{2})!}, \tag{4.6}$$

where L_n^α are Laguerre polynomials.

One can apply perturbation theory to the Hamiltonian (4.3). Only diagonal terms contribute to first order. Denoting $|nl\!> = u_{nlm}$ and using (4.5), one has

$$<\!nl|x_i^2 p_j^2 + p_i^2 x_j^2|nl\!> = \ 2 <\!nl|\, 2mE_{nl} r^2 - m^2\omega^2 r^4 |nl\!>, \tag{4.7}$$

and then, from the results of [21],

$$<\!nl|x_i^2 p_j^2 + p_i^2 x_j^2|nl\!> = 2n^2 + 2nl + 3n + l^2 + 2l - \frac{3}{4}. \tag{4.8}$$

It follows that at first order

$$E_{nl} = \omega \left(2n + l + \frac{3}{2} \right) - A \left(2n^2 + 2nl + 3n + l^2 + 2l + \frac{3}{4} \right) + \frac{\gamma}{2} l(l+1), \tag{4.9}$$

with A given by (3.10). Defining a new quantum number $\bar{n} = 2n + l$, this can also be written

$$E_{\bar{n}l} = \omega \left(\bar{n} + \frac{3}{2} \right) - \frac{\alpha^2 + \beta^2 m^2 \omega^2}{4m} \left(\bar{n}^2 + 3\bar{n} + \frac{3}{2} \right) + \frac{\gamma - A}{2} l(l+1). \tag{4.10}$$

Notice that the choice $\gamma = A$ in the Hamiltonian eliminates the dependence on the quantum number l. In fact, with this choice the symmetry is enhanced, since the Hamiltonian is invariant under transformations generated by $\hat{p}_i$ and $\hat{x}_i$. In particular, this is the natural choice for $\beta = 0$.

The spectrum can be compared with that of TSR [16]: like in the one-dimensional case the correction have a similar structure, but opposite sign.

5. Final remarks

The Yang model is based on a linear 15-parameters algebra that includes phase space operators, Lorentz generators and an additional scalar operator that rotates positions into momenta.

We have shown that the Yang algebra can be reduced to a 14-parameter nonlinear algebra, since the scalar generator can be written in terms of the squares of the other operators in such a way that all the commutation relations are satisfied. This is important for the physical applications, since it allows to discuss the physics on ordinary phase space, avoiding the introduction of extra degrees of freedom.

The dynamics of such model of course differs from that of the standard quantum mechanics. As an example, we have considered the elementary case of a nonrelativistic harmonic oscillator in 1 and 3 dimensions. The energy spectrum can be calculated perturbatively and contains corrections proportional to the deformation parameters α and β. In particular, at first order it coincides with that of the ordinary Yang model [17]. It is likely that this is true also at higher orders. Experiments aiming to test perturbations of the frequency spectrum of quantum harmonic oscillators have been performed [18], but the size of the effects predicted from the Yang model (with the standard identifications of the coupling constants) is still too tiny to be detected.

The results of this paper could also be useful in the context of quantum field theory, for example in the construction of a model based on a gauge group generated by the Yang algebra.

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