



Design of interface modes in canonical phononic waveguides

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ABSTRACT

An interface mode is a localised vibration field at the interface between two waveguides that may be excited at a frequency sitting in a band gap that is in common between the two structures. For electromagnetic waves, the condition for the mode to occur is associated with certain properties of either the surface impedances of the two waveguides or the value of the Zak phase of the adjacent pass bands. In this work, we propose a novel, rigorous and simple method to predict the presence of interface modes at the join between two dissimilar, one-dimensional, periodic, two-phase phononic waveguides. In particular, we show that when the two rods have a *canonical configuration* it is possible to determine the band gaps of the frequency spectrum where this condition is satisfied. The value of the impedance for all band gaps of the spectrum is analysed through an extended version of the method of the universal toroidal manifold, recently adopted by the Authors to describe the dynamic properties of canonical structures. In terms of prediction, the outcome of the proposed approach is identical to that derived by calculating the Zak phase of the bulk bands for both the waveguides composing the system. By considering two specific combinations of finite-sized canonical rods and studying the associated reflection coefficients, we also determine the frequency of the interface mode in closed form. Our approach provides significant new insight to the mechanics of structured waveguides in order to design and optimise systems able to support interface modes avoiding the challenging numerical calculations normally required to estimate topological invariants.

1. Introduction

In the last two decades, the concepts of topologically-protected edge wave propagation and topological insulator, originally developed in quantum mechanics (Kane and Mele, 2005; Hasan and Kane, 2010), have been rapidly generalised to other classical areas of physics, including acoustics (Brendel et al., 2018), photonics (Khanikaev et al., 2013; Lu et al., 2014) and mechanics (Wang et al., 2015; Huber, 2016; Pal and Ruzzene, 2017; Chen et al., 2018; Davies and Craster, 2022; Wang et al., 2023; Yang and Wang, 2024). In all of these different domains, phenomena such as the existence of waves limited to a boundary or an interface, immunity to back-scattering, and localisation in the presence of defects and imperfections are associated with the notion of topological invariants related to the bulk-band structure, such as the Chern number (Thouless et al., 1982; Moore and Balents, 2007) and the Zak phase for one-dimensional systems (Zak, 1989). Recently, existence of interface states between two one-dimensional photonic crystals has been related to the connection between their surface impedances and the values of the Zak phase of the pass bands of their dispersion spectra (Xiao et al., 2014; Kalozoumis et al., 2018). In these studies, a condition to obtain an interface mode depending only on the surface impedance of the two structures composing the system is introduced for both semi-infinite and finite photonic

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crystals. The sign of the surface impedance, and then the fulfilment of this general condition, is rigorously related to the sum of the values of the Zak phase of branches at frequencies below the band gap considered on either the sides of the interface. Lately, these results have been experimentally demonstrated for acoustic waveguides (Xiao et al., 2015), and generalised to study mechanical systems (Meng et al., 2018; Chen et al., 2021).

In this work, we adopt the approach developed by Xiao et al. (2014), Meng et al. (2018) and Chen et al. (2021) to study the presence of localised modes at the interface between periodic two-phase phononic rods that possess *canonical* configurations (Gei et al., 2020; Farhat et al., 2022). Canonical phononic waveguides are characterised by unit cells designed according to rational ratios between mechanical and geometrical properties of the different phases, and their frequency spectra display periodic properties (Gei et al., 2020). We take advantage of the periodicity features, together with the general representation of the spectra in terms of universal toroidal manifold proposed by Morini et al. (2019), to predict the sign of the surface impedance inside each band gap displayed by the two canonical rods composing the system. This allows us to determine the band gaps for which the condition of existence of an interface mode is satisfied for any system composed of an arbitrary pair of semi-infinite canonical structures. Our approach provides results identical to those achieved by determining, after a lengthy calculation, the Zak phase of the relevant pass bands for both the rods composing the system.

We also extend our approach to the study of interface modes between two finite-sized waveguides. By examining two specific combinations of canonical rods, we derive their reflection coefficients adopting the representation of the transfer matrix in terms of Chebyshev polynomials (Yeh et al., 1977). In turn, they are finally employed to obtain a set of equations determining the frequency of the interface mode. For one of the two studied cases, the excitation frequency of the interface mode is accomplished in closed form.

Our results show that canonical rods can be suitably adopted to design systems exhibiting interface modes at a frequency within a prescribed band gap, and that for some specific configurations it is even possible to determine exactly its frequency. The proposed formulation, based on the representation of the spectrum through the universal toroidal surface, represents a novel tool to determine the presence of interface modes in sets of arbitrary band gaps avoiding the –sometimes cumbersome– numerical calculations required to estimate topological invariants such as the Zak phase.

2. Preliminaries

2.1. Waves in finite and infinite two-phase periodic rods

Assume a two-phase periodic rod composed of a number $\mathcal{N}$ of repeated unit cells, as shown in Fig. 1.(a), whose internal structure is set by elementary bars S (in red) and L (in blue). Here, we postulate the mid-point of phase S coinciding with the centre of the cell. It can be noted that the latter corresponds to the element F_2 of Fibonacci's *golden mean* sequence analysed in detail in Morini and Gei (2018), Gei et al. (2020); this observation, though not essential at this point, will become important later. The lengths of the two bars S and L are indicated with l_S and l_L , respectively, whereas A_X , E_X and ρ_X ($X \in \{L, S\}$) denote the cross-section area of each rod, Young's modulus and mass density per unit volume, respectively. Being z the co-ordinate describing the longitudinal axis, for both elements we define the displacement function $u(z)$ and the axial force $N(z) = EAu'(z)$.

We are interested in harmonic axial waves, therefore, the displacement u_{X_n} in phase X of the n^{th} unit cell is given by the solution of the governing equation

$$u_{X_n}''(z) + \frac{\rho_X}{E_X} \omega^2 u_{X_n}(z) = 0, \quad (1)$$

and assumes the form

$$u_{X_n}(z) = a_n^X \exp\left(i\sqrt{\frac{\rho_X}{E_X}}\omega z\right) + b_n^X \exp\left(-i\sqrt{\frac{\rho_X}{E_X}}\omega z\right), \quad (2)$$

where ω is the circular frequency, and a_n^X and b_n^X are the undetermined –complex– amplitudes of right-going and left-going waves, respectively. Coefficients a_n^S and b_n^S are related to a_n^L and b_n^L by continuity of functions $u_{X_n}(z)$ and $N_{X_n}(z)$ across the interface between two elements; therefore, for any cell, we can define a state vector $\mathbf{a}_n = [a_n^L, b_n^L]^T = [a_n, b_n]^T$, with state vectors of two adjacent unit cells that are linked though the matrix equation

$$\begin{bmatrix} a_{n+1} \\ b_{n+1} \end{bmatrix} = M \begin{bmatrix} a_n \\ b_n \end{bmatrix}, \quad (3)$$

where M is the 2×2 unimodular transfer matrix of the unit cell; its components are reported in Appendix A. Relationship (3) can be easily generalised to study wave propagation across the whole finite rod sketched in Fig. 1.(a):

$$\begin{bmatrix} a_{\mathcal{N}} \\ b_{\mathcal{N}} \end{bmatrix} = M^{\mathcal{N}} \begin{bmatrix} a_0 \\ b_0 \end{bmatrix}. \quad (4)$$

Here, a_0 and b_0 stand for the amplitudes of the incident and reflected waves at the input side of the structure (the left-hand side in Fig. 1.(a)), respectively, whereas $b_{\mathcal{N}}$ and $a_{\mathcal{N}}$ are their counterparts at the output boundary, respectively.

We now focus on the case of an infinite (or semi-infinite) periodic rod ($\mathcal{N} \rightarrow +\infty$, see Fig. 1.(b)). By applying the Floquet–Bloch conditions at the boundaries of the unit cell and recalling Eq. (3), the familiar dispersion equation is

$$\det[M - \exp(iKL_r)I] = 0, \quad (5)$$

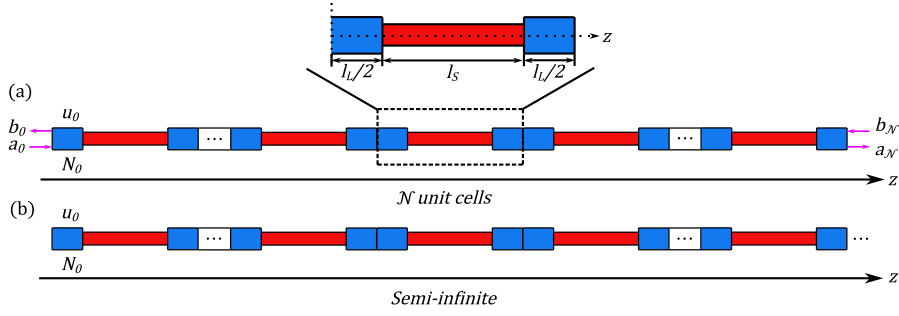


Fig. 1. (a) Finite two-phase periodic rod composed of $\mathcal{N}$ unit cells; a_0, b_0 are the amplitudes of the incident and reflected waves, respectively, at the input side of the structure, whereas b_N and a_N are their counterparts at the output boundary; (b) semi-infinite periodic rod ($\mathcal{N} \rightarrow +\infty$). In both plots, u_0 and N_0 represent displacement and axial force at the input side of the waveguide, respectively.

where I is the 2×2 identity matrix, $L_t = l_L + l_S$ is the total length of the unit cell, and KL_t the normalised Bloch wave number. The solution to Eq. (5) provides the frequency spectrum and, in particular, the layout of pass and stop bands of the problem. Its final form is

$$\frac{1}{2} \text{Tr} M(\omega) = \cos(KL_t), \quad (6)$$

where the trace of the transfer matrix is given by

$$\text{Tr} M(\omega) = 2 \cos \xi_L(\omega) \cos \xi_S(\omega) - \frac{Z_L^2 + Z_S^2}{Z_S Z_L} \sin \xi_L(\omega) \sin \xi_S(\omega), \quad (7)$$

the mechanical impedance $Z_X = A_X \sqrt{E_X \rho_X}$ ($X \in \{L, S\}$) is introduced and

$$\xi_X(\omega) = \omega \sqrt{\frac{\rho_X}{E_X}} l_X \quad (8)$$

is the ‘flow’ of each phase, whose role will be clarified later. Note that elegant analytical methods have been recently formulated for the calculation of the dispersion relation, see e.g. [García-Suárez \(2022\)](#), [González-Carbajal et al. \(2025\)](#).

Waves are transmitted along the waveguide for intervals of frequency $f = \omega/2\pi$ for which real values of KL_t exist ($|\text{Tr} M(\omega)| < 2$), whereas band gaps (stop bands) occur at frequency for which KL_t is imaginary ($|\text{Tr} M(\omega)| > 2$). The specific values $KL_t = 0$ and $KL_t = \pi$, i.e. $|\text{Tr} M(\omega)| = 2$, correspond to standing waves, which are referred here as *band-edge modes*.

As further discussed in the paper, the *symmetry properties of the band-edge modes* are the key elements to predict interface states between two joined semi-infinite dissimilar rods, and they depend on both the values of axial force and displacement at the boundary of the unit cell, i.e. at $z = 0$, namely

$$N_0 = N(0) = i\omega Z_L(a_0 - b_0), \quad u_0 = u(0) = a_0 + b_0. \quad (9)$$

The amplitudes can be evaluated through the components of M , together with the dispersion relation (6), namely

$$a_0 = \frac{i}{2} \left(\frac{Z_S}{Z_L} - \frac{Z_L}{Z_S} \right) \sin \xi_S, \quad b_0 = e^{iKL_t} - e^{i\xi_L} \left[\cos \xi_S + \frac{i}{2} \left(\frac{Z_S}{Z_L} + \frac{Z_L}{Z_S} \right) \sin \xi_S \right], \quad (10)$$

and substituting expressions (10) into Eqs. (9) their final form is attained, i.e.

$$N_0 = -i\omega \frac{2Z_S Z_L \cos \xi_S \sin \xi_L + [Z_S^2 - Z_L^2 + (Z_L^2 + Z_S^2) \cos \xi_L] \sin \xi_S}{2Z_S}, \quad (11)$$

$$u_0 = -i \frac{2Z_S Z_L \cos \xi_S \sin \xi_L + [Z_L^2 - Z_S^2 + (Z_L^2 + Z_S^2) \cos \xi_L] \sin \xi_S}{2Z_S Z_L}. \quad (12)$$

We now proceed to show the relationships that are in place between functions (11) and (12) and the symmetry features of band-edge modes of the frequency spectrum.

2.2. General condition for interface modes between two bonded dissimilar periodic rods

The focus is now turned to a system composed of two dissimilar either finite-sized or semi-infinite periodic rods as those illustrated in Fig. 2. A general condition for the presence of an interface mode between two distinct joined sub-systems in similar one-dimensional structures has been pointed out for electromagnetic waves in both semi-infinite and finite cases ([Xiao et al., 2014](#); [Kalozoumis et al., 2018](#)), and then extended to the study of topological mechanical states in phononic waveguides ([Yin et al., 2018](#); [Meng et al., 2018](#)). This condition is given by the relationship

$$Z_{\text{rhs}} + Z_{\text{lhs}} = 0, \quad (13)$$

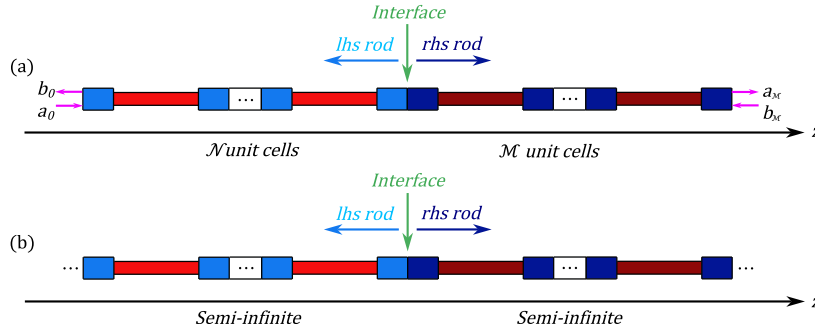


Fig. 2. (a) Two one-dimensional periodic rods with N unit cells on the left and M unit cells on the right, connected to each other at the interface; (b) same as (a), but now the rods are both semi-infinite.

where Z_{rhs} and Z_{lhs} are the surface impedances of the rods of the right-hand side and the left-hand side sketched in Fig. 2, respectively. A frequency satisfying condition (13) is commonly associated with a wave field (in our case, either displacement or axial force) that is both localised and amplified at the bonding section between the two rods.

The surface impedance Z_J ($J \in \{\text{lhs}, \text{rhs}\}$) is related to the reflection coefficient r_J through the expression (Xiao et al., 2014)

$$Z_J = \frac{1 + r_J}{1 - r_J} Z_0 \quad (J \in \{\text{lhs}, \text{rhs}\}), \quad (14)$$

where Z_0 is the vacuum impedance.

Following the approach proposed by Xiao et al. (2014), we analyse the case of an interface mode excited at a frequency inside a band gap that is in common in the spectra of the two joined rods. Consequently, only the imaginary part of both the surface impedances is non zero ($\text{Re}(Z_{\text{rhs}}) = \text{Re}(Z_{\text{lhs}}) = 0$), and then Eq. (13) becomes

$$Z_{\text{rhs}} + Z_{\text{lhs}} = i(\text{Im}(Z_{\text{rhs}}) + \text{Im}(Z_{\text{lhs}})) = 0 \quad \Rightarrow \quad \text{Im}(Z_{\text{rhs}}) = -\text{Im}(Z_{\text{lhs}}). \quad (15)$$

Therefore, the control of the sign of $\varsigma_J = \text{Im}(Z_J)$ ($J \in \{\text{rhs}, \text{lhs}\}$) is the pivotal information that reveals whether an interface mode is present or not. The simplest way to obtain such a mode is to design a system for which the quantities ς_J at the two sides of the interface are opposite for a frequency sitting in a band gap.

Several studies (see, e.g. Xiao et al. (2015), Chen et al. (2021), Muhammad et al. (2019)) have shown that, for systems composed of two semi-infinite rods, the symmetry properties of band-edge modes are strongly related to the two quantities commonly used to predict the presence of an interface state: on the one hand, the value of $\text{sign}[\varsigma_J]$ of a selected band gap and, on the other, the Zak phase of the adjacent pass bands.

The properties of symmetry/antisymmetry of band-edge modes depend on the value of the wave fields (11) and (12) at the boundary of the unit cells. A symmetric edge mode, henceforth denoted by (S), corresponds to $N_0 \neq 0$ and $u_0 = 0$, whereas an antisymmetric one, (A) from now on, occurs for $N_0 = 0$ and $u_0 \neq 0$. An illustrative example is reported in Fig. 3. In particular, in Fig. 3(a), two edge modes at $KL_t = 0$ are indicated with O and P , and two edge modes at $KL_t = \pi$ are indicated with Q and R . For all the four modes, the axial force field is colour mapped along the whole unit cell in Fig. 3(b), showing a symmetric behaviour for modes P and Q and an antisymmetric one for O and R .

In Xiao et al. (2014), the following relationships between the types of mode of the edge state and ς_J are found:

- (1) the property of the edge mode swaps either from (S) to (A) or from (A) to (S) when the frequency goes across a band gap;
- (2) if a (A) mode is at the lower edge of a band gap and a (S) mode is at the upper edge, then $\varsigma_J < 0$, otherwise $\varsigma_J > 0$.

In the next Sections, we will use this classification to predict and control the presence of an interface mode in systems alike those illustrated in Fig. 2 characterised by a particular ratio between the geometrical and physical properties of the two phases, named *canonical structures* (Gei et al., 2020).

3. Interface modes in systems composed of semi-infinite canonical rods

Canonical two-phase composite rods have been firstly introduced and thoroughly studied by Morini and Gei (2018) and Gei et al. (2020) as a special subclass of two-phase structures described by the Fibonacci *golden-mean* sequence. They are periodic waveguides whose basic elements of the unit cell, such as S and L , possess geometric and mechanical features such that the parameter

$$C = \sqrt{\frac{\rho_S E_L}{\rho_L E_S}} \frac{l_S}{l_L} = \frac{\xi_S}{\xi_L}, \quad (16)$$

termed the *canonical ratio* C , satisfies one of the following three conditions:

$$C^{(1)} = \frac{1+2j}{1+2k}, \quad C^{(2)} = \frac{1+2j}{2q}, \quad C^{(3)} = \frac{2q}{1+2k} \quad (j, k, q \in \mathbb{N}), \quad (17)$$

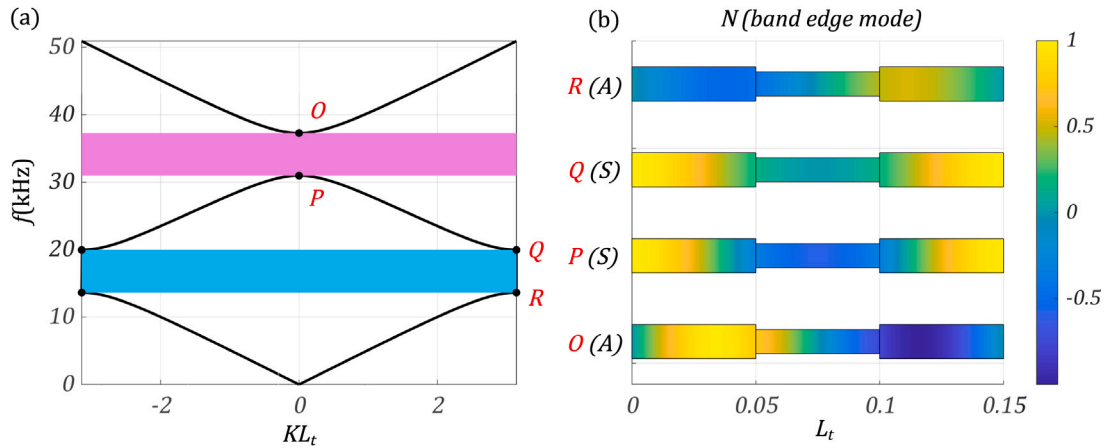


Fig. 3. (a) Dispersion curves of a two-phase, one-dimensional, periodic waveguide and points R , Q , P and O at which band-edge modes take place; cyan and magenta colours highlight band gaps where the sign of ζ_j is negative and positive, respectively; (b) diagrams of the axial force field for modes R , Q , P and O mapped along the whole unit cell (values are normalised). The following values of mechanical and geometrical parameters are assumed: $E_S = E_L = 70$ GPa, $\rho_S = \rho_L = 2700$ kg m⁻³, $A_L = 2A_S = 2 \times 10^{-4}$ m², $l_S = 0.05$ m, $l_L = 0.1$ m and $L_t = 0.15$ m.

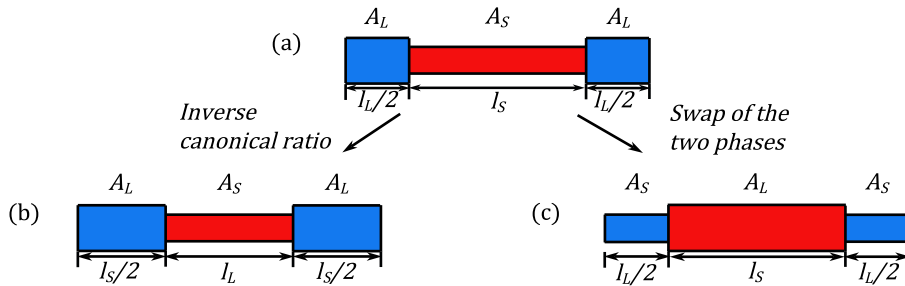


Fig. 4. Combinations of unit cells considered for joined canonical rods: (a) reference unit cell with canonical ratio C ; (b) cell with canonical ratio $1/C$ (Case I); (c) cell with canonical ratio C , but where the two constituents are swapped w.r.t. (a) (Case II). The total length L_t is the same for all the three cells.

in which it is required that indices j , k and q are such that fractions in the right-hand sides of the three conditions are to lowest terms. Superscripts (1), (2) and (3) indicate **three distinct families of canonical sets**. For family no. 1 the ratio is odd/odd, whereas odd/even and even/odd ratios belong to families no. 2 and no. 3, respectively. By observing formulae (17), we can deduce that the main requirement of a canonical structure is that the ratio C is a rational number.

For a canonical structure, the trace of the transfer matrix $\text{Tr}M(f)$ is a periodic function of the frequency. Consequently, the dispersion features, determined by Eq. (6), are also periodic in the frequency. The Authors (Morini and Gei, 2018; Gei et al., 2020) found that the period of the trace of the transfer matrix, f_t , is given by $f_t = 4f_c$, where f_c is the (fundamental) canonical frequency, defined assuming $\ell = 0$ in the relationship $f_{c\ell} = f_c^{(h)}(1 + 2\ell)$ ($\ell \in \mathbb{N}$, $h \in \{1, 2, 3\}$), where:

$$\begin{aligned} f_c^{(1)} &= \frac{1}{4l_L} \sqrt{\frac{E_L}{\rho_L}} (1 + 2k) = \frac{1}{4l_S} \sqrt{\frac{E_S}{\rho_S}} (1 + 2j), & f_c^{(2)} &= \frac{1}{4l_L} \sqrt{\frac{E_L}{\rho_L}} 2q = \frac{1}{4l_S} \sqrt{\frac{E_S}{\rho_S}} (1 + 2j), \\ f_c^{(3)} &= \frac{1}{4l_L} \sqrt{\frac{E_L}{\rho_L}} (1 + 2k) = \frac{1}{4l_S} \sqrt{\frac{E_S}{\rho_S}} 2q. \end{aligned} \quad (18)$$

Examples of frequency spectra of canonical configurations are displayed in Figs. 5–7. In all these diagrams, the lowest period $[0, f_t]$ of the spectrum is represented, being understood that the subsequent ones coincide with the lowest one. The positions of the canonical frequencies are indicated with red arrows. In all plots, the lowest arrow is, strictly speaking, the fundamental frequency f_c , whereas the higher arrow marks $f_{c1} = 3f_c$. We stress that, although at first glance it would seem that the period was $2f_c$, its value is $4f_c$. For configurations that fall within family no. 1, f_c and f_{c1} are the mid-frequencies of band gaps (both diagrams of Fig. 5), whereas for both families no. 2 (Fig. 6.(a)) and no. 3 (Fig. 6.(b)) the two quantities are the mid-frequency of the relevant pass band.

We use the periodicity of the frequency spectrum, together with the definitions of frequencies f_t and f_c , to predict and control the symmetry properties of band-edge modes in canonical rods, and then study the presence of interface states in waveguides composed of two dissimilar semi-infinite structures of this type. In particular, the following two specific cases are addressed:

Case I: a system composed of two semi-infinite rods, one with canonical ratio C ('rod C ') and the other with canonical ratio equal to $1/C$ ('rod $1/C$ ');
 Case II: a system composed of two semi-infinite rods sharing the same canonical ratio, but in one of the two the phases in the unit cell are swapped w.r.t. the former.

The unit cells for both cases are shown in Fig. 4. For numerical calculations and illustration of results we assume homogeneous materials for the two phases ($E_S = E_L = E$ and $\rho_S = \rho_L = \rho$) so that ratios C and $1/C$ depend only on l_S/l_L . The total length of the unit cell $L_t = l_S + l_L$ is assumed to be the same for both the structures composing the system. This ensures that the general condition for two periodic rods with different unit cells to possess identical dispersive spectra, illustrated in Appendix B, is satisfied. As shown further in the paper, systems composed of two rods with identical spectra provide significant advantages in terms of predicting the presence of interface modes inside specific band gaps. To conclude this brief introduction to canonical rods, it is worth mentioning that when the value of C is irrational the configuration is no longer canonical: periodicity of the frequency spectrum is lost. The interested reader is referred to Morini et al. (2019) where this case is addressed.

3.1. Systems composed of two rods with canonical ratios C and $1/C$

We turn now our attention to Case I (Figs. 4.(a) and .(b)). For 'rod C ', expressions (11) and (12) for both axial force and displacement at $z = 0$ specialise to

$$N_0^C = -\omega H_2 / (2Z_S), \quad u_0^C = -iH_1 / (2Z_S Z_L), \quad (19)$$

respectively, where H_1 and H_2 are given by

$$\begin{aligned} H_1 &= 2Z_S Z_L \cos \xi_S \sin \xi_L + [Z_L^2 - Z_S^2 + (Z_L^2 + Z_S^2) \cos \xi_L] \sin \xi_S, \\ H_2 &= 2Z_S Z_L \cos \xi_S \sin \xi_L + [Z_S^2 - Z_L^2 + (Z_L^2 + Z_S^2) \cos \xi_L] \sin \xi_S. \end{aligned} \quad (20)$$

Similarly, for 'rod $1/C$ ', quantities (11) and (12) can be written as follows:

$$N_0^{1/C} = -\omega W_2 / (2Z_S), \quad u_0^{1/C} = -iW_1 / (2Z_S Z_L), \quad (21)$$

where

$$\begin{aligned} W_1 &= 2Z_S Z_L \cos \xi_L \sin \xi_S + [Z_L^2 - Z_S^2 + (Z_L^2 + Z_S^2) \cos \xi_S] \sin \xi_L, \\ W_2 &= 2Z_S Z_L \cos \xi_L \sin \xi_S + [Z_S^2 - Z_L^2 + (Z_L^2 + Z_S^2) \cos \xi_S] \sin \xi_L. \end{aligned} \quad (22)$$

From expressions (20) and (22), it can be easily noted that W_1 and W_2 do coincide with H_1 and H_2 if the positions of variables ξ_S and ξ_L are swapped, respectively. The reason of this feature lies in the symmetry of the unit cell and in the fact that, as previously pointed out, spectra of canonical rods are symmetric w.r.t. the value of the frequency $2f_c$, that is half of the canonical period.

Recalling that the conditions for the symmetry of the band-edge modes are $u_0 = 0$ and $N_0 \neq 0$, and those for the antisymmetry are $u_0 \neq 0$ and $N_0 = 0$, from Eqs. (19) and (21) we can infer that, for 'rod C ', a symmetric mode at the band edge can be attained if $H_1 = 0$, whereas an antisymmetric one is achieved if $H_2 = 0$. Similarly, if we consider 'rod $1/C$ ', on the basis of Eqs. (21), the edge mode is symmetric if $W_1 = 0$, whereas it is antisymmetric if $W_2 = 0$. Quantities H_1 , H_2 , W_1 and W_2 can be factorised as

$$H_1 = 4Q_1 Q_4, \quad H_2 = 4Q_2 Q_3, \quad W_1 = 4Q_3 Q_4, \quad W_2 = 4Q_1 Q_2, \quad (23)$$

where Q_i ($i \in \{1, \dots, 4\}$) are

$$\begin{aligned} Q_1 &= Z_S \sin \frac{\xi_L}{2} \cos \frac{\xi_S}{2} + Z_L \cos \frac{\xi_L}{2} \sin \frac{\xi_S}{2}, \quad Q_2 = Z_S \cos \frac{\xi_L}{2} \cos \frac{\xi_S}{2} - Z_L \sin \frac{\xi_L}{2} \sin \frac{\xi_S}{2}, \\ Q_3 &= Z_L \sin \frac{\xi_L}{2} \cos \frac{\xi_S}{2} + Z_S \cos \frac{\xi_L}{2} \sin \frac{\xi_S}{2}, \quad Q_4 = Z_L \cos \frac{\xi_L}{2} \cos \frac{\xi_S}{2} - Z_S \sin \frac{\xi_L}{2} \sin \frac{\xi_S}{2}. \end{aligned} \quad (24)$$

By using (24) into the dispersion relation (6), we finally derive the two equivalent forms:

$$4Q_1 Q_3 / (Z_L Z_S) = 2 - 2 \cos(KL_t), \quad 4Q_2 Q_4 / (Z_L Z_S) = 2 + 2 \cos(KL_t). \quad (25)$$

Observing expressions (19), (21), (23) and (25), we note how the values of Q_i determine both the symmetry properties of an edge mode for the two canonical rods composing the system and whether it is located either at the centre or at the boundary of the Brillouin zone. Indeed, from Eq. (25) it can be deduced that either $Q_1 = 0$ or $Q_3 = 0$ correspond to $KL_t = 0$ (centre of the Brillouin zone), while either $Q_2 = 0$ or $Q_4 = 0$ occurs at $KL_t = \pi$ (boundary of the Brillouin zone). The change of the properties of edge modes of the two canonical structures with the values of Q_i is schematised in Table 1. In the last two columns, namely 'edge mode^C' and 'edge mode^{1/C}', we report the type of symmetry [either (S) or (A)] of modes detected in the two sub-systems for each combination of factor Q_i . It can be observed that for a band edge at the centre of the Brillouin zone ($KL_t = 0$), the symmetry conditions are inverse, for instance $u_0^C = 0$ and $u_0^{1/C} \neq 0$. Thus, if we consider a band gap delimited by two edge modes situated at the centre of the Brillouin zone, the value of $\text{sign}[\zeta_J]$ is opposite in the two adjacent rods, and then an interface state can be found for a frequency inside that band gap. Conversely, for a band edge at the boundary of the Brillouin zone ($KL_t = \pi$), the symmetry conditions are the same, for instance $u_0^C \neq 0$ and $u_0^{1/C} \neq 0$. Consequently, a band gap delimited by two edge modes at the border of the Brillouin zone share the same sign of ζ_J for both rods, and an interface mode cannot be found for a frequency belonging to the comprised gap.

Table 1Values for KL_t , u_0^C , N_0^C , $u_0^{1/C}$, $N_0^{1/C}$ and band-edge mode symmetry properties.

Q_i	KL_t	$Q_1 Q_3$	$Q_2 Q_4$	u_0^C	N_0^C	$u_0^{1/C}$	$N_0^{1/C}$	Edge mode ^C	Edge mode ^{1/C}
$Q_1 = 0$	0	0	$Z_L Z_S$	0	$\neq 0$	$\neq 0$	0	S	A
$Q_2 = 0$	π	$Z_L Z_S$	0	$\neq 0$	0	$\neq 0$	0	A	A
$Q_3 = 0$	0	0	$Z_L Z_S$	$\neq 0$	0	0	$\neq 0$	A	S
$Q_4 = 0$	π	$Z_L Z_S$	0	0	$\neq 0$	0	$\neq 0$	S	S

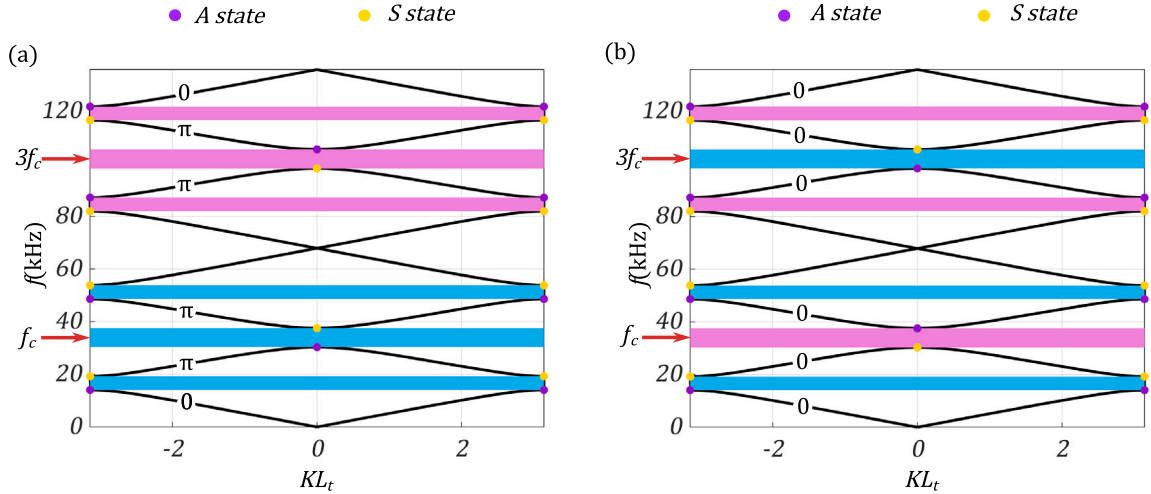


Fig. 5. Frequency spectrum (solid black lines) of infinite periodic waveguides whose unit cells are set by (a) $C = 1/3$ and (b) $C = 3$ (both belong to family no. 1 of canonical rods). The adopted parameters are $E = 70$ GPa, $\rho = 2700$ kg/m³, $A_L = 2A_S = 2 \times 10^{-4}$ m² and $L_t = 0.15$ m. Magenta fills gaps with $\zeta_J > 0$, whereas cyan indicates gaps for which $\zeta_J < 0$; the value of the Zak phase of each individual isolated branch is indicated; a solid purple circle marks a (A) state at the band edge, whereas a yellow one indicates a (S) state.

3.1.1. Interface between two semi-infinite canonical rods of family no. 1

We now analyse the periodicity of the edge symmetry properties and then of the sign of ζ_J inside the different band gaps for a system composed of two semi-infinite canonical rods of family no. 1. To this end, the frequency spectra for two structures whose canonical ratio are $C = 3$ and $C = 1/3$ are reported in Fig. 5.(a) and .(b), respectively, where it is adopted $E = 70$ GPa, $\rho = 2700$ kg m⁻³, $A_L = 2A_S = 2 \times 10^{-4}$ m² and total length $L_t = 0.15$ m (therefore, as mentioned, $C = l_S/l_L$). The spectra are sketched for an interval of frequency that coincides to the lowest period $[0, f_i]$, and symmetric and antisymmetric band-edge modes are marked with yellow and purple circles, respectively. Due to the periodicity of the functions (20) and (22), and then of the expressions of displacements and axial forces at the edge of the unit cell w.r.t. the variables ξ_S and ξ_L (see Eqs. (19) and (21)), it is easy to infer that the symmetry of the different edge modes does not change if the latter are translated by a canonical period. Therefore, $\text{sign}[\zeta_J]$ inside the band gaps is also periodic with the same period. This property holds true for canonical configurations of all the three families.

Observing Fig. 5.(a) and .(b), we note that the symmetry properties of the edge modes are symmetric w.r.t. the frequency $f_i/2 = 2f_c$. This means that if, for $0 \leq f \leq f_i/2$, there is a band gap delimited by a symmetric edge mode on the bottom and an antisymmetric one on the top, then the corresponding gap for $f_i/2 \leq f \leq f_i$ features an antisymmetric mode on the bottom and a symmetric one on the top. Consequently, the sign of the imaginary part of the impedance inside the two band gaps is opposite, as shown in the figure, where band gaps for which $\text{sign}[\zeta_J] = +1$ are depicted in magenta, whereas cyan fills band gaps for which $\text{sign}[\zeta_J] = -1$. We can summarise what noted saying that for *canonical rods of family no. 1 the properties of the band edge modes are symmetric w.r.t. $f_i/2$* , and this implies an antisymmetry of the imaginary part of the impedance of the band gaps w.r.t. this frequency.

Conversely, the properties of band-edge modes are antisymmetric w.r.t. the canonical frequency f_c , and then $\text{sign}[\zeta_J]$ inside the band gaps is symmetric. The predictions of $\text{sign}[\zeta_J]$ obtained through the analysis of symmetries of band-edge modes have been verified by the calculation of the Zak phase for the different branches of the frequency spectrum, as shown in Appendix C. The values of the Zak phase are reported on the dispersion curves.

We can conclude that, for family no. 1 of canonical rods, the value of $\text{sign}[\zeta_J]$ inside the different band gaps is antisymmetric w.r.t. the frequency $f_i/2$ (half canonical period), while it is symmetric w.r.t. the canonical frequency f_c . It is crucial to note that, in agreement with the results reported in Table 1, band gaps limited by points at the edge of the Brillouin zone have the same $\text{sign}[\zeta_J]$ for both ‘rod C’ and ‘rod 1/C’. Conversely, opposite values of $\text{sign}[\zeta_J]$ are in place for band gaps limited by two points at the centre of the Brillouin zone, for which their mid-frequency coincides with the canonical frequency f_c ; therefore, *an interface mode can be found at a frequency within them*.

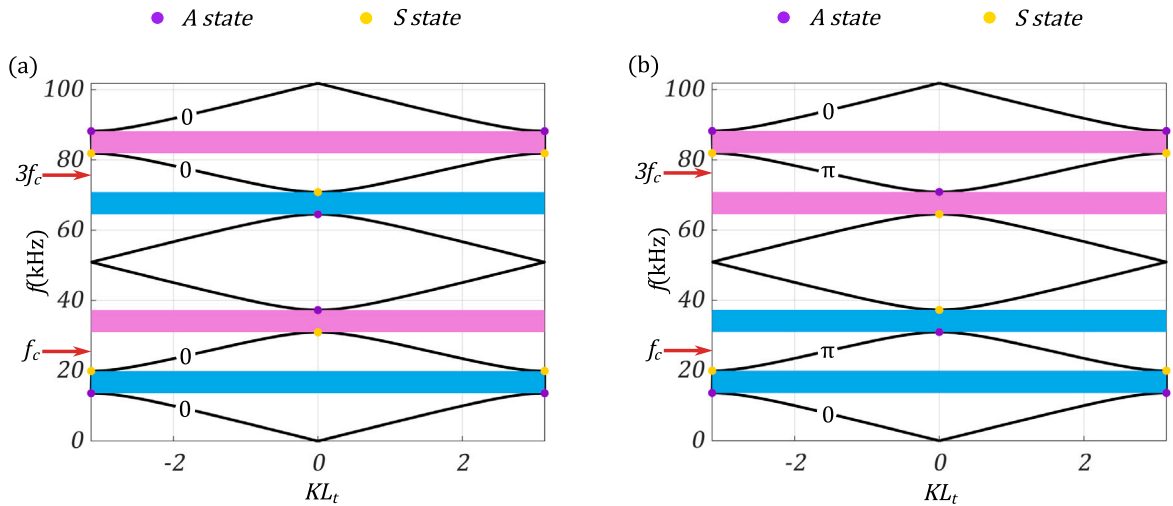


Fig. 6. Frequency spectrum (solid black lines) of infinite waveguides whose unit cells are set by (a) $C = 1/2$ (family no. 2) and (b) $C = 2$ (family no. 3). The adopted parameters are $E = 70$ GPa, $\rho = 2700$ kg m $^{-3}$, $A_L = 2A_S = 2 \times 10^{-4}$ m 2 and $L_t = 0.15$ m. Magenta fills gaps with $\zeta_j > 0$, whereas cyan indicates gaps for which $\zeta_j < 0$; the value of the Zak phase of each isolated branch is indicated; a solid purple circle marks a (A) state at the band edge, whereas a yellow one indicates a (S) state.

3.1.2. Interface between two semi-infinite canonical rods, one of family no. 2 and the other of family no. 3

Recalling the classification of canonical ratios (17), we note that assuming the same value for the two indices $j = k$, two canonical rods, one belonging to family no. 2 with ratio C and the other to family no. 3 with ratio $1/C$, can be defined. In order to study a system bonding those, we consider ‘rod C ’ with $C = 1/2$ (family no. 2) and ‘rod $1/C$ ’ with $C = 2$ (family no. 3). The same properties of phases adopted in Section 3.1.1 are here considered.

The dispersion diagrams are reported in Fig. 6, again for the frequency interval $[0, f_i]$. First, we observe that, similarly to the case of an interface between two canonical rods of family no. 1, for both families no. 2 and 3 edge modes are symmetric w.r.t. the half-canonical period $f_i/2$ and anti-symmetric w.r.t. the canonical frequency f_c . As a consequence, we note that for both the spectra reported in Fig. 6, $\text{sign}[\zeta_j]$ is antisymmetric w.r.t. $f_i/2$. Focusing our analysis on the interval $0 \leq f \leq 2f_c$, we can see that band-edge modes are symmetric w.r.t. f_c for $C = 1/2$, while they are antisymmetric for $C = 2$. This implies that in the same interval of frequencies the sign of the imaginary part of the impedance inside the band gaps is antisymmetric w.r.t. f_c . Differently from the case analysed in Section 3.1.1, we can observe that for families no. 2 and no. 3 the canonical frequency f_c is at the centre of a pass band, as predicted by the accurate spectral characterisation illustrated by Gei et al. (2020). Considering the band gap just beyond f_c , we have $\text{sign}[\zeta_j] > 0$ for the rod with $C = 1/2$, and $\text{sign}[\zeta_j] < 0$ for the one with $C = 2$. Due to the antisymmetry of $\text{sign}[\zeta_j]$ w.r.t. f_c , we have the opposite situation for the consecutive band gap, immediately below the pass band centred at $3f_c$. Consequently, both these two band gaps can be used to obtain a mode localised at the interface joining the two rods composing the system. Note that also in this case, according with the results summarised in Table 1, both the gaps associated with an interface state are delimited by points at the centre of the Brillouin zone. Since all the properties of the spectrum are periodic w.r.t. the canonical period $f_i = 4f_c$, for this configuration we can conclude that in any band gap lying between two points at the centre of the Brillouin zone an interface mode can be found.

3.2. Interface between two structures with the same canonical ratio whose phases are swapped

As a final step, we are going to investigate now the dynamic properties of a system constructed according to Case II. We consider two homogeneous semi-infinite rods for which, as usual, $E_S = E_L$ and $\rho_S = \rho_L$. The cross-section areas of the two phases are swapped between the two unit cells, i.e. $(A_S, A_L) \rightarrow (A_L, A_S)$. This means that the impedances and the variables ξ_S and ξ_L are also swapped $[(\xi_S, \xi_L) \rightarrow (\xi_L, \xi_S)]$. Due to the symmetry properties of the trigonometric functions determining both the values of the relationships (22) and the displacements and axial force on the boundary of the unit cells given Eqs. (19), it is easy to deduce that, at a given frequency, if $N_0 = 0$ and $u_0 \neq 0$ in one rod, in the other $N_0 \neq 0$ and $u_0 = 0$. Consequently, as shown in Fig. 7, the symmetry properties of band-edge modes in the dispersion diagrams of the two rods are fully reversed: at a given frequency, a symmetric mode in one of the rod is associated with an anti-symmetric mode in the other, and vice versa. Therefore, for such a waveguide, an interface mode can be found in each band gap.

4. Predicting the presence of interface modes through the toroidal representation of the spectrum

The general representation of the frequency spectrum introduced for two-phase periodic laminates by Shmuel and Band (2016), and generalised to axial waves in quasicrystalline-generated periodic rods by Morini et al. (2019), is now used to predict the values

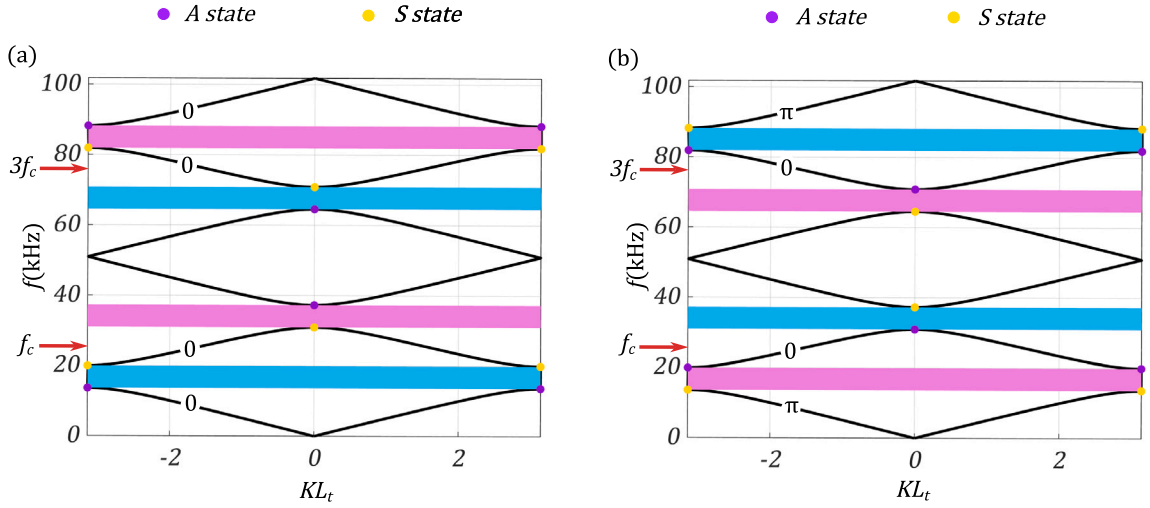


Fig. 7. Frequency spectrum (solid black lines) of infinite waveguides whose unit cells are set by $C = 1/2$ and $E = 70$ GPa, $\rho = 2700$ kg m $^{-3}$ and $L_t = 0.15$ m; (a) $A_L = 2A_S = 2 \times 10^{-4}$ m 2 , (b) $A_S = 2A_L = 2 \times 10^{-4}$ m 2 (the two phases are swapped). Magenta fills gaps with $\zeta_J > 0$, whereas cyan indicates gaps for which $\zeta_J < 0$; the value of the Zak phase of each isolated branch is indicated; a solid purple circle marks a (A) state at the band edge, whereas a yellow one indicates a (S) state.

of $\text{sign}[\zeta_J]$ in all band gaps and the possible occurrence of interface modes for the two cases introduced in Section 3 (Fig. 4). With this approach, the dispersion properties of a waveguide can be inferred following a ‘flow line’ parametrised with the frequency on a *reduced* two-dimensional *torus* described by coordinates (ξ_S, ξ_L) , in which band-gap and pass-band intervals are regions of the plane univocally identified. The domain of the reduced torus is $\xi_S, \xi_L \in [0, \pi]$.

The canonical rods composing the systems reported in Fig. 4 possess unit cells generated according to the element F_2 of the Fibonacci’s sequence. For this specific case,¹ the boundaries of the band-gap region in the reduced torus follow by the analytical expression (Morini et al., 2019)

$$\tan \frac{\xi_L}{2} = (\gamma \pm \sqrt{\gamma^2 - 1}) \cot \frac{\xi_S}{2}, \quad (26)$$

where $\gamma = (Z_L^2 + Z_S^2)/(2Z_S Z_L)$. The curve determined by sign ‘+’ in Eq. (26) is the top boundary of the band-gap region next to the origin of the axis, coloured in cyan in Fig. 8.(a) (referred, from now on, as *low-frequency band gap*) whereas that with sign ‘−’ is the bottom boundary. The same expression can be adopted to describe the boundaries of other band-gap domains by applying the transformation

$$\xi'_S = \xi_S + n\pi, \quad \xi'_L = \xi_L + m\pi \quad (m, n \in \mathbb{N}). \quad (27)$$

Any point $P(\xi_S, \xi_L)$ lying on the curves (26) is associated with a band-edge mode. We now use Eq. (26), together with the symmetry conditions (20), with the goal to predict $\text{sign}[\zeta_J]$ inside the band gap. Further in the Section, we will show that, due to the periodicity of the spectrum and of the relationships (20), once we know $\text{sign}[\zeta_J]$ inside the low-frequency band gap, its value in all the others can be easily deduced. By means of a simple algebraic procedure, Eq. (26) with sign ‘−’ becomes²

$$2Z_L Z_S \sin \frac{\xi_L}{2} \sin \frac{\xi_S}{2} = \left(Z_L^2 + Z_S^2 - \sqrt{(Z_L^2 - Z_S^2)^2} \right) \cos \frac{\xi_L}{2} \cos \frac{\xi_S}{2}. \quad (28)$$

Recalling that $\sqrt{(Z_L^2 - Z_S^2)^2} = |Z_L^2 - Z_S^2|$, we can split relationship (28) into the following couple of expressions:

$$Z_L \cos \frac{\xi_L}{2} \cos \frac{\xi_S}{2} - Z_S \sin \frac{\xi_L}{2} \sin \frac{\xi_S}{2} = 0 \quad \Rightarrow \quad Q_4 = 0 \quad (\text{for } Z_S > Z_L), \quad (29)$$

$$Z_S \cos \frac{\xi_L}{2} \cos \frac{\xi_S}{2} - Z_L \sin \frac{\xi_L}{2} \sin \frac{\xi_S}{2} = 0 \quad \Rightarrow \quad Q_2 = 0 \quad (\text{for } Z_L > Z_S). \quad (30)$$

Observing the dispersion relation (6) and using the properties of trigonometric functions, we can easily deduce that the low-frequency band gap is associated with $\cos(KL_t) < -1$. Recalling the variation of the symmetry properties of the edge modes determined by the functions Q_i and summarised in Table 1, we can infer that:

¹ For higher order indices the equations of the boundaries of band gaps are not in closed form and can be studied numerically (see Chen et al. (2022)).

² For our goal, the analysis of Eq. (26) with sign ‘+’ does not provide further useful information.

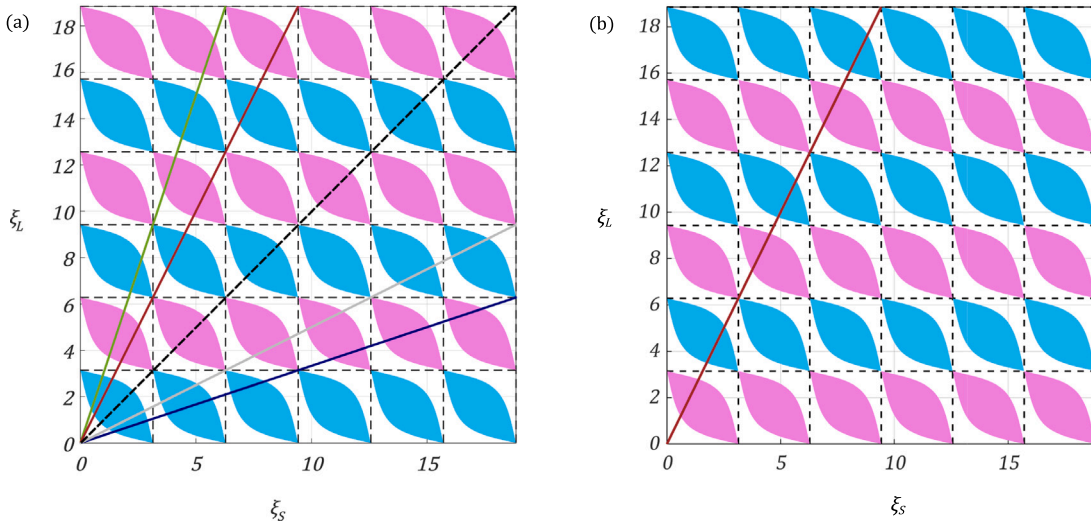


Fig. 8. (a) Extended torus of the frequency spectrum reported for $0 \leq \xi_S, \xi_L \leq 6\pi$. The band-gap regions with $\zeta_J > 0$ are coloured in magenta, while those with $\zeta_J < 0$ are in cyan. The reported case is $Z_L > Z_S$, then inside the low-frequency band gap ζ_J is negative. Four different flow lines corresponding to different canonical ratios are shown: $C = 1/3$ (green line), $C = 3$ (blue line), $C = 1/2$ (brown line) and $C = 2$ (light grey line); (b) extended torus for the case $Z_S > Z_L$, then for the low-frequency band gap ζ_J is positive; the flow line for $C = 1/2$ is reported in brown.

- for cases in which $Z_S > Z_L$, the boundary of the low-frequency band gap is associated with the condition (29) ($Q_4 = 0$), and then the lower band-edge mode is symmetric. Therefore, the upper band-edge mode is antisymmetric and the imaginary part of the impedance is positive within the band gap (i.e. $\text{sign}[\zeta_J] > 0$);
- for cases in which $Z_L > Z_S$, the same boundary of the band-gap region corresponds to the condition (30) ($Q_2 = 0$); therefore, the lower band-edge mode is antisymmetric, the upper band edge mode is symmetric, and the imaginary part of the impedance is negative within the band gap (i.e. $\text{sign}[\zeta_J] < 0$).

These results demonstrate that, for any two-phase periodic rod, $\text{sign}[\zeta_J]$ of the low-frequency band gap depends only on the impedances Z_S and Z_L .

Starting from the knowledge of $\text{sign}[\zeta_J]$ of the low-frequency band gap and taking advantage of the periodicity of the spectrum w.r.t. variables ξ_S and ξ_L , it is possible to predict the value of the same quantity for any band gap of any canonical rod. Still consider point $P(\xi_S, \xi_L)$. Due to the periodicity of the frequency spectrum, P can be coupled to $P'(\xi'_S, \xi'_L)$ belonging to the boundary of a subsequent gap through relationships (27). The variation of axial force and displacement at the boundary of the unit cell, and then of the symmetry features of the band-edge modes, is studied for all possible combinations of odd and even numbers m and n by using Eqs. (19) and taking advantage of the periodicity of functions (20). With this analysis, we can determine whether or not the value of $\text{sign}[\zeta_J]$ for an arbitrary band gap coincides with that of the low-frequency band gap. The results of this study show that for cases where m is an even number, independently of n , at P' the values of axial force and displacement on the boundary of the unit cell are identical to those at P , while if m is odd these quantities have opposite values. For example, by assuming $u_0 = 0$ and $N_0 \neq 0$ at P , if m is odd then $u_0 \neq 0$ and $N_0 = 0$ at P' . Consequently, a band-gap domain obtained by translating the low-frequency one of an interval $m\pi$ along the ξ_L -axis possesses the same sign of the imaginary impedance of the low-frequency band gap if m is even. Conversely, band-gap domains selected by odd values of m show opposite values of $\text{sign}[\zeta_J]$ w.r.t. the low-frequency band gap. This result is illustrated in Fig. 8, where the band gaps for which $\text{sign}[\zeta_J] > 0$ are coloured in magenta, while those with $\text{sign}[\zeta_J] < 0$ are filled in cyan.

We are now able to predict $\text{sign}[\zeta_J]$ inside any band gap for any arbitrary canonical rod by representing the spectrum through the two-dimensional plots reported in Fig. 8. We call this representation *extended torus* in order to distinguish it from the *reduced torus*. The various flow lines drawn in Fig. 8.(a) correspond to the dispersion spectra analysed in Section 3 and shown in Figs. 5 and 6. The four lines are for $C = 1/3$ (green), $C = 3$ (deep blue), $C = 1/2$ (brown) and $C = 2$ (light grey). From Eq. (16) we know that the slope of each flow line is equal to $1/C$, and all lines pass through at least one band-gap vertex (that is a Dirac point in the frequency spectrum). On the extended torus, these points have coordinates $(n\pi, m\pi)$ ($n/m \in \mathbb{Q}$). It is clear from the colours of band-gap regions that $\text{sign}[\zeta_J]$ is antisymmetric w.r.t. said vertices. As an illustrative example, follow the flow line for $C = 3$, reported in deep blue in Fig. 8.(a). Before passing through the point $(3\pi, \pi)$, it crosses three cyan band gaps with $\text{sign}[\zeta_J] < 0$, and then it goes through three magenta sub-regions with $\text{sign}[\zeta_J] > 0$.

The representation of the spectrum illustrated in Fig. 8.(a) can be suitably exploited to design systems composed of two canonical rods that do display interface modes. Starting from a structure with ratio C , we are able to draw the corresponding flow line on the torus plane. The line for a rod with $1/C$ is the symmetric one w.r.t. the bisector (the dashed line). From the analysis developed

just above, we know the value of $\text{sign}[\zeta_J]$ inside any band-gap region, and then we can predict which combination can provide a localised mode at the interface between a ‘rod C ’ and a ‘rod $1/C$ ’. Fig. 8.(b) shows how it is possible to design a system composed of two structures with the same canonical ratio (in this case $C = 1/3$) able to display an interface mode in any band gap. This can be explained with the fact that, on the basis of the proposed analysis, if the phases are swapped then $(Z_S, Z_L) \rightarrow (Z_L, Z_S)$, and, as a consequence, $\text{sign}[\zeta_J]$ is inverted in any band-gap sub-domain. The extended tori of Figs. 8.(a) and .(b) can also be used to determine the number $g^{(i)}$ ($i = 1, 2, 3$) of band gaps inside a canonical period $[\ell f_i, (\ell + 1)f_i]$ ($\ell \in \mathbb{N}$), of any two-phase canonical rod with unit cells similar to those sketched in Fig. 4. For the three different families of canonical structures, this number is given by

$$g^{(1)} = 4(j + k) + 2, \quad g^{(2)} = 4(j + q), \quad g^{(3)} = 4(k + q) \quad (j, k, q \in \mathbb{N}). \quad (31)$$

We have shown how the proposed approach to study the dispersion spectrum can be used to predict the presence of a mode at the interface between two semi-infinite canonical rods with ratios C and $1/C$, or, alternatively, between two structures with the same canonical ratio, but where phases S and L are swapped. In the next Section, an exact formula for the frequency of the interface mode is derived for finite-sized specimens for both cases.

5. Systems composed of finite-sized canonical rods: exact formulae for the frequency of the interface modes

We move now to the analysis of finite-sized systems alike that shown in Fig. 2.(a), characterised by an interface between two finite rods whose unit cells are F_2 . Both rods are assumed to be composed of $\mathcal{N}$ unit cells (i.e. $\mathcal{N} = \mathcal{M}$) and have the same frequency spectrum as explained in detail in Appendix B.

Substituting the expression for the surface impedance (14) into the general condition for the existence of an interface mode (15), we obtain the following equation valid for any system composed of two finite structures:

$$\text{Im} \left(\frac{1 + r_{\text{lhs}}}{1 - r_{\text{lhs}}} \right) + \text{Im} \left(\frac{1 + r_{\text{rhs}}}{1 - r_{\text{rhs}}} \right) = 0. \quad (32)$$

Therefore, in order to determine explicitly the frequency at which the interface mode is excited, the reflection coefficients r_{lhs} and r_{rhs} are needed for both rods, considered separately. Assuming that there is no incident wave at the output boundary for both structures ($b_{\mathcal{N}} = 0$), their reflection and transmission coefficients, denoted respectively by r and t , are given in the form (see, e.g., Markos and Soukoulis (2008))

$$r = \frac{b_0}{a_0} = -\frac{M_{21}^{\text{gl}}}{M_{22}^{\text{gl}}}, \quad t = \frac{a_{\mathcal{N}}}{a_0} = M_{11}^{\text{gl}} - \frac{M_{12}^{\text{gl}} M_{21}^{\text{gl}}}{M_{22}^{\text{gl}}}, \quad (33)$$

where $M_{11}^{\text{gl}}, M_{12}^{\text{gl}}, M_{21}^{\text{gl}}$ and M_{22}^{gl} are the components of the ‘global’ transfer matrix of the rod concerned $M^{\text{gl}} = M^{\mathcal{N}}$ (see Eq. (4)). Since M^{gl} is defined as the $\mathcal{N}^{\text{th}}$ power of a unimodular matrix, it can be expressed by means of the Chebyshev’s identity, reported in Appendix A. Consequently, the reflection coefficient becomes

$$r = \frac{b_0}{a_0} = \frac{\frac{i}{2} \left(\frac{Z_S}{Z_L} - \frac{Z_L}{Z_S} \right) \sin(\mathcal{N} K L_t) \sin \xi_S}{e^{-i\xi_L} \left[\cos \xi_S - \frac{i}{2} \left(\frac{Z_S}{Z_L} + \frac{Z_L}{Z_S} \right) \sin \xi_S \right] \sin(\mathcal{N} K L_t) - \sin((\mathcal{N} - 1) K L_t)}. \quad (34)$$

We now use Eq. (34) to derive an explicit expression for the imaginary quantities involved in relationship (32) for finite systems that are constructed according to rules dictated by Case I and Case II, respectively.

5.1. Frequency of modes at the interface between two finite rods with canonical ratios C and $1/C$

Assuming that the rod at the left-hand side is ‘rod C ’ and recalling that, as we are considering a frequency inside a band gap satisfying condition (32), $\sin(\mathcal{N} K L_t)$ is pure imaginary and $H_1 H_2 = 4 Z_L^2 Z_S^2 \sin^2(K L_t)$ is real, we substitute expression (34) into Eq. (32) obtaining

$$\text{Im} \left(\frac{1 + r_{\text{lhs}}}{1 - r_{\text{lhs}}} \right) = \frac{4 Z_L Z_S \sin(\mathcal{N} K L_t) \cos(\mathcal{N} K L_t) \sin(K L_t) (Z_L^2 - Z_S^2) \sin \xi_S}{H_1^2 + 2 H_1 (Z_L^2 - Z_S^2) \cos^2(\mathcal{N} K L_t) \sin \xi_S}, \quad (35)$$

and

$$\text{Re} \left(\frac{1 + r_{\text{lhs}}}{1 - r_{\text{lhs}}} \right) = -\frac{4 Z_L^2 Z_S^2 \sin^2(K L_t)}{H_1^2 + 2 H_1 (Z_L^2 - Z_S^2) \cos^2(\mathcal{N} K L_t) \sin \xi_S}. \quad (36)$$

Since Eq. (32) should be satisfied for a frequency inside a band gap, the dimensionless wave number $K L_t$ can be written in the form

$$K L_t = s\pi + i\bar{K} \quad (s \in \mathbb{Z}), \quad (37)$$

in which $\bar{K}$, that describes the decay length inside the band gap, is positive. Therefore, $\cos^2(\mathcal{N} K L_t) = \cosh^2(\mathcal{N} \bar{K})$, and due to the properties of $\cosh x$ (i.e. $\cosh^2(\mathcal{N} \bar{K}) \rightarrow \infty$ for $\mathcal{N} \rightarrow \infty$), the real quantity (36) vanishes, in agreement with the model formulated for semi-infinite rods.

As illustrated in the previous Section, the two joined rods possess identical dispersion spectra with different values of $\text{sign}[\zeta_J]$ inside the corresponding band gaps. Consequently, KL_t has identical value for both the dispersion relations of the two rods composing the system. Moreover, as the number of unit cells in the two structures is the same, the quantities $\sin(\mathcal{N}KL_t)$, $\sin((\mathcal{N}-1)KL_t)$ and $\cos(\mathcal{N}KL_t)$ appearing in expressions (33) for both rods share the same values.

Taking into account these equalities, we can now introduce an expression specular to (35) for ‘rod 1/C’ on the right-hand side, by applying the transformation $\xi_S \rightarrow \xi_L$ ($H_1 \rightarrow W_1$):

$$\text{Im} \left(\frac{1 + r_{\text{rhs}}}{1 - r_{\text{rhs}}} \right) = \frac{4Z_L Z_S \sin(\mathcal{N}KL_t) \cos(\mathcal{N}KL_t) \sin(KL_t) (Z_L^2 - Z_S^2) \sin \xi_L}{W_1^2 + 2W_1(Z_L^2 - Z_S^2) \cos^2(\mathcal{N}KL_t) \sin \xi_L}. \quad (38)$$

Substituting (35) and (38) into (32), we obtain

$$H_1^2 \sin \xi_L + W_1^2 \sin \xi_S + (H_1 + W_1)[2(Z_S^2 - Z_L^2) \cos^2(\mathcal{N}KL_t) \sin \xi_S \sin \xi_L] = 0. \quad (39)$$

The frequency satisfying Eq. (39) excites an interface mode inside a band gap. According to results derived by Xiao et al. (2014), ζ_J ($J \in \{\text{lhs}, \text{rhs}\}$) is a monotonic decreasing function from ∞ to 0 when $\zeta_J > 0$, and from 0 to $-\infty$ when $\zeta_J < 0$. Therefore, if an interface mode exists for this type of finite system, Eq. (39) has only one solution. It is easy to deduce that this unique solution corresponds to the pair (ξ_S, ξ_L) which satisfies

$$\sin \xi_S = -\sin \xi_L \quad \text{and} \quad \cos \xi_S = \cos \xi_L \quad (\text{with } \sin \xi_L \neq 0). \quad (40)$$

Eqs. (40)₁ and (40)₂ can be re-written equivalently

$$\sin \xi_S + \sin \xi_L = 0 \quad \Rightarrow \quad 2 \sin \frac{\xi_S + \xi_L}{2} \cos \frac{\xi_L - \xi_S}{2} = 0, \quad (41)$$

$$\cos \xi_S - \cos \xi_L = 0 \quad \Rightarrow \quad 2 \sin \frac{\xi_S + \xi_L}{2} \sin \frac{\xi_L - \xi_S}{2} = 0. \quad (42)$$

The solution of the coupled Eqs. (41) and (42) is given by the values of ξ_S and ξ_L that fulfil $\sin((\xi_S + \xi_L)/2) = 0$, a condition that leads to the following value of the frequency

$$\hat{f} = \frac{n_1}{(1+C)l_L} \sqrt{\frac{E_L}{\rho_L}} = \frac{Cn_1}{(1+C)l_S} \sqrt{\frac{E_S}{\rho_S}} \quad (n_1 \in \mathbb{N}). \quad (43)$$

Although it seems at first sight that both $2n_1/(1+C)$ and $2Cn_1/(1+C)$ are also solutions, they are not admissible because they result from the identity $\sin \xi_L = \sin \xi_S = 0$ that violates the hypothesis stated in (40). Moreover, they correspond to points $P(n\pi, m\pi)$ on the extended torus reported in Fig. 8. These are Dirac points and do not lie inside a band gap as required by our approach. It is also worth stressing that the total length L_t of the two joined rods must be set equal in order to satisfy the condition for identical dispersion diagrams according to the analysis conducted in Appendix B.

The results shown in Fig. 9 are about a system composed of a l.h.s. ‘rod C’ and a r.h.s. ‘rod 1/C’, with $C = 1/3$, both assembled with $\mathcal{N} = 6$ unit cells (both rods belong to family no. 1 of canonical configurations). The adopted mechanical and geometrical parameters in all figures of this Section are identical to those employed for the specimens analysed in Fig. 5. Fig. 9.(a) displays the transmission coefficient for the whole finite system for the range of frequencies $[0, f_t/2]$. The imaginary part of the mechanical impedance for both rods, as well as for the whole system, is plotted as a function of the frequency in Fig. 9.(b). In Fig. 9.(c), functions $\cos \xi_S$, $\cos \xi_L$, $\sin \xi_S$, $\sin \xi_L$ and $\text{Tr}M(\omega)/2$ are displayed. In particular, the plot in Fig. 9.(a) shows that the frequency of the interface mode corresponds to the peak in the transmission coefficient occurring at the canonical frequency within the median band gap. In Fig. 9.(b), this peak is associated with a zero value of the sum of the imaginary parts of the mechanical impedance of the two structures, in accordance with Eq. (32). The value of the frequency of the interface mode is exactly that evaluated by the analytical formula (43) that coincides, for family no. 1, to the canonical frequency (18)₁. The functions displayed in Fig. 9.(c) show that the frequency of the localised mode fulfils Eq. (40).

Fig. 10 illustrates results for an analogous system with $C = 1/2$ (now ‘rod C’ belongs to family no. 2, ‘rod 1/C’ to family no. 3) for which (43) does not coincide with f_c .

The absolute value of the axial force $|N|$ reported in Fig. 11 refers to the same system studied in Fig. 9. The diagrams are obtained by imposing a harmonic axial force with amplitude equal to 1 N at the l.h.s. edge of the waveguide. Fig. 11.(a) shows the interface mode at the frequency of to the peak of the transmission coefficient displayed in Fig. 9.(a) ($f = f_c = 33.95$ kHz); in .(b), the considered interface mode is the second one along the frequency spectrum, at $f = 3f_c = 101.84$ kHz. For both figures, it is visible that $|N|$ is localised at the interface between the two rods, and decays dramatically towards the ends of the structure. The amplification factor is approximately sixty-four. The same field for values of the frequency sitting either in a pass band or in a band gap is shown in Figs. 11.(c) and .(d), respectively, where any interfacial amplification does occur; rather, propagation and decay are the exhibited behaviours, as expected for the selected frequencies.

For the system examined in Fig. 10, the amplification of field $|N|$ at two interface modes is sketched in Fig. 12. Here, the enhancement of the axial force reaches approximately the factor thirty-eight.

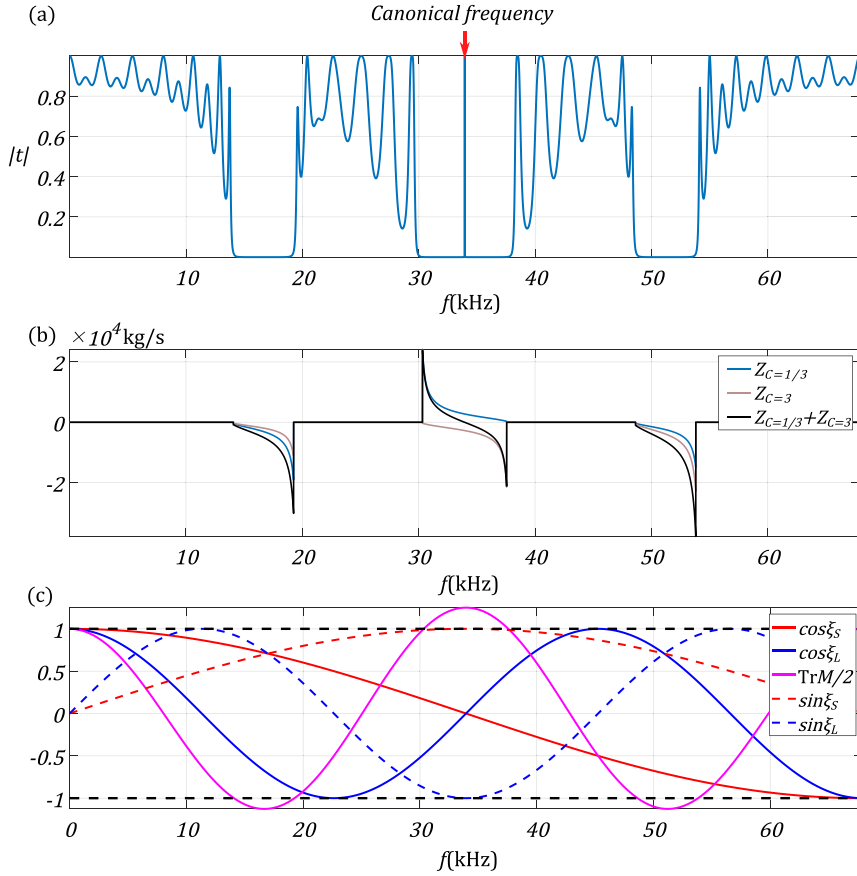


Fig. 9. System composed of finite-sized 'rod C' and 'rod 1/C' ($C = 1/3$); (a) transmission spectrum in the domain $[0, f_i/2]$ where it is evident a peak at the canonical frequency f_c at which the interface mode occurs; (b) imaginary parts of the relative surface impedance of 'rod C' (light blue) and 'rod 1/C' (light brown), and the sum of the two (black) that is vanishing at f_c ; (c) plot of functions $\cos \xi_S$, $\cos \xi_L$, $\sin \xi_S$, $\sin \xi_L$ and $\text{Tr}M/2$.

5.2. A method to obtain an interface mode at the centre of a band gap

The excitation frequency of an interface mode for a system composed of two finite-sized rods with canonical ratios C and $1/C$ is given by Eq. (40). Under the same configuration, we now determine the condition helpful to design a waveguide exhibiting an interface mode at the centre of the band gap. To this end, we define the small perturbation parameter δ and introduce the perturbed variables $\xi'_L = \xi_L - \delta$ and $\xi'_S = \xi_S - C\delta$ corresponding to a point on the extended torus lying on the edge of the band gap domain. Substituting ξ'_L and ξ'_S in the dispersion relation (6), and recalling that $\xi_S = C\xi_L$, we obtain

$$\cos(\xi_L - \delta) \cos(C(\xi_L - \delta)) - \gamma \sin(\xi_L - \delta) \sin(C(\xi_L - \delta)) = \pm 1. \quad (44)$$

If the excitation frequency of the interface mode is at the centre of the band gap, then the additional perturbed variables $\xi''_L = \xi_L + \delta$ and $\xi''_S = \xi_S + C\delta$ also satisfy the dispersion relation (6), yielding the following relationship

$$\cos(\xi_L + \delta) \cos(C(\xi_L + \delta)) - \gamma \sin(\xi_L + \delta) \sin(C(\xi_L + \delta)) = \pm 1. \quad (45)$$

Substituting $\xi_S = C\xi_L$ into the interface mode conditions (40) yields

$$\sin \xi_L = -\sin(C\xi_L) \quad \text{and} \quad \cos \xi_L = \cos(C\xi_L). \quad (46)$$

Subtracting (44) from (45) and using the relationship (46), the following condition for the occurrence of an interface mode at the centre of a band gap is obtained, i.e.

$$2(\gamma - 1) \sin((C - 1)\delta) \sin \xi_L \cos \xi_L = 0. \quad (47)$$

Eq. (47) possesses non-trivial solutions only for $\cos \xi_L = 0$. For $\gamma = 1$, the waveguide becomes homogeneous. For γ approaching 1, the frequency of the interface mode tends to the centre of the band gap. The case where $\sin \xi_L = 0$ corresponds to $\xi_L = m\pi$, and then, by observing the extended torus in Fig. 8, it may be inferred that it selects a Dirac point or, alternatively, a point inside a pass band.

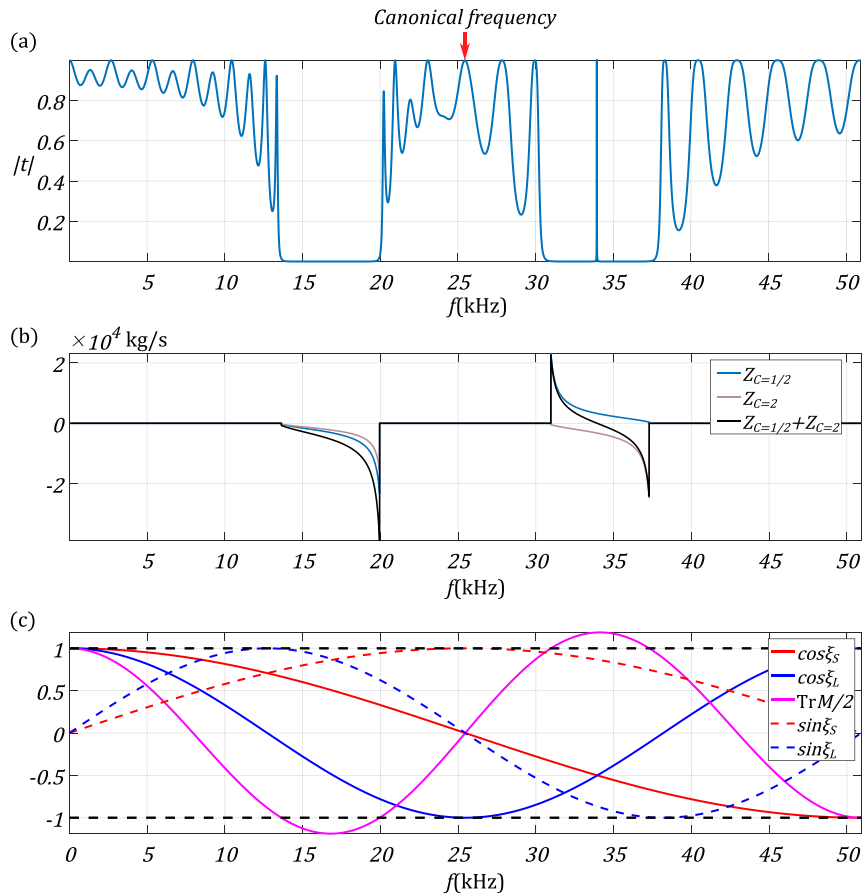


Fig. 10. System composed of finite-sized ‘rod C’ and ‘rod 1/C’ ($C = 1/2$); (a) transmission spectrum in the domain $[0, f_i/2]$ where it is evident a peak on the r.h.s. band gap at which the interface mode occurs; (b) imaginary parts of the relative surface impedance of ‘rod C’ (light blue) and ‘rod 1/C’ (light brown), and the sum of the two (black) that is vanishing at the interface mode; (c) plot of functions $\cos \xi_S$, $\cos \xi_L$, $\sin \xi_S$, $\sin \xi_L$ and $\text{Tr} M/2$.

5.3. Frequency of modes at the interface between two finite rods whose phases are swapped

We derive, at last, an equation for evaluating the frequency of interface modes in a finite-sized system where the two rods are built following the prescriptions of Case II. This means that the mechanical impedances of the two phases are swapped in the two unit cells, i.e. $Z_L \rightarrow Z_S$ and $Z_S \rightarrow Z_L$ (in the case of two homogeneous rods with modulated cross-sections $A_L \rightarrow A_S$ and $A_S \rightarrow A_L$). For this type of system, it is not possible to simplify Eq. (32) in order to obtain an analytical formula similar to (43). By applying the transformation for impedances in (35), we obtain the following expression valid for the left hand-side rod

$$\text{Im} \left(\frac{1 + r_{\text{rhs}}}{1 - r_{\text{rhs}}} \right) = \frac{4Z_L Z_S \sin(\mathcal{N}KL_t) \cos(\mathcal{N}KL_t) \sin(KL_t)(Z_S^2 - Z_L^2) \sin \xi_S}{H_2^2 + 2H_2(Z_L^2 - Z_S^2) \cos^2(\mathcal{N}KL_t) \sin \xi_S}. \quad (48)$$

If the interface state inside the band gap existed, the corresponding frequency would satisfy the following relationship given by the sum of Eqs. (35) and (48):

$$H_1^2 Z_S - H_2^2 Z_L + 2(Z_S^2 - Z_L^2) \cos^2(\mathcal{N}KL_t) (H_1 Z_S + H_2 Z_L) = 0. \quad (49)$$

This implicit condition can be easily solved numerically. In Fig. 13 we report the transmission coefficient and the corresponding imaginary part of the impedance for a system with $C = 1/2$. The solution to Eq. (49), associated with a zero value of the sum of the imaginary parts of the mechanical impedances, is denoted by the green line in part (b) of the figure.

Moving now to Fig. 14, where four interface modes of the structure investigated in Fig. 13 are studied, we can clearly note that the trend of $|N|$ confirms the localisation at the interface with amplification factors that range approximately between twenty and thirty.

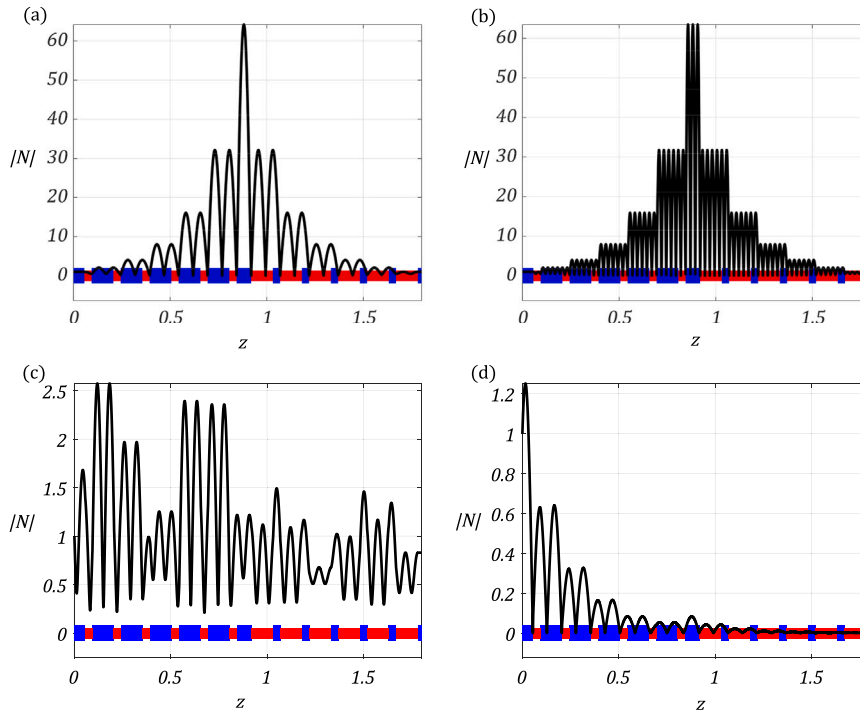


Fig. 11. Plot of the absolute value of the axial force field $|N|$ along the whole finite-sized structure analysed in Fig. 9; (a) at the interface mode $f = 33.95$ kHz; (b) at the interface mode $f = 101.84$ kHz; (c) at the generic frequency in a pass band $f = 40$ kHz; (d) at the generic frequency in a band gap $f = 33$ kHz. All simulations are conducted by imposing a harmonic axial force at the l.h.s. edge with amplitude equal to 1 N.

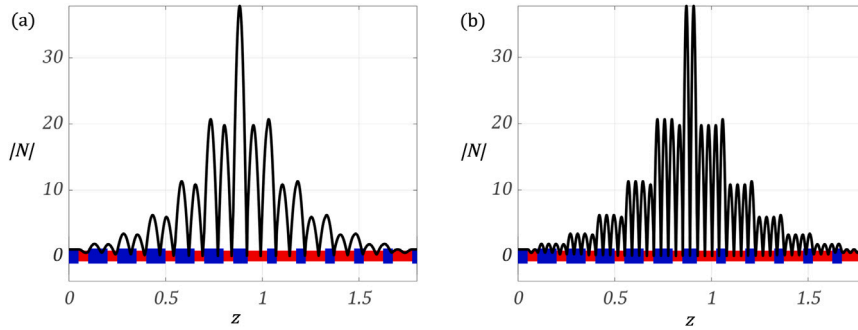


Fig. 12. Plot of the absolute value of the axial force field $|N|$ along the whole finite-sized structure analysed in Fig. 10; (a) at the interface mode $f = 33.95$ kHz; (b) at the interface mode $f = 67.89$ kHz. All simulations are conducted by imposing a harmonic axial force at the l.h.s. edge with amplitude equal to 1 N.

6. Conclusions

In this work, we study the presence of localised modes at the interface between two dissimilar periodic rods whose unit cells are designed according to *canonical* layouts, after the definition provided by the Authors in their previous research. The core of the investigation focuses on the application of the periodic and symmetric dynamic properties of canonical rods in order to predict the presence of a localised interface mode in both infinite and finite-sized systems. The main outcome of the paper is a general procedure, easy to implement, to determine the existence of an interface mode at a frequency inside a prescribed spectral band gap. The developed method allows one to know in which band gap the condition of existence of such a mode, i.e. that the sum of the surface impedances of the two rods composing the system is zero, is satisfied.

More in detail:

- we show how the symmetry properties of the edge modes at the boundary of a band gap are related to the sign of the imaginary part of the surface impedance of the band gap itself;
- we introduce a representation of the spectrum based on the *extended* torus very useful to predict both the symmetry properties of band-edge modes and the sign of the imaginary part of the surface impedance of any band gap of an arbitrary semi-infinite

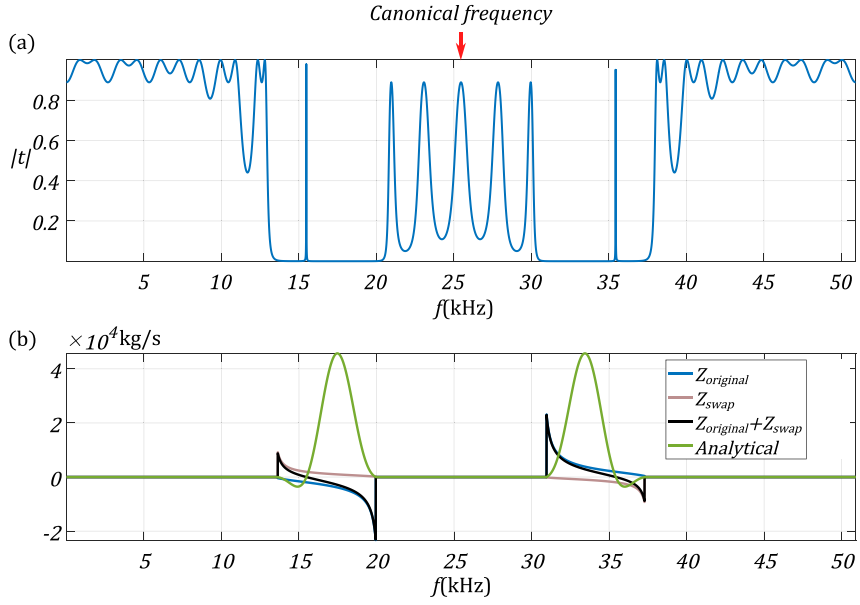


Fig. 13. Finite-sized structure composed of two rods whose phases are swapped, $C = 1/2$; (a) transmission spectrum in the domain $[0, f_i/2]$ where the two peaks inside the band gaps at which interface modes occur are evident; (b) imaginary parts of the relative surface impedance of the two rods (light blue and light brown) and the sum of the two (black) that is vanishing at the interface mode; the solid green line represents the solution of Eq. (49).

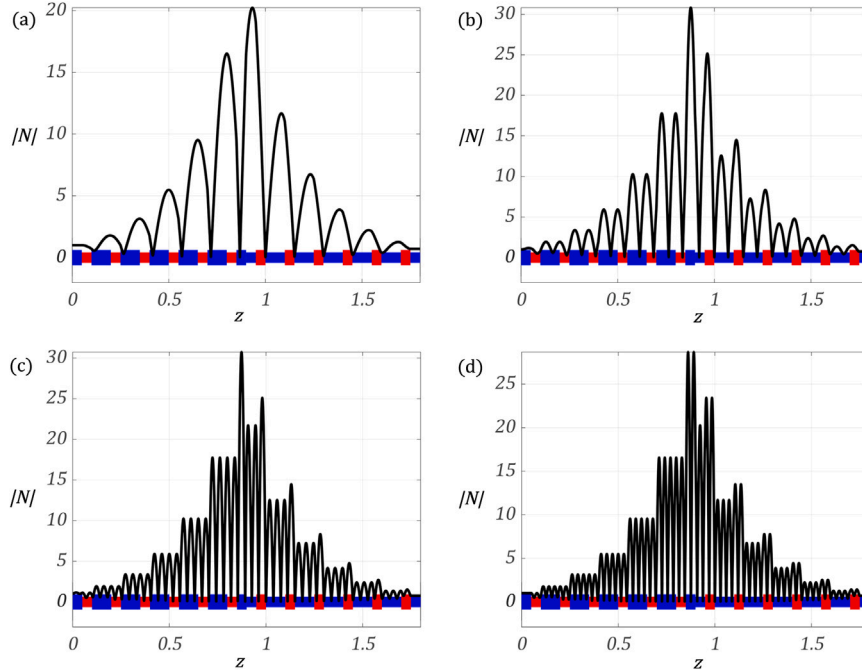


Fig. 14. Plot of the absolute value of the axial force field $|N|$ along the whole finite-sized structure analysed in Fig. 13 for the interface modes (a) $f = 15.48$ kHz; (b) $f = 35.44$ kHz; (c) $f = 66.40$ kHz; (d) $f = 86.35$ kHz. All simulations are conducted by imposing a harmonic axial force at the l.h.s. edge with amplitude equal to 1 N.

canonical waveguide; the extended torus is employed to identify systems composed of two canonical structures possessing the same pass-band/band-gap layout, but with some corresponding gaps where the imaginary part of surface impedances are opposite. Therefore, these gaps satisfy the condition of existence of an interface mode whose excitation frequency lies within their frequency range;

- for systems composed of two finite-sized canonical rods, we derive an equation to evaluate the frequency of the localised mode. For a specific configuration, consisting of two finite rods whose canonical ratios are reciprocal, the excitation frequency can be determined in closed form. All the derived formulae are benchmarked against numerical calculations.

The results we achieve provide new insight into the design and optimisation of phononic waveguides based on canonical rods that display localised interface modes in a prescribed band gap, avoiding the calculation of topological invariants, such as the Zak phase, and the related computational challenges. Due to the universality of the concept of surface impedance and to the versatility of the extended toroidal representation of the spectrum, the established methodology can be extended to investigate interface modes and localised states in two-dimensional systems such as plates and discrete architected lattices.

A final remark is necessary regarding the concept of canonical structure, whose properties are intimately connected to the Fibonacci's golden mean sequence (Gei et al., 2020). In this paper, we have focused on symmetric unit cells that correspond to element F_2 of said sequence to compare our novel methodology and results to those presented in some relevant works (Xiao et al., 2014, 2015; Kalozoumis et al., 2018; Meng et al., 2018). With the exception of F_3 , that is basically a re-scaling of F_2 , cells constructed starting from F_4 , F_5 , etc., are non symmetric. The extension of the strict connections between the symmetry properties of band-edge modes and the topological features of band gaps for non symmetric cells is not trivial, therefore, we leave its investigation for a future work.

CRedit authorship contribution statement

Z. Chen: Formal analysis, Methodology, Writing – original draft, Conceptualization, Investigation, Visualization, Funding acquisition, Validation. **L. Morini:** Conceptualization, Methodology, Investigation, Validation, Formal analysis, Supervision, Writing – review & editing. **M. Gei:** Funding acquisition, Supervision, Formal analysis, Methodology, Writing – review & editing, Investigation, Validation.

Declaration of competing interest

The authors declare the following financial interests/personal relationships which may be considered as potential competing interests: Zhijiang Chen reports financial support was provided by National Natural Science Foundation of China. Zhijiang Chen reports financial support was provided by China Postdoctoral Science Foundation. Massimiliano Gei reports financial support was provided by Italian Minister of University and Research. If there are other authors, they declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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Appendix A

The explicit form of the transfer matrix M for a single unit cell and used in Eq. (3) is given by

$$M = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix},$$

where

$$M_{11} = e^{i\xi_L} \left[\cos \xi_S + \frac{i}{2} \left(\frac{Z_S}{Z_L} + \frac{Z_L}{Z_S} \right) \sin \xi_S \right], \quad M_{22} = e^{-i\xi_L} \left[\cos \xi_S - \frac{i}{2} \left(\frac{Z_S}{Z_L} + \frac{Z_L}{Z_S} \right) \sin \xi_S \right],$$

$$M_{12} = -M_{21} = \frac{i}{2} \left(\frac{Z_S}{Z_L} - \frac{Z_L}{Z_S} \right) \sin \xi_S.$$

As M is unimodular ($\det M = 1$), its $\mathcal{N}^{\text{th}}$ power can be expressed by means of the following Chebyshev's identity:

$$M^{\mathcal{N}} = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix}^{\mathcal{N}} = \begin{bmatrix} M_{11} U_{\mathcal{N}-1} - U_{\mathcal{N}-2} & M_{12} U_{\mathcal{N}-1} \\ M_{21} U_{\mathcal{N}-1} & M_{22} U_{\mathcal{N}-1} - U_{\mathcal{N}-2} \end{bmatrix},$$

with $U_{\mathcal{N}} = \sin((\mathcal{N} + 1)KL_t) / \sin(KL_t)$ (see Markos and Soukoulis (2008) for the demonstration of this identity). According to the definition reported in Yariv and Yeh (1984) and Yeh et al. (1977), the reflection coefficient r for a wave coming from the left is given as a function of the elements of M as

$$r = -\frac{M_{21}^{\mathcal{N}}}{M_{22}^{\mathcal{N}}} = \frac{M_{21} \sin(\mathcal{N} KL_t)}{\sin((\mathcal{N} - 1)KL_t) - M_{22} \sin(\mathcal{N} KL_t)}.$$

Appendix B

We illustrate the general condition for two semi-infinite periodic rods with different unit cells to have identical dispersive spectra. We first focus on a system composed of two structures whose canonical ratios are reciprocal (Case I). The expressions for the two ‘flow’ variables (8) of ‘rod C’ can be written as

$$\xi_L^C = \frac{l_L^C \omega^C}{c_L}, \quad \xi_S^C = \frac{l_S^C \omega^C}{c_S}, \quad (50)$$

where $c_L = \sqrt{E_L/\rho_L}$ and $c_S = \sqrt{E_S/\rho_S}$. Recalling that the total length of the unit cell is $l_L^C + l_S^C = L_t^C$, and using the definition of canonical ratio (16), we can express the quantities l_L^C/c_L and l_S^C/c_S as follows

$$\frac{l_L^C}{c_L} = \frac{L_t^C}{c_L + C c_S}, \quad \frac{l_S^C}{c_S} = \frac{C L_t^C}{c_L + C c_S}. \quad (51)$$

Substituting (51) in (50) and inverting the equations, we obtain an expression for the radian frequency ω^C corresponding to values of the flow variables ξ_L^C and ξ_S^C , namely

$$\omega^C = \frac{\xi_L^C (c_L + C c_S)}{L_t^C} = \frac{\xi_S^C (c_L + C c_S)}{C L_t^C}. \quad (52)$$

Similarly, for ‘rod $1/C$ ’, the two variables $\xi_L^{1/C}$ and $\xi_S^{1/C}$, and the associated radian frequency $\omega^{1/C}$ take the forms

$$\xi_L^{1/C} = \frac{l_L^{1/C} \omega^{1/C}}{c_L}, \quad \xi_S^{1/C} = \frac{l_S^{1/C} \omega^{1/C}}{c_S}, \quad (53)$$

$$\omega^{1/C} = \frac{\xi_L^{1/C} (C c_L + c_S)}{C L_t^{1/C}} = \frac{\xi_S^{1/C} (C c_L + c_S)}{L_t^{1/C}}. \quad (54)$$

We now turn to Case II. The two variables $\tilde{\xi}_L^C$ and $\tilde{\xi}_S^C$, and the radian frequency $\tilde{\omega}^C$ for the rod in which phases L and S are swapped can be introduced by imposing the transformation $(c_L, c_S) \rightarrow (c_S, c_L)$ to expressions (50) and (52), i.e.

$$\tilde{\xi}_L^C = \frac{\tilde{l}_L^C \tilde{\omega}^C}{c_S}, \quad \tilde{\xi}_S^C = \frac{\tilde{l}_S^C \tilde{\omega}^C}{c_L}, \quad (55)$$

$$\tilde{\omega}^C = \frac{\tilde{\xi}_L^C (C c_L + c_S)}{\tilde{L}_t^C} = \frac{\tilde{\xi}_S^C (C c_L + c_S)}{C \tilde{L}_t^C}. \quad (56)$$

Identical frequency spectra for both rods in either Case I or Case II are achieved if $\omega^C = \omega^{1/C}$ and $\omega^C = \tilde{\omega}^C$, respectively. For the former system, it should be $\xi_L^C = \xi_S^{1/C}$ and $\xi_S^C = \xi_L^{1/C}$, for the latter it should be $\xi_L^C = \tilde{\xi}_L^C$ and $\xi_S^C = \tilde{\xi}_S^C$. Consequently, by imposing $\omega^C = \omega^{1/C}$ and $\omega^C = \tilde{\omega}^C$, respectively, we finally obtain:

$$\omega^C = \omega^{1/C} \Leftrightarrow \frac{L_t^{1/C}}{L_t^C} = \frac{C c_L + c_S}{c_L + C c_S}, \quad \omega^C = \tilde{\omega}^C \Leftrightarrow \frac{\tilde{L}_t^C}{L_t^C} = \frac{C c_L + c_S}{c_L + C c_S}. \quad (57)$$

Waveguides which satisfy conditions (57) share the same frequency spectrum. Therefore, in order to change the properties of the unit cells maintaining for both the same spectrum, the total lengths L_t^C , $L_t^{1/C}$ and $\tilde{L}_t^C$ could be modified provided the ratios (57) remain unchanged. For the examples displayed in this work, we assumed homogeneous materials for the two phases, e.g. $c_L = c_S = c$. Consequently, Eqs. (57)₁ and (57)₂ simply become $L_t^{1/C}/L_t^C = 1$ and $\tilde{L}_t^C/L_t^C = 1$, respectively; for this reason, for both Case I and Case II, the total length of all unit cells are set to L_t .

Appendix C

For any one-dimensional periodic rod, the Zak phase of the n^{th} pass band can be numerical calculated though the sums (Zak, 1989)

$$\theta_n^{\text{Zak}} = i \sum_{i=1}^p \left[\ln \left(\int_{\text{unitcell}} \frac{1}{2\rho c^2} d\mathbf{r} dz \mathcal{W}_{n,K_i}^* \mathcal{W}_{n,K_i+\Delta K} \right) \right], \quad (58)$$

where the first Brillouin zone $K L_t \in [-\pi, \pi]$ is divided in p sub-intervals, each wide ΔK , $\mathbf{r}$ denotes the position within the cross-section and $\mathcal{W}_{n,K}$ is the periodic part of the axial force field with the same interval.

As illustrated by Xiao et al. (2014), for the case of one-dimensional periodic photonic crystals, the topological properties inside the g^{th} band gap can be alternatively characterised by the sign of the impedance ς , which is related to the Zak phase by means of the following exact relationship:

$$\text{sign}[\varsigma^{(g)}] = (-1)^g (-1)^h \exp \left(i \sum_{p=1}^g \theta_p^{\text{Zak}} \right), \quad (59)$$

where the integer h denotes the number of branch crossings below the considered gap. The Zak phase of the lowest pass band is determined by Xiao et al. (2015)

$$\exp(i\theta_1^{\text{Zak}}) = \text{sign}(1 - Z_S/Z_L). \quad (60)$$

We have verified numerically that the sign of the impedance obtained through Eq. (59) coincides with that evaluated using the symmetric features of band-edge modes together with the symmetry properties inferred from inspection of the extended torus.

Data availability

No data was used for the research described in the article.

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