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Abstract: The primary concern of the present chapter is to address the distributed leader-following consensus tracking problem for a network of agents governed by uncertain diffusion PDEs with the Neumann-type boundary actuation and uncertain spatially varying diffusivity. Except for the "leader" agent that generates the reference profile to be tracked, all remaining agents, called "followers," are required to asymptotically track the infinite-dimensional time-varying leader state. The dynamics of the follower agents are affected by smooth boundary disturbances unbounded in magnitude and with a bounded derivative. The proposed local interaction rule is developed by assuming that only neighboring collocated boundary sensing is available, and it consists of a nonlinear sliding-mode-based protocol. The performance and stability properties of the resulting infinite-dimensional networked system are then formally studied by means of Lyapunov analysis. The analysis demonstrates the global exponential stability of the resulting error boundary-value problem in a suitable Sobolev space. The effectiveness of the developed control scheme is supported by simulation results.

Second-order sliding mode leader-follower consensus tracking for a network of uncertain diffusion PDEs with spatially-varying diffusivity

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Abstract The primary concern of the present chapter is to address the distributed leader-following consensus tracking problem for a network of agents governed by uncertain diffusion PDEs with Neumann-type boundary actuation and uncertain spatially-varying diffusivity. Except for the "leader" agent that generates the reference profile to be tracked, all remaining agents, called "followers", are required to asymptotically track the infinite-dimensional time-varying leader state. The dynamics of the follower agents is affected by smooth boundary disturbances which are, possibly, unbounded in magnitude. The proposed local interaction rule is developed assuming that only boundary sensing is available, and it consists of a nonlinear sliding-mode based protocol. The performance and stability properties of the resulting infinite-dimensional networked system are then formally studied by means of Lyapunov analysis. The analysis demonstrates the global exponential stability of the resulting error boundary-value problem in a suitable Sobolev space. The effectiveness of the developed control scheme is supported by simulation results.

1 Introduction

Since its first formulation, the problem of understanding when individual actions of interacting dynamical systems give rise to a coordinated *emergent behaviour* continues receiving considerable attention in academic research, see for instance [Zampieri (2008), Huang et al. (2020)] and reference therein.

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In the most typical setup, a local interaction rule is found such that a number of dynamical agents reach an agreement condition among the respective state values that is called *consensus*. The local interation rule is distributed in the sense that it can only use information from the local agent and from its neighbours. Application of the consensus concept can also be found in the distributed optimization setting, see e.g. [Pilloni et al. (2020)].

A particular class of *consensus problems* is the *leader-following distributed tracking*, where a specific agent in the network plays the role of *leader*, and all remaining agents, called *followers*, are wanted to synchronize their state with that of the leader, see for details [Cao et al. (2012), Huang et al. (2020)]. Relevant applications of the leader-following setting to real problems can be found, e.g., in [Shao et al. (2020)], [Davila (2014)] and [Li et al. (2017)] respectively dealing with flocking, coordinated tracking of mechanical systems, and opinion dynamics. The reader can refer to [Cao et al. (2013)] for tutorial overviews of consensus-based control in the finite-dimensional setup.

It can be noticed that deep connections between the consensus problem, and certain partial differential equations (PDEs), the *diffusion* equation in particular, exist. Indeed, discretizing in the spatial domain the one dimensional diffusion equation yields a high-dimensional system of networked first-order continuous-time integrators interacting through a linear consensus protocol. Owing on this fact, some authors, see for instance [Chapman & Mesbahi (2011), Sardellitti et al. (2010)], have exploited (discretized forms of) different distributed parameter systems, such as *advection* and *diffusion-advection equations*, to derive more effective consensus protocols with improved convergence features. In spite of this intimate relationship between consensus algorithms and certain discretized PDEs, the consensus problem for networks of distributed parameter systems has not received yet the same attention than its finite-dimensional counterpart.

The leaderless consensus (i.e., synchronization) problem for coupled hyperbolic processes governed by the wave equation was studied in [Li & Rao (2014)], [Li et al. (2017)], and more recently in [Aguilar et al. (2021)]. In the parabolic PDEs setting, it is worth mentioning the seminal work [Demetriou (2013)] where the consensus problem for a network of agents modeled by a class of parabolic PDEs was firstly studied. In the more recent paper [Pilloni et al. (2015)] the leader-less consensus problem for a network of agents governed by the diffusion equation was studied. It was interestingly shown there that using a linear boundary local interaction rule the state of the agents converges towards the spatial average of their initial conditions, thereby establishing an infinite-dimensional counterpart of the *average consensus* algorithm for single-integrator finite-dimensional agents. In addition, the same paper proposed a sliding-mode based boundary local interaction rule, based on boundary sensing, yielding consensus in the presence of matched boundary perturbations.

The distributed leader following tracking problem for multi-agent network agents governed by parabolic PDEs subject to expeternal disturbances was dealt with in [He (2018)] and [Orlov et al. (2016)]. In [He (2018)] the goal was achieved by means of an adaptive unit-vector sliding mode local interaction rule and by exploiting distributed state measurements taken on the entire spatial domain. On the contrary, in [Orlov et al. (2016)] the same goal was achieved using collocated measurement only along with a second-order sliding mode based local interaction rule.

Motivated by the previous state-of-the-art analysis, in this chapter we make a step beyond the existing literature by addressing the leader-following consensus problem with reference to a more general class of agents' dynamics as compared to that considered in earlier works, in particular [Orlov et al. (2016)]. More precisely, we are going to consider a network of infinite-dimensional agents governed by the diffusion PDE having an uncertain and spatially varying diffusion coefficient. This extended class of agents' dynamics was never considered in previous works on the leader-less or leader-following consensus for parabolic PDEs. The presence of a spatially varying diffusivity characterizes several relevant physical situations such as diffusion in non-homogeneous media [Carslaw & Jaeger (1959)].

We propose a sliding mode based local interaction rule similar to that considered in [Orlov et al. (2016)] and we present here a novel Lyapunov analysis demonstrating the global exponential achievement of the leader following consensus for this wider class of agent's dynamics, that also includes possibly unbounded external boundary perturbations. The proposed design is accompanied by a set of simple tuning rules for setting the control parameters. The provided analysis demonstrates the global exponential stability of the synchronization error boundary value problem in the $H^2(0,1)$ Sobolev space, which in turns implies the point-wise achievement of leader-following consensus.

The correct functioning of the proposed local interaction rule is supported by numerical simulations. In the simulations it is also shown how to control the leader agent by means of motion planning strategies, as those discussed for instance in [Krstic & Smyshlyaev (2008), Siranosian et al. (2009)], to impose a prescribed profile to XXXX

1.1 Organization

The remainder of the chapter is organized as follows. In Section 2 the mathematical notation along with some preliminaries on Sobolev spaces, spatial norms, and graph theory are recalled. Then, in the Section 3 the considered distributed leaader-following consensus tracking problem is formulated, the proposed second-order sliding-mode local interaction protocol is presented, and the convergence and stability features are investigated by means of Lyapunov analysis. Simulation results supporting the proposed theoretical results are then discussed in Section 4. Finally, the last Section 6 reports some concluding remarks and perspectives for next research.

2 Preliminaries and Notations

2.1 *Mathematical notation and norm properties*

The sets of natural and real numbers are denoted by $\mathbb{N}$, and $\mathbb{R}$. For $n \in \mathbb{N}$, a real column vector is denoted by $X = (x_1, \dots, x_n)^\intercal \in \mathbb{R}^n$. The transpose of X is denoted by $X^\intercal \in \mathbb{R}^{1\times n}$.

Let $\mathbf{1}_n = (1, \dots, 1)^\intercal \in \mathbb{R}^n$ and $\mathbf{0}_n = (0, \dots, 0)^\intercal \in \mathbb{R}^n$ the n-dimensional all-one, and all-zero column vectors, whereas $\boldsymbol{I}_n = \mathrm{diag}(\mathbf{1}_n) \in \mathbb{R}^{n\times n}$ denotes the n-dimensional identity matrix, and the operator $\mathrm{diag}(X)$ returns a square diagonal matrix with the elements of vector X on the main diagonal.
The 1-, 2-, and ∞-norms of X are defined as follows

$$\|X\|_1 = \sum_{i=1}^{n} |x_i|\,, \quad \|X\|_2 = \left(\sum_{i=1}^{n} x_i^2\right)^{\frac{1}{2}}, \quad \|X\|_\infty = \max_{1\le i\le n}\{x_i\}\,. \tag{1}$$

Moreover, for the norms in (1) the next chain of inequalities hold [Khalil (2002)]

$$\|X\|_\infty \le \|X\|_2 \ \le\ \|X\|_1 \ \le\ \sqrt{n}\ \|X\|_2, \quad \text{with} \quad X \in \mathbb{R}^n. \tag{2}$$

Consider a n-dimensional positive definite matrix $\boldsymbol{M} = [m_{ij}] \succ 0$, with entries $m_{ij} \in \mathbb{R}$, and let $0 < \mathrm{m}_1 \le \mathrm{m}_2 \le \cdots \le \mathrm{m}_n \in \mathbb{R}_{>0}$ be the collection of its ordered eigenvalues, then the 2-norm of the vector $\boldsymbol{M}X \in \mathbb{R}^n$ satisfies the next relation

$$\mathrm{m}_1\|X\|_2 \le \|\boldsymbol{M}X\|_2 \le \mathrm{m}_n\|X\|_2, \quad \text{where} \quad \mathrm{m}_n \ge \mathrm{m}_1 > 0. \tag{3}$$

Let $\|X\|_\mathrm{p}$ and $\|Y\|_\mathrm{q}$ be the p- and q-norms of X and $Y \in \mathbb{R}^n$, and let $\mathrm{p} \ge 1$ and $\mathrm{q} \ge 1$ be given such that $1/\mathrm{p} + 1/\mathrm{q} = 1$. Then, the next chain of inequalities is in force [Khalil (2002)]

$$|X^\intercal Y| \le \|X\|_\mathrm{p}\ \|Y\|_\mathrm{q} \le \frac{\|X\|_\mathrm{p}^2}{\mathrm{p}} + \frac{\|Y\|_\mathrm{q}^2}{\mathrm{q}}. \tag{4}$$

The $\mathrm{Sign}(X)$ operator with argument $X \in \mathbb{R}^n$ denotes the set-valued component-wise operator

$$\mathrm{sign}(x_i) = \begin{cases} 1 & \text{if } x_i > 0 \\ [-1, 1] & \text{if } x_i = 0\ . \\ -1 & \text{if } x_i < 0 \end{cases} \tag{5}$$

2.2 Preliminaries on Sobolev spaces and integral norms

The *Sobolev space* of absolutely continuous scalar functions $z(\varsigma)$ with square integrable derivatives $z^{(k)}(\varsigma)$ on $(0,1)$ up to the order $\ell \in \mathbb{N}$, and with $k \leq \ell$, is denoted by $\mathrm{H}^{\ell}(0,1)$. It follows that the space $\mathrm{H}^{0}(0,1)$ corresponds to the well-known *Hilbert space*, which is commonly denoted also by $L^2(0,1)$. The H^{ℓ}-norm of $z(\zeta)$ is

$$\|z(\cdot)\|_{\mathrm{H}^{\ell}(0,1)} = \left(\int_0^1 \sum_{k=0}^{\ell} \left[z^{(k)}(\xi)\right]^2 \, d\xi\right)^{\frac{1}{2}}. \tag{6}$$

In the remainder of the chapter the first and second order derivatives $z^{(1)}(\varsigma)$ and $z^{(2)}(\varsigma)$ will be also denoted as $z^{(1)}(\varsigma) = z_{\varsigma}(\varsigma)$ and $z^{(2)}(\varsigma) = z_{\varsigma\varsigma}(\varsigma)$. For later use we further introduce the explicit expression of $\|z(\cdot)\|_{\mathrm{H}^2(0,1)}$

$$\|z(\cdot)\|_{\mathrm{H}^2(0,1)} = \sqrt{\|z(\cdot)\|^2_{\mathrm{H}^0(0,1)} + \|z_{\xi}(\cdot)\|^2_{\mathrm{H}^0(0,1)} + \|z_{\xi\xi}(\cdot)\|^2_{\mathrm{H}^0(0,1)}}, \tag{7}$$

and that of the "weighted" $\mathrm{H}^{2,\varphi}(0,1)$-norm defined as follows

$$\|z(\cdot)\|_{\mathrm{H}^{2,\varphi}(0,1)} = \sqrt{\|z(\cdot)\|^2_{\mathrm{H}^0(0,1)} + \|z_{\xi}(\cdot)\|^2_{\mathrm{H}^0(0,1)} + \|[\varphi(\cdot) z_{\xi}(\cdot)]_{\xi}\|^2_{\mathrm{H}^0(0,1)}}, \tag{8}$$

where $\varphi(\xi) \in \mathrm{H}^1(0,1)$ is a sufficiently smooth weighting function.

Then, we introduce the notation

$$\left[\mathrm{H}^{\ell}(0,1)\right]^n = \underbrace{\mathrm{H}^{\ell}(0,1) \times \mathrm{H}^{\ell}(0,1) \times \cdots \times \mathrm{H}^{\ell}(0,1)}_{n\ \texttt{-times}} \tag{9}$$

and

$$\|Z(\cdot)\|^2_{[\mathrm{H}^{\ell}(0,1)]^n} = \sum_{i=1}^{n} \|z_i(\cdot)\|^2_{\mathrm{H}^{\ell}(0,1)} \tag{10}$$

for the norm of $Z(\varsigma) = [z_1(\varsigma), \ldots, z_n(\varsigma)]^{\intercal} \in \left[\mathrm{H}^{\ell}(0,1)\right]^n$.
We further define

$$\|Z(\cdot)\|^2_{[\mathrm{H}^2(0,1)]^n} = \|Z(\cdot)\|^2_{[\mathrm{H}^0(0,1)]^n} + \|Z_{\xi}(\cdot)\|^2_{[\mathrm{H}^0(0,1)]^n} + \|Z_{\xi\xi}(\cdot)\|^2_{\mathrm{H}^0(0,1)}, \tag{11}$$

and

$$\|Z(\cdot)\|^2_{[\mathrm{H}^{2,\varphi}(0,1)]^n} = \|Z(\cdot)\|^2_{[\mathrm{H}^0(0,1)]^n} + \|Z_{\xi}(\cdot)\|^2_{[\mathrm{H}^0(0,1)]^n} + \|[\varphi(\cdot) Z_{\xi}(\cdot)]_{\xi}\|^2_{\mathrm{H}^0(0,1)}. \tag{12}$$

Lemma 1. *Let $Z(\varsigma) \in [\mathrm{H}^1(0,1)]^n$. Then, the next inequality holds*

$$\|Z(\cdot)\|^2_{[\mathrm{H}^0(0,1)]^n} \leq 2\left(\|Z(k)\|_2^2 + \|Z_{\varsigma}(\cdot)\|^2_{[\mathrm{H}^0(0,1)]^n}\right), \quad \forall \quad k = 0,1. \tag{13}$$

Proof of Lemma 1: Following [Pisano & Orlov (2012), Lemma 1], it results the generic scalar function $z(\varsigma) \in \mathrm{H}^1(0,1)$ satisfies

$$\|z(\cdot)\|^2_{\mathrm{H}^0(0,1)} \leq 2(z(k)^2 + \|z_\varsigma(\cdot)\|^2_{\mathrm{H}^0(0,1)}), \quad \forall \quad k = 0,1. \tag{14}$$

Thus, by invoking definitions (6) and (10), and by virtue of (14) specified with $z(\cdot) = z_i(\cdot)$, the following chain of inequalities can be further derived

$$\begin{aligned}\|Z(\cdot)\|^2_{[\mathrm{H}^0(0,1)]^n} &= \sum_{i=1}^{n} \|z_i(\cdot)\|^2_{\mathrm{H}^0(0,1)} \leq 2\sum_{i=1}^{n} \left(z_i(k)^2 + \|Z_{i,\varsigma}(\cdot)\|^2_{\mathrm{H}^0(0,1)} \right) \\ &= 2\left(\|Z(k)\|_2^2 + \|Z_\varsigma(\cdot)\|^2_{[\mathrm{H}^0(0,1)]^n} \right), \quad \forall \quad k = 0,1.\end{aligned}$$

Lemma 1 is proved. □

2.3 *Algebraic graph theory*

Consider a network of n dynamical subsystems, referred to as *agents*. Let $\boldsymbol{\mathcal{V}} = \{1,\ldots,n\}$ be the agent set. The interaction among these subsystems can be described by means of a *weighted graph* $\boldsymbol{\mathcal{G}}(\boldsymbol{\mathcal{V}},\boldsymbol{\mathcal{E}},\boldsymbol{A})$, where $\boldsymbol{\mathcal{V}}$ is referred to as the *node set* and $\boldsymbol{\mathcal{E}} \subseteq \{\boldsymbol{\mathcal{V}} \times \boldsymbol{\mathcal{V}}\}$ denotes the so-called *edge set* collecting all the enabled oriented interactions among pairs of nodes, namely $(i,j) \in \boldsymbol{\mathcal{E}}$ if node (agent) i can receive information from node (agent) j. Finally, the real square matrix $\boldsymbol{A} = [a_{ij}] \in \mathbb{R}^{n\times n}$ denotes the so-called *weighted adjacency matrix* associated with $\boldsymbol{\mathcal{G}}$, which entry a_{ij} is positive if $(i,j) \in \boldsymbol{\mathcal{E}}$ and zero otherwise. The *neighbors set* of node i is denoted by $\boldsymbol{\mathcal{N}}_i = \{j \in \boldsymbol{\mathcal{V}} \;:\; (i,j) \in \boldsymbol{\mathcal{E}}\}$.

A *directed path* over $\boldsymbol{\mathcal{G}}$ denotes a finite or infinite sequence of edges joining two distinct nodes. A node is called *root*, if it can be reached from any other node by traversing a directed path.

A weighted graph is called *undirected* if and only if $a_{ij} = a_{ji} \; \forall \; i \neq j$, which implies that, if $(i,j) \in \boldsymbol{\mathcal{E}}$, then also $(j,i) \in \boldsymbol{\mathcal{E}}$. An undirected graph is said to be *connected* if there exists a path between every pair of nodes.

The *Laplacian Matrix* associated with $\boldsymbol{\mathcal{G}}$ is denoted by $\boldsymbol{L} = [l_{ij}] \in \mathbb{R}^{n\times n}$, where $l_{ii} = \sum_{j=1,i\neq j}^{n} a_{ij}$ and $l_{ij} = -a_{ij}$, $i \neq j$, such that $\boldsymbol{L}\mathbf{1}_n = \mathbf{0}_n$. If $\boldsymbol{\mathcal{G}}$ is undirected then $\boldsymbol{L}$ is positive semi-definite and symmetric, thus satisfying also $1_n^{\mathsf{T}}\boldsymbol{L} = \mathbf{0}_n^{\mathsf{T}}$. Finally, if $\boldsymbol{\mathcal{G}}$ is also connected then $\boldsymbol{L}$ has a simple zero eigenvalue which corresponding eigenvector is the all-one vector $\mathbf{1}_n$.

3 Leader-following consensus for diffusion PDEs

3.1 Problem formulation

Consider a set of n infinite-dimensional dynamical systems, identified as *follower agents*, and communicating throughout a static undirected topology described by the graph $\mathcal{G}_f(\mathcal{V}_f, \mathcal{E}_f, \mathcal{A}_f)$, where $\mathcal{V}_f = \{1, 2, \ldots, n\}$.

The state of each follower is the space- and time-dependent function $q_i(\varsigma, t)$ with $i = 1, 2, \ldots, n$, which is governed by the diffusion equation

$$q_{i,t}(\varsigma, t) = \left[\theta(\varsigma) \cdot q_{i,\varsigma}(\varsigma, t)\right]_{\varsigma}, \tag{15}$$

where the subscripts "t" and "ς" denote, respectively, the temporal and spatial derivatives, whereas $\theta(\varsigma) \in \mathscr{C}^1$ is a positive, and possibly uncertain, smooth function known as *diffusivity*.

The initial condition for the state of each follower is

$$Q_{i0}(\varsigma) = q_i(\varsigma, 0) \in \mathrm{H}^4(0,1). \tag{16}$$

Neumann-type boundary conditions are considered of the form

$$q_{i,\varsigma}(0,t) = 0, \tag{17}$$
$$q_{i,\varsigma}(1,t) = u_i(t) + \psi_i(t), \tag{18}$$

where $u_i(t) \in \mathbb{R}$ is the manipulable boundary control input of the $i-th$ follower agent and $\psi_i(t)$ is a persistent time-varying matching disturbance.

By stacking all the follower agents' states into the vector

$$Q(\varsigma, t) = [q_1(\varsigma, t), q_2(\varsigma, t), \ldots, q_n(\varsigma, t)] \in [\mathrm{H}^4(0,1)]^n,$$

the following compact representation of the followers' boundary value problem (15)-(18) is derived

$$Q_t(\varsigma, t) = \left[\theta(\varsigma) Q_{\varsigma}(\varsigma, t)\right]_{\varsigma} \tag{19}$$
$$Q(\varsigma, 0) \in [\mathrm{H}^4(0,1)]^n \tag{20}$$
$$Q_{\varsigma}(0,t) = \mathbf{0}_n \tag{21}$$
$$Q_{\varsigma}(1,t) = \Psi(t) + U(t) \tag{22}$$

where

$$U(\varsigma, t) = [u_1(t), u_2(t), \ldots, u_n(t)] \in \mathbb{R}^n$$

is the vector collecting all the local control inputs to be designed, and

$$\Psi(\varsigma, t) = [\psi_1(t), \psi_2(t), \ldots, \psi_n(t)] \in \mathbb{R}^n$$

collects all the boundary disturbances.

In addition to the n followers, a *leader* agent is also considered. The leader state $q_0(\varsigma,t) \in \mathrm{H}^4(0,1)$, which without loss of generality can possibly be virtually generated, is governed by the following boundary-value problem

$$q_{0,t}(\varsigma,t) = \left[\theta(\varsigma) q_{0,\varsigma\varsigma}(\varsigma,t)\right]_\varsigma \tag{23}$$

$$q_0(\varsigma,0) \in \mathrm{H}^4(0,1) \tag{24}$$

$$q_{0,\varsigma}(0,t) = 0 \tag{25}$$

$$q_{0,\varsigma}(1,t) = u_0(t) \tag{26}$$

where $u_0(t)$ is the leader control input.

3.2 Control objective and operating assumptions

The control objective consists of designing, for each follower agent, a local boundary control input $u_i(t)$ such that the local states $q_i(\varsigma,t)$ will *asymptotically* track the leader state $q_0(\varsigma,t)$ *point-wisely* in space,i.e. such that the next relation holds

$$\lim_{t\to\infty} |q_i(\varsigma,t) - q_0(\varsigma,t)| = 0, \quad \forall\, t \geq 0,\ \varsigma \in (0,1),\ i \in \boldsymbol{\mathcal{V}}. \tag{27}$$

Let us note that the above point-wise synchronization requirement can be ensured provided that the deviation error

$$E(\varsigma,t) = Q(\varsigma,t) - \mathbf{1}_n \otimes q_0(\varsigma,t). \tag{28}$$

is asymptotically stable in the space $\left[\mathrm{H}^1(0,1)\right]^n$, i.e. that

$$\lim_{t\to\infty} \|E(\varsigma,t)\|_{[\mathrm{H}^1(0,1)]^n} = 0, \tag{29}$$

Note also that the achievement of (29), is equivalent to the following conditions

$$\lim_{t\to\infty} \|E(\varsigma,t)\|_{\mathrm{H}^0(0,1)} = 0, \quad \lim_{t\to\infty} \|E_\varsigma(\varsigma,t)\|_{\mathrm{H}^0(0,1)} = 0. \tag{30}$$

The following assumptions are made.

Assumption 1 *The initial condition* $Q(\varsigma,0) \in [\mathrm{H}^4(0,1)]^n$ *in* (20) *is compatible to the given perturbed boundary conditions, i.e.*

$$Q_\varsigma(0,0) = 0, \quad Q_\varsigma(1,0) = \Psi(0). \tag{31}$$

Assumption 2 *The spatially-varying diffusivity coefficient is uncertain and there exists a-priori known constants Θ_m and Θ_M such that*

$$0 < \Theta_m \leq \theta(\varsigma) \leq \Theta_M, \forall \quad \varsigma \in (0,1). \tag{32}$$

Assumption 3 *There exist known positive constants Π, Ξ such that*

$$\left\|\dot{\Psi}(t)\right\|_{\infty} < \Pi < \infty, \quad |\dot{u}_0(t)| < \Xi < \infty. \tag{33}$$

Remark 1 *With the assumptions above, the collective motion of the overall (leader-plus-followers) network is studied in a high-order Sobolev space being specified to* $[\mathrm{H}^2(0,1)]^{n+1}$. *As a consequence, the domain* $\mathscr{D}(\mathscr{A})$ *of the infinitesimal operator* $\mathscr{A} = \partial^2/\partial\varsigma^2$ *in the boundary value problems* (19)-(22) *and* (23)-(26) *is confined to the Sobolev space* $[\mathrm{H}^4(0,1)]^{n+1}$. *For details the reader may refer to [Curtain & Zwart (1995)].*

4 Main result

4.1 Control synthesis

To construct the local interaction law, it is assumed the leader state is available only to a non-empty subset of followers such that node 0 is as a *root node* of the overall communication graph $\boldsymbol{\mathcal{G}}(\boldsymbol{\mathcal{V}},\boldsymbol{\mathcal{E}},\boldsymbol{A})$ with $\boldsymbol{\mathcal{V}} = \{0\} \cup \mathbf{V}_f$. Moreover, the leader input $u_0(\varsigma,t)$ is assumed to be independent on the state of the followers'.

Under the given communication constraints the overall communication network comprising both the leader and the followers is thus characterized by the following adjacency matrix

$$\boldsymbol{A} = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 \\ a_{10} & & & & \\ a_{20} & & & & \\ \vdots & & \boldsymbol{A}_f & & \\ a_{n0} & & & & \end{bmatrix} \tag{34}$$

where $a_{i0} > 0$ if the leader position is available to the i-th follower, and $a_{i0} = 0$ otherwise. It is further assumed each follower can only access its own state and the boundary measurement $q_h(1,t)$ of its neighbouring follower agents i.e. $h \in \{\ i\ \} \cup \boldsymbol{\mathcal{N}}_i$.

To achieve the control goal (29), or equivalently (30), the state of each follower will be augmented through a *dynamic input extension* by inserting an integrator at the controlled boundary input. Then, the time-derivative of the local control signal is regarded as a fictitious control input to be constructed by a suitable feedback mechanism. In particular, the next local interaction protocol

$$\begin{aligned}\dot{u}_i(t) = & -a\cdot \mathrm{sign}\left(\sum_{j=0}^{n} a_{ij}\left(q_i(1,t)-q_j(1,t)\right)\right)\\ & -b\cdot \mathrm{sign}\left(\sum_{j=0}^{n} a_{ij}\left(q_{i,t}(1,t)-q_{j,t}(1,t)\right)\right)\\ & -c\cdot \left(\sum_{j=0}^{n} a_{ij}\left(q_i(1,t)-q_j(1,t)\right)\right)\\ & -d\cdot \left(\sum_{j=0}^{n} a_{ij}\left(q_{i,t}(1,t)-q_{j,t}(1,t)\right)\right) \end{aligned} \tag{35}$$

is proposed, where $i=1,2,\dots,n$, and which initial values $u_i(0)$ are all set to zero compatibly with Assumption 1. Coefficient a_{ij} denotes the (i,j) entry of the adjacency matrix associated with the overall communication graph $\boldsymbol{\mathcal{G}}$. It is thus apparent that each local controller only requires information from the neighbouring agents. Finally, a, b, c, and, d are the non-negative design parameters which will be subject to proper design tuning inequalities to be derived in the sequel.

The next auxiliary boundary-value problem

$$E_{tt}(\varsigma,t) = \left[\theta(\varsigma)E_{t\varsigma}(\varsigma,t)\right]_{\varsigma} \tag{36}$$

$$E_{t\varsigma}(0,t) = \mathbf{0}_n \tag{37}$$

$$E_{t\varsigma}(1,t) = \Psi_t(t) - \mathbf{1}_n\otimes\dot{u}_0(t) + U_t(t) \tag{38}$$

governing the error variable $E(\varsigma,t)$ is thus derived. The boundary input vector $U_t(t)=[\dot{u}_1(t),\dot{u}_2(t),\dots,\dot{u}_n(t)]^T$, can be rewritten in compact form as follows

$$\begin{aligned}U_t(t) = & -a\,\mathrm{Sign}\left(\boldsymbol{M}E(1,t)\right) - b\,\mathrm{Sign}\left(\boldsymbol{M}E_t(1,t)\right)\\ & -c\,\boldsymbol{M}E(1,t) - d\,\boldsymbol{M}E_t(1,t),\end{aligned} \tag{39}$$

$$\boldsymbol{M} = \boldsymbol{L}_f - \mathrm{diag}\left(\left[a_{10}\ a_{20}\ \dots\ a_{n0}\right]\right)\in\mathbb{R}^{n\times n},$$

where $\boldsymbol{L}_f$ is the Laplacian matrix associated with $\boldsymbol{\mathcal{G}}_f$. Note that $\boldsymbol{M}=\boldsymbol{M}^\intercal\succ 0$ is a positive definite matrix (see [Cao et al. (2012)]).

Since the dynamic control vector $U(t)$ is governed by the discontinuous right-hand side ordinary differential equation (39), then the solutions of the resulting closed-loop boundary value problem (36)-(38), driven by the dynamic controller (39), need to be specified in the so-called *Filippov sense*.

Nonetheless, as established in [Levaggi (2002)], such a generalized solution for an infinite-dimensional system is a *mild solution* of the closed-loop system (36)-(38) with the multi-valued discontinuous feedback (39).

In the remainder the meaning of the auxiliary boundary-value problem (36)-(39) will thus be viewed in the *mild sense* [Curtain & Zwart (1995)]. Moreover, in accordance with [Buthovskiy (1982)], it should be noticed that these mild solutions coincide with the corresponding *weak solutions* of the so-called standardizing PDE in distributions

$$E_t(\varsigma,t) = W(\varsigma,t) \tag{40}$$

$$W_t(\varsigma,t) = \left[\theta(\varsigma)W_\varsigma(\varsigma,t)\right]_\varsigma + \theta(1)\left(\Psi_t(t) - \mathbf{1}_n \otimes \dot{u}_0(t) + U_t(t)\right)\delta(\varsigma-1) \tag{41}$$

$$E_\varsigma(0,t) = E_\varsigma(1,t) = \mathbf{0}_n \tag{42}$$

$$W_\varsigma(0,t) = W_\varsigma(1,t) = \mathbf{0}_n \tag{43}$$

where

$$E(\varsigma,t) = Q(\varsigma,t) - \mathbf{1}_n \otimes q_0(t), \tag{44}$$

$$W(\varsigma,t) = E_t(\varsigma,t) = Q_t(\varsigma,t) - \mathbf{1}_n \otimes \dot{q}_0(t). \tag{45}$$

It is also worth mentioning that, as in the finite-dimensional case, a motion along the discontinuity manifolds

$$E(1,t) = \mathbf{0}_n \quad \text{and} \quad W(1,t) = \mathbf{0}_n$$

generates a sliding motion on it. The rigorous demonstration of the well-posedness of the closed-loop network dynamics goes beyond the scope of this paper. Anyway, the well-posedness of the system in question is actually verifiable, in accordance with [Curtain & Zwart (1995), Theorem 3.3.3], by taking into account that the effective control input $U(t)$ is twice piece-wise continuously differentiable along the solutions $E(\varsigma,t)$.

By virtue of (36), the next relation

$$\|W(\cdot,t)\|_{\mathrm{H}^0(0,1)} = \|E_t(\cdot,t)\|_{\mathrm{H}^0(0,1)} = \left\|\left[\theta(\varsigma)E_\varsigma(\varsigma,t)\right]_\varsigma\right\|_{\mathrm{H}^0(0,1)}, \tag{46}$$

which will be instrumental in the next derivations is finally introduced.

4.2 *Convergence analysis*

The main result of the chapter and the resulting convergence analysis will be now discussed in detail. The following preliminary instrumental result is demonstrated.

Lemma 2. *Let Assumption 1, 2 and 3 be satisfied, and let $W(\varsigma,t)$ be defined as in* (40). *Then, the functional $V : [\mathrm{H}^2(0,1)]^n \times [\mathrm{H}^0(0,1)]^n \mapsto \mathbb{R}$*

$$\mathrm{V}(E,W) = a\theta(1)\|\boldsymbol{M}E(1,t)\|_1 + \frac{1}{2}c\theta(1)\|\boldsymbol{M}E(1,t)\|_2^2 + \frac{1}{2}\int_0^1 W(\zeta,t)^\intercal \boldsymbol{M} W(\zeta,t)d\zeta, \tag{47}$$

being computed on the mild solutions $(E(\varsigma,t),W(\varsigma,t))$ of the closed-loop boundary-value problem (40)-(43) *is non-negative, and it upper estimates the weighted $\left[\mathrm{H}^{2,\theta}(0,1)\right]^n \times \left[\mathrm{H}^0(0,1)\right]^n$-norm of these solutions, in the sense that*

$$\exists\, \lambda > 0 \,:\, \mathrm{V}(E,W) \geq \lambda \left(\|E(\cdot,t)\|^2_{[\mathrm{H}^{2,\theta}(0,1)]^n} + \|W(\cdot,t)\|^2_{[\mathrm{H}^0(0,1)]^n} \right) \;\; \forall t \geq 0 \tag{48}$$

□

Proof of Lemma 2: Let $0 < \mathrm{m}_1 < \mathrm{m}_2 < \cdots < \mathrm{m}_n$ be the ordered sequence of the real positive eigenvalues of the symmetric positive-definite matrix $\boldsymbol{M}$. By applying (2) and (3), it results that

$$\|\boldsymbol{M}E(1,t)\|_1 \geq \frac{\mathrm{m}_1}{\sqrt{n}}\,\|E(1,t)\|_1, \tag{49}$$

$$\|\boldsymbol{M}E(1,t)\|_2^2 \geq \mathrm{m}_1^2\,\|E(1,t)\|_2^2, \tag{50}$$

$$W(\varsigma,t)^\intercal \boldsymbol{M} W(\varsigma,t) \geq \mathrm{m}_1\,\|W(\varsigma,t)\|_2^2. \tag{51}$$

Spatial integration of all terms in (51) yields

$$\int_0^1 W(\varsigma,t)^\intercal \boldsymbol{M} W(\varsigma,t)d\varsigma \geq \mathrm{m}_1\,\|W(\cdot,t)\|^2_{[\mathrm{H}^0(0,1)]^n}. \tag{52}$$

Thus, from (49)-(51) and (52), one concludes that

$$\mathrm{V}(E,W) \geq a\theta(1)\mathrm{m}_1\,\|E(1,t)\|_1 + \frac{c\theta(1)\mathrm{m}_1^2}{2}\|E(1,t)\|_2^2 + \frac{\mathrm{m}_1}{2}\|W(\cdot,t)\|^2_{[\mathrm{H}^0(0,1)]^n} \tag{53}$$

Successively applying relation (13) specified with $i = 1$ to the mild solutions $Z(\varsigma,t) = E(\varsigma,t)$ and $Z(\varsigma,t) = \theta(\varsigma)E_\varsigma(\varsigma,t)$, and because of the homogenous boundary conditions (42)-(43), one derives that

$$\|E(\cdot,t)\|^2_{[\mathrm{H}^0(0,1)]^n} \leq 2\left(\|E(1,t)\|_2^2 + \|E_\varsigma(\cdot,t)\|^2_{[\mathrm{H}^0(0,1)]^n}\right), \tag{54}$$

$$\begin{aligned}\|\theta(\cdot)E_\varsigma(\cdot,t)\|^2_{[\mathrm{H}^0(0,1)]^n} &\leq 2\left(\theta^2(1)\|E_\varsigma(1,t)\|_2^2 + \|[\theta(\cdot)E_\varsigma(\cdot,t)]_\varsigma\|^2_{[\mathrm{H}^0(0,1)]^n}\right) \\ &= 2\|[\theta(\cdot)E_\varsigma(\cdot,t)]_\varsigma\|^2_{[\mathrm{H}^0(0,1)]^n}.\end{aligned} \tag{55}$$

Moreover, by exploiting the next chain of inequalities

$$\Theta_m^2 \|E(\cdot,t)\|^2_{[\mathrm{H}^0(0,1)]^n} \le \|\theta(\cdot)E_\varsigma(\cdot,t)\|^2_{[\mathrm{H}^0(0,1)]^n} \le \Theta_M^2 \|E(\cdot,t)\|^2_{[\mathrm{H}^0(0,1)]^n}, \tag{56}$$

which is a trivial consequence of (32), relation (55) can be further manipulated so as to obtain

$$\|E_\varsigma(\cdot,t)\|^2_{[\mathrm{H}^0(0,1)]^n} \le \beta \|[\theta(\cdot)E_\varsigma(\cdot,t)]_\varsigma\|^2_{[\mathrm{H}^0(0,1)]^n} \quad \text{with} \quad \beta = \frac{2}{\Theta_m^2}. \tag{57}$$

Now, by employing relation (46), it follows from (54), and (57), that

$$\begin{aligned} \|E(\cdot,t)\|^2_{[\mathrm{H}^{2,\theta}(0,1)]^n} &= \|E(\cdot,t)\|^2_{[\mathrm{H}^0(0,1)]^n} + \|E_\varsigma(\cdot,t)\|^2_{[\mathrm{H}^0(0,1)]^n} \\ &\quad + \|[\theta(\cdot)E_\varsigma(\cdot,t)]_\varsigma\|^2_{[\mathrm{H}^0(0,1)]^n} \\ &\le 2\|E(1,t)\|_2^2 + (3\beta+1)\|[\theta(\cdot)E_\varsigma(\cdot,t)]_\varsigma\|^2_{[\mathrm{H}^0(0,1)]^n} \\ &= 2\|E(1,t)\|_2^2 + (3\beta+1)\|W(\cdot,t)\|^2_{[\mathrm{H}^0(0,1)]^n}. \end{aligned} \tag{58}$$

Now, taking into account (53), and (58), the validity of (48) is finally concluded for all $t \ge 0$ and for some positive constant λ. The Proof of Lemma 2 is thus concluded. □

We are now in position to state the main result of the paper.

Theorem 1. *Consider the perturbed network of followers* (19)-(22) *communicating along the undirected topology* $\mathcal{G}_f(\mathcal{V}_f, \mathcal{E}_f, A_f)$*, with* $\mathcal{V}_f = \{1,2,\ldots,n\}$*.*
Let Assumption 1, 2 and 3 be in force. If the leader node "0" is a root node for the overall communication graph $\mathcal{G}(\mathcal{V}, \mathcal{E}, A)$*, with* $\mathcal{V} = \{0\} \cup \mathcal{V}_f$*, then the boundary second-order sliding mode consensus protocol* (35) *with the tuning parameters selected according to*

$$a > b + \Pi + \Xi, \quad b > \frac{\Theta_M}{\Theta_m}(\Pi + \Xi), \quad c > 0, \quad d > 0, \tag{59}$$

ensures the global exponential stability of the mild solutions $(E(\varsigma,t), W(\varsigma,t))$ *of the error boundary-value problem* (40)-(43) *in the space* $[\mathrm{H}^2(0,1)]^n \times [\mathrm{H}^0(0,1)]^n$*.* □

4.3 *Proof of Theorem 1*

The proof is divided into two separated steps. Firstly, by means of Lyapunov arguments it will be demonstrated the *uniform stability* of the solutions of the boundary-value problem (40)-(43). Then, by relying on this preliminary result, the exponential stability of the corresponding solutions in the in the space $[\mathrm{H}^2(0,1)]^n \times [\mathrm{H}^0(0,1)]^n$ is finally proven.

Uniform stability

By Lemma 2, functional (47) is positive definite along the mild solutions $(E,W) \in [\mathrm{H}^2(0,1)]^n \times [\mathrm{H}^0(0,1)]^n$ of the boundary-value problem (40)-(43). The time derivative of $\mathrm{V}(t)$ along these solutions is

$$\begin{aligned}\dot{\mathrm{V}}(t) = {} & a\theta(1)W(1,t)^\intercal \boldsymbol{M}\mathrm{Sign}(\boldsymbol{M}E(1,t)) + c\theta(1)E(1,t)^\intercal \boldsymbol{M}^2 W(1,t) \\ & + \int_0^1 W(\varsigma,t)^\intercal \boldsymbol{M}[\theta(\varsigma)W_\varsigma(\varsigma,t)]_\varsigma d\varsigma \\ & + \theta(1)W(1,t)\left(\Psi_t(t) - \mathbf{1}_n \otimes \dot{u}_0(t) + U_t(t)\right) \end{aligned} \tag{60}$$

The integral term in the right hand side of (60), being integrated by parts by taking into account the boundary conditions (43), is expressed as follows

$$\begin{aligned}\int_0^1 W(\varsigma,t)^\intercal \boldsymbol{M}[\theta(\varsigma)W_\varsigma(\varsigma,t)]_\varsigma d\varsigma = {} & \left[\theta(1)W_\varsigma(1,t) - \theta(0)W_\varsigma(0,t)\right] \\ & - \int_0^1 \theta(\varsigma)W_\varsigma(\varsigma,t)^\intercal \boldsymbol{M} W_\varsigma(\varsigma,t)\, d\varsigma \\ = {} & - \int_0^1 \theta(\varsigma)W_\varsigma(\varsigma,t)^\intercal \boldsymbol{M} W_\varsigma(\varsigma,t)\, d\varsigma \end{aligned} \tag{61}$$

In accordance with (51), the negative definite term (61) can further be upper-estimated as follows

$$- \int_0^1 \theta(\varsigma)W_\varsigma(\varsigma,t)^\intercal \boldsymbol{M} W_\varsigma(\varsigma,t) d\varsigma \le -m_1\Theta_m \|W_\varsigma(\cdot,t)\|^2_{[\mathrm{H}^2(0,1)]^n}. \tag{62}$$

By substituting (61) and (62) into (60), then one obtains

$$\begin{aligned}\dot{\mathrm{V}}(t) \le {} & a\theta(1)W(1,t)^\intercal \boldsymbol{M}\mathrm{Sign}(\boldsymbol{M}E(1,t)) + c\theta(1)E(1,t)^\intercal \boldsymbol{M}^2 W(1,t) \\ & - m_1\Theta_m \|W_\varsigma(\cdot,t)\|^2_{[\mathrm{H}^2(0,1)]^n} + \theta(1)W(1,t)\left(\Psi_t(t) - \mathbf{1}_n \otimes \dot{u}_0(t) + U_t(t)\right). \end{aligned} \tag{63}$$

Now, by replacing (39) into (63), and making simple manipulations, one obtains

$$\begin{aligned}\dot{\mathrm{V}}(t) \le {} & -m_1\Theta_m \|W_\varsigma(\cdot,t)\|^2_{[\mathrm{H}^2(0,1)]^n} - b\theta(1)\|\boldsymbol{M}W(1,t)\|_1 \\ & - d\theta(1)\|\boldsymbol{M}W(1,t)\|_2^2 + \theta(1)W(1,t)^\intercal \boldsymbol{M}(\Psi_t(t) - \mathbf{1}_n \otimes \dot{u}_0(t)). \end{aligned} \tag{64}$$

By (4), and taking into account Assumption 3 and relation (33), the last term in the right hand side of (64) is upper-estimated as follows

$$\begin{aligned}\mid W(1,t)^\intercal \boldsymbol{M}(\Psi_t(t) - \mathbf{1}_n \otimes \dot{u}_0(t)) \mid \le {} & \left(\|\Psi_t(t)\|_\infty + \|\mathbf{1}_n \otimes \dot{u}_0(t)\|_\infty\right) \|\boldsymbol{M}W(1,t)\|_1 \\ \le {} & (\Pi + \Xi)\, \|\boldsymbol{M}W(1,t)\|_1. \end{aligned} \tag{65}$$

Finally, substituting (65) into (63), and taking into account (50), one derives, after some manipulations, the next expression

$$\begin{aligned}\dot{\mathrm{V}}(t) \leq & -[b\Theta_m - \Theta_M(\Pi+\Xi)]\|\boldsymbol{M}W(1,t)\|_1 \\ & -m_1\Theta_m\|W_\varsigma(\cdot,t)\|^2_{[\mathrm{H}^0(0,1)]^n} - d\Theta_m m_1^2\|W(1,t)\|_2^2\end{aligned} \tag{66}$$

Thus, due to (66) and (59), the time derivative of the Lyapunov functional $\mathrm{V}(t)$, being computed along the mild solutions of the closed-loop boundary-value problem (40)-(43) is negative semi-definite.

This implies that $\mathrm{V}(t)$ is a non-increasing function of time and the compact domain

$$\mathscr{D}_R^{\mathrm{V}} = \left\{ (E,W) \in [\mathrm{H}^2(0,1)]^n \times [\mathrm{H}^0(0,1)]^n : \mathrm{V} \leq R \right\}, \tag{67}$$

specified for an arbitrary large ball of radius $R \geq \mathrm{V}(0)$ is thus invariant.

It then follows that the closed-loop solutions will be confined in the invariant domain $\mathscr{D}_R^{\mathrm{V}}$ forever. Due to the invariance of the domain $\mathscr{D}_R^{\mathrm{V}}$, and (47), the inequality $\mathrm{V}(t) \leq R$ yields the next estimates

$$\|\boldsymbol{M}E(1,t)\|_1 \leq \frac{R}{a\Theta_m}, \quad \|E(1,t)\|_2^2 \leq \frac{2R}{c\Theta_m m_1^2}, \quad \|W(\cdot,t)\|^2_{[\mathrm{H}^0(0,1)]^n} \leq \frac{2R}{m_1}. \tag{68}$$

Global exponential stability

Consider the augmented functional

$$\mathrm{V}_R(t) = \mathrm{V}(t) + \kappa_R\, \bar{\mathrm{V}}(t) \tag{69}$$

where κ_R is a sufficiently small positive constant to be specified later, and $\bar{\mathrm{V}}(t)$ is the following sign-indefinite function

$$\bar{\mathrm{V}}(t) = \frac{1}{2}d\theta(1)\, \|\boldsymbol{M}E(1,t)\|_2^2 + \int_0^1 E(1,t)^\intercal \boldsymbol{M}W(\varsigma,t)d\varsigma. \tag{70}$$

By invoking inequalities (2) and (4), and the first relation in (68), it results that, within the domain $\mathscr{D}_R^{\mathrm{V}}$ in (67), the second term in the right-hand side of (70) can be lower-estimated as follows

$$\begin{aligned}\int_0^1 E(1,t)^\intercal \boldsymbol{M}W(\varsigma,t)d\varsigma &\geq -\frac{1}{2}\left(\|\boldsymbol{M}E(1,t)\|_2^2 + \|W(\cdot,t)\|^2_{[\mathrm{H}^0(0,1)]^n}\right) \\ &\geq -\frac{1}{2}\left(\|\boldsymbol{M}E(1,t)\|_1^2 + \|W(\cdot,t)\|^2_{[\mathrm{H}^0(0,1)]^n}\right) \\ &\geq -\frac{1}{2}\left(\frac{R}{a\Theta_m}\|\boldsymbol{M}E(1,t)\|_1 + \|W(\cdot,t)\|^2_{[\mathrm{H}^0(0,1)]^n}\right).\end{aligned} \tag{71}$$

Considering (70), (71), and (50), along with (47), it results the right-hand side of (69) can be lower-estimated as follows

$$\mathrm{V}_R(t) \geq \left(a\Theta_m - \kappa_R \frac{R}{2a\Theta_m}\right)\|\boldsymbol{M}E(1,t)\|_1 + \frac{1}{2}(\mathrm{m}_1 - \kappa_R)\|W(\cdot,t)\|^2_{[\mathrm{H}^0(0,1)]^n}$$
$$+\frac{1}{2}\Theta_m \mathrm{m}_1^2 (c+\kappa_R d)\|E(1,t)\|_2^2. \quad (72)$$

Thus, the positive definitiveness of $V_R(t)$ is guaranteed by selecting the positive constant κ_R small enough in accordance with the next expression

$$\kappa_R \leq \min\left\{\tfrac{2a^2\Theta_m^2}{R}\ ,\ \mathrm{m}_1\right\}. \quad (73)$$

As a consequence, in the invariant domain (67) the augmented functional $\mathrm{V}_R(t)$ turns out to be lower estimated by $\mathrm{V}(t)$ as follows

$$\mathrm{V}_R(t) \geq \gamma\,\mathrm{V}(t) \succ 0 \quad (74)$$

where

$$\gamma = \min\left\{1 - \tfrac{\kappa_R R}{2a^2\Theta_m^2}\ ,\ \mathrm{m}_1 - \kappa_R\ ,\ \tfrac{c}{c+\kappa_R d}\right\}. \quad (75)$$

Let us now evaluate the time-derivative of (69) along the solutions of (40)-(43). By simple manipulation it results that

$$\begin{aligned}\dot{V}_R(t) &= \dot{V}(t) + \kappa_R d\theta(1)W^T(1,t)\boldsymbol{M}^2E(1,t) + \kappa_R \frac{d}{dt}\int_0^1 E(1,t)^\intercal \boldsymbol{M}W(\varsigma,t)d\varsigma \\ &= \dot{V}(t) + \kappa_R d\theta(1)W^T(1,t)\boldsymbol{M}^2E(1,t) + \kappa_R \int_0^1 E(1,t)^\intercal \boldsymbol{M}W_t(\varsigma,t)d\varsigma \\ &\quad + \kappa_R \int_0^1 W(1,t)^\intercal \boldsymbol{M}W(\varsigma,t)d\varsigma. \end{aligned} \quad (76)$$

By solving the the first integral term in the right hand side of (76), in light of the homogenous boundary conditions (40)-(43) and by (35), it yields that

$$\begin{aligned}\int_0^1 E(1,t)^\intercal \boldsymbol{M}W_t(\varsigma,t)d\varsigma &= E(1,t)^\intercal \boldsymbol{M}\int_0^1 \left[\theta(\varsigma)W_\varsigma(\varsigma,t)\right]_\varsigma d\varsigma \\ &\quad + E(1,t)^\intercal \boldsymbol{M}\ \theta(1)\left(\Psi_t(t) - \mathbf{1}_n \otimes \dot{u}_0(t) + U_t(t)\right) \\ &= E(1,t)^\intercal \boldsymbol{M}\ \left(\theta(1)W_\varsigma(1,t) - \theta(0)W_\varsigma(0,t)\right) \\ &\quad + \theta(1)E(1,t)^\intercal \boldsymbol{M}\left(\Psi_t(t) - \mathbf{1}_n \otimes \dot{u}_0(t) + U_t(t)\right) \\ &= -a\theta(1)\|\boldsymbol{M}E(1,t)\|_1 - b\theta(1)E(1,t)^\intercal \boldsymbol{M}\mathrm{Sign}(\boldsymbol{M}W(1,t)) \\ &\quad - c\theta(1)\|\boldsymbol{M}E(1,t)\|_2^2 - d\theta(1)E(1,t)^\intercal \boldsymbol{M}^2W(1,t) \\ &\quad + \theta(1)E(1,t)^\intercal \boldsymbol{M}(\Psi_t(t) + 1_n\dot{u}_0(t)). \end{aligned} \quad (77)$$

Then, by invoking (4), and (33), and (68), the sign-undefined terms of (77) can be upper-estimated as next

$$|-b\,E(1,t)\boldsymbol{M}\mathrm{Sign}(\boldsymbol{M}W(1,t))| \leq b\,\|\boldsymbol{M}E(1,t)\|_1, \quad (78)$$
$$|\,E(1,t)^\intercal \boldsymbol{M}(\Psi_t(t) - \mathbf{1}_n \otimes \dot{u}_0(t))| \leq (\Pi + \Xi)\ \|\boldsymbol{M}E(1,t)\|_1 \quad (79)$$

Let m_{ij} be the (i,j) entry of matrix $\boldsymbol{M}$. Then, the last term in the right hand side of (76) is estimated by means of the next chain of inequalities, which is constructed on the basis of (68) and of the Holder integral inequality, as follows

$$\begin{aligned}
\left|\kappa_R \int_0^1 W(1,t)^\top \boldsymbol{M} W(\varsigma,t) d\varsigma\right| &= \kappa_R \left|\int_0^1 \sum_{i=1}^n \sum_{j=1}^n W_i(1,t) m_{ij} W_j(\varsigma,t) d\varsigma\right| \\
&\le \kappa_R \sum_{j=1}^n \int_0^1 \mid W_j(\varsigma,t) \mid d\varsigma \left(\sum_{i=1}^n \mid W_i(1,t) m_{ij} \mid\right) \\
&\le \kappa_R \left(\sum_{j=1}^n \|W_j(\cdot,t)\|_{\mathrm{H}^0(0,1)}\right)\left(\sum_{i=1}^n \mid W_i(1,t) m_{ij} \mid\right) \\
&\le \kappa_R \sum_{i,j} \sqrt{\frac{2R}{m_1}} \mid m_{ij} W_i(1,t) \mid \\
&= \kappa_R \sqrt{\frac{2R}{m_1}} \|\boldsymbol{M} W(1,t)\|_1
\end{aligned} \tag{80}$$

By (77)-(80), and (66), relation (76) can further be manipulated as follows:

$$\begin{aligned}
\dot{\mathrm{V}}_R(t) \le &-\left(b\Theta_m - \Theta_M(\Pi+\Xi) - \kappa_R\sqrt{\frac{2R}{m_1\Theta_m^2}}\right)\|\boldsymbol{M} W(1,t)\|_1 \\
&-\kappa_R\Theta_m\left(a-b-\Pi-\Xi\right)\|\boldsymbol{M} E(1,t)\|_1 - m_1^2\Theta_m d \ \|W(1,t)\|_2^2 \\
&-m_1\Theta_m \ \|W_\varsigma(\cdot,t)\|^2_{[\mathrm{H}^0(0,1)]^n} - \kappa_R m_1^2 \Theta_m c \ \|E(1,t)\|_2^2.
\end{aligned} \tag{81}$$

Let us now consider Lemma 1 specialized for $i=1$ and $Z(\varsigma,t)=W(\varsigma,t)$. From the resulting relation, the next inequality is derived

$$\|W(1,t)\|_2^2 + \|W_\varsigma(\cdot,t)\|^2_{[\mathrm{H}^0(0,1)]^n} \ge \frac{1}{2}\|W(\cdot,t)\|^2_{[\mathrm{H}^0(0,1)]^n}. \tag{82}$$

Then, in light of the above relation, one further obtains that

$$-m_1^2\Theta_m d\|E(1,t)\|_2^2 - m_1\Theta_m\|W_\varsigma(\cdot,t)\|_2^2 \le -\upsilon_4\|W(\cdot,t)\|^2_{[\mathrm{H}^0(0,1)]^n} \tag{83}$$

where

$$\upsilon_4 = \min\left\{2m_1\Theta_m \ , \ 2m_1^2\Theta_m d\right\}.$$

Now, by substituting (83) into the right-hand side of (81), the inequality

$$\begin{aligned}
\dot{\mathrm{V}}_R(t) \le &-\upsilon_1 \ \|\boldsymbol{M} E(1,t)\|_1 - \upsilon_2 \ \|\boldsymbol{M} W(1,t)\|_1 \\
&-\upsilon_3 \ \|E(1,t)\|_2^2 - \upsilon_4 \ \|W(\cdot,t)\|^2_{[\mathrm{H}^0(0,1)]^n}
\end{aligned} \tag{84}$$

is finally obtained, with the coefficients

$$\upsilon_1 = \kappa_R \Theta_m (a - b - \Pi - \Xi), \tag{85}$$

$$\upsilon_2 = \left(b\Theta_m - \Theta_M(\Pi + \Xi) - \kappa_R \sqrt{\frac{2R}{m_1 \Theta_m^2}} \right), \tag{86}$$

$$\upsilon_3 = \kappa_R m_1^2 \Theta_m c, \tag{87}$$

$$\upsilon_4 = \min\{2m_1\Theta_m, 2m_1^2\Theta_m d\}. \tag{88}$$

Thus, it is clear that all terms in the right-hand side of (84) are non-positive provided that the tuning conditions (59) imposed on the controller parameters hold and, in place of (73), the next more restrictive condition on the coefficient κ_R is additionally satisfied

$$\kappa_R \leq \min\left\{ \tfrac{2a^2}{R} \ , \ m_1 \ , \ (b\Theta_m - \Theta_M(\Pi + \Xi)) / \sqrt{\tfrac{2R}{m_1\Theta_m^2}} \right\}. \tag{89}$$

From (84)-(89), it then follows that

$$\dot{\mathrm{V}}_R(t) \leq -\gamma_1 \left(\|\boldsymbol{M}E(1,t)\|_1 + \|E(1,t)\|_2^2 + \|W(\cdot,t)\|^2_{[\mathrm{H}^0(0,1)]^n} \right) \tag{90}$$

where $\gamma_1 = \min\{\upsilon_1, \upsilon_3, \upsilon_4\} \in \mathbb{R}^+$. Now, by (47) combined with the second inequality of (3), it yields

$$\mathrm{V}(t) \leq a\Theta_M \ \|\boldsymbol{M}E(1,t)\|_1 + \frac{1}{2} c\Theta_M m_n^2 \ \|E(1,t)\|_2^2 + \frac{1}{2} m_n \|W(\cdot,t)\|_{[\mathrm{H}^0(0,1)]^n}, \tag{91}$$

whereas using (3), (4) and (71), relation (91) can further be estimated as follows

$$\bar{\mathrm{V}}(t) \leq \frac{1}{2} m_n^2 \Theta_M d \|E(1,t)\|_2^2 + \frac{R}{2a\Theta_m} \|\boldsymbol{M}E(1,t)\|_1 + \frac{1}{2} m_n \|W(\cdot,t)\|_{[\mathrm{H}^0(0,1)]^n}. \tag{92}$$

Substituting the last two equations in (69), one finally obtains that

$$\mathrm{V}_R(t) \leq \gamma_2 \left(\|\boldsymbol{M}E(1,t)\|_1 + \|E(1,t)\|_2^2 + \|W(\cdot,t)\|^2_{\mathrm{H}^{0,n}} \right) \tag{93}$$

where

$$\gamma_2 = \min\left\{ \tfrac{2a^2\Theta_M^2 - \kappa_R R}{2a\Theta_m} \ , \ \tfrac{1}{2} m_n^2 \Theta_M (c + \kappa_R d) \ , \ \tfrac{1}{2} m_n (1 - \kappa_R) \right\} > 0.$$

Thus, from (90) and (93), it can be concluded that

$$\dot{\mathrm{V}}_R(t) \leq -\rho_R \ \mathrm{V}_R(t) \quad \text{with} \quad \rho_R = \frac{\gamma_1}{\gamma_2} \tag{94}$$

thereby confirming the exponential decay of $V_R(t)$, initialized within the invariant set $\mathcal{D}_R^V$ in (67).

To complete the proof, note that, due to the upper-estimation (74) of the functional $\mathrm{V}(t)$ by the functional $\mathrm{V}_R(t)$, it follows that $\mathrm{V}(t)$, being com-

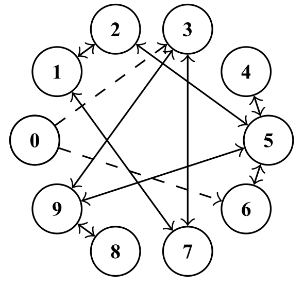

Fig. 1 Communication graph $\mathcal{G}$ describing the enabled interactions among agents.

puted on the mild solutions $(E(\varsigma,t),W(\varsigma,t))$ of the boundary value problem (40)-(43) exponentially decays, too, and by virtue of Lemma 2, the exponential stability of (40)-(43) with the augmented state $(E(\varsigma,t),W(\varsigma,t))$ in the $[\mathrm{H}^2(0,1)]^n \times [\mathrm{H}^0(0,1)]^n$-space is established for any point within the initial set $\mathscr{D}_R^{\mathrm{V}}$ defined in (67). Since $\boldsymbol{\mathcal{D}}_R^V$ can be specified with an arbitrarily large radius $R > 0$, and the tuning rules (59) do not depend by R, condition (30) is thus guaranteed. The proof of Theorem 1 is concluded. □

5 Simulation Results

Let us consider a networked system consisting of $n = 9$ follower agents indexed from "1" to "9", and a leader agent identified by the index "0". The interaction topology associated with the considered case study is encoded by the graph $\mathcal{G}$ depicted in Figure 1, and where, for the sake of simplicity, all the edges have a unitary weight.

Each follower is governed by the boundary-value problem (15)-(18), which initial condition is arbitrarily set equal to

$$q_i(\varsigma,0) = i + \varphi_i \cos(4\pi\varsigma), \quad \text{with} \quad i = 1,2,\ldots,9,$$

whereas the unknown disturbances acting at the controlled boundary are chosen as next

$$\psi_i(t) = \sin(\phi_i\, t) \quad \text{with} \quad \phi_i \in (\pi, 2\pi),\ i = 1,2,\ldots,9,$$

where the coefficient ϕ_i is randomly selected, for each follower, within the range $(\pi,2\pi)$, and thus it satisfies, in accordance with Assumption 3, the following constraint

$$\max_i \|\dot{\psi}_i\|_\infty \leq \Theta = 2\pi,$$

thus implying, respectively, $\|\Psi(t)\|_\infty = 1$ and $\|\Psi_t(t)\|_\infty = 2\pi$.

The scalar field $q_0(\varsigma,t)$ associated with the leader agent, which without loss of generality can also be numerically generated at higher control levels, is the solution of the boundary-value problem (23)-(26), and where its local Neumann-type control signal $u_0(t)$ is ad-hoc designed to enforce an admissible desired profile to be tracked by the collection of followers.

For simulation purposes, in the remainder the leader control signal $u_0(t)$ is selected in accordance with the motion planning strategies proposed in [Siranosian et al. (2009)] and [Krstic & Smyshlyaev (2008)] to enforce at the under-actuated extremity, namely at $\varsigma = 0$, the following "reference" waveform

$$q_0(0,t) = q_0^r(0,t) = 10 + \sin(\omega_1 t) + \sin(\omega_2 t) \quad \text{with} \quad \omega_1 = \pi,\ \omega_2 = \pi\sqrt{2}.$$

In particular, similarly to [Krstic & Smyshlyaev (2008), Examples 12.1 and 12.3], the leader state profile is firstly expressed by means of the next power series of the spatial variable

$$\begin{aligned} u_0^r(\varsigma,t) = \sum_{k=0}^{\infty} \phi_k(t)\varsigma^k/k! = \mathrm{Im}\Bigg\{ & 10 + \cosh\left((1+j)\sqrt{\frac{\omega_1}{2}}\,\varsigma\right) \\ & \times e^{j\omega_1 t} + \cosh\left((1+j)\sqrt{\frac{\omega_2}{2}}\,\varsigma\right) \times e^{j\omega_2 t} \Bigg\}, \end{aligned} \tag{95}$$

and where the field initial condition is arbitrary, and set to $q_0(\varsigma,0) = 11.24$.

Thus, from (95), and in accordance with [Siranosian et al. (2009), Section 3.1], as well as a Dirichlet actuation can be found by computing the previous function (95) at $\varsigma = 1$, thus obtaining

$$q(1,t) = u_0^r(1,t),$$

in (26), the resulting Neumann-type boundary actuation is found by evaluating the first spatial derivative of (95) at $\varsigma = 1$ as next

$$u_0(t) = \left.\frac{\partial q_0^r(\varsigma,t)}{\partial \varsigma}\right|_{\varsigma=1}.$$

The resulting temporal profile of leader control $u_0(t)$, and that of its first time derivative $\dot{u}_0(t) = u_{0,t}(t)$, which satisfies the next constraint $\|\dot{u}_0(t)\| \leq \Xi < 35$, are shown in Figure 2.

In order to achieve the control objective, namely the asymptotic synchronization of the perturbed network of followers to the leader state $q_0(\varsigma,t)$ point-wisely, in accordance with the statement of Theorem 1, the following set of control gains

$$a = 70, \quad b = 35, \quad c = 5, \quad d = 5$$

is used as tuning for the proposed non-smooth control protocol (35).

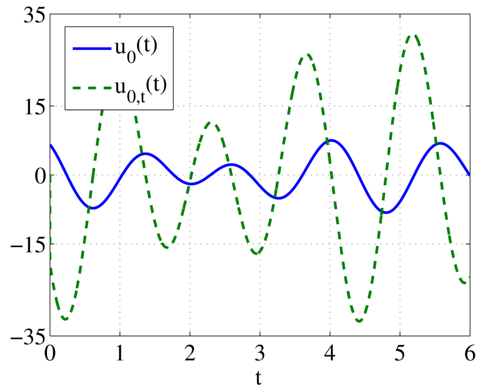

Fig. 2 Temporal profile of the leader's Neumann-type boundary control input $u_0(t)$ (solid line) and its time derivative $q_{0,t}(t)$ (dashed line).

To implement numerically the considered case of study, the resulting coupled system of PDEs has been spatially discretized by means of the finite-difference approximation method by considering $n = 30$ uniformly spaced solution nodes, with coordinates $\varsigma_\kappa = \kappa h$, where $h = 1/(n+1)$, and $\kappa = 1, 2, \ldots, n$.

Then resulting finite dimensional system of coupled ordinary differential equations, having order $N \times n = 300$, is then solved by means of the standard Euler fixed-step solver with a step-size $T = 5 \times 10^{-4}$ seconds.

Figure 3 and Figure 4 show the spatio-temporal profiles of the scalar field associated, respectively, to the leader agent $q_0(\varsigma, t)$ and to the 8-th follower $q_8(\varsigma, t)$. The deviation between the scalar field $q_8(\varsigma, t)$ and $q_0(\varsigma, t)$ is instead depicted in Figure 5. As we can see it tends asymptotically to zero as expected.

In Figure 6 is shown the temporal decay to zero of the $[\mathrm{H}^2(0,1)]^n$-norm of the tracking deviation vector $E(\varsigma, t) = Q(\varsigma, t) - \mathbf{1}_n \otimes q_0(\varsigma, t)$ as well, which implies, in accordance to the statement of Theorem 1, the achievement of the control objective point-wisely in the so-called $[\mathrm{H}^2(0,1)]^n$ higher-order Sobolev space.

Finally, in Figure 7 is shown a comparison of the temporal profiles of the leader and followers states evaluated at the under-actuated extremity $\varsigma = 0$, along with that of $q_0^r(0, t)$ which explicit expression is in (95). As we can see, as the time variable t progresses, the asymptotic tracking of the reference waveform $u_0^r(t)(0, t)$ at the under-actuated extremity is achieved. From that, it can be further noticed that all the followers states $q_i(0, t)$ converge to the desired waveform approximately after 3 seconds.

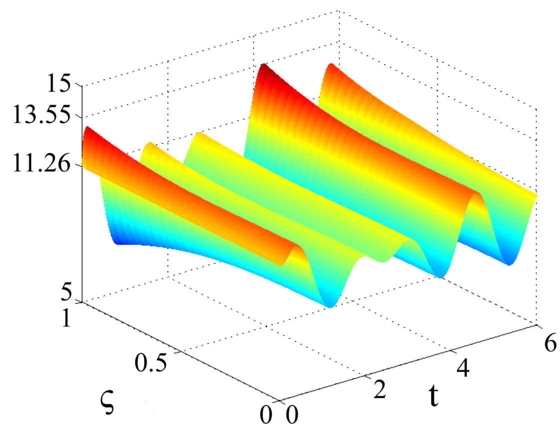

Fig. 3 Spatio-temporal profile of the scalar field associated with leader agent $q_0(\varsigma,t)$.

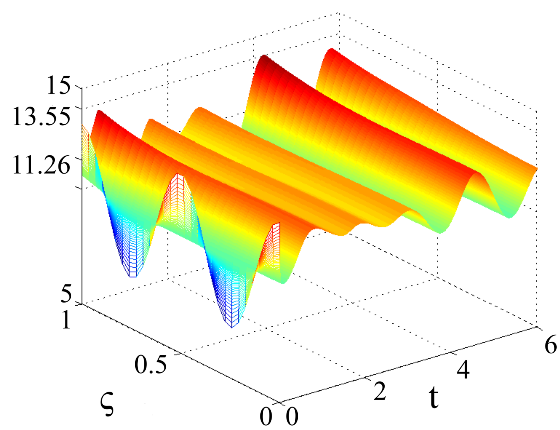

Fig. 4 Spatio-temporal profile of the scalar field associated with the 8-th follower system $q_8(\varsigma,t)$.

6 Conclusion

In this work the leader-follower distributed tracking problem for a network of heat PDEs with spatially varying, uncertain, diffusivity coefficient, and under persistent smooth boundary disturbance has been addressed and solved by means of a dynamic second-order sliding mode based local interaction rule. By exploiting only boundary sensing and Neumann-type boundary actuation, it is proven by Lyapunov analysis that consensus between the

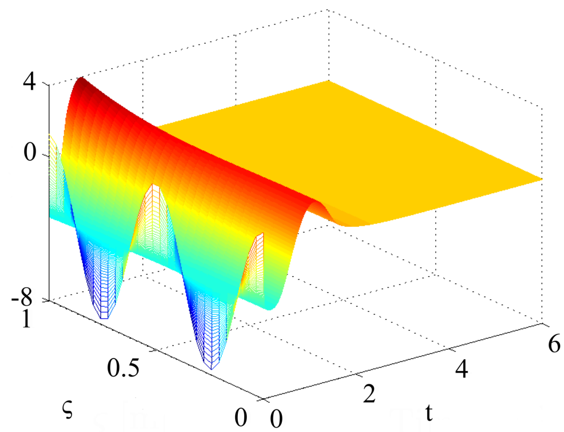

Fig. 5 Spatio-temporal profile of the tracking deviation field associated to the 8-th follower system, namely $q_8(\varsigma,t) - q_0(\varsigma,t)$.

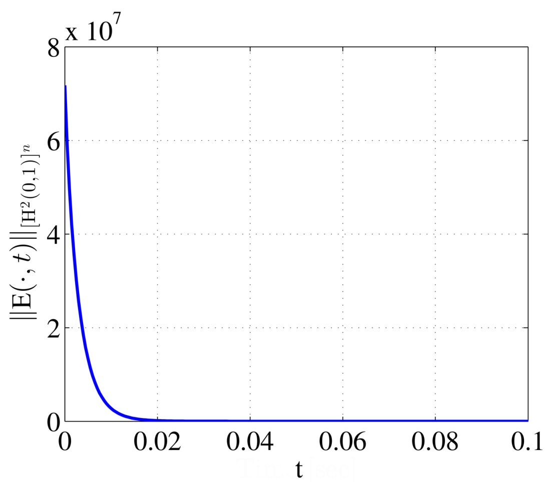

Fig. 6 Temporal profile of the $[\mathrm{H}^2(0,1)]^n$-norm of the tracking deviation vector $E(\varsigma,t) = Q(\varsigma,t) - \mathbf{1}_n \otimes q_0(\varsigma,t)$.

leader and followers' trajectories is exponentially achieved point-wise in space.

Future activities will be targeted towards the extension of these results to more general classes of distributed parameter dynamical agents such as, for instance, reaction-diffusion equations. In addition, relaxing the topological restrictions of the communication graph by covering directed and/or

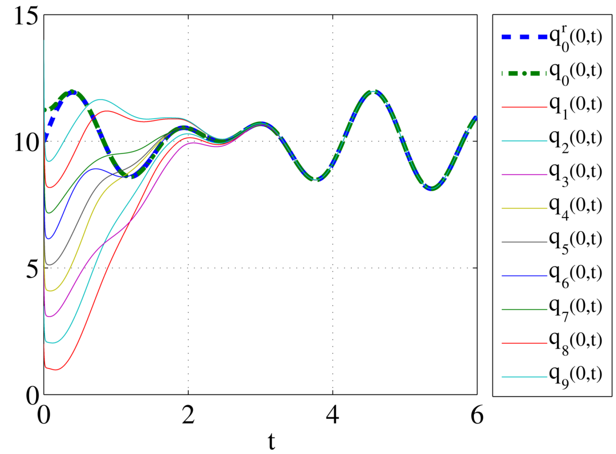

Fig. 7 Comparison of the temporal profiles of the reference waveform $q_0^r(0,t)$ (bold dashed line), the leader profile $q_0(0,t)$ (bold dot-dashed line) and followers profiles $q_i(0,t)$ (solid lines), evaluated at the under-actuated boundary $\varsigma = 0$, with $i = 1,2,\ldots,9$.

switching communication topologies is another interesting line of activity for next research.

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