Novel Stability Conditions for Nonlinear Monotone Systems and Consensus in Multi-Agent Networks

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***Abstract*— We introduce a novel definition of monotonicity, termed "type-K" in honor of Kamke, and study nonlinear type-K monotone dynamical systems possessing the plus-subhomogeneity property, which we call "K-subtopical" systems after Gunawardena and Keane.**

We show that type-K monotonicity, which is weaker than strong monotonicity, is also equivalent to monotonicity for smooth systems evolving in continuous-time, but not in discrete-time. K-subtopical systems are proved to converge toward equilibrium points, if any exists, generalizing the result of Angeli and Sontag about convergence of topical systems' trajectories toward the unique equilibrium point when strong monotonicity is considered.

The theory provides an new methodology to study the consensus problem in nonlinear multi-agent systems (MASs). Necessary and sufficient conditions on the local interaction rule of the agents ensuring the K-subtopicality of MASs are provided, and consensus is proven to be achieved asymptotically by the agents under given connectivity assumptions on directed graphs. Examples in continuous-time and discrete-time corroborate the relevance of our results in different applications.

***Index Terms*— Nonlinear, Monotone, Order-preserving, Type-K, Plus-subhomogeneous, subtopical, Kamke, Consensus, Multi-Agent, Networks.**

I. Introduction

Dynamical systems whose trajectories preserve a partial order have represented a fruitful topic of research in numerous fields: such systems are usually called *monotone* [1]. Among different classes of monotone systems, special attention has been paid to those that are also *plus-homogeneous*. These systems, called *topical* by Gunawardena and Keane [2], have solutions or *flows* φ satisfying, for $t > 0$,

$$x \leq z \Rightarrow \varphi(t,x) \leq \varphi(t,z), \quad \forall x, z \in \mathbb{R}^n, \tag{1}$$

$$\varphi(t, x + \alpha\mathbf{1}) = \varphi(t,x) + \alpha\mathbf{1}, \quad \forall x \in \mathbb{R}^n, \alpha \in \mathbb{R}, \tag{2}$$

where x, z are initial conditions. We refer to the property in eq. (1) as *monotonicity* and to the property in eq. (2) as *plus-homogeneity*. Topical dynamical systems have been a subject of interest of two different yet close domains: monotone dynamical systems theory [3], [4], where plus-homogeneity is referred to as "translation invariance", and nonlinear Perron–Frobenius theory [5], where monotonicity is referred to as "order-preservation".

In this paper, we introduce a variation of monotonicity, called *type-K monotonicity* in honor of Kamke, satisfying eq. (1) and also the following,

$$x \leq y \wedge x_i < z_i \Rightarrow \varphi_i(t,x) < \varphi_i(t,z), \quad \forall i = 1, \ldots, n,$$

which will be shown to be an important bridging link between the above-mentioned theories. We also consider the more general property of *plus-subhomogeneity* given next

$$\varphi(t, x + \alpha\mathbf{1}) \leq \varphi(t,x) + \alpha\mathbf{1}, \quad \forall x \in \mathbb{R}^n, \alpha \in \mathbb{R}_{\geq 0}.$$

Type-K monotone and plus-subhomogeneous systems, which we call *K-subtopical*, are the object of study of this paper.

A. Main contributions

The main goal of this paper is to give a self-contained introduction to smooth K-subtopical systems both in continuous-time, where "smooth" denotes the continuous differentiability of the vector field, and in discrete-time, where "smooth" denotes the continuous differentiability of the map. Within this goal, our first main result is the following:

- Trajectories of smooth K-subtopicalsystems are proved to asymptotically converge toward an equilibrium point, if any exists (Theorem 1).

A further contribution is the derivation of necessary and sufficient conditions for type-K monotonicity:

- A smooth continuous-time system is type-K monotone if and only if its Jacobian matrix is Metzler (Proposition 3);
- A smooth discrete-time system is type-K monotone if and only if its Jacobian matrix is Metzler with a strictly positive diagonal (Theorem 5).

A knowledgeable reader may recognize the similarity of these conditions to the well-known *Kamke condition* for continuous-time system [1], [6], which we have shown to be necessary and sufficient for type-K monotonicity of smooth systems:

- Smooth monotone systems in continuous-time in continuous-time are also type-K monotone (Theorem 3).

A second goal consists in exploiting the convergence result and the characterization of K-subtopical systems presented

This work was partially supported by Fondazione Banco di Sardegna with the grant "Formal Methods and Technologies for the Future of Energy Systems", cup F72F20000350007.

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above to solve the consensus problem in K-subtopical Multi-Agent Systems (MASs). Most of the results for achieving consensus in linear MAS have been derived by considering row-substochastic matrices, for which the celebrated Perron-Frobenius theory provides a thorough spectral characterization [7]–[9]. Since subtopical maps generalize linear maps defined by row-substochastic matrices, our results lay the groundwork for a systematic analysis of general MAS with nonlinear interaction rules among agents. Within this goal is our last result:

- A K-subtopical MAS achieves consensus asymptotically if the consensus states are equilibrium points and if the graph contains a globally reachable node (Theorem 1 and Corollary 6).

B. Literature review

In the theory of monotone dynamical systems, emphasis is put on the class of continuous-time systems being strongly monotone [4], i.e., whose flows possess the following property:

$$x \leq z \wedge x \neq z \Rightarrow \varphi(t,x) < \varphi(t,z), \qquad \forall x, z \in \mathbb{R}^n.$$

Pioneering work in this field was done by Hirsch, who first showed that if solutions of continuous-time strongly monotone dynamical systems exist and are bounded, then they converge to a set of equilibrium points [10]. On the other hand, for discrete-time strongly monotone dynamical systems, Polavcik showed that their iterative behavior converges to periodic points under appropriate additional conditions [11]. An extensive overview of these results was given by Hirsch and Smith [1], [12]. Remarkably, generic convergence to equilibria can be made global, as in the case of contractive systems with a unique equilibrium point [13].

In contrast, in nonlinear Perron–Frobenius theory one usually considers discrete-time dynamical systems that are only monotone [5]. However, the relaxation of the assumption of strong monotonicity makes unenforceable most of the theory of monotone systems which then requires some additional assumptions. An interesting branch of research has focused on monotone systems evolving on the positive orthant $\mathbb{R}^n_+$ with the additional subhomogeneity property [14]–[20], given by,

$$\varphi(t, \alpha x) \leq \alpha\varphi(t,x), \quad \forall x \in \mathbb{R}_+, \alpha > 1,$$

as well as its extension to the multi-homogeneous systems [21], [22]: a unified framework has been recently provided by Gautier et. al. in [23]. Homogeneous systems on the positive orthant $\mathbb{R}^n_+$ are in one-to-one relationship with plus-subhomogeneous systems on the whole space $\mathbb{R}^n$ [5, Section 2.7], which we call *subtopical systems*, object of this paper. Consequently, the results provided in this paper for subtopical systems in $\mathbb{R}^n$ have equivalent multiplicative formulations, i.e., they apply to monotone and subhomogeneous systems in $\mathbb{R}^n_+$, both in discrete-time [24] and continuous-time.

The pioneering work of Nussbaum [25] showed that topical systems are nonexpansive under the sup-norm, contrary to the strong monotonicity assumption which causes the system to be contractive, thus ensuring the convergence of all trajectories to an equilibrium point by a direct application of the Banach fixed point theorem [13]. Indeed, when the system is merely nonexpansive, such a nice global convergence result is lost and one can only show that the trajectories converge to periodic points and thus not necessarily to an equilibrium point. Nussbaum has also shown that the primitiveness of the Jacobian matrix is a sufficient condition ensuring the convergence of a differentiable discrete-time system to its positive eigenvector; this result holds also for multi-homogeneous systems [23].

The control community has recently recognized the importance of bridging the two above-mentioned approaches. Angeli and Sontag were the first to consider topical systems [3], [4]. In particular, they have proved that every solution of continuous-time topical systems possessing the strong monotonicity property converges to an equilibrium point if the trajectory is bounded. If one wishes to get a global convergence result only assuming that the dynamical system is monotone without a stronger assumption, one meets several difficulties when applying any known methods used in the strongly monotone case. Afterward, Hu and Jiang provided a similar result for the restricted class of time-periodic systems while getting rid of the strong monotonicity assumption [26]. Their proof methodology is interesting: they provide a global convergence result of discrete-time systems ruled by the Poincaré map associated with a time-periodic topical system, which is, in turn, a topical system possessing the property of *type-K monotonicity*. The type-K monotonicity property, which encompasses strong monotonicity, has been proposed for the first time by Jiang in [27], and it has been recently exploited in the context of multi-agent systems by us in [24], [28].

There are many authors currently investigating the consensus problem over nonlinear monotone networks and systems, which sometimes intrinsically possess the plus-subhomogeneity property. Among them, Manfredi and Angeli have studied the case of monotone networks with unilateral interactions [29]. Como and Lovisari have considered monotone dynamical flow networks [30], [31], a topic of interest for Coogan and Arcak as well [32]. In particular, Coogan has recently presented a tutorial paper on mixed monotonicity, which extends the usual notion of monotonicity [33]. Worthy of mention is also the line of research on eventually monotone systems pursued by Altafini and Mauroy [34], [35], as well as the framework of differentially positive systems drawn up by Forni and Sepulchre [36], and also the operator-theoretic perspective adopted by Belgioioso and Grammatico [37]. For insights on new advances and applications of monotone systems, we refer the reader to the recent work of Smith [38].

C. Structure of the paper

Section II introduces the notation of the paper along with some preliminary results. Section III provides a global convergence result for K-subtopical systems. Section IV provides necessary and sufficient conditions to verify K-subtopicality. Section V study the consensus problem over K-subtopical multi-agent systems. In Section VI some examples are discussed and in Section VII final remarks are given and potential future directions are discussed.

This article has been accepted for publication in IEEE Transactions on Automatic Control. This is the author's version which has not been fully edited and content may change prior to final publication. Citation information: DOI 10.1109/TAC.2023.3246419

II. Notation and Preliminaries

The set of real and integer numbers are denoted by $\mathbb{R}$ and $\mathbb{Z}$, while their restriction to nonnegative values are denoted with $\mathbb{R}_{\geq 0}$ and $\mathbb{N}$, respectively. Matrices are denoted by uppercase letters, vectors and scalars are denoted by lowercase letters, while sets are denoted by uppercase calligraphic letters. We denote by $\mathbf{0}_n$ and $\mathbf{1}_n$ the vector of zeros and ones of dimension n, respectively. The identity matrix of dimension n is denoted by I_n. If clear from the context, subscripts are omitted.

A. Dynamical systems

We consider *autonomous dynamical systems* with an euclidean *state space* $\mathcal{X} \subseteq \mathbb{R}^n$ and denote the *state* of the system at a generic time t by $x(t) \in \mathcal{X}$.

Assumption 1. *The domain $\mathcal{X} \subseteq \mathbb{R}^n$ is open and convex, i.e., $(1-\alpha)x + \alpha y \in \mathcal{X}$ for all $x, y \in \mathcal{X}$.*

When time is a continuous variable, $t \in \mathbb{R}$, the system is described by a set of ordinary differential equations arising from,

$$\dot{x}(t) = f(x(t)), \qquad t \in \mathbb{R}.$$

When time is a discrete variable, $k \in \mathbb{Z}$, the system is described by a set of difference equations,

$$x(k+1) = f(x(k)), \qquad k \in \mathbb{Z}.$$

Function f determines the evolution of the state in time: in continuous-time, $f : \mathcal{X} \to \mathbb{R}^n$ is a vector field; in discrete-time, $f : \mathcal{X} \to \mathcal{X}$ is a map. We limit our study to smooth systems, which are systems satisfying the following assumption.

Assumption 2. *The function f is continuously differentiable, i.e., of class C^1, both in continuous- and discrete-time.*

Since we consider both continuous-time and discrete-time systems, it is convenient to describe a dynamical system in terms of its flow. Such description applies to both frameworks and allows us to use a general uniform notation throughout the paper. To this aim, we denote the time domain $\mathbb{T}$ which has to be intended as follows:

- $\mathbb{T} = \mathbb{R}$ for continuous-time systems;
- $\mathbb{T} = \mathbb{Z}$ for discrete-time systems;

To emphasize the dependence of the evolution $x(t)$ on the initial state $x(0) = \xi$, we denote the corresponding evolution by $\varphi(t, \xi)$, i.e.,

$$\varphi(t, \xi) = x(t), \quad \text{if} \quad x(0) = \xi.$$

The map $\varphi(t, \xi) : \mathbb{T} \times \mathcal{X} \to \mathcal{X}$ is called the *flow* of the system at time $t \in \mathbb{T}$ starting at ξ. The sequence of all consecutive states of the system is called the *trajectory* of the system, and it is denoted by $(\varphi(t, \xi))_{t \geq 0}$. A trajectory is said to be *bounded* if there exists $\ell, u \in \mathcal{X}$ such that for all $x \in (\varphi(t, \xi))_{t \geq 0}$ it holds $\ell \leq x \leq u$; otherwise it is said to be *unbounded*.

A point $\xi \in \mathcal{X}$ is called *periodic* if there exists a positive T such that $\varphi(T, \xi) = \xi$. The minimal such T is called the period of ξ. If the relation holds for any $T \in \mathbb{R}_{\geq 0}$, we call ξ an *equilibrium point*. We denote by $\mathcal{F}(\varphi) = \{\xi \in \mathcal{X} : \varphi(t, \xi) = \xi, \forall t \in \mathbb{T}\}$, the set of equilibrium points, or simply $\mathcal{F}$ when clear from the context. An equilibrium point $x_e \in \mathcal{F}(\varphi)$ is said to be *stable* if for every $\varepsilon > 0$ there is $\delta > 0$ such that $\|\xi - x_e\| < \delta$ implies $\|\varphi(t, \xi) - x_e\| < \varepsilon$ for any $\xi \in \mathcal{X}$ and $t \geq 0$, where $\|\cdot\|$ denotes the norm of a vector.

B. Multi-agent systems

We consider Multi-Agent Systems (MASs) wherein the $n \in \mathbb{N}$ agents are modeled as autonomous dynamical systems with scalar state $x_i(t) \in \mathbb{R}$, for $i = 1, \ldots, n$.

The interconnections among the agents are given by a graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ where $\mathcal{V} = \{1, \ldots, n\}$ is the set of nodes representing the agents and $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ is a set of directed edges. A *directed edge* $(i, j) \in \mathcal{E}$ exists if agent i is influenced by agent j: in this case, agent j is said to be a *neighbor* of agent i. The set of neighbors of the i-th node is denoted by $\mathcal{N}_i = \{j \in \mathcal{V} : (i, j) \in \mathcal{E}\}$. Each agent $i \in \mathcal{V}$ updates its own state according to a local interaction protocol, which, in continuous-time, takes the form

$$\dot{x}_i(t) = f_i\left(x_i(t), x_j(t) : j \in \mathcal{N}_i\right), \qquad t \in \mathbb{R},$$

and, in discrete-time, it takes the form

$$x_i(k+1) = f_i\left(x_i(k), x_j(k) : j \in \mathcal{N}_i\right), \qquad k \in \mathbb{Z}.$$

A *directed path* between two nodes p and q in a graph is a finite sequence of m edges $e_k = (j_k, i_k) \in \mathcal{E}$ that joins node p to node q, i.e., $j_1 = p$, $i_m = q$ and $i_k = j_{k+1}$ for $k = 1, \ldots, m-1$. The node i is said to be *reachable* from node j if there exists a directed path from node i to node j. A node is said to be *globally reachable* if it is reachable from all nodes $j \in \mathcal{V}$.

A MAS is said to *achieve consensus asymptotically* if the agents' states converge to the same constant value, called the *consensus state*, i.e., there is $c \in \mathbb{R}$ such that

$$\lim_{t \to \infty} x(t) = c\mathbf{1}, \qquad \text{or} \qquad \lim_{k \to \infty} x(k) = c\mathbf{1},$$

for any initial condition $x(0) \in \mathcal{X}$. We denote the consensus equilibrium set by

$$\mathcal{C} = \{c\mathbf{1} : c \in \mathbb{R}\}.$$

C. K-subtopical systems

Consider the Euclidean space $\mathbb{R}^n$ equipped with the standard partial order $\leq$. Given two vectors $u, v \in \mathbb{R}^n$, we can write the partial ordering relations as follows

$$\begin{aligned} &u \leq v \Leftrightarrow u_i \leq v_i \quad \forall i = 1, \ldots, n, \\ &u \lneq v \Leftrightarrow u \leq v \text{ and } u \neq v, \\ &u < v \Leftrightarrow u_i < v_i \quad \forall i = 1, \ldots, n. \end{aligned}$$

Dynamical systems in $(\mathcal{X}, \leq)$, with $\mathcal{X} \subseteq \mathbb{R}^n$, whose flow preserves such order are referred to as *order-preserving* or *monotone* dynamical systems [5], [12], [39]; we use the latter denomination. We formally define the monotonicity property in Definition 1, along with the *type-K* and *strong* variations that have been considered in the current literature. In particular, in this manuscript we propose the *type-K* monotonicity as a more natural property to work with, instead of *strong* monotonicity. Type-K monotonicity has been recently introduced by us for dynamical systems in discrete-time [24], [28], while here it is presented also for systems evolving in continuous-time.

Definition 1 (Monotonicity, type-K, and strong). *A dynamical system on $\mathcal{X} \subseteq \mathbb{R}^n$ with flow $\varphi : \mathbb{T} \times \mathcal{X} \to \mathcal{X}$ is:*

- *"monotone" if for all $\xi_1, \xi_2 \in \mathcal{X}$ it holds*

$$\xi_1 \leq \xi_2 \Rightarrow \varphi(t, \xi_1) \leq \varphi(t, \xi_2), \quad \forall t \geq 0,$$

- *"type-K monotone" if it is monotone and if for all $\xi_1, \xi_2 \in \mathcal{X}$ and for all $i = 1, \ldots, n$ it holds*

$$\xi_1 \leq \xi_2 \,\wedge\, \xi_{1,i} < \xi_{2,i} \Rightarrow \varphi_i(t, \xi_1) < \varphi_i(t, \xi_2), \quad \forall t \geq 0$$

- *"strongly monotone" if for all $\xi_1, \xi_2 \in \mathcal{X}$ it holds*

$$\xi_1 \lneq \xi_2 \Rightarrow \varphi(t, \xi_1) < \varphi(t, \xi_2), \quad \forall t \geq 0,$$

where $\xi_{1,i}$, $\xi_{2,i}$ and φ_i denote the i-th components. Correspondingly, the map φ is said to be "monotone", "type-K monotone", or "strongly monotone", respectively.

The properties in Definition 1 are related as follows.

Proposition 1. *Strongly monotone ⇒ Type-K monotone ⇒ monotone. The reverse implications do not hold.*

Proof: The fact that the implications hold follows from Definiton 1. We prove the reverse implications do not hold by means of counterexamples. For systems in continuous-time:

- $f(x) = -\text{sign}(x)$: monotone but not type-K;
- $f(x_1, x_2) = \begin{bmatrix} x_2 - x_1 & -x_2 \end{bmatrix}^\top$: type-K but not strong;

For systems in discrete-time:

- $f(x_1, x_2) = \begin{bmatrix} x_2 & x_1 \end{bmatrix}^\top$: monotone but not type-K;
- $f(x_1, x_2) = \begin{bmatrix} \frac{1}{2}(x_1 + x_2) & x_2 \end{bmatrix}^\top$: type-K but not strong. ■

We consider type-K monotone systems that also satisfy the property of plus-subhomogeneity[1] as defined next [5], [12].

Definition 2 (Plus-(sub)homogeneity). *A dynamical system on $\mathcal{X} \in \mathbb{R}^n$ is said to be "plus-subhomogeneous" if the flow $\varphi : \mathbb{T} \times \mathcal{X} \to \mathcal{X}$ satisfies*

$$\varphi(t, \xi + \alpha \mathbf{1}) \leq \varphi(t, \xi) + \alpha \mathbf{1}, \quad \forall \alpha \in \mathbb{R}_{\geq 0}, \, \forall t \geq 0$$

for all initial conditions $\xi \in \mathcal{X}$. When the inequality holds strictly, i.e.,

$$\varphi(t, \xi + \alpha \mathbf{1}) = \varphi(t, \xi) + \alpha \mathbf{1}, \quad \forall \alpha \in \mathbb{R}, \, \forall t \geq 0$$

then the system is said to be "plus-homogeneous". Correspondingly, the map φ is said to be plus-(sub)homogeneous.

[1] The name *plus-subhomogeneity* comes from the fact that the homogeneity is intended with respect to the addition operation, while simple *subhomogeneity* is usually intended with respect to the multiplication operation, i.e., $\varphi(t, \alpha\xi) \leq \alpha\varphi(t, \xi)$, cfr. [5]

Monotone systems satisfying also the plus-homogeneity property are known in the literature as *topical* systems [5], [40]–[42]. Since we consider the more general class of plus-subhomogeneous systems but require the stricter type-K property, we next define the class of *K-subtopical* systems.

Definition 3 (K-(sub)topicality). *A dynamical system on $\mathcal{X} \subseteq \mathbb{R}^n$ is called "K-(sub)topical" if it is type-K monotone and plus-(sub)homogeneous. Correspondingly, the map $\varphi : \mathbb{T} \times \mathcal{X} \to \mathcal{X}$ is said to be K-(sub)topical.*

A nice feature of K-subtopical systems is that they are non-expansive w.r.t. the sup-norm; this property is widely known in the discrete-time framework, see [5, Lemma 2.7.2] that builds upon the former results of Crandall and Tartar [43], while in Lemma 1 we prove it also for the continuous-time framework.

Definition 4 (Non-expansiveness). *A dynamical system on $\mathcal{X} \subseteq \mathbb{R}^n$ is said to be "non-expansive" w.r.t. a metric $d : \mathcal{X} \times \mathcal{X} \to \mathbb{R}_{\geq 0}$ if the flow φ satisfy*

$$d(\varphi(t, \xi_1), \varphi(t, \xi_2)) \leq d(\xi_1, \xi_2), \quad \forall t \geq 0$$

for all initial conditions $\xi_1, \xi_2 \in \mathcal{X}$. Correspondingly, the map $\varphi : \mathbb{T} \times \mathcal{X} \to \mathcal{X}$ is said to be non-expansive.

Lemma 1. *K-subtopical systems on $\mathcal{X} \subseteq \mathbb{R}^n$ are "non-expansive" w.r.t. the sup-metric $d_\infty : \mathcal{X} \times \mathcal{X} \to \mathbb{R}_{\geq 0}$ induced by the sup-norm, i.e.,*

$$d_\infty(\xi_1, \xi_2) = \|\xi_1 - \xi_2\|_\infty, \quad \forall \xi_1, \xi_2 \in \mathcal{X}.$$

Proof: For each fixed $t \geq 0$, we define a map $\phi^t(x) = \varphi(t, x) : \mathcal{X} \to \mathcal{X}$. According to [5, Lemma 2.7.2], each $\phi^t(\xi)$ is such that

$$\left\|\phi^t(\xi_1) - \phi^t(\xi_2)\right\|_\infty \leq \|\xi_1 - \xi_2\|_\infty, \quad \forall t \geq 0,$$

for any pair of initial conditions $\xi_1, \xi_2 \in \mathcal{X}$. By replacing $\phi^t(x) = \varphi(t, x)$, the proof is complete. ■

III. K-SUBTOPICAL DYNAMICAL SYSTEMS

The main result of this section, given in Theorem 1, is that for smooth K-subtopical systems in continuous or discrete-time, each trajectory converges to some stable equilibrium point, if any exists. For the convenience of the reader, we state here this result and postpone its proof to Section III-A, which makes use of several intermediate results discussed next.

Theorem 1. *Consider a K-subtopical dynamical system on $\mathcal{X} \subseteq \mathbb{R}^n$ under Assumptions 1-2. If the set of equilibrium points $\mathcal{F}$ is not empty, then for any initial condition $\xi \in \mathcal{X}$ there exists an equilibrium point $x_\xi \in \mathcal{F}$ such that*

$$\lim_{t \to \infty} \varphi(t, \xi) = x_\xi. \quad ■$$

Topical systems have been considered for decades in discrete-time,

$$x(k + 1) = f(x(k)), \qquad k \in \mathbb{Z}. \tag{3}$$

In this case, the properties of the flow φ directly translate into properties of the map f since $\varphi(k, \xi) = f^k(\xi)$ for any initial condition $\xi \in \mathcal{X}$ and time $k \in \mathbb{Z}$. Thus, the asymptotic behavior of the system is studied by considering the iterative behavior of the map $\varphi(1,\xi) \equiv f(\xi)$.

On the other hand, less attention has been paid to continuous-time systems,

$$\dot{x}(t) = f(x(t)), \qquad t \in \mathbb{R}. \tag{4}$$

We show in Lemma 3 that, similarly to the discrete-time case, the asymptotic behavior of the continuous-time system can be inferred from the iterative behavior of its flow φ^T for any time discretization $T \geq 0$, under the assumption of a continuously differentiable vector field. To this aim, we first prove in the next lemma that their flow is defined and unique at all times.

Lemma 2. *Consider a continuous-time dynamical system on $\mathcal{X} \subseteq \mathbb{R}^n$ under Assumptions 1-2. If it is subtopical, then for any initial condition $\xi \in \mathcal{X}$ the flow $\varphi(t,\xi)$ exists for all $t \geq 0$ and it is unique.*

Proof: Since $f \in C^1$, then we have the following facts[2]:

(i) For any initial condition $\xi \in \mathcal{X}$ the flow $\varphi(t,\xi)$ exists and it is unique in an interval $[0,T]$ with $T > 0$;

(ii) the flow $\varphi(t,\xi)$ is C^1, i.e., its partial derivatives with respect to time and initial conditions exists and are continuous in the interval of existence $[0,T]$.

By subtopicality of the system, we can exploit the non-expansiveness property given by Lemma 1 to ensure that solutions exist for all $t \geq 0$. For any initial condition $\xi \in \mathcal{X}$ and any subsequent state $\varphi(t^*,\xi)$ with $t^* > 0$ it holds

$$\|\varphi(t,\xi) - \varphi(t,\varphi(t^*,\xi))\|_\infty \leq \|\xi - \varphi(t^*,\xi)\|_\infty, \quad \forall t \geq 0.$$

i.e., solutions do not diverge in finite time. This, jointly with fact (ii) that the flow is continuous and differentiable in the interval of existence, ensures that the existence and uniqueness of the solutions stated in (i) hold in the interval $[0,\infty)$, thus completing the proof. ■

Lemma 3. *Consider a subtopical continuous-time dynamical system on $\mathcal{X} \subseteq \mathbb{R}^n$ under Assumptions 1-2. Let φ be the flow of the system and consider the family of discrete-time dynamical systems defined by*

$$y_T(k+1) = \varphi(T, y_T(k)), \qquad \forall T > 0. \tag{5}$$

If the initial states of the continuous-time and discrete-time systems coincide, i.e., $x(0) = y_T(0)$, and if solutions $y_T(k)$ admit a finite limit for all choices of $T > 0$, then also the solution $x(t)$ does, and moreover these limits coincide, i.e.,

$$\lim_{t\to\infty} x(t) = \lim_{k\to\infty} y_T(k), \qquad \forall T > 0.$$

Proof: Having shown the existence and uniqueness of flows in Lemma 2, the systems satisfies the so-called *group law* (cfr. [44, Section 7.1]),

$$\varphi(q,\varphi(p,\xi)) = \varphi(p+q,\xi)),$$

By selecting $p = q = T > 0$, we can write $\varphi(T,\varphi(T,\xi)) = \varphi(2T,\xi)$. Thus, considering the family of discrete-time systems in eq. (5) such that $x(0) = y_T(0) = \xi$, we can write

$$x(kT) = \varphi(kT,\xi) = \varphi^k(T,\xi) = y_T(k), \qquad \forall T > 0, k \in \mathbb{N}.$$

Now, if all $y_T(k)$ converges to a finite limit, then, by construction, these limits must all coincide since for all $T > 0$ trajectories $(y_T(k))_{k\in\mathbb{N}}$ are sampled from the trajectory $(x(t))_{t\in\mathbb{R}_{\geq 0}}$. This, in turn, implies that also $x(t)$ converges to the same limit, completing the proof. ■

Due to Lemma 3, which can be generalized to arbitrary dynamical systems for which global existence and uniqueness of the solutions hold, regardless of whether the system under consideration is continuous or discrete in time, one can equivalently study its asymptotic behavior by means of the family of discrete-time systems as in eq. (5). This enables us to prove in the next Lemma 4 that each equilibrium point of subtopical systems is stable and, consequently, to prove the main result in Theorem 1 anticipated at the beginning of this section.

Lemma 4. *Consider a dynamical system on $\mathcal{X} \subseteq \mathbb{R}^n$ under Assumptions 1-2. If it is subtopical, then every equilibrium point $x_e \in \mathcal{F}$ is stable.*

Proof: Let $x_e \in \mathcal{F}$ be an equilibrium point, then for any neighborhood $\mathcal{W}$ of x_e, one can find two points belonging to this neighborhood $a, b \in \mathcal{W}$ such that $a + \alpha\mathbf{1} = x_e = b - \alpha\mathbf{1}$ and $[a,b] \subset \mathcal{W}$. By plus-subhomogeneity, for all $t \geq 0$,

$$\begin{aligned}\varphi(t, a+\alpha\mathbf{1}) &\leq \varphi(t,a) + \alpha\mathbf{1}\\ \varphi(t,x_e) &\leq \varphi(t,a) + \alpha\mathbf{1}\\ x_e - \alpha\mathbf{1} &\leq \varphi(t,a)\\ a &\leq \varphi(t,a)\end{aligned}$$

and also

$$\begin{aligned}\varphi(t, x_e+\alpha\mathbf{1}) &\leq \varphi(t,x_e) + \alpha\mathbf{1}\\ \varphi(t,b) &\leq x_e + \alpha\mathbf{1}\\ \varphi(t,b) &\leq b\end{aligned}$$

The proof is completed by exploiting the monotonicity property, which implies that the set $[a,b]$ is forward invariant,

$$a \leq \varphi(t,a) \leq \varphi(t,x) \leq \varphi(t,b) \leq b, \ \forall x \in [a,b], t \geq 0. \quad ■$$

We now provide a simple example that shows the non-trivial behavior of K-subtopical dynamical systems: one should be aware that type-K monotonicity of a system does not imply that the trajectories are either monotonically increasing, decreasing or constant, but may exhibit more complex behaviors. Indeed, all variations of monotonicity in Definition 1 imply that $\varphi(t,\xi)$ is monotone in ξ but not necessarily in t.

Example 1. *Consider a dynamical system in continuous-time with state $x(t) \in \mathbb{R}^4$ and dynamics*

$$\dot{x}(k) = \text{atan}\Big(Ax(t)\Big), \quad A = \begin{bmatrix} -9 & 5 & 0 & 0\\ 0 & -2 & 2 & 0\\ 0 & 0 & -5 & 5\\ 0 & 3 & 0 & -3\end{bmatrix}, \tag{6}$$

where atan$(\cdot)$ *denotes the arctangent function to be intended as component-wise. Fig. 1 (in the next page) shows the trajectory of the system for the initial condition $x(0) = \begin{bmatrix}0 & 1 & 2 & 3\end{bmatrix}^\top$ and reveals the non-trivial behavior of the trajectories of such systems, indeed, the 3-rd component (solid blue curve) has a non-monotonic behavior.*

[2]The proof of these standard results can be found in Section 17.2 and Section 17.6 of [44], respectively.

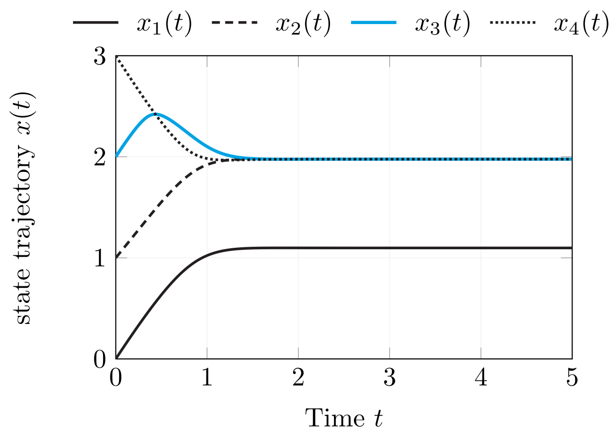

Fig. 1. State trajectory of the K-subtopical linear system in eq. (6).

A. Proof of Theorem 1

Discrete-time: Let the system be $x(k+1) = f(x(k))$ with $x(k) \in \mathbb{R}^n$. The map $f : \mathbb{R}^n \to \mathbb{R}^n$ is subtopical by assumption, and thus by Lemma 1 it is non-expansive under the sup-norm. Trajectories of sup-norm non-expansive maps have been proved either to be all unbounded or to *converge to a periodic point*[3]: the trajectory $(f^k(\xi))_{k\in\mathbb{N}}$ starting at $\xi \in \mathcal{X}$ is said to converge to the periodic point x_ξ if

$$\lim_{k\to\infty} f^{kp}(\xi) = x_\xi,$$

where $p \in \mathbb{N} \setminus \{0\}$ is the period of x_ξ. Since by assumption there exists at least one equilibrium point, i.e., $\mathcal{F} \neq \emptyset$, then for any point $\xi \in \mathcal{X}$, there exists a periodic point x_ξ which the trajectory through ξ converges to.

We now prove that for a discrete-time subtopical system possessing the additional property of type-K monotonicity, all periodic points are equilibrium points. In so doing, we make use of the concept of *limit set* of a periodic point x_ξ with period p, given by

$$\Omega = \{x_\xi, f(x_\xi), f^2(x_\xi), \cdots, f^{p-1}(x_\xi)\}. \tag{7}$$

Moreover, we consider the tightest lower bound ℓ to such limit set, defined component-wise by

$$\ell_i = \min_{y\in\Omega} y_i, \quad \forall i = 1, \dots, n, \tag{8}$$

which means that for all $y \in \Omega$ it holds $\ell \leq y$ and that each component ℓ_i of ℓ is also a component y_i of some $y \in \Omega$. Finally, we also consider the furthest point $y^* \in \Omega$ from ℓ w.r.t. the sup-norm, with i^* being the component attaining such maximum distance, i.e.,

$$y^* = \operatorname*{argmax}_{y\in\Omega} \|y - \ell\|_\infty, \quad i^* = \operatorname*{argmax}_{i=1,\cdots,n} |y^*_i - \ell_i|. \tag{9}$$

With this notation, proving that "all periodic points are equilibrium points" is equivalent to proving $\Omega = \{\ell\}$. The proof[4] makes use of the following intermediate results:

[3]This was proved by Lemmens in [45] and can be found in his book [5].

[4]We thank an anonymous Reviewer that contributed to the technical development of this proof.

(a) The trajectory $(f^k(\ell))_{k\in\mathbb{N}}$ is non-increasing w.r.t. k, i.e., $f^{k+1}(\ell) \leq f^k(\ell)$ for all $k \geq 0$;
(b) The trajectory $(f^k(\ell))_{k\in\mathbb{N}}$ stays constant along the component i^* as in eq. (9), i.e., $f^k_{i^*}(\ell) = \ell_{i^*}$ for all $k \geq 0$;
(c) The i^*-th component of y^* and ℓ are the same, i.e., $y^*_{i^*} = \ell_i$, where y^* and i^* are defined as in eq. (9);

We now show that (a) ⇒ (b) ⇒ (c), from which the final result follows.

Proof of claim (a): By monotonicity of f, for any $y \in \Omega$ it holds that $\ell \leq y \Rightarrow f(\ell) \leq f(y)$. Since $f(y) \in \Omega$, then also $f(\ell)$ is a lowerbound to Ω. On the other hand, given that ℓ is the tightest lower bound, it holds $f(\ell) \leq \ell$. Applying the monotone map f to both sides of $f(\ell) \leq \ell$ for $k \geq 0$ times yields $f^{k+1}(\ell) \leq f^k(\ell)$.

Proof of claim (b): As a particular case of claim (a), for any component i and time k, it holds that $f^k_i(\ell) \leq \ell_i$. Moreover, by definition of ℓ in eq. (8) it holds that $\ell_i \leq y^*_i$, which yields

$$f^k_i(\ell) \leq \ell_i \leq y^*_i, \quad \forall i,\ k \geq 0. \tag{10}$$

The following chain of inequalities holds

$$\|y^*-\ell\|_\infty \overset{(i)}{\leq} \|y^*-f^p(\ell)\|_\infty \overset{(ii)}{\leq} \|f^p(y^*)-f^p(\ell)\|_\infty \overset{(iii)}{\leq} \|y^*-\ell\|_\infty$$

where (i) follows from eq. (10), (ii) follows from the p-periodicity of y^*, and (iii) follows from non-expansiveness of f w.r.t. $\|\cdot\|_\infty$. This shows that $\|y^* - \ell\|_\infty = \|y^* - f^p(\ell)\|_\infty$, which, by eq. (10) and the definition of i^* in eq. (9), implies $f^p_{i^*}(\ell) = \ell_{i^*}$. By claim (a) follows that $f^k_{i^*}(\ell) = \ell_{i^*}$ for $k \geq 0$.

Proof of claim (c): Since ℓ is the tightest lower bound of Ω by definition in eq. (8), then

$$\forall i,\ \exists k \geq 0 : \qquad \ell_i = f^k_i(y^*). \tag{11}$$

Assume by contradiction that $\ell_{i^*} < y^*_{i^*}$. By type-K monotonicity of the map f, it holds $\ell \leq y^* \Rightarrow f(\ell) \leq f(y^*)$ with $\ell_{i^*} < y^*_{i^*} \Rightarrow f_{i^*}(\ell) < f_{i^*}(y^*)$, and, by claim (b), $\ell_{i^*} < f_{i^*}(y^*)$. Repeating this reasoning for all $k \geq 0$ yields $\ell_{i^*} < f^k_{i^*}(y^*)$, which means that it does not exist $k \geq 0$ such that $\ell_{i^*} = f^k_{i^*}(y^*)$, leading to a contradiction with eq. (11). Therefore it must hold that $y^*_{i^*} = \ell_{i^*}$.

Conclusion: Since i^* is the component attaining the maximum distance between ℓ and y^* according to eq. (9), by claim (c) it follows that $\|y^* - \ell\|_\infty = |y^*_{i^*} - \ell_{i^*}| = 0$, i.e., ℓ and y^* are the same point. Moreover, since y^* is the furthest point from ℓ w.r.t. the $\|\cdot\|_\infty$, it follows that the set Ω is a singleton and contains only ℓ, i.e., $\Omega = \{\ell\}$. This leads to the conclusion that the system always converges to an equilibrium point, which is stable according to Lemma 4.

Continuous-time: We now apply Lemma 3 to infer the same result for continuous-time systems $\dot{x}(t) = f(x(t))$. Consider the family of discrete-time systems as in eq. (5) given by $y_T(k+1) = \varphi(T, y_T(k))$ for any choice of $T > 0$. Since all maps $\varphi(T, \cdot)$ are subtopical, from the previous derivations we conclude that any solution $y_T(k)$ converges to a stable equilibrium point. Moreover, by Lemma 3, all these solutions converge to the same equilibrium point, to which also the continuous-time solution $x(t)$ converges, completing the proof.

This article has been accepted for publication in IEEE Transactions on Automatic Control. This is the author's version which has not been fully edited and content may change prior to final publication. Citation information: DOI 10.1109/TAC.2023.3246419

IV. How to verify K-subtopicality

A. K-subtopicality in continuous-time

The next remark provides a way to verify plus-subhomogeneity of a continuous-time system by only looking at the function f, which is a direct generalization of the result of Angeli and Sontag for plus-homogeneity [3, Lemma 3.1].

Remark 1. *A continuous-time system as in eq.* (4) *on* $\mathcal{X} \in \mathbb{R}^n$ *is plus-subhomogeneous if and only if*

$$f(\xi + \alpha \mathbf{1}) \leq f(\xi), \qquad \forall \alpha \in \mathbb{R}_{\geq 0}, \forall \xi \in \mathcal{X},$$

and it is plus-homogeneous if and only if

$$f(\xi + \alpha \mathbf{1}) = f(\xi), \qquad \forall \alpha \in \mathbb{R}, \forall \xi \in \mathcal{X}.$$

Examples 1-2 deal with plus-subhomogeneous systems in continuous-time and discrete-time. The plus-homogeneity property is usually assumed in networked systems when the agents have not access to a common reference but to only relative information depending on their state differences; see Section VI-C.

On the other hand, verifying the monotonicity of a system is not an easy task. For continuous-time dynamical systems whose vector field is continuously differentiable, a necessary and sufficient condition to ensure monotonicity is the well-known *Kamke condition*, which dates back to the 30s and the work of Kamke in [6], see Proposition 1.1 in [33][5].

Theorem 2 (Kamke condition). *A continuous-time system as in eq.* (4) *on* $\mathcal{X} \in \mathbb{R}^n$ *is monotone if and only if for any two points* $a, b \in \mathcal{X}$ *such that* $a \leq b$ *the following holds*

$$\forall i: \; a_i = b_i \quad \Rightarrow \quad f_i(a) \leq f_i(b).$$

It should be noted that from the previous theorem it follows that any scalar continuous-time system is monotone, since the condition is satisfied trivially for $n = 1$.

Remark 2. *Any scalar continuous-time system as in eq.* (4) *on* $\mathcal{X} \in \mathbb{R}$ *is monotone.*

For continuous-time systems with a continuously differentiable vector field, the Kamke condition turns out to be equivalent to a specific sign structure of the Jacobian matrix.

Proposition 2. *Consider a continuous-time system* (4) *on* $\mathcal{X} \subseteq \mathbb{R}^n$ *whose vector field* f *is* C^1*. The system is monotone if and only if Jacobian matrix is Metzler, i.e.,*

$$\frac{\partial f_i(x)}{\partial x_j} \geq 0, \qquad i \neq j, \quad x \in \mathcal{X}$$

Proof: See [1, Remark 1.1] and [33, Proposition 1]. ■

In the following, we show that for a continuous-time system whose vector field is continuously differentiable, monotonicity is equivalent to type-K monotonicity, thus proving that the same sign structure of the Jacobian matrix is also a necessary and sufficient condition for type-K monotonicity.

[5]Note that in standard books, such as those of Smith [1] and Coppel [46], "monotonicity" is referred to as "type-K", even if this notation has been lost in the current literature. In this paper, we recover the notation "type-K" with a different meaning.

Theorem 3. *Consider a continuous-time system in* $\mathcal{X} \subseteq \mathbb{R}^n$ *with dynamics*

$$\dot{x}(t) = f(x(t)), \qquad t \in \mathbb{R}_{\geq 0}. \tag{12}$$

If the system is monotone and its vector field f *is* C^1*, then the system is type-K monotone.*

Proof: The monotonicity of the system implies that two solutions $x(t)$, $z(t)$ of the system in eq. (12) are ordered at all times $t \geq 0$ if their initial conditions $x(0)$, $z(0)$ are ordered, i.e., $x(0) \leq z(0) \Rightarrow x(t) \leq z(t)$; if instead $x(0) \not\leq z(0)$ then nothing can be said about the order between the two trajectories $x(t)$ and $z(t)$. Thus, consider the case $x(0) \leq z(0)$ and let $v \in \mathbb{R}^n_{\geq 0}$ be the non-negative vector such that $z(0) = x(0) + v$, and write the solutions as follows

$$x(t) = \varphi(t, x(0)), \qquad z(t) = \varphi(t, x(0) + v),$$

where φ is the flow of the monotone system in eq. (12). Without loss of generality, assume that both solutions $x(t)$ and $z(t)$ exists in an interval $[0, T^*]$ with $T^* \in \mathbb{R}_{>0}$.

The monotonicity of the system implies that the order between the initial conditions, $x(0) \leq z(0)$, must be preserved by the solutions at all times, i.e.,

$$x(t) \leq z(t), \qquad t \in [0, T^*]. \tag{13}$$

To prove the type-K monotonicity of the system, we need to show that if there is a strict order in the i-th component, i.e., $x_i(0) < z_i(0)$, which is equivalent to $v_i > 0$, then such order is preserved at all times, i.e., for $t \in [0, T^*]$ it holds

$$v_i > 0 \quad \Rightarrow \quad x_i(t) < z_i(t). \tag{14}$$

At $t = 0$ eq. (14) holds because $x_i(0) < x_i(0) + v_i = z_i(0)$. Now, since f is C^1, then both solutions $x(t)$ and $z(t)$ are also C^1, and thus there exists an interval of time $[0, t^*)$ of positive measure, i.e., $0 < t^* \leq T^*$, in which eq. (14) holds.

Finally, we aim to prove that eq. (14) holds for all $t \in [0, T^*]$ by contradicting the following

$$\exists\, T \in [t^*, T^*]: \quad x_i(T) = z_i(T). \tag{15}$$

Denoting $a_{-i} \in \mathbb{R}^{n-1}$ the vector of $(n-1)$ elements obtained from vector $a \in \mathbb{R}^n$ by removing the i-th component, i.e., $a_{-i} = [a_1, \ldots, a_{i-1}, a_{i+1}, \ldots, a_n]^\intercal$, we can say that the i-th component of $x(t)$ is solution of the differential equation

$$\dot{x}_i(t) = f_i(x_i(t), x_{-i}(t)). \tag{16}$$

where $x_{-i}(t)$ is treated as an exogenous input. Similarly, the i-th component of $z(t)$ is solution of

$$\dot{z}_i(t) = f_i(z_i(t), z_{-i}(t)).$$

Moreover, from the monotonicity of the system in eq. (13), which implies $z_{-i}(t) \geq x_{-i}(t)$, and from Proposition 2, which states that the map f_i is a nondecreasing function in all variables other than the i-th, i.e., $f_i(z_i(t), z_{-i}(t)) \geq f_i(z_i(t), x_{-i}(t))$, it follows that $z_i(t)$ is also a solution of the differential inequality

$$\dot{z}_i(t) \geq f_i(z_i(t), x_{-i}(t)). \tag{17}$$

We now operate a time reversal and a time shift by letting $\tau = T - t$. We denote $x_i^{rev}(\tau) = x_i(T-\tau)$ and $z_i^{rev}(\tau) = z_i(T-\tau)$ the reversed solutions. By this change of variables we compute

$$\dot{x}_i^{rev}(\tau) = \frac{dx_i^{rev}(\tau)}{d\tau} = \frac{dx_i(T-\tau)}{d\tau} = -\dot{x}_i(T-\tau),$$

and from eq. (16) we derive that $x_i^{rev}(\tau)$ is solution of

$$\dot{x}_i^{rev}(\tau) = -f_i(x_i^{rev}(\tau), x_{-i}^{rev}(\tau)). \tag{18}$$

With similar steps, from eq. (17) we derive that $z_i^{rev}(\tau)$ is a solution of

$$\dot{z}_i^{rev}(\tau) \le -f_i(z_i^{rev}(\tau), x_{-i}^{rev}(\tau)). \tag{19}$$

Assuming that eq. (15) holds, at $\tau = 0$ the two solutions are equal, namely $x_i^{rev}(0) = z_i^{rev}(0)$, in fact

$$x_i^{rev}(0) = x_i(T) = z_i(T) = z_i^{rev}(0),$$

and since $z_i^{rev}(\tau)$ is a solution of the differential inequality (19), while $x_i^{rev}(\tau)$ is solution of (18), then

$$z_i^{rev}(\tau) \le x_i^{rev}(\tau), \qquad \tau \in [0, T^*], \tag{20}$$

which, at $\tau = T$, leads to

$$z_i(0) = z_i^{rev}(T) \le x_i^{rev}(T) = x_i(0).$$

This leads to a contradiction since $v_i > 0$ by eq. (14) and therefore $z_i(0) = x_i(0) + v_i > x_i(0)$. Therefore, eq. (15) does not hold, and eq. (14) holds instead for all $t \in [0, T^*]$, completing the proof of the theorem. ∎

Remark 3. *Proposition 1 and Theorem 3 lead to the following statements:*

i) f *is type-K monotone* $\Rightarrow$ f *is monotone;*
ii) f *is type-K monotone* $\not\Leftarrow$ f *is monotone;*
iii) If f *is* C^1*, then* f *is type-K monotone* $\Leftrightarrow$ f *is monotone.*

Remark 3 emphasizes the role of Theorem 3, which states that if a monotone continuous-time system has a continuously differentiable vector field, then it is type-K monotone. In other words, under the assumption of a continuously differentiable vector field, all monotone systems are also type-K monotone. Consequently, all results provided in this paper for type-K monotone systems apply to smooth monotone systems.

As a consequence of Theorem 3, we restate Proposition 2 in the particular case of type-K monotone systems with continuously differentiable vector fields.

Proposition 3. *Consider a continuous-time system* (4) *on* $\mathcal{X} \subseteq \mathbb{R}^n$ *whose vector field* f *is* C^1*. The system is type-K monotone if and only if Jacobian matrix is Metzler, i.e.,*

$$\frac{\partial f_i(x)}{\partial x_j} \ge 0, \qquad i \ne j, \quad x \in \mathcal{X}.$$

B. K-subtopicality in discrete-time

Verifying the plus-subhomogeneity of a discrete-time system by only looking at the function f can be done by directly applying Definition 2, as remarked next.

Remark 4. *A discrete-time system as in eq.* (3) *on* $\mathcal{X} \subseteq \mathbb{R}^n$ *is plus-subhomogeneous if and only if*

$$f(\xi + \alpha\mathbf{1}) \le f(\xi) + \alpha\mathbf{1}, \qquad \forall\alpha \in \mathbb{R}_{\ge 0}, \forall\xi \in \mathcal{X},$$

and it is plus-homogeneous if and only if

$$f(\xi + \alpha\mathbf{1}) = f(\xi) + \alpha\mathbf{1}, \qquad \forall\alpha \in \mathbb{R}, \forall\xi \in \mathcal{X}.$$

As a counterpart to the Kamke condition given in Theorem 2, we provide a necessary and sufficient condition for type-K monotonicity in the case of discrete-time systems, which we denote *Kamke-like condition*.

Theorem 4 (Kamke-like condition). *A discrete-time system as in eq.* (4) *on* $\mathcal{X} \in \mathbb{R}^n$ *is monotone if and only if for any two points* $a, b \in \mathcal{X}$ *the following holds*

$$a \le b \Rightarrow f(a) \le f(b), \tag{21}$$

and it is type-K monotone if and only if it further satisfies

$$\forall i: \quad a_i < b_i \Rightarrow f_i(a) < f_i(b). \tag{22}$$

Proof: The solution of a discrete-time system at time $k \in \mathbb{N}$ is equal to the k-th composition of the map f, i.e., $\varphi(k, \xi) = f^k(\xi)$ for any $\xi \in \mathcal{X}$. With this notion, the proof is a consequence of Definition 1. ∎

For discrete-time systems with a continuously differentiable vector field, the Kamke-like condition turns out to be equivalent to a specific sign structure of the Jacobian matrix, similar to what happens in continuous-time. A preliminary sufficient condition was given in [24, Proposition 9], while next, we provide a necessary and sufficient condition.

Theorem 5. *Consider a discrete-time system as in eq.* (3) *on* $\mathcal{X} \subseteq \mathbb{R}^n$ *whose map* f *is* C^1*. The system is monotone if and only if the Jacobian matrix is non-negative, i.e.,*

$$\frac{\partial f_i(x)}{\partial x_j} \ge 0, \qquad x \in \mathcal{X}, \tag{23}$$

and it is type-K monotone if and only if the Jacobian is non-negative as in eq. (23) *with a strictly positive diagonal almost everywhere, i.e.,*

$$\frac{\partial f_i(x)}{\partial x_i} > 0 \qquad x \in \mathcal{X} \setminus S, \tag{24}$$

where S *is a set of measure zero.*

Proof: We rewrite monotonicity in eq. (21) as follows

$$f_i(a) \le f_i(a + v), \qquad \forall a \in \mathcal{X},\ \forall v \ge 0,\ \forall i. \tag{25}$$

We start with the first part of the proof, i.e., (25) $\Leftrightarrow$ (23). Being f continuously differentiable, each directional derivative $\nabla_v f_i(a)$ can be computed by means of either the limit definition or the partial derivatives,

$$\nabla_v f_i(a) = \lim_{\varepsilon \to 0} \frac{f_i(a + \varepsilon v) - f_i(a)}{\varepsilon} = \sum_{j=1}^{n} \frac{\partial f_i(a)}{\partial x_j} \cdot \frac{v_j}{|v|}.$$

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When v is an arbitrary canonical vector $v = e_j$ with all zero entries except for the j-th entry that is equal to one, then

$$\lim_{\varepsilon\to 0} \frac{f_i(a+\varepsilon e_j) - f_i(a)}{\varepsilon} = \frac{\partial f_i(a)}{\partial x_j}, \quad \forall i, j.$$

Therefore, (25) $\Rightarrow$ (23) is proven by

$$f_i(a) \le f_i(a+\varepsilon e_j) \Rightarrow \frac{\partial f_i(a)}{\partial x_j} \ge 0, \quad \forall i, j,$$

while (23) $\Rightarrow$ (25) , given $v \ge 0$, is proven by

$$\frac{\partial f_i(a)}{\partial x} \ge 0 \Leftrightarrow \sum_{j=1}^{n} \frac{\partial f_i(a)}{\partial x_j} \cdot \frac{v_j}{|v|} \ge 0 \Leftrightarrow f_i(a) \le f_i(a+v).$$

The second part of the proof is completed by noticing that eq. (22) is equivalent to the fact that function f_i is a strictly increasing function with respect to x_i, which in turn is equivalent to the requirement that the partial derivative of f_i with respect to x_i can be zero at most in a set S of measure zero in $\mathcal{X}$, cfr. [47, Section I.1]. ■

V. K-SUBTOPICAL MULTI-AGENT SYSTEMS

We consider the consensus problem in Multi-Agent Systems (MASs) composed of $n \in \mathbb{N}$ agents modeled as dynamical systems with scalar state $x_i \in \mathbb{R}$, whose pattern of interaction is described by a directed graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, and evolve either in discrete-time,

$$x_i(k+1) = f_i\left(x_i(k), x_j(k) : \mathcal{N}_i\right), \qquad k \in \mathbb{Z}. \tag{26}$$

or in continuous-time,

$$\dot{x}_i(t) = f_i\left(x_i(t), x_j(t) : \mathcal{N}_i\right), \qquad t \in \mathbb{R}, \tag{27}$$

We first state our main result for discrete-time MASs in Theorem 6, which provides necessary and sufficient conditions on the local interaction rule f_i of the single agent ensuring that the overall MAS is a K-subtopical dynamical system. Due to Theorem 1, K-subtopicality implies stability of the system and convergence toward the equilibrium point set $\mathcal{F} \ne \emptyset$ of its state trajectories, while accounting for heterogeneous local interaction rules. Moreover, we provide some extra sufficient conditions ensuring that the equilibrium point set $\mathcal{F}$ coincides with the so-called consensus set

$$\mathcal{C} = \{\alpha \mathbf{1} : \alpha \in \mathbb{R}\}, \tag{28}$$

thus solving the consensus problem for nonlinear K-subtopical MASs. The proposed sufficient condition is graph theoretical and based on the graph $\mathcal{G}$ describing the pattern of interconnections among the agents: it requires that there exists a globally reachable node in $\mathcal{G}$ and that consensus states are equilibrium points.

We also provide the continuous-time counterpart of Theorem 6 in Corollary 1. These results are particularly interesting from a control perspective when addressing the problem of steering a MAS toward specific equilibrium points, by only relying on partial and relative information, without the intervention of a central controller, as in the case of formation control [48] or distributed optimization [49].

Theorem 6 (Discrete-time MAS). *Consider a discrete-time MAS as in eq.* (26) *on* $\mathcal{X} \subseteq \mathbb{R}^n$ *whose map is continuously differentiable. If the set of local interaction rules* $f_i : \mathcal{X} \to \mathbb{R}$, *with* $i = 1, \dots, n$, *satisfies the next conditions:*

(i) $\partial f_i / \partial x_i > 0$ *and* $\partial f_i / \partial x_j \ge 0$ *a.e. for* $i \ne j$*;*

(ii) $f_i(x + \alpha \mathbf{1}) \le f_i(x) + \alpha$ *for any* $\alpha \in \mathbb{R}_{\ge 0}$*;*

then the MAS converges asymptotically to one of its equilibrium points, if any, for any initial state $x(0) \in \mathcal{X}$.

If it further satisfies

(iii) $f_i(x) = x_i$ *if* $x_i = x_j$ *for all* $j \in \mathcal{N}_i$*;*

(iv) *The graph* $\mathcal{G}$ *has a globally reachable node;*

then the MAS converges asymptotically to a consensus state for any initial state $x(0) \in \mathcal{X}$.

Proof: The MAS is a K-topical system: condition (i) implies type-K monotonicity by Theorem 5 and condition (ii) implies plus-subhomogeneity, as underlined in Remark 4. When the system has at least one equilibrium point, we can exploit the result in Theorem 1 to establish that for any initial conditions $x(0) \in \mathcal{X}$, the state trajectories of the MAS converge to one of its equilibrium points in $\mathcal{F}$, completing the first part of the proof.

Condition (iii) implies that the consensus space contains only equilibrium points, i.e., $\mathcal{C} \subseteq \mathcal{F}$. Now, we are going to prove that condition (iv) further implies that there are no other equilibrium points, i.e., $\mathcal{C} \equiv \mathcal{F}$. The graph $\mathcal{G}$ is aperiodic due to condition (i) which ensures the presence of a self-loop at each node, and it contains a globally reachable node due to condition (iv). Since the Jacobian matrix J_f is row-stochastic at any consensus point $c\mathbf{1}$ with $c \in \mathbb{R}$, indeed, by means of the definition of directional derivative we can derive

$$\begin{aligned} J_f(c\mathbf{1})\mathbf{1} &= \lim_{h\to 0} \frac{f(c\mathbf{1} + h\mathbf{1}) - f(c\mathbf{1})}{h} \\ &= \lim_{h\to 0} \frac{c\mathbf{1} + h\mathbf{1} - c\mathbf{1}}{h} = \lim_{h\to 0} \frac{h\mathbf{1}}{h} = \mathbf{1}, \end{aligned}$$

we can exploit the widely known Theorem 5.1 in [9] and conclude that $J_f(c\mathbf{1})$ has a simple unitary eigenvalue $\lambda = 1$ with corresponding eigenvector equal to $v = \mathbf{1}$, unique up to a scaling factor.

Since K-topical systems are nonexpansive by Lemma 1, then the set of equilibrium points $\mathcal{F}$ is either empty or closed and convex by [50, Theorem 1]. Now, if there exists $x_e \in \mathcal{F} \setminus \mathcal{C}$, then $\mathcal{F}$ is not empty and all points $c\mathbf{1} + hx_e$ with $h \in [0, 1]$ and $c \in \mathbb{R}$ are also equilibrium points, and thus

$$\begin{aligned} J_f(c\mathbf{1})x_e &= \lim_{h\to 0} \frac{f(c\mathbf{1} + hx_e) - f(c\mathbf{1})}{h} \\ &= \lim_{h\to 0} \frac{c\mathbf{1} + hx_e - c\mathbf{1}}{h} = \lim_{h\to 0} \frac{hx_e}{h} = x_e, \end{aligned}$$

This means that x_e is a second eigenvector (other than $v = \mathbf{1}$) of the unitary eigenvalue $\lambda = 1$ of Jacobian matrix $J(c\mathbf{1})$, which is a contradiction with respect to [9, Theorem 5.1]. Therefore, $\mathcal{F} \equiv \mathcal{C}$, completing the proof. Therefore, there does not exist any point $x_e \ne c\mathbf{1}$, completing the second part of the proof. ■

By means of Lemma 3, one can generalize the previous result to continuous-time MASs, which is made explicit in the next corollary of Theorem 6.

This article has been accepted for publication in IEEE Transactions on Automatic Control. This is the author's version which has not been fully edited and content may change prior to final publication. Citation information: DOI 10.1109/TAC.2023.3246419

Corollary 1 (Continuous-time MAS). *Consider a continuous-time MAS (27) on $\mathcal{X} \subseteq \mathbb{R}^n$ whose vector field is continuously differentiable. If the set of local interaction rules $f_i : \mathcal{X} \to \mathbb{R}$, with $i = 1, \ldots, n$, satisfies the next conditions:*

(i) $\partial f_i / \partial x_j \geq 0$ for $i \neq j$;

(ii) $f_i(x + \alpha \mathbf{1}) \leq f_i(x)$ for any $\alpha \in \mathbb{R}_{\geq 0}$;

then the MAS converges asymptotically to one of its equilibrium points, if any, for any initial state $x(0) \in \mathcal{X}$.

If it further satisfies

(iii) $f_i(x) = 0$ if $x_i = x_j$ for all $j \in \mathcal{N}_i$;

(iv) The graph $\mathcal{G}$ has a globally reachable node;

then the MAS converges asymptotically to a consensus state for any initial state $x(0) \in \mathcal{X}$.

Corollary 1 represents a special case of Theorem 12 in [29], where Manfredi and Angeli consider monotone continuous-time system without the plus-subhomogeneity property. On the other hand, it is interesting to notice that all the examples discussed in [29] satisfy the plus-homogeneity property, thus suggesting that it is a natural property to work with in the context of MASs. Moreover, Theorem 6 constitutes the most general result in the current literature for discrete-time MASs, and it could be further generalized by relaxing the plus-homogeneity property in (ii), while keeping the type-K monotonicity property in (i) that seems to be key property to generalize results in [29] to the discrete-time framework.

VI. Examples of Application

A. Dynamical systems in continuous-time: chemical reactions

An emblematic case of K-topical systems in continuous-time is that of well-mixed and isothermal chemical reactions [3], [4], [26].

Let $s(t) \in \mathbb{R}^n$ denote the vector specifying the concentrations of n chemical species, $h : \mathbb{R}^n_{\geq 0} \to \mathbb{R}^m$ be a function which provides the vector of m reaction rates for any given vector of concentrations, $\Gamma \in \mathbb{R}^{n \times m}$ be the stoichiometry matrix, then the dynamics of the system is given by

$$\dot{s}(t) = \Gamma h(s(t)).$$

Using the reaction coordinates $x(t)$ such that $s(t) = \Gamma x(t)$, the system dynamics becomes $\Gamma \dot{x}(t) = \Gamma h(\Gamma x(t))$, and one can infer the stability of this system by studying the system

$$\dot{x}(t) = h(\Gamma x(t)).$$

which is plus-homogeneous if Γ has zero row-sums (see Remark 1) and type-K monotone if the Jacobian matrix of function h is Metzler (see Proposition 3), and thus it is K-topical. An example of this kind of systems is given in Example 1 in Section III.

By relaxing the assumption of strong monotonicity to that of type-K monotonicity, one can generalize the results in [3], [4] by means of Theorem 1, proving the convergence of the system's trajectories to an equilibrium point (not necessarily unique) for any initial condition. In other words, type-K monotonicity allows one to study also the Michaelis-Menten type of reactions that occur when some pairs of chemical species do not appear in the reaction.

B. Dynamical systems in discrete-time: max-plus maps

Important examples of K-subtopical systems in discrete-time are those ruled by max-plus maps. Applications of max-plus maps arise in several fields, such as optimal control [51], decentralized estimation [52], discrete event systems [53].

To introduce these maps let $\mathbb{R}_\infty = \mathbb{R} \cup \{-\infty\}$ denote the *max-plus semi-ring* and let $A = \{a_{ij}\}$ be a $n \times n$ matrix with entries from $\mathbb{R}_\infty$ and suppose that for each i there exists j such that $a_{ij} \neq -\infty$, and let $u = [u_1, \ldots, u_2]$ a vector with entries $u_i \in \mathbb{R}_\infty$. A max-plus map $f : \mathbb{R}^n \to \mathbb{R}^n$ is defined by

$$f_i(x) = \max_j \{a_{ij} + x_j, u_i\}, \quad \forall x \in \mathbb{R}^n, i = 1, \ldots, n.$$

It is easy to verify that discrete-time max-plus systems are monotone and plus-subhomogeneous, hence subtopical.

Remark 5. *A smooth version of the max-function that not only preserves monotonicity, but forces type-K monotonicity, while not affecting plus-subhomogeneity, thus making the system K-subtopical, can be obtained through the approximation shown next, usually called "softmax" [54],*

$$\alpha\text{-}\max(x) = \frac{\sum_{i=1}^n x_i e^{\alpha x_i}}{\sum_{i=1}^n e^{\alpha x_i}}, \quad \alpha > 0,$$

and we define $\infty\text{-}\max(x) = \max(x)$.

The following example shows how the type-K property prevents the system from oscillating while forcing it to converge to an equilibrium point.

Example 2. *Consider the discrete-time dynamical system with state $x(k) \in \mathbb{R}^2$ and dynamics*

$$\begin{aligned} x_1(k+1) &= \alpha\text{-}\max\{x_1(k) - 2,\ x_2(k), 1\} \\ x_2(k+1) &= \alpha\text{-}\max\{x_2(k) - 2,\ x_1(k), 4\} \end{aligned}. \tag{29}$$

For the initial condition $x(0) = [5, 1]^\top$, Fig. 2 shows the evolution of the system when the max-function (left) and when the softmax-function (right) are employed. It can be noticed that when $\alpha = \infty$ and thus the max-function if employed, then the system is not type-K monotone and the trajectory converges to a periodic point. On the other hand, when α is finite and the softmax-function is used instead of the max-function, the system becomes type-K monotone according to Remark 5, and the trajectory converges to an equilibrium point.

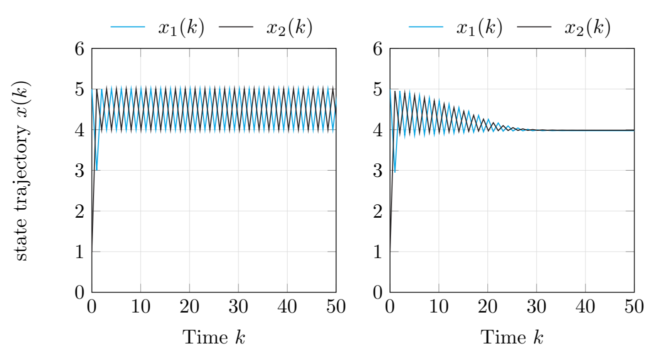

Fig. 2. State trajectories of the system (29): (left) the system is not type-K monotone for $\alpha = \infty$ and (right) it is type-K monotone for $\alpha = 3$.

C. Multi-agent systems

The most common consensus algorithms for discrete and continuous-time single-integrator multi-agent systems

$$\dot{x}_i(t) = u_i, \qquad x_i(k+1) = x_i(k) + \varepsilon_i u_i. \tag{30}$$

with $\varepsilon > 0$ are given by the following control input

$$u_i = \sum_{j\in\mathcal{N}_i} (x_j - x_i)\,. \tag{31}$$

It can be verified that the standard consensus protocol makes the system K-topical, thus known results with linear interactions constitute a special case of the theory developed in this paper. Indeed, the translation-invariance property is a common assumption in MASs when there is not a common reference point for all the agents, i.e., their interactions only depend on state differences rather than on the explicit values of their states. Many variations of eq. (31) have been proposed in several applications, such as formation control in multi-vehicle systems [55], the modeling of the emergent flocking behavior [56], optimization algorithms [57], and many others.

It is remarkable that K-topicality is preserved if one considers nonlinearities of the following type [58][6],

$$u_i = \sum_{j\in\mathcal{N}_i} h_{ij}\left(x_j - x_i\right), \tag{32}$$

under some mild conditions discussed next. We point out that the generality of our approach allows the local interaction rule of the agents to be different from the others, thus enabling the study of heterogeneous multi-agent systems, which is still today a topic of great interest in our community [20], [60], [61].

(*Continuous-time*) It has been proved that the system in eq. (30) with the linear protocol in eq. (31) converges to a consensus state if the graph $\mathcal{G}$ possesses a globally reachable node [9, Theorem 7.4]. By means of Theorem 1 we directly generalize this result by considering the nonlinear protocol in eq. (32) couplings $h_{ij} : \mathbb{R} \to \mathbb{R}$ satisfying

- $\dfrac{\partial}{\partial x_j} h_{ij} \geq 0$ for all $j \neq i$ and $i \in \mathcal{V}$;
- $h_{ij}(0) = 0$ for all $i, j \in \mathcal{V}$.

A similar result is given in [62], where in addition the vector field of the global system is required to meet an extra strict sub-tangentiality condition. It is clear that if the maps are taken as the identity map $h_{ij}(x) = x$, then protocol reduces to the linear one in eq. (31).

(*Discrete-time*) The convergence properties of the system in eq. (30) with the linear protocol in eq. (31) depend on the parameter ε and the topological structure of $\mathcal{G}$ [9, Theorem 5.1]. In particular, the system reaches consensus if the graph possesses a globally reachable node belonging to an aperiodic component, and if $\varepsilon_i < \left|\mathcal{N}_i^{-1}\right|$. The condition on ε ensures that the state transition matrix is row-stochastic and nonnegative. In a similar way, one can find a condition on ε ensuring that the map f given the nonlinear protocol in eq. (32) is plus-homogeneous and type-K monotone, given by

$$\varepsilon_i < \left[|\mathcal{N}_i| \frac{\partial}{\partial x_i} h_{ij}\right]^{-1}.$$

Such property, jointly with the two presented in the previous paragraph, allows to exploit Theorem 6 and prove convergence to a consensus state of the system.

Bounded control inputs. As the first example of application, consider the case wherein the control inputs are constrained by a saturating effect [63]–[65]. The problem of designing proper saturating functions h_{ij} in eq. (32) such that the consensus protocols are yet qualifiable can be solved by the use of the following function

$$h_{ij}(x) = s_i\left(\frac{1 - e^{-m_i x}}{1 + e^{-m_i x}}\right), \quad \forall j \in \mathcal{N}_i$$

with $s_i, m_i > 0$, which is easily proved to be K-topical[7].

Notably, the proposed function encompasses several well-known saturating functions:

- $h_{ij}(x) = \tanh(x)$ if $s_i = 1$ and $m = 2$;
- $h_{ij}(x) = \operatorname{sign}(x)$ if $s_i = 1$ and $m \to \infty$;

Theorem 6 and Corollary 1 ensures that a multi-agent system wherein the agents are subject to the above described saturated control action achieves consensus if the underlying graph contains a globally reachable node.

Oscillator Networks. The emergence of synchronization or desynchronization in networks of coupled oscillators is another interesting example [66], [67]. Here, we consider a network of oscillators with the same natural frequency whose angular velocities are coupled through their phase differences according to a graph $\mathcal{G}$ and coupling functions h_{ij}. Weakly-coupled identical limit-cycle oscillators can be well approximated by this canonical model through a phase reduction and averaging analysis, with appropriate coupling functions h_{ij} that are closely related to the phase response curve of the oscillators. Since the phase response curve is a function computed on the periodic limit cycle, it is 2π-periodic and so are the coupling functions h_{ij}.

Theorem 1 constitutes a new analysis tool for studying synchronization in such networks, where the couplings can be directed and heterogeneous, while they must met the next condition,

$$\frac{d}{d\theta} h_{ij}(\theta) = \begin{cases} > 0 & \theta \in (-\alpha, \alpha) \\ < 0 & \theta \in (-\pi, -\alpha) \cup (\alpha, \pi) \end{cases}, \tag{33}$$

with $\alpha \in [0, \pi]$ and $h_{ij}(0) = 0$. It can be noticed that letting a, b be any real numbers such that $0 \leq b - a \leq \alpha$, then Theorem 1 holds for $\mathcal{X} = [a, b]^n \subset \mathbb{R}^n$. In fact, $\mathcal{X}$ is an invariant space wherein all conditions of the theorem are satisfied if the graph is also assumed to contain a globally reachable node.

[6]Similar results hold also if the nonlinearity is applied after the summation is operated, $u_i = h_i\left(\sum_{j\in\mathcal{N}_i}(x_j - x_i)\right)$, [59].

[7]Note that for the discrete-time case it is further required that $\varepsilon_i < [0.5 \cdot m_i \cdot s_i|\mathcal{N}_i|]^{-1}$.

This article has been accepted for publication in IEEE Transactions on Automatic Control. This is the author's version which has not been fully edited and content may change prior to final publication. Citation information: DOI 10.1109/TAC.2023.3246419

VII. Future directions and concluding remarks

In this work, we have introduced the property of type-K monotonicity, which is weaker than strong monotonicity and stronger than standard monotonicity. We have shown that this property comes for free in continuously differentiable continuous-time monotone systems (see Theorem 3), but this is not the case in discrete-time (See Theorem 4). Moreover, we have shown that type-K monotonicity can be verified by the sign-structure of the Jacobian matrix (see Proposition 3 and Theorem 5). In our opinion, this suggests that many of the generalizations for monotone systems may potentially be considered for type-K monotonicity instead, which is a future direction that the authors will investigate. For instance, we believe that it could be possible to generalize the notion of type-K monotonicity to general orderings (other than the usual one induced by the positive orthant considered in this manuscript), thus allowing for competitive interactions among the state variables other than only cooperative effects, as in the case of mixed-monotone systems which have recently attracted much attention [68]–[71].

This manuscript provides a self-contained analysis of smooth type-K monotone dynamical systems with the additional plus-subhomogeneity property. These systems, which we have called them K-subtopical systems, have been proved to have very nice behavior, avoiding periodic trajectories and eventually converging to equilibrium points, if any exists (see Theorem 1). These results provide a generalization of the convergence result presented by Angeli and Sontag in [3] for monotone and plus-homogeneous systems in continuous-time, whose trajectories have been shown to converge to a unique equilibrium point.

We have also investigated the application of these results in the context of multi-agent systems (MASs) for this class: K-subtopicality is often a direct consequence of local interaction rules of the agents. Moreover, standard connectivity conditions on the interaction graph have been proved to be sufficient to solve the consensus problem in nonlinear K-subtopical MASs (see Theorem 6 and Corollary 1). Thus, this manuscript paves the way to a variety of lines of research in the context of multi-agent systems which will retrace those investigated for standard linear consensus.

Acknowledgments

We thank an anonymous Reviewer for the insightful comments and for the contribution to the technical development of the proof of Theorem 1.

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