A 16 Moments Model in Relativistic Extended Thermodynamics of Rarefied Polyatomic Gas

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Abstract

A model with 16 moments is here presented in the framework of RET of polyatomic gases. It furnishes as principal subsystem the relativistic counterpart of a work by Arima T., Ruggeri T., Sugiyama M.; this is present in literature and treats the non relativistic case which incorporates relaxation processes of molecular rotation and vibration. Another principal subsystem is the natural extension of the 14 moments model by Proffs. Pennisi S. and Ruggeri T.; this is also present in literature in the relativistic framework but where the trace of the third balance equation is neglected. Its extension is found here for the case when this trace isn't neglected.
Keyword:Relativistic Extended Thermodynamics, Polyatomic gases.

1 Introduction

We aim to discuss here the following set of balance equations for the description of relativistic polyatomic gases:

$$\partial_\alpha V^\alpha = 0\,, \quad \partial_\alpha T^{\alpha\beta} = 0\,, \quad \partial_\alpha A^{\alpha\beta\gamma} = P^{\beta\gamma}\,, \quad \partial_\alpha H^\alpha_V = P\,. \tag{1}$$

In [1], the authors considered only the first two of these equations and the traceless part of $(1)_3$, i.e., $\partial_\alpha A^{\alpha<\beta\gamma>} = I^{<\beta\gamma>}$; the reason behind this choice was that they wanted to find, in the non relativistic limit and in the monoatomic limit, the results of the 14 moments models of the articles [2]-[6]. Moreover, in [1] the case of polyatomic gases was considered when only one microscopic energy of internal modes is present. Still remaining in this framework, the article [1] can be extended by considering all the components of the triple tensor $A^{\alpha\beta\gamma}$ including the trace that was messing in [1]. So we have an equation more, i.e. 15 moments. The opportunity of this extension is evident from the article [7] where the case of an arbitrary but fixed number of moments is considered but only to discover an optimal choice of moments. So it is useful to have a confirmation of the results obtained in [7] for the simpler case of 15 moments. This purpose is realized in the present article.
A second purpose we want to achieve here is to give the relativistic counterpart of the work

[8]; here the authors consider two microscopic energies of internal modes because they want describe the relaxation processes of rotational and vibrational modes separately. To reach this end they decompose the energy of internal mode and the energy of vibrational mode. For this reason they have an equation more with respect to the article [2], i.e., 15 moments also in this case. It is natural to wonder if this additional moment is not the same as that obtained above by considering all the components of the triple tensor $A^{\alpha\beta\gamma}$. The answer is negative because its expression at the non-relativistic limit is completely different, as it can be seen below in eq. $(3)_7$. So we need one additional equation. In such a way one arrives at a thery of 16 moments. The new equation must be a scalar one, otherwise the total system will have more than 16 moments. We have reported it in eq. $(1)_4$. It can be justified as the other equations starting from the kinetic theory. In fact, the relativistic counterpart of the Boltzmann equation (1) of [8] is $p^\alpha \partial_\alpha f = Q$ where the distribution function f depends on the position x^α, momentum p^α, energy of rotational mode $\mathcal{I}^R$, energy of vibrational mode $\mathcal{I}^V$. If we multiply it by $m\left(1+\frac{2\mathcal{I}^V}{m\,c^2}\right)\varphi(\mathcal{I}^\mathcal{R})\,\psi(\mathcal{I}^\mathcal{V})$ and integrate in $d\mathcal{I}^\mathcal{R}\,d\mathcal{I}^\mathcal{V}\,d\vec{P}$, then we obtain the present eq. $(1)_4$. We can see that it is appropriate because at the non-relativistic limit it gives just eq. $(3)_5$ of [8]. Its generalization to the case of many moments is straightforward but we don't report it here for the sake of simplicity; the interested reader can request it and we will send it to him.
So the present model reaches 2 purposes:

- One is to find the relativistic counterpart of [8], which isn't present in literature. This is realized by putting equal to zero the Lagrange multiplier corresponding to the trace of $(1)_3$; in other words, we consider the subsystem of eqs. (1) according to the definition of [9].
- Another one is to see what happens if we don' t drop the trace of $(1)_3$, i.e., the subsystem of eqs. (1) obtained by putting equal to zero the Lagrange multiplier corresponding to $(1)_4$.

The field equations are expressed in terms of the tensors

$$V^\alpha = mc\int_{\Re^3}\int_0^{+\infty}\int_0^{+\infty} f p^\alpha \varphi(\mathcal{I}^\mathcal{R})\,\psi(\mathcal{I}^\mathcal{V})\,d\mathcal{I}^\mathcal{R}\,d\mathcal{I}^\mathcal{V}\,d\vec{P},$$
$$T^{\alpha\beta} = c\int_{\Re^3}\int_0^{+\infty}\int_0^{+\infty} f p^\alpha p^\beta \left(1+\frac{\mathcal{I}}{mc^2}\right)\varphi(\mathcal{I}^\mathcal{R})\,\psi(\mathcal{I}^\mathcal{V})\,d\mathcal{I}^\mathcal{R}\,d\mathcal{I}^\mathcal{V}\,d\vec{P}\,, \tag{2}$$

$$A^{\alpha\beta\gamma} = \frac{c}{m}\int_{\Re^3}\int_0^{+\infty}\int_0^{+\infty} f p^\alpha p^\beta p^\gamma \left(1+\frac{2\,\mathcal{I}}{mc^2}\right)\varphi(\mathcal{I}^\mathcal{R})\,\psi(\mathcal{I}^\mathcal{V})\,d\mathcal{I}^\mathcal{R}\,d\mathcal{I}^\mathcal{V}\,d\vec{P}\,,$$
$$H_V^\alpha = m\int_{\Re^3}\int_0^{+\infty}\int_0^{+\infty} f p^\alpha \left(1+\frac{2\,\mathcal{I}^V}{m\,c^2}\right)\varphi(\mathcal{I}^\mathcal{R})\,\psi(\mathcal{I}^\mathcal{V})\,d\mathcal{I}^\mathcal{R}\,d\mathcal{I}^\mathcal{V}\,d\vec{P}\,,$$

where $d\vec{P} = \frac{dp^1\,dp^2\,dp^3}{p^0}$ and $\mathcal{I} = \mathcal{I}^\mathcal{R} + \mathcal{I}^\mathcal{V}$.
In the next section we will calculate the non relativistic limit of the full set of eqs. (1), finding a 16 moments model for classical extended thermodynamics of polyatomic gases. It encloses

two important subsystems in the sense of [9]: The natural extension of [1] which is obtained neglecting eq. $(1)_4$ and the model [8] which comes out by neglecting the trace of eq. $(1)_3$. In sect. 3 we will impose the Maximum Entropy Principle for these field equations and compare the results with those of [1]. The resulting system is hyperbolic for every timelike congruence and the characteristic velocities dont't exceed the speed of light, We have already proved these properties and we aim to publish these results in a subsequent article.

2 The non relativistic limit

If we compare eqs. $(2)_{1-3}$ with those of [1], we see that they are formally the same, except to substitute $\mathcal{I}$ with $\mathcal{I}^R + \mathcal{I}^V$, $\phi(\mathcal{I})$ with $\varphi(\mathcal{I}^R)\,\psi(\mathcal{I}^V)$ and to integrate in $d\mathcal{I}^R\,d\mathcal{I}^V$ instead of $d\mathcal{I}$. Now the passages used in [1], to obtain the non relativistic limit of its field equations, aren't affected by these changes; so we can say that the non relativistic limit of eqs. $(1)_{1,2}$ and of the traceless part of $(1)_3$, give exactly the eqs. in the first and second block of the following set

$$
\begin{aligned}
&\partial_t\, F + \partial_k\, F^k = 0\\
&\partial_t\, F^i + \partial_k\, F^{ki} = 0\\
&\partial_t\, F^{ij} + \partial_k\, F^{ki} = P_F^{ij} \qquad \partial_t\, G^{ll} + \partial_k\, G^{kll} = 0 \qquad \partial_t\, H_V^{ll} + \partial_k\, H_V^{kll} = P_V^{ij}\,,\\
&\qquad\qquad \partial_t\, G^{ill} + \partial_k\, G^{kill} = Q^{ill}\\
&\qquad \partial_t\,\left(2G^{pppp} - F^{pppp}\right) + \partial_i\,\left(2G^{ppppi} - F^{ppppi}\right) = I^{pppp}\,,
\end{aligned} \tag{3}
$$

with

$$
F^A = \int_{\Re^3}\int_0^{+\infty}\int_0^{+\infty} m\, f\, \xi^A\, d\tilde{\mathcal{I}}\, d\vec{\xi},\; G^{Bll} = \int_{\Re^3}\int_0^{+\infty}\int_0^{+\infty} m\, f\, \xi^B\left(\xi^2 + 2\,\frac{\mathcal{I}}{m}\right) d\tilde{\mathcal{I}}\, d\vec{\xi},
$$

$$
H_V^{Cll} = \int_{\Re^3}\int_0^{+\infty}\int_0^{+\infty} f\, \xi^C\; 2\,\mathcal{I}^V\, d\tilde{\mathcal{I}}\, d\vec{\xi},
$$

and $d\tilde{\mathcal{I}}$ is an abbreviation for $\phi(\mathcal{I}^R)\,\psi(\mathcal{I}^V)\,d\mathcal{I}^R\,d\mathcal{I}^V$. They are also the equations of the 14 moments model [2].

We proceed now to calculate the non relativistic limit of the new component of eq. $(1)_3$ which is now present because we don' t take its traceless part. For the sake of simplicity, let us maintain the notation of [1] reserving to reconvert the results at the end by distinguishing the contribution of the two energies. To this end, let us recall that in pages 420, 421 of [1] we take into account that $x^0 = c\,t$, $p^0 = m\,\Gamma\,c$, $p^i = m\,\Gamma\,\xi^i$,
$\Gamma = \left(1 - \frac{\xi^2}{c^2}\right)^{-\frac{1}{2}}$, $\lim_{c\to+\infty} f = \frac{1}{m^3}\, f^C$, $d\,\vec{P} = \frac{m^2\Gamma^4}{c}\, d\,\vec{\xi}$.

Ater that we took the 3-dimensional components of V^α, $T^{\alpha\beta}$, $A^{\alpha\beta\gamma}$ except for A^{000} which we now evaluate here and is

$$
A^{000} = \frac{c}{m}\int_{\Re^3}\int_0^{+\infty}\int_0^{+\infty} f\, m^5\,\Gamma^7\, c^2\left(1 + \frac{2\,\mathcal{I}}{m\,c^2}\right) d\vec{\xi}\, d\tilde{\mathcal{I}}\,. \tag{4}
$$

In page 420 of [1] we noted also that

$$\frac{1}{c^2}\left(T^{00}-c\,V^0\right)=\int_{\Re^3}\int_0^{+\infty}\int_0^{+\infty} f\,m^4\left[\left(1+\frac{\mathcal{I}}{m\,c^2}\right)\Gamma^6-\Gamma^5\right]d\vec{\xi}\,d\tilde{\mathcal{I}}$$

whose non relativistic limit is zero. So we multiplied it times $2c^2$ before taking the limit and find

$$\lim_{c\to+\infty} 2\left(T^{00}-c\,V^0\right)=\int_{\Re^3}\int_0^{+\infty}\int_0^{+\infty} f\,m^4\left(\xi^2+\frac{2\,\mathcal{I}}{m}\right)d\vec{\xi}\,d\tilde{\mathcal{I}}\,. \quad (5)$$

Now we note that also $\frac{1}{c^3}\left(A^{000}-c^2\,V^0\right)$ has limit zero. So we multiply it times c^2 before taking the limit and find $\lim_{c\to+\infty}\frac{1}{c}\left(A^{000}-c^2\,V^0\right)=\int_{\Re^3}\int_0^{+\infty}\int_0^{+\infty} f\,m^4\left(\xi^2+\frac{2\,\mathcal{I}}{m}\right)d\vec{\xi}\,d\tilde{\mathcal{I}}$,

which is the same limit of (5). So we take now the difference of their left hand sides and multiply the result times c^2 obtaining

$$c\,A^{000}+c^3\,V^0-2\,c^2\,T^{00}=$$

$$=\int_{\Re^3}\int_0^{+\infty}\int_0^{+\infty} f\,m^4\,\Gamma^5\left[\left(1+\frac{2\mathcal{I}}{m\,c^2}\right)\Gamma^2\,c^4+c^4-2c^4\,\Gamma\left(1+\frac{\mathcal{I}}{m\,c^2}\right)\right]d\vec{\xi}\,d\tilde{\mathcal{I}} \quad (6)$$

To take the limit of this expression we use the well known Taylor's serie

$$\frac{1}{\sqrt{1+x}}=1+\sum_{n=1}^{+\infty}(-1)^n\frac{(2n-1)!!}{(2n)!!}\,x^n\,,$$

which is convergent for $-1<x<1$. By applying it to the Lorentz's factor, we obtain

$$\Gamma=1+\sum_{n=1}^{+\infty}\frac{(2n-1)!!}{(2n)!!}\,\frac{\xi^{2n}}{c^{2n}}=1+\frac{1}{2}\,\frac{\xi^2}{c^2}+\frac{3}{8}\,\frac{\xi^4}{c^4}+\frac{1}{c^6}\,(\cdots)\,.$$

So we have $c\,A^{000}+c^3\,V^0-2\,c^2\,T^{00}=$

$$\begin{aligned}&=\int_{\Re^3}\int_0^{+\infty}\int_0^{+\infty} f\,m^4\,\Gamma^5\left[\left(1+\frac{2\mathcal{I}}{m\,c^2}\right)\left(1+\frac{1}{4}\,\frac{\xi^4}{c^4}+\frac{\xi^2}{c^2}+\frac{3}{4}\,\frac{\xi^4}{c^4}+\frac{1}{c^6}\,(\cdots)\right)c^4+c^4-\right.\\&\left.2c^4\left(1+\frac{1}{2}\,\frac{\xi^2}{c^2}+\frac{3}{8}\,\frac{\xi^4}{c^4}+\frac{1}{c^6}\,(\cdots)\right)\left(1+\frac{\mathcal{I}}{m\,c^2}\right)\right]d\vec{\xi}\,d\tilde{\mathcal{I}}=\\&=\int_{\Re^3}\int_0^{+\infty}\int_0^{+\infty} f\,m^4\,\Gamma^5\left[\frac{1}{4}\,\xi^4+\frac{1}{c^2}\,(\cdots)+\right.\\&\left.+\frac{\mathcal{I}}{m}\left(\frac{1}{2}\,\frac{\xi^4}{c^2}+2\,\xi^2+\frac{3}{2}\,\frac{\xi^4}{c^2}+\frac{1}{c^4}\,(\cdots)-\xi^2-\frac{3}{4}\,\frac{\xi^4}{c^2}\right)\right]d\vec{\xi}\,d\tilde{\mathcal{I}}\end{aligned} \quad (7)$$

whose non relativistic limit is $\frac{1}{4}\left(2G^{pppp}-F^{pppp}\right)$ with

$$G_{pppp}=\int_{\Re^3}\int_0^{+\infty}\int_0^{+\infty} f\,m\left(\xi^2+\frac{2\,\mathcal{I}}{m}\right)\xi^2\,d\vec{\xi}\,d\tilde{\mathcal{I}}\,,\quad F^{pppp}=\int_{\Re^3}\int_0^{+\infty}\int_0^{+\infty} f\,m\,\xi^4\,d\vec{\xi}\,d\tilde{\mathcal{I}}\,.$$

Similarly, we have $\quad c\left(c\,A^{i00}+c^3\,V^i\;-2\,c^2\,T^{i0}\right)=$

$$= \int_{\Re^3}\int_0^{+\infty}\int_0^{+\infty} f\,m^4\,\Gamma^5\left[\left(1+\frac{2\mathcal{I}}{m\,c^2}\right)\Gamma^2\,c^4+c^4-2c^4\,\Gamma\,\left(1+\frac{\mathcal{I}}{m\,c^2}\right)\right]\xi^i\,d\vec{\xi}\,d\tilde{\mathcal{I}} \tag{8}$$

whose limit for c going to ∞ is $\frac{1}{4}\,(2G^{ppppi}\;-\;F^{ppppi})$ with

$$G^{ppppi}=\int_{\Re^3}\int_0^{+\infty}\int_0^{+\infty} f\,m\,\left(\xi^2+\frac{2\,\mathcal{I}}{m}\right)\xi^2\,\xi^i\,d\vec{\xi}\,d\tilde{\mathcal{I}}\,,\;\; F^{ppppi}=\int_{\Re^3}\int_0^{+\infty}\int_0^{+\infty} f\,m\,\xi^4\;\xi^i\,d\vec{\xi}\,d\tilde{\mathcal{I}}.$$

Consequently, the 15^{th} equation is $(3)_7$ which is an hybrid between the mass block and the energy block in the balance laws. (Note that F^{pppp} and F^{ppppi} are enclosed in the definition $(3)_8$ with $A=pppp$ and $A=ppppi$ respectively; similarly, G^{pppp} and G^{ppppi} are enclosed in the definition $(3)_9$ with $B=pppp$ and $B=ppppi$ respectively).
There remains only to do the non relativistic limit of the equation in the third block, i.e., eq. $(3)_{10}$. With similar passages, we obtain

$$c^2\,H^0_V-c\,V^0=m^4\,\Gamma^5\int_{\Re^3}\int_0^{+\infty}\int_0^{+\infty} f\,2\,\mathcal{I}^V\,d\tilde{\mathcal{I}}\,d\vec{\xi}\,,$$
$$c^3\,H^k_V-c^2\,V^k=m^4\,\Gamma^5\int_{\Re^3}\int_0^{+\infty}\int_0^{+\infty} f\,\xi^k\,2\,\mathcal{I}^V\,d\tilde{\mathcal{I}}\,d\vec{\xi}\,,$$

and the non relativistic limits of the right hand sides are H^{ll}_V and H^{kll}_V, respectively. In other words, eq. $(1)_4$ multiplied by c^3 minus $(1)_1$ multiplied by c^2 give an equation whose non relativistic limit is $(3)_5$.
We see now that eqs. $(3)_{1-6}$ are those of [8] (To be true, instead of $(3)_4$ they have $\partial_t\,H^{ll}_R+\partial_k\,H^{kll}_R=P^{ij}_R$ with $H^{ll}_R=G^{ll}-F^{ll}-H^{ll}_V\quad,\,H^{kll}_R=G^{kll}-F^{kll}-H^{kll}_V\quad,\,P^{ll}_R=-P^{ll}_F-P^{ll}_V\quad.$
In our result we have obtained the sum of this equation, of eq. $(3)_5$ and of the trace of $(3)_3$) and this is an equivalent system.
It is obvious that the system (3) satisfies the Galilean Relativity Principle because it was obtained from a relativistic version which satisfies the Einsteinian Relativity Principle thanks to its covariant form [10]. We have also found a direct proof of this property.
We note that, in the limiting case of monoatomic gases, $(3)_5$ disappears, while $(3)_4$ becomes equal to the trace of $(3)_3$, the system $(3)_{1-3,6}$ gives the 13 moments model [6], while the system $(3)_{1-3,6,7}$ gives the 14 moments model by Kremer [11]. So, eq. $(3)_7$ isn't completely new and we need it to catch the previous article as a limiting case.
Obviously, this methodology can be extended to enclose an arbitrary but fixed number of moments in the relativistic version; this idea has been considered in [7] finding an interesting hyerarchy of moments also for the classical case; the present article is an intermediate easier step but, in any case, it is more complete because it encloses the closure of the balance equations, which is the result of the next section.

3 The closure of the balance equations

We note that many considerations in [1] don' t take into account that the authors will take subsequently only the traceless part of $(1)_3$. So those results hold also in the present case.

For example, they impose the Maximum Entropy Principle and find a distribution function, in eq. (31) of that article, which we can adapt also for the present more general case. It reads

$$f = e^{-1-\frac{\chi}{k_B}} \quad , \quad \text{with} \tag{9}$$
$$\chi = m\,\lambda + \left(1 + \frac{\mathcal{I}}{m\,c^2}\right) \lambda_\beta\, p^\beta + \frac{1}{m} \left(1 + \frac{2\,\mathcal{I}}{m\,c^2}\right) \lambda_{\alpha_1\alpha_2} p^{\alpha_1} p^{\alpha_2} + \frac{m}{c} \left(1 + \frac{2\,\mathcal{I}^V}{m\,c^2}\right) \nu\,,$$

where k_B is the Boltzmann constant and the last term has been enclosed for taking into account eq. $(1)_4$. Moreover, λ, λ_β, $\lambda_{\alpha_1\alpha_2}$ and ν are the Lagrange multipliers.
From this distribution function we can desume the 4-potential defined as

$$h'^\alpha = -\, k_B\, c \int_{\Re^3} \int_0^{+\infty} \int_0^{+\infty} e^{-1-\frac{\chi}{k_B}} p^\alpha d\,\tilde{\mathcal{I}}\, d\vec{P}\,, \tag{10}$$

from which it follows that the fields in eqs. (1) can be written simply as

$$V^\alpha = \frac{\partial\, h'^\alpha}{\partial\, \lambda}\,, \quad T^{\alpha\beta} = \frac{\partial\, h'^\alpha}{\partial\, \lambda_\beta}\,, \quad A^{\alpha\beta\gamma} = \frac{\partial\, h'^\alpha}{\partial\, \lambda_{\beta\gamma}}\,, \quad H^\alpha_V = \frac{\partial\, h'^\alpha}{\partial\, \nu}\,. \tag{11}$$

Moreover, equilibrium is defined as the state where $\lambda_{\alpha\beta} = 0$, $\nu = 0$ and eqs. (11) calculated at equilibrium become

$$m\, n\, U^\alpha = \frac{\partial\, h'^\alpha_E}{\partial\, \lambda}\,, \quad p\, h^{\alpha\beta} + \frac{e}{c^2}\, U^\alpha U^\beta = \frac{\partial\, h'^\alpha_E}{\partial\, \lambda_\beta}\,, \quad A^{\alpha\beta\gamma}_E = \left(\frac{\partial\, h'^\alpha}{\partial\, \lambda_{\beta\gamma}}\right)_E\,, \quad H^\alpha_{VE} = \left(\frac{\partial\, h'^\alpha}{\partial\, \nu}\right)_E\,. \tag{12}$$

The first three of these equations have ben exploited in [1] and those considerations still hold also in the present case, so that we can report here simply the modified results, where an overlined term denotes that this term is multiplied by $\phi(\mathcal{I}^R)\, \psi(\mathcal{I}^V)$ and, after that integrated in $d\mathcal{I}^R\, d\mathcal{I}^V$ for $\mathcal{I}^R \in [0\,, +\infty[$, $\mathcal{I}^V \in [0\,, +\infty[$. We will use this notation also in the subsequent part of the article. The results are:

$$e^{1+\frac{m}{k_B}\lambda_E} = \frac{4\,\pi\, m^3\, c^3}{n}\, \overline{J_{2,1}(\gamma^*)} \quad \text{(which gives } \lambda_E) \quad \text{with} \quad \gamma^* = \left(1 + \frac{\mathcal{I}}{m\, c^2}\right) \gamma\,,$$
$$\lambda_{E\beta} = \frac{k_B\, \gamma}{m\, c^2}\, U_\beta\,,\ p = \frac{n\, m\, c^2}{\gamma}\,,\ e = n\, m\, c^2\, \gamma\, \frac{\overline{J_{2,2}(\gamma^*)\, \left(1 + \frac{\mathcal{I}}{m\, c^2}\right)}}{\overline{J_{2,1}(\gamma^*)}}\,,$$
$$A^{\alpha\beta\gamma}_E = A^0_1\, U^\alpha U^\beta U^\gamma + 3\, A^0_{11}\, h^{(\alpha\beta} U^{\gamma)}\,, \tag{13}$$
$$A^0_1 = n\, m\, \gamma\, \frac{\overline{J_{2,3}(\gamma^*)\, \left(1 + \frac{2\,\mathcal{I}}{m\, c^2}\right)}}{\overline{J_{2,1}(\gamma^*)}}\,, \quad A^0_{11} = \frac{n\, m\, c^2 \gamma}{3}\, \frac{\overline{J_{4,1}(\gamma^*)\, \left(1 + \frac{2\,\mathcal{I}}{m\, c^2}\right)}}{\overline{J_{2,1}(\gamma^*)}}\,,$$

while $(12)_4$ gives $\quad H^\alpha_{VE} = H_V\, U^\alpha$ with $H_V = \dfrac{m\, n}{c^3}\, \dfrac{\overline{J_{2,1}(\gamma^*)\, \left(1 + \frac{2\,\mathcal{I}^V}{c^2}\right)}}{\overline{J_{2,1}(\gamma^*)}}$.

3.1 The first order deviation from equilibrium

The first order deviation of eqs. (11) from their equilibrium values is

$$
\begin{aligned}
& V_E^{\alpha}(\lambda-\lambda_E) + T_E^{\alpha\mu}\,(\lambda_\mu-\lambda_{E\mu}) + A_E^{\alpha\mu\nu}\lambda_{\mu\nu} + H_{VE}^{\alpha}\,\nu = 0\,, \\
& T_E^{\alpha\beta}(\lambda-\lambda_E) + m\,A_{11}^{\alpha\beta\mu}\,(\lambda_\mu-\lambda_{E\mu}) + m\,A_{12}^{\alpha\beta\mu\nu}\lambda_{\mu\nu} + T_V^{\alpha\beta}\,\nu = \\
& \qquad = -\frac{k_B}{m}\left(t^{<\alpha\beta>_3} + \pi h^{\alpha\beta} + \frac{2}{c^2}\,U^{(\alpha}q^{\beta)}\right)\,, \\
& A_E^{\alpha\beta\gamma}(\lambda-\lambda_E) + m\,A_{12}^{\alpha\beta\gamma\nu}\,(\lambda_\mu-\lambda_{E\mu}) + m\,A_{22}^{\alpha\beta\gamma\mu\nu}\lambda_{\mu\nu} + A_V^{\alpha\beta\gamma}\,\nu = -\frac{k_B}{m}(A^{\alpha\beta\gamma}-A_E^{\alpha\beta\gamma})\,, \\
& H_{VE}^{\alpha}(\lambda-\lambda_E) + T_V^{\alpha\mu}\,(\lambda_\mu-\lambda_{E\mu}) + A_V^{\alpha\mu\nu}\lambda_{\mu\nu} + V_{VV}^{\alpha}\,\nu = -\frac{k_B}{m}(H_V^{\alpha}-H_{VE}^{\alpha})\,,
\end{aligned}
\tag{14}
$$

where the new tensors appear

$$
\begin{aligned}
A_{11}^{\alpha\beta\mu} &= \frac{c}{m^4}\int_{\Re^3}\int_0^{+\infty}\!\!\int_0^{+\infty} f_E p^\alpha p^\beta p^\mu \left(1+\frac{\mathcal{I}}{m\,c^2}\right)^2 d\tilde{\mathcal{I}}\,d\vec{P}\,, \\
A_{12}^{\alpha\beta\mu\nu} &= \frac{c}{m^5}\int_{\Re^3}\int_0^{+\infty}\!\!\int_0^{+\infty} f_E p^\alpha p^\beta p^\mu p^\nu \left(1+\frac{\mathcal{I}}{m\,c^2}\right)\left(1+\frac{2\mathcal{I}}{m\,c^2}\right) d\tilde{\mathcal{I}}\,d\vec{P}\,, \\
A_{22}^{\alpha\beta\gamma\mu\nu} &= \frac{c}{m^6}\int_{\Re^3}\int_0^{+\infty}\!\!\int_0^{+\infty} f_E p^\alpha p^\beta p^\gamma p^\mu p^\nu \left(1+\frac{2\mathcal{I}}{m\,c^2}\right)^2 d\tilde{\mathcal{I}}\,d\vec{P}\,.
\end{aligned}
\tag{15}
$$

$$
\begin{aligned}
T_V^{\alpha\beta} &= \int_{\Re^3}\int_0^{+\infty}\!\!\int_0^{+\infty} f_E p^\alpha p^\beta \left(1+\frac{\mathcal{I}}{m\,c^2}\right)\left(1+\frac{2\mathcal{I}^V}{c^2}\right) d\tilde{\mathcal{I}}\,d\vec{P}\,, \\
A_V^{\alpha\beta\gamma} &= \frac{1}{m}\int_{\Re^3}\int_0^{+\infty}\!\!\int_0^{+\infty} f_E p^\alpha p^\beta p^\gamma \left(1+\frac{2\mathcal{I}}{m\,c^2}\right)\left(1+\frac{2\mathcal{I}^V}{c^2}\right) d\tilde{\mathcal{I}}\,d\vec{P}\,, \\
V_{VV}^{\alpha} &= \frac{m}{c}\int_{\Re^3}\int_0^{+\infty}\!\!\int_0^{+\infty} f_E p^\alpha \left(1+\frac{2\mathcal{I}^V}{c^2}\right)^2 d\tilde{\mathcal{I}}\,d\vec{P}\,.
\end{aligned}
$$

The expressions of the first three of these tensor have been calculated in [1], while the others can be calculated in a similar manner and they are

$$
\begin{aligned}
A_{11}^{\alpha\beta\mu} &= B_4 h^{(\alpha\beta}U^{\mu)} + B_5 U^\alpha U^\beta U^\mu\,, \\
A_{12}^{\alpha\beta\mu\nu} &= \frac{1}{5}B_1 h^{(\alpha\beta}h^{\mu\nu)} + 2B_2 h^{(\alpha\beta}U^\mu U^{\nu)} + B_3 U^\alpha U^\beta U^\mu U^\nu\,, \\
A_{22}^{\alpha\beta\gamma\mu\nu} &= B_6 h^{(\alpha\beta}h^{\gamma\mu}U^{\nu)} + \frac{10}{3}B_7 h^{(\alpha\beta}U^\gamma U^\mu U^{\nu)} + B_8 U^\alpha U^\beta U^\gamma U^\mu U^\nu\,.
\end{aligned}
\tag{16}
$$

$$
T_V^{\alpha\beta} = B_9\frac{U^\alpha U^\beta}{c^2} + B_{10}h^{\alpha\beta}\,,\; A_V^{\alpha\beta\gamma} = A_{1V}^0 U^\alpha U^\beta U^\gamma + 3A_{11V}^0 h^{(\alpha\beta}U^{\gamma)}\,, \quad V_{VV}^{\alpha} = B_{11}U^\alpha\,.
$$

with

$$B_1 = n\gamma c^4 \, \frac{\overline{J_{6,0}(\gamma^*) \left(1+\frac{\mathcal{I}}{mc^2}\right)\left(1+\frac{2\mathcal{I}}{mc^2}\right)}}{\overline{J_{2,1}(\gamma^*)}} \,, \quad B_2 = n\gamma c^2 \, \frac{\overline{J_{4,2}(\gamma^*) \left(1+\frac{\mathcal{I}}{mc^2}\right)\left(1+\frac{2\mathcal{I}}{mc^2}\right)}}{\overline{J_{2,1}(\gamma^*)}} \,,$$

$$B_3 = n\gamma \, \frac{\overline{J_{2,4}(\gamma^*) \left(1+\frac{\mathcal{I}}{mc^2}\right)\left(1+\frac{2\mathcal{I}}{mc^2}\right)}}{\overline{J_{2,1}(\gamma^*)}} \,, \quad B_4 = n\gamma c^2 \, \frac{\overline{J_{4,1}(\gamma^*) \left(1+\frac{\mathcal{I}}{mc^2}\right)^2}}{\overline{J_{2,1}(\gamma^*)}} \,, \tag{17}$$

$$B_5 = n\gamma \, \frac{\overline{J_{2,3}(\gamma^*) \left(1+\frac{\mathcal{I}}{mc^2}\right)^2}}{\overline{J_{2,1}(\gamma^*)}} \,, \quad B_6 = n\gamma c^4 \, \frac{\overline{J_{6,1}(\gamma^*) \left(1+\frac{2\mathcal{I}}{mc^2}\right)^2}}{\overline{J_{2,1}(\gamma^*)}} \,,$$

$$B_7 = n\gamma c^2 \, \frac{\overline{J_{4,3}(\gamma^*) \left(1+\frac{2\mathcal{I}}{mc^2}\right)^2}}{\overline{J_{2,1}(\gamma^*)}} \,, \; B_8 = n\gamma \, \frac{\overline{J_{2,5}(\gamma^*) \left(1+\frac{2\mathcal{I}}{mc^2}\right)^2}}{\overline{J_{2,1}(\gamma^*)}} \,, \quad B_{10} = \frac{c^4}{\gamma}\, H_V \,,$$

$$B_9 = n\, m\, c\, \gamma \, \frac{\overline{J_{2,2}(\gamma^*) \, \left(1+\frac{\mathcal{I}}{m\,c^2}\right)\left(1+\frac{2\mathcal{I}^V}{c^2}\right)}}{\overline{J_{2,1}(\gamma^*)}} \,, \; A^0_{1V} = \frac{n\, m\, \gamma}{c} \, \frac{\overline{J_{2,3}(\gamma^*) \, \left(1+\frac{2\,\mathcal{I}}{m\,c^2}\right)\left(1+\frac{2\,\mathcal{I}^V}{c^2}\right)}}{\overline{J_{2,1}(\gamma^*)}} \,,$$

$$A^0_{11V} = \frac{n\, m\, c \gamma}{3} \, \frac{\overline{J_{4,1}(\gamma^*) \, \left(1\,+\,\frac{2\,\mathcal{I}}{m\,c^2}\right)\left(1\,+\,\frac{2\,\mathcal{I}^V}{c^2}\right)}}{\overline{J_{2,1}(\gamma^*)}} \,, \quad B_{11} = \frac{n\, m}{c^2} \, \frac{\overline{J_{2,1}(\gamma^*) \, \left(1\,+\,\frac{2\,\mathcal{I}^V}{c^2}\right)^2}}{\overline{J_{2,1}(\gamma^*)}} \,,$$

We have now to obtain $(\lambda - \lambda_E)$, $\left(\lambda_\mu - \frac{U_\mu}{T}\right)$, $\Sigma_{\mu\nu}$ from eqs. $(14)_{1,2}$ and substitute them in $(14)_{3,4}$ to obtain the requested closure, that is the expression of $A^{\alpha\beta\gamma} - A_E^{\alpha\beta\gamma}$ and of $H_V^\alpha - H_{VE}^\alpha$. To this end, we define $\Sigma_{\mu\nu} = \lambda_{<\mu\nu>}$ and $\mu = \frac{1}{4}\, \lambda_{\mu\nu} g^{\mu\nu}$ so that $\lambda_{\mu\nu} = \Sigma_{\mu\nu} + \mu\, g_{\mu\nu}$ and we have that $\Sigma_{\mu\nu}$ is traceless.

After that, we consider firstly eq. $(14)_1$ contracted by U_α and eq. $(14)_2$ contracted a first time by $U_\alpha\, U_\beta$ and a second time by $h_{\alpha\beta}$, so obtaining the system

$$\begin{aligned} & nc^2(\lambda - \lambda_E) + \frac{e}{m}\, U^\mu \left(\lambda_\mu - \frac{U_\mu}{T}\right) + \frac{1}{m}\,(A_1^0 c^2 + A_{11}^0)\, U^\mu U^\nu \Sigma_{\mu\nu} = \\ & \qquad = -\frac{c^2}{m}\,(A_1^0 c^2 - 3A_{11}^0)\,\mu - \frac{c^2}{m}\, H_{VE} \quad \nu \,, \\ & \frac{e}{m}\, c^2(\lambda - \lambda_E) + c^4 B_5\, U^\mu \left(\lambda_\mu - \frac{U_\mu}{T}\right) + \left(\frac{1}{3} B_2 c^2 + B_3 c^4\right) U^\mu U^\nu \Sigma_{\mu\nu} = \\ & \qquad = \left(B_2 - B_3 c^2\right) c^4 \mu \, - \frac{c^2}{m}\, B_9\, \nu \,, \end{aligned} \tag{18}$$

$$\frac{p}{m}\left(\lambda-\lambda_E\right)+\frac{1}{3}B_4U^\mu\left(\lambda_\mu-\frac{U_\mu}{T}\right)+\left(\frac{1}{3}B_2+\frac{1}{9}\frac{B_1}{c^2}\right)U^\mu U^\nu\Sigma_{\mu\nu}=$$
$$=-\frac{k_B}{m^2}\,\pi+\frac{1}{3}\left(B_1-B_2c^2\right)\mu-\frac{1}{m}\,B_{10}\,\nu\,.$$

If we calculate this system in $\mu=0$, $\nu=0$, we obtain exactly the system $(A.10)_{1-3}$ of [1]. Obviously, the matrix of coefficients is the same of that reported in $(A.11)_1$ of [1], i.e.,

$$\tilde{D}_1^\pi=\begin{vmatrix} nc^2 & \frac{e}{m} & \frac{1}{m}\left(A_1^0c^2+A_{11}^0\right) \\ \frac{e}{m}\,c^2 & c^4\,B_5 & \frac{1}{3}B_2c^2+B_3c^4 \\ \frac{p}{m} & \frac{1}{3}B_4 & \frac{1}{3}B_2+\frac{1}{9}\frac{B_1}{c^2}\end{vmatrix}\,. \tag{19}$$

So, by using the Kramer' s theorem, we find

$$\lambda-\lambda_E=-\frac{k_B\,\pi}{m^2\,\tilde{D}_1^\pi}\begin{vmatrix}\frac{e}{m} & \frac{1}{m}\left(A_1^0c^2+A_{11}^0\right) \\ c^4\,B_5 & \frac{1}{3}B_2c^2+B_3c^4\end{vmatrix}+\mu\,\Delta_1+\nu\,\Delta_2\,, \tag{20}$$
$$U^\mu\left(\lambda_\mu-\lambda_{E\mu}\right)=\frac{k_B\,\pi}{m^2\,\tilde{D}_1^\pi}\begin{vmatrix} nc^2 & \frac{1}{m}\left(A_1^0c^2+A_{11}^0\right) \\ \frac{e}{m}\,c^2 & \frac{1}{3}B_2c^2+B_3c^4\end{vmatrix}+\left(\mu\,X_1+\nu\,Y_1\right)\,c^2\,,$$
$$U^\mu U^\nu\Sigma_{\mu\nu}=-\frac{k_B\,\pi}{m^2\,\tilde{D}_1^\pi}\begin{vmatrix} nc^2 & \frac{e}{m} \\ \frac{e}{m}\,c^2 & c^4\,B_5\end{vmatrix}+\left(\mu\,X_2+\nu\,Y_2\right)\,c^2\quad\text{with}$$

$$\Delta_1=\frac{1}{\tilde{D}_1^\pi}\begin{vmatrix} -\frac{c^2}{m}\left(A_1^0c^2-3A_{11}^0\right) & \frac{e}{m} & \frac{1}{m}\left(A_1^0c^2+A_{11}^0\right) \\ \left(B_2-B_3c^2\right)c^4 & c^4\,B_5 & \frac{1}{3}B_2c^2+B_3c^4 \\ \frac{1}{3}\left(B_1-B_2c^2\right) & \frac{1}{3}B_4 & \frac{1}{3}B_2+\frac{1}{9}\frac{B_1}{c^2}\end{vmatrix}\,,$$

$$\Delta_2=\frac{1}{\tilde{D}_1^\pi}\begin{vmatrix} -\frac{c^2}{m}\,H_V & \frac{e}{m} & \frac{1}{m}\left(A_1^0c^2+A_{11}^0\right) \\ -\frac{c^2}{m}\,B_9 & c^4\,B_5 & \frac{1}{3}B_2c^2+B_3c^4 \\ -\frac{1}{m}\,B_{10} & \frac{1}{3}B_4 & \frac{1}{3}B_2+\frac{1}{9}\frac{B_1}{c^2}\end{vmatrix}\,,$$

$$X_1 = \frac{1}{c^2\,\tilde{D}_1^\pi} \begin{vmatrix} nc^2 & -\frac{c^2}{m}\left(A_1^0 c^2 - 3A_{11}^0\right) & \frac{1}{m}\left(A_1^0 c^2 + A_{11}^0\right) \\ \frac{e}{m}\,c^2 & \left(B_2 - B_3 c^2\right) c^4 & \frac{1}{3}B_2 c^2 + B_3 c^4 \\ \frac{p}{m} & \frac{1}{3}\left(B_1 - B_2 c^2\right) & \frac{1}{3}B_2 + \frac{1}{9}\frac{B_1}{c^2} \end{vmatrix},$$

$$Y_1 = \frac{1}{c^2\,\tilde{D}_1^\pi} \begin{vmatrix} nc^2 & -\frac{c^2}{m}\,H_V & \frac{1}{m}\left(A_1^0 c^2 + A_{11}^0\right) \\ \frac{e}{m}\,c^2 & -\frac{c^2}{m}\,B_9 & \frac{1}{3}B_2 c^2 + B_3 c^4 \\ \frac{p}{m} & -\frac{1}{m}\,B_{10} & \frac{1}{3}B_2 + \frac{1}{9}\frac{B_1}{c^2} \end{vmatrix},$$

$$X_2 = \frac{1}{c^2\,\tilde{D}_1^\pi} \begin{vmatrix} nc^2 & \frac{e}{m} & -\frac{c^2}{m}\left(A_1^0 c^2 - 3A_{11}^0\right) \\ \frac{e}{m}\,c^2 & c^4\,B_5 & \left(B_2 - B_3 c^2\right) c^4 \\ \frac{p}{m} & \frac{1}{3}B_4 & \frac{1}{3}\left(B_1 - B_2 c^2\right) \end{vmatrix}, \; Y_2 = \frac{1}{c^2\,\tilde{D}_1^\pi} + \nu \begin{vmatrix} nc^2 & \frac{e}{m} & -\frac{c^2}{m}\,H_V \\ \frac{e}{m}\,c^2 & c^4\,B_5 & -\frac{c^2}{m}\,B_9 \\ \frac{p}{m} & \frac{1}{3}B_4 & -\frac{1}{m}\,B_{10} \end{vmatrix}.$$

If we calculate these expressions in $\mu = 0$, $\nu = 0$, we obtain exactly those reported in the equations subsequent to (61) of [1].
We consider now eq. $(14)_1$ contracted by h_α^δ and eq. $(14)_2$ contracted $h_\alpha^\delta\,U_\beta$. So we obtain the system $(A.14)_{1,2}$ of [1], i.e.,

$$\begin{pmatrix} \frac{p}{m} & 2\,\frac{A_{11}^0}{m} \\ \frac{1}{3}B_4 c^2 & \frac{2}{3}B_2\,c^2 \end{pmatrix} \begin{pmatrix} h^{\delta\mu}\left(\lambda_\mu - \frac{U_\mu}{T}\right) \\ h^{\delta\mu}U^\nu\Sigma_{\mu\nu} \end{pmatrix} = \begin{pmatrix} 0 \\ -\frac{k_B}{m^2}\,q^\delta \end{pmatrix}$$

By calling $\tilde{D}_3$ the determinant of the coefficients, we see that the solution is

$$h^{\delta\mu}\left(\lambda_\mu - \frac{U_\mu}{T}\right) = \frac{2}{\tilde{D}_3}\frac{k_B}{m^3}\,A_{11}^0\,q^\delta\,, \quad h^{\delta\mu}U^\nu\Sigma_{\mu\nu} = -\frac{1}{\tilde{D}_3}\frac{k_B}{m^3}\,p\,q^\delta\,. \qquad .$$

Finally, eq. $(14)_2$ contracted $h_\alpha^{<\delta}\,h_\beta^{\theta>_3}$ gives $(A.15)_1$ of [1], i.e.,

$$\frac{2}{15}\,B_1\,h_\mu^{<\delta}\,h_\nu^{\theta>_3}\,\Sigma^{\mu\nu} = -\frac{k_B}{m^2}\,t^{<\delta\theta>_3} \quad \text{from which} \quad h_\mu^{<\delta}\,h_\nu^{\theta>_3}\,\Sigma^{\mu\nu} = -\frac{15}{2\,B_1}\frac{k_B}{m^2}\,t^{<\delta\theta>_3}\,,$$

as in the beginning of page 431 of [1].
Now we have to substitute all these results in $(14)_{3,4}$. To this end let us first note that the following identity holds:

$$\Sigma_{\beta\gamma} = \Sigma_{\mu\nu}\,h_{<\beta}^\mu\,h_{\gamma>_3}^\nu - \frac{2}{c^2}\,\Sigma_{\mu\nu}\,U^\nu\,h_{(\beta}^\mu\,U_{\gamma)} + \frac{1}{c^4}\,\left(\Sigma_{\mu\nu}\,U^\mu U^\nu\right)\,U_\beta\,U_\gamma + \frac{1}{3}\,h_{\beta\gamma}\,\left(\Sigma_{\mu\nu}\,h^{\mu\nu}\right)\,.$$

Since $\Sigma_{\mu\nu}$ is traceless, it follows

$$\Sigma_{\beta\gamma} = \Sigma_{\mu\nu}\,h_{<\beta}^\mu\,h_{\gamma>_3}^\nu - \frac{2}{c^2}\,\Sigma_{\mu\nu}\,U^\nu\,h_{(\beta}^\mu\,U_{\gamma)} + \frac{1}{c^2}\,\left(\Sigma_{\mu\nu}\,U^\mu U^\nu\right)\left(\frac{1}{c^2}\,U_\beta\,U_\gamma + \frac{1}{3}\,h_{\beta\gamma}\right)\,,$$

and it is easy to verify that this $\Sigma_{\beta\gamma}$ is traceless.
After that, we can compact the above results in the form

$$\lambda - \lambda_E = (\lambda - \lambda_E)^{(14)} + \mu\,\Delta_1 + \nu\,\Delta_2\,, \quad \lambda_\beta - \lambda_{E\beta} = (\lambda_\beta - \lambda_{E\beta})^{(14)} + (\mu\,X_1 + \nu\,Y_1)\,U_\beta\,,$$
$$\Sigma_{\beta\gamma} = (\Sigma_{\beta\gamma})^{(14)} + \,(\mu\,X_2 + \nu\,Y_2)\,\left(\frac{1}{c^2}\,U_\beta\,U_\gamma + \frac{1}{3}\,h_{\beta\gamma}\right)\,,$$

where the notation $(\cdots)^{(14)}$ denotes the expression of $(\cdots)$ in the 14 moments model. By inserting these expressions in $(14)_{3,4}$, we obtain

$$A^{\alpha\beta\gamma} - A_E^{\alpha\beta\gamma} = \left(A^{\alpha\beta\gamma} - A_E^{\alpha\beta\gamma}\right)^{(14)} - \frac{m}{k_B}\,\left[A_E^{\alpha\beta\gamma}\,\left(\mu\,\Delta_1 + \nu\,\Delta_2\right)\,+ m\,A_{12}^{\alpha\beta\gamma\nu}\,U_\mu\,(\mu\,X_1 + \nu\,Y_1) + \right.$$
$$\left. + m\,A_{22}^{\alpha\beta\gamma\mu\nu}\,\left(\frac{1}{c^2}\,U_\mu\,U_\nu + \frac{1}{3}\,h_{\mu\nu}\right)(\mu\,X_2 + \nu\,Y_2) + \,\mu\,m\,A_{22}^{\alpha\beta\gamma\mu\nu}g_{\mu\nu} + \nu\,\,A_V^{\alpha\beta\gamma}\right]\,,$$
$$H_V^\alpha - H_{EV}^\alpha = (H_V^\alpha - H_{EV}^\alpha)^{(14)} - \frac{m}{k_B}\,\left[H_{EV}^\alpha\,\left(\mu\,\Delta_1 + \nu\,\Delta_2\right)\,+\,T_V^{\alpha\mu}\,U_\mu\,(\mu\,X_1 + \nu\,Y_1) + \right.$$
$$\left. + A_V^{\alpha\mu\nu}\,\left(\frac{1}{c^2}\,U_\mu\,U_\nu + \frac{1}{3}\,h_{\mu\nu}\right)(\mu\,X_2 + \nu\,Y_2) + \,\mu\,\,A_V^{\alpha\mu\nu}g_{\mu\nu} + \nu\,\,V_{VV}^\alpha\right]\,, \tag{21}$$

where

$$\left(A^{\alpha\beta\gamma} - A_E^{\alpha\beta\gamma}\right)^{(14)} = -\frac{m}{k_B}\,\left[A_E^{\alpha\beta\gamma}\,\left(\lambda - \lambda_E\right)^{(14)}\,+ m\,A_{12}^{\alpha\beta\gamma\mu}\,(\lambda_\mu - \lambda_{E\mu})^{(14)}\,+ m\,A_{22}^{\alpha\beta\gamma\mu\nu}\Sigma_{\mu\nu}^{(14)}\right]\,,$$
$$(H_V^\alpha - H_{VE}^\alpha)^{(14)} = -\frac{m}{k_B}\,\left[H_{VE}^\alpha\,\left(\lambda - \lambda_E\right)^{(14)}\,+ T_V^{\alpha\nu}\,(\lambda_\mu -\,\lambda_{E\mu})^{(14)}\,+\,A_V^{\alpha\mu\nu}\Sigma_{\mu\nu}^{(14)}\right]\,, \tag{22}$$

and we have that this $\left(A^{\alpha\beta\gamma} - A_E^{\alpha\beta\gamma}\right)^{(14)}$ is exactly the expression for $\left(A^{\alpha\beta\gamma} - A_E^{\alpha\beta\gamma}\right)$ reported in [1], even if it is not traceless.
Since $(H_V^\alpha - H_{VE}^\alpha)^{(14)}$ is missing in [1], we evaluate it now and find

$$(H_V^\alpha - H_{VE}^\alpha)^{(14)} = \frac{\tilde{N}_{111}^\pi}{\tilde{D}_1^\pi}\,\frac{\pi}{m}\,U^\alpha\,+\,\frac{N_{33}}{D_3}\,q^\alpha\,, \tag{23}$$

where $\tilde{D}_1^\pi$ and D_3 are the determinants in eq. $(A.11)_1$ and in page 444 of [1], with the pertinent adjustments in this article indicated, while

$$\tilde{N}_{111} = \begin{vmatrix} n\,c^2 & \frac{e}{m} & \frac{A_1^0\,c^2 + A_{11}^0}{m} \\ \frac{e}{m} & B_5\,c^2 & B_3\,c^2\,+\frac{1}{3}\,B_2 \\ H_V\,c^2 & B_0 & A_{1V}^0\,c^2 + A_{11V}^0 \end{vmatrix}\,,\; N_{33} = \frac{9}{2}\,\frac{1}{m\,n^2\,c^6}\,\left(\overline{J_{2,1}(\gamma^*)}\right)^2 \begin{vmatrix} \frac{p}{m} & 2\,\frac{A_1^0}{m} \\ B_{10} & 2\,\frac{A_{11V}^0}{m} \end{vmatrix}\,.$$

So the complete closure is given by (21), where $\left(A^{\alpha\beta\gamma} - A_E^{\alpha\beta\gamma}\right)^{(14)}$ can be taken from [1] with the pertinent adjustments in the integrals above indicated and $(H_V^\alpha - H_{VE}^\alpha)^{(14)}$ from (23).

4 Principal Subsystems

- We note that the subsystem, in the sense of [9], of eqs. (1) with $\Sigma_{\mu\nu} = 0$, $\nu = 0$, i.e. the eqs. $\partial_\alpha V^\alpha = 0$, $\partial_\alpha T^{\alpha\beta} = 0$, $\partial_\alpha A^{\alpha\beta}_\beta = I^\beta_\beta$ is the relativistic counterpart of the model called ET_6 in [8]. In fact, the non relativistic limit of the first two of these equations is $(2)_{1-2,4}$, as before. Moreover, in sect. 2 we have seen that $\partial_\alpha \left(c\,A^{\alpha 00} + c^3\,V^\alpha - 2\,c^2\,T^{\alpha 0}\right)$ has a finite limit; it follows that $\lim_{c\to+\infty}\left[\frac{1}{c}\,\partial_\alpha\left(c\,A^{\alpha 00} + c^3\,V^\alpha - 2\,c^2\,T^{\alpha 0}\right) = 0\right]$. Since $A^{\alpha\beta}_\beta = A^{\alpha\beta\gamma}g_{\beta\gamma} = A^{\alpha 00} - A^{\alpha ll}$, we have $\lim_{c\to+\infty}\left[\frac{1}{c}\,\partial_\alpha\left(c\,A^{\alpha\beta}_\beta + c^3\,V^\alpha - 2\,c^2\,T^{\alpha 0}\right)\right] = -\lim_{c\to+\infty}\partial_\alpha\,A^{\alpha ll}$. Thanks to this fact and to eqs. $(17)_{3,4}$ of [1], we see that the non relativistic limit of $\partial_\alpha\,A^{\alpha\beta}_\beta = I^\beta_\beta$ is the trace of $(2)_3$, i.e., the third eq. reported in the middle of page 7 of [8]. We note that, in the full 14 moments model, this equation already exists as trace of the eq. $\partial_\alpha\,A^{\alpha ij} = I^{ij}$, so that it is necessary to multyply their difference by c^2 and, after that, calculate the limit so obtaining $(3)_7$; but now we don't need this further passage because we have only the trace of $\partial_\alpha\,A^{\alpha\beta\gamma} = I^{\beta\gamma}$. This fact shows the importance to consider also this trace, to which it corresponds the Lagrange multiplier μ, otherwise this subsystem ET_6 couldn't be obtained.
- We note that the subsystem of eqs. (1) with $\Sigma_{\mu\nu} = 0$, i.e. the eqs. $\partial_\alpha V^\alpha = 0$, $\partial_\alpha T^{\alpha\beta} = 0$, $\partial_\alpha A^{\alpha\beta}_\beta = I^\beta_\beta$, $\partial_\alpha H^\alpha_V = P$ is the relativistic counterpart of the model called ET_7 in [8]. In fact, for the first three of these equation we can proceed as in the previous case and the last one gives $(3)_5$. In this way we obtain the eqs. reported in the middle of page 7 of [8]. Also in this case we see the necessity to maintain the Lagrange multiplier μ.

5 The non equilibrium temperatures

We note that in the equation before (24) of [8] the authors have defined the non equilibrium temperatures as

$$\vartheta^K = \frac{3}{2\,\lambda^G_{Ill}} \quad , \quad \vartheta^R = \frac{1}{2\,\mu^G_{IR}} \quad , \quad \vartheta^V = \frac{1}{2\,\mu^G_{IV}}\,, \tag{24}$$

where λ^G_{Ill}, μ^G_{IR} μ^G_{IV} are the Lagrange multipliers of their balance equations calculated for zero velocity. To see their correspondent for the present equations we have firstly to consider the non relativistic limits of our Lagrange multipliers, as we have already done for the fields. To this end, we see that from (11) it follows $d\,h'^\alpha = V^\alpha d\,\lambda + T^{\alpha\beta}d\,\lambda_\beta + A^{\alpha\beta\gamma}d\,\lambda_{\beta\gamma} + H^\alpha_V d\,\nu$, from which we desume

$$d\left(\frac{h'^0}{c}\right) = \frac{V^0}{c}\,d\left(\lambda + c^2\,\lambda_{00} + \frac{\nu}{c} + c\,\lambda_0\right) + \frac{T^{0i}}{c}\,d\left(\lambda_i + 2\,c\,\lambda_{0i}\right) + \frac{A^{0ij}}{c}\,d\left(\lambda_{ij}\right) +$$
$$+\left(2\,T^{00} - 2\,c\,V^0\right)\,d\left(\lambda_{00} + \frac{1}{2\,c}\,\lambda_0\right) + \left(c^2\,H^0_V - c\,V^0\right)\,d\,\frac{\nu}{c^3} + \left(2\,A^{00i} - 2\,c\,T^{0i}\right)\,d\,\frac{\lambda_{0i}}{c} +$$
$$+4\left(c\,A^{000} + c^3\,V^0 - 2\,c^2\,T^{00}\right)\,d\,\frac{\lambda_{00}}{4\,c^2}\,.$$

Now $\left(\frac{h'^0}{c}\right)$ in the non relativistic limit behaves like $\left(\frac{V^0}{c}\right)$ so that it tends to $h'^{classic}$; morever, the coefficients of the differentials in the right hand side tend respectively to F, F^i, F^{ij}, G^{ll}, H_V^{ll}, G^{ill}, A_2 (as seen above). So we deduce that the Lagrange multipliers of eqs. (3) are λ^{cl}, λ_i^{cl}, λ_{ij}^{cl}, μ^{cl}, μ_V^{cl}, μ_i^{cl}, η^{cl} which are respectively the non relativistic limits of

$$\left(\lambda + c^2\,\lambda_{00} + \frac{\nu}{c} + c\,\lambda_0\right)\,,\ (\lambda_i + 2\,c\,\lambda_{0i})\,,\ (\lambda_{ij})\,,\ \left(\lambda_{00} + \frac{1}{2\,c}\,\lambda_0\right)\,,\ \frac{\nu}{c^3}\,,\ \frac{\lambda_{0i}}{c}\,,\ \frac{\lambda_{00}}{4\,c^2}\,. \tag{25}$$

But in [8] the authors have not considered the present eq. $(3)_7$ and, instead of the $(3)_4$ they have considered
$\partial_t\,\left(G^{ll} - F^{ll} - H_V^{ll}\right) + \partial_i\,\left(G^{ill} - F^{ill} - H_V^{ill}\right) = -\,P_F^{ll} - P_V^{ll}\,.$
So, instead of

$$d\,h' = F\,d\,\lambda^{cl} + F^i\,d\,\lambda_i^{cl} + F^{ij}\,d\,\lambda_{ij}^{cl} + G^{ll}\,d\,\mu^{cl} + H_V^{ll}\,d\,\mu_V^{cl} + G^{ill}\,d\,\mu_i^{cl} + A_2\,d\,\eta^{cl}\,,$$

with $A_2 = 2G^{pppp} - F^{pppp}$ they have

$$\begin{aligned} d\,h' = {} & F\,d\,\lambda^G + F^i\,d\,\lambda_i^G + F^{ij}\,d\,\lambda_{ij}^G + \left(G^{ll} - F^{ll} - H_V^{ll}\right)\,d\,\mu_R^G + H_V^{ll}\,d\,\mu_V^G + G^{ill}\,d\,\mu_i^G + \\ & + A_2\,d\,\eta^G = F\,d\,\lambda^G + F^i\,d\,\lambda_i^G + F^{ij}\,d\,\left(\lambda_{ij}^G - \mu_R^G\,\delta_{ij}\right) + \\ & + G^{ll}\,d\,\mu_R^G + H_V^{ll}\,d\,\left(\mu_V^G - \mu_R^G\right) + G^{ill}\,d\,\mu_{ill}^G + A_2\,d\,\eta^G\,. \end{aligned}$$

By comparing the two expressions we obtain the relation

$$\lambda^{cl} = \lambda^G\,,\ \lambda_i^{cl} = \lambda_i^G\,,\ \lambda_{ij}^{cl} = \lambda_{ij}^G - \mu_R^G\,\delta_{ij}\,,\ \mu^{cl} = \mu_R^G\,,\ \mu_V^{cl} = \mu_V^G - \mu_R^G\,,\ \mu_i^{cl} = \mu_{ill}^G\,,\ \eta^{cl} = \eta^G\,.$$

From this result we desume $\mu_R^G = \mu^{cl}\,,\ \lambda_{ij}^G = \mu^{cl}\,\delta_{ij} + \lambda_{ij}^{cl}\,,\ \mu_V^G = \mu^{cl} + \mu_V^{cl}\,.$

We substitute now these expressions in (24) and, after that, the scalars in (25) of which they are the non relativistic limits; so we find

$$\vartheta^K = \frac{1}{2\,\lambda_{00} + \frac{1}{c}\,\lambda_0 + \frac{2}{3}\,\lambda_{ll}}\quad,\quad \vartheta^R = \frac{1}{2\,\lambda_{00} + \frac{1}{c}\,\lambda_0}\quad,\quad \vartheta^V = \frac{1}{2\,\lambda_{00} + \frac{1}{c}\,\lambda_0 + 2\,\frac{\nu}{c^3}}\,,$$

which can be written in covariant form as

$$\begin{aligned} \vartheta^K &= \frac{1}{\frac{2}{c^2}\,\Sigma_{\alpha\beta}U^\alpha U^\beta + \frac{1}{c^2}\,\lambda_\alpha U^\alpha + \frac{2}{3}\,\Sigma_{\alpha\beta}h^{\alpha\beta}} = \frac{1}{\frac{8}{3\,c^2}\,\Sigma_{\alpha\beta}U^\alpha U^\beta + \frac{1}{c^2}\,\lambda_\alpha U^\alpha}\quad, \\ \vartheta^R &= \frac{1}{\frac{2}{c^2}\,\Sigma_{\alpha\beta}U^\alpha U^\beta + \frac{1}{c^2}\,\lambda_\alpha U^\alpha}\quad,\quad \vartheta^V = \frac{1}{\frac{2}{c^2}\,\Sigma_{\alpha\beta}U^\alpha U^\beta + \frac{1}{c^2}\,\lambda_\alpha U^\alpha + 2\,\frac{\nu}{c^3}}\,, \end{aligned} \tag{26}$$

and we assume these expressions as definitions of the relativistic non equilibrium temperatures. (We have substituted $\lambda_{\alpha\beta}$ with $\Sigma_{\alpha\beta}$ because [8] is obtained in the non relativistic limit only if $\lambda_{\alpha\beta}$ is traceless and we aim to find a definition corresponding to that of [8]).

After that, we see that at equilibrium these temperatures have the same value $\vartheta^K = T$, $\vartheta^R = T$, $\vartheta^V = T$, while at first order with respect to equilibrium they are

$$\begin{aligned} \vartheta^K - T &= -T^2 \left[\frac{8}{3\,c^2} \, \Sigma_{\alpha\beta} U^\alpha U^\beta + \frac{1}{c^2} \, (\lambda_\alpha - \lambda_{E\alpha}) \, U^\alpha \right] \quad , \\ \vartheta^R - T &= -T^2 \left[\frac{2}{c^2} \, \Sigma_{\alpha\beta} U^\alpha U^\beta + \frac{1}{c^2} \, (\lambda_\alpha - \lambda_{E\alpha}) \, U^\alpha \right] \quad , \\ \vartheta^V - T &= -T^2 \left[\frac{2}{c^2} \, \Sigma_{\alpha\beta} U^\alpha U^\beta + \frac{1}{c^2} \, (\lambda_\alpha - \lambda_{E\alpha}) \, U^\alpha + 2 \, \frac{\nu}{c^3} \right] , \end{aligned} \tag{27}$$

from which

$$\begin{aligned} &\nu = \frac{c^3}{2\,T^2} \, \left(\vartheta^R - \vartheta^V \right) \, , \; \Sigma_{\alpha\beta} U^\alpha U^\beta = \frac{3\,c^2}{2\,T^2} \, \left(\vartheta^R - \vartheta^k \right) \, , \\ &(\lambda_\alpha - \lambda_{E\alpha}) \, U^\alpha = \frac{3\,c^2}{T^2} \, \left[\vartheta^K - T - \frac{4}{3} \left(\vartheta^R - T \right) \right] . \end{aligned} \tag{28}$$

As consequence of this result, if we think that the non equilibrium temperatures have physical meaning, then also the Lagrange multipliers ν, $\Sigma_{\alpha\beta} U^\alpha U^\beta$ and $(\lambda_\alpha - \lambda_{E\alpha}) \, U^\alpha$ assume a corrsponding physical meaning and it is not necessary to desume them from (18).
At this point there are different possible choices:
1) The first one consists in continuing as above by taking n, U^α, T, π, q^α, $t^{<\alpha\beta>_3}$, $U_\alpha \left(H^\alpha_V - H^\alpha_{EV} \right)$, $U_\alpha U_\beta U_\gamma \left(A^{\alpha\beta\gamma} - A^{\alpha\beta\gamma}_E \right)$ as independent variables desuming ν and μ from these last two scalars; in this case eqs. (27) are simply definitions of non equilibrium temperatures and are not used.
2) The second one considers n, U^α, T, π, q^α, $t^{<\alpha\beta>_3}$, $\vartheta^R - \vartheta^V$, $U_\alpha U_\beta U_\gamma \left(A^{\alpha\beta\gamma} - A^{\alpha\beta\gamma}_E \right)$ as independent variables desuming μ from this last scalar; in this case the expression of $H^\alpha_V - H^\alpha_{EV}$ is explicitly obtained and eqs. $(28)_{2,3}$ are not used.
3) In the third one we take n, U^α, T, q^α, $t^{<\alpha\beta>_3}$, $\vartheta^R - \vartheta^V$, $\vartheta^R - \vartheta^K$, $U_\alpha U_\beta U_\gamma \left(A^{\alpha\beta\gamma} - A^{\alpha\beta\gamma}_E \right)$ as independent variables desuming μ from this last scalar; in this case $(18)_{1,2}$ are used to desume $\lambda - \lambda_E$ and $(\lambda_\alpha - \lambda_{E\alpha}) \, U^\alpha$, while $(18)_3$ gives π. Moreover, $(28)_3$ is not used.
4) In the last one the independent variables are n, U^α, T, q^α, $t^{<\alpha\beta>_3}$, $\vartheta^R - \vartheta^V$, $\vartheta^R - \vartheta^K$, $\vartheta^K - T$. In this case $(18)_{1,2}$ are used to desume $\lambda - \lambda_E$ and μ. By using the above expressions to calculate e, $A^0_1 c^2 - 3 A^0_{11}$, $B_2 - B_3 c^2$, we see that the determinant of coefficients of the unknowns is $n^2 \gamma \, c^8 \left(\overline{J_{2,1}(\gamma^*)} \right)^{-2}$ multiplied by

$$\begin{vmatrix} \overline{J_{2,1}(\gamma^*)} & \overline{J_{2,1}(\gamma^*) \, \left(1 + \frac{2\mathcal{I}}{mc^2}\right)} \\ \overline{J_{2,2}(\gamma^*) \, \left(1 + \frac{\mathcal{I}}{mc^2}\right)} & \overline{J_{2,2}(\gamma^*) \, \left(1 + \frac{\mathcal{I}}{mc^2}\right) \, \left(1 + \frac{2\mathcal{I}}{mc^2}\right)} \end{vmatrix} .$$

By adding the first coulumn to the second one, it becomes

$$2\begin{vmatrix} \overline{J_{2,1}(\gamma^*)} & \overline{J_{2,1}(\gamma^*)\left(1+\frac{\mathcal{I}}{mc^2}\right)} \\ & \\ \overline{J_{2,2}(\gamma^*)\left(1+\frac{\mathcal{I}}{mc^2}\right)} & \overline{J_{2,2}(\gamma^*)\left(1+\frac{\mathcal{I}}{mc^2}\right)^2} \end{vmatrix} .$$

So this last choice can be adopted only if this determinant is different from zero. (In any case, the other 3 possible choices are surely mathematically correct).
Ater that, $(18)_3$ gives π and all the other dependent functions are determined.

Conclusions: We have completed the closure of our balance equations (1). It still remains the problem of the hyperbolicity requirement. If no approximation is introduced it surely holds for every value of the independent variables; but in this case the closure is expressed in terms of the integral (10) whose integrability doesn't hold for every value of the fields. If we introduce the Taylor's expansion of this integral up to whatever order with respect to equilibrium, then other integrals appear whose integrability has been proved in [12]. But if we stop this Taylor's expansion at a given order M with respect to equilibrium, then the zone of hyperbolicity holds within a zone called "hyperbolicity region" whose radius is an increasing function of M, as it can be seen in [13] (In the abstract it is here written that, in the case of one-dimensional space, with a second-order approximation the radius of the hyperbolicity region is larger than the corresponding radius of the first-order approximation). This result was also recently confirmed in [14]. Unfortunately, a Taylor's expansion around equilibrium is necessary when we want to express the closure in terms of physical variables instead of the Lagrange multipliers due to human calculation problems. So the question remains on how to perform this Taylor's expansion without restricting the hyperbolicity zone. This can be object of future investigations.

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