International Journal of Non–Linear Mechanics 178 (2025) 105154

Contents lists available at ScienceDirect

International Journal of Non-Linear Mechanics

journal homepage: www.elsevier.com/locate/nlm

Dissipative relativistic extended thermodynamics of polyatomic gases in the Landau–Lifshitz description

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ARTICLE INFO

Keywords:
Rational Extended Thermodynamics
Rarefied polyatomic gases
Relativistic fluids

ABSTRACT

This article presents and analyses a relativistic model for polyatomic gases within the Landau–Lifshitz framework, incorporating the contribution of internal energy arising from interactions between gas molecules. This model generalizes the one proposed by Cercignani and Kremer in 2001 for monatomic gases within the context of the Marle expansion. Specifically, in this paper a moment model is derived and its closure is determined by using the methods of Rational Extended Thermodynamics. This is compared with the one recently proposed by Arima, Carrisi, Pennisi, and Ruggeri, which employs a variant of the Anderson–Witting model in the Eckart frame. The analysis demonstrates that the balance equations and the collision term are equivalent up to first order with respect to equilibrium.

1. Introduction

The study of relativistic gases under dissipative conditions is essential for understanding a wide range of physical phenomena, from astrophysical plasmas to high-energy particle flows in relativistic hydrodynamics. In extreme environments such as those found in neutron star mergers, accretion disks around black holes, and the early universe, dissipative effects, including heat conduction and viscosity, play a crucial role in determining the macroscopic behaviour of relativistic fluids [1,2]. Traditional equilibrium models fail to capture these effects accurately, necessitating the development of more comprehensive frameworks. The first step towards a non-equilibrium thermodynamic in relativistic setting is the Eckart extension of the Navier–Stokes equations in 1940 [3]. Since then, numerous efforts have been made to establish a consistent relativistic theory that incorporates dissipative effects like, for example, [4,5] and, more recently, Extended Irreversible Thermodynamics [6] and Rational Extended Thermodynamics (RET) [7,8].

RET has emerged as a powerful framework for modelling non-equilibrium thermodynamic processes in both classical and relativistic regimes. RET has been successfully applied to a variety of physical problems, demonstrating strong agreement with experimental data [7]–[8]. The key feature of this approach is the inclusion of an extended set of independent variables beyond those used in conventional thermodynamics, incorporating non-equilibrium quantities such as dynamic pressure, heat flux, and stress tensor. Within RET, thermodynamic processes are described through field equations in the form of balance laws, derived from a truncated moment system based on kinetic theory, while ensuring compliance with relativity and entropy principles. These constraints lead to an hyperbolic system that, when expressed in terms of specific variables called *main field* [9]–[10], acquires a symmetric form. This structure guarantees key properties such as well-posedness of the Cauchy problem and the finite propagation speed of disturbances, an essential requirement for relativistic models, ensuring that perturbations do not travel faster than the speed of light.

A key challenge in relativistic dissipative thermodynamics is the selection of the flow frame, or equivalently, the definition of the four velocity U^α. Two commonly used are the Eckart (or particle) frame [3] and the Landau–Lifshitz (or energy) frame [11]. In the Eckart frame observers detect no particle flux but can measure heat flux whereas in the Landau–Lifshitz frame the energy flux is set to zero, allowing the observation of particle drift. As a result, the deviation from equilibrium of key quantities is defined differently in these two reference frames. The choice of frame largely depends on the specific problem being investigated.

In RET, both frames have been employed to describe monatomic relativistic gases (since 1986 with the first relativistic extended model by Liu, Müller, and Ruggeri [12]). The majority of the papers in relativistic RET are developed in the Eckart context, because it allows to obtain more manageable equations. However, some scenarios necessitate the use of the Landau–Lifshitz frame, such as relativistic heavy ion collisions at high energies where the net number of baryon is approximately null [13] or the study of dissipative Friedmann–Robertson–Walker models [14]. For this reason, in 2001, Cercignani and Kremer proposed an extended model for monatomic gas in the Landau–Lifshitz framework [15].

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https://doi.org/10.1016/j.ijnonlinmec.2025.105154
Received 27 March 2025; Received in revised form 7 May 2025; Accepted 7 May 2025
Available online 23 May 2025

Polyatomic gases introduce additional complexities due to their internal degrees of freedom, requiring more sophisticated moment models to account for energy due to internal modes of the molecule. Studies on relativistic polyatomic gases within RET have been limited to the Eckart frame (see [16] – [17]) until 2021 with the first proposal of a Landau–Lifshitz description for relativistic polyatomic gases [18]. Recently, there have been improvements in modelling polyatomic relativistic gases. In particular a new model has been proposed that takes into account the total energy composed by the rest energy and the energy of the molecular internal mode [19]. This work aims to extend the Landau–Lifshitz description of polyatomic relativistic gases, incorporating such recent developments to improve the accuracy and wider applicability of the model. In Sections 2 and 3, the fundamentals of RET are summarized, and the differences between the Eckart and Landau–Lifshitz representations are outlined. In Section 4, the model for relativistic polyatomic gases within the Landau–Lifshitz framework is introduced, taking into account the contribution of the molecule's total energy, and its closure is determined. In Section 5, the present model is compared with the corresponding one in the Eckart frame [19], demonstrating that the two approaches are equivalent in the sense that there exists an invertible transformation of the independent variables that maps the results of one onto those of the other, at least up to first order with respect to equilibrium. Finally, conclusions are drawn.

2. Relativistic rational extended thermodynamics

In RET, gases are described through an extended system of balance equations [12,15,16,19]:

$$\partial_\alpha A^{\alpha\alpha_1\cdots\alpha_n} = I^{\alpha_1\cdots\alpha_n} \qquad \text{with n=0,}\ldots\text{,N,} \tag{1}$$

where the partial derivatives ∂_α are calculated with respect to the space–time coordinates x^α with $\alpha = 0, 1, 2, 3$. These equations can be interpreted as a truncated system of moments equations [15] derived from the relativistic Boltzmann–Chernikov equation

$$p^\alpha \partial_\alpha f = Q\,,$$

where f is the distribution function, p^α is the four-momentum satisfying $p^\alpha p_\alpha = m^2c^2$, with c being the speed of light and m the rest mass of the particle, and Q is the collision term which accounts particles interactions. In this sense Eqs. (1) are suggested and justified at a mesoscopic level through kinetic theory.

For monatomic gases the distribution function $f(x^\alpha, p^\beta)$ depends on space–time coordinates and four-momentum. The tensors present in Eqs. (1) are defined as follows:

$$A^{\alpha\alpha_1\cdots\alpha_n} = \frac{c}{m^{n-1}}\int_{\Re^3} f\, p^\alpha p_1^\alpha \cdots p_n^\alpha \, d\boldsymbol{P} \qquad\qquad I^{\alpha_1\cdots\alpha_n} = \frac{c}{m^{n-1}}\int_{\Re^3} Q\, p_1^\alpha \cdots p_n^\alpha \, d\boldsymbol{P},$$

where the integral measure is $d\boldsymbol{P} = \frac{dp^1\,dp^2\,dp^3}{p^0}$. The tensors assume the following physical meaning:

- $A^\alpha = V^\alpha$ is the particle–particle flux vector (or rest-mass density current)
- $A^{\alpha\beta} = T^{\alpha\beta}$ is the energy–momentum tensor
- $I = I^\alpha = 0$ ensuring that Eqs. (1) with $n = 0, 1$ are respectively the conservation laws of particle number and energy–momentum.

For $n = 2$ Eq. (1) yields an extended equation for fluxes, where the tensors satisfy the following constraints:

$$A^{\alpha\beta}_{\beta} = c^2 V^\alpha\,, \qquad\qquad I^\alpha_\alpha = I = 0\,,$$

justified by their definition. Additionally, the third-order tensor $A^{\alpha\beta\gamma}$ is symmetric.

For polyatomic gases, we adopt the most recent model proposed in [19], which builds upon the generalization of the Boltzmann equation for polyatomic gases presented in [20]. In this approach, the distribution function is extended to depend on an additional variable $\mathcal{I}$, representing the energy of the internal modes of the molecule: $f(x^\alpha, p^\beta, \mathcal{I})$.

The system of balance equations retains the same structure as in (1), but the moment definitions are modified to incorporate the additional internal energy contribution:

$$A^{\alpha\alpha_1\cdots\alpha_n} = \left(\frac{1}{mc}\right)^{2n-1}\int_{\mathbb{R}^3}\int_0^{+\infty} f\, p^\alpha p^{\alpha_1}\cdots p^{\alpha_n}\left(mc^2+\mathcal{I}\right)^n \phi(\mathcal{I})\, d\mathcal{I}\, d\boldsymbol{P}\,,$$
$$I^{\alpha_1\cdots\alpha_n} = \left(\frac{1}{mc}\right)^{2n-1}\int_{\mathbb{R}^3}\int_0^{+\infty} Q\, p^{\alpha_1}\cdots p^{\alpha_n}\left(mc^2+\mathcal{I}\right)^n \phi(\mathcal{I})\, d\mathcal{I}\, d\boldsymbol{P}. \tag{2}$$

The function $\phi(\mathcal{I})$ represents the state density of the internal modes and satisfies the condition

$$\lim_{c\to+\infty}\phi(\mathcal{I}) = \mathcal{I}^a \qquad \text{with } a = \frac{D-5}{2}\,,$$

where D are the degrees of freedom of the molecule. This condition, established in [21], ensures the correct caloric equation in the classical context. The explicit expression of $\phi(\mathcal{I})$ is still an open research question. However, in some studies, it has been assumed that $\phi(\mathcal{I}) = \mathcal{I}^a$ also in relativistic context, in analogy with the classical case. In this work, we maintain $\phi(\mathcal{I})$ in its general form to preserve the broad applicability of our results.

3. Eckart and Landau–Lifshitz frame

In the case of a relativistic ideal fluid, it is possible to define a precise bulk velocity that characterizes the motion of the fluid, allowing for the adoption of a co-moving reference frame. However, for dissipative fluids, the definition of a local rest frame, and consequently the velocity, is not uniquely determined. Several approaches have been proposed in the literature, based on both microscopic and macroscopic perspectives [22–24], yet no single choice has been universally recognized as the most appropriate. Nevertheless, in relativistic hydrodynamics, a macroscopic approach is commonly employed, relying on balance equations such as (1) and imposing the second law of thermodynamics, as in RET. Within this framework, the four-velocity can be naturally defined in two distinct ways:

1. As a time-like vector parallel to the rest-mass density current V^α;
2. As an eigenvector of the energy–momentum tensor $T^{\alpha\beta}$.

From the first perspective, the observer follows the motion of the particles, meaning that no particle flux or energy flux is detected, as the particle is considered within its equilibrium configuration. This viewpoint corresponds to the formulation introduced by Eckart [3]. The physical quantities are expressed in terms of the field variables V^α and $T^{\alpha\beta}$ through the following relations:

$$V^\alpha = m\,n\,U^\alpha\,, \qquad T^{\alpha\beta} = \frac{e}{c^2}U^\alpha U^\beta + (p+\Pi)\,h^{\alpha\beta} + \frac{2}{c^2}U^{(\alpha}q^{\beta)} + t^{<\alpha\beta>_3}\,, \tag{3}$$

where n represents the particle number density, U^α is the four-velocity, e is the energy density, p the pressure, Π the dynamic pressure and parentheses around indices indicate symmetrization. The quantity $h^{\alpha\beta} = -g^{\alpha\beta} + \frac{1}{c^2}U^\alpha U^\beta$ acts as the projection operator onto the subspace orthogonal to U^α, with $g^{\alpha\beta} = \mathrm{diag}(1\,, -1\,, -1\,, -1)$ being the metric tensor. Additionally, q^α denotes the heat flux, while $t^{<\alpha\beta>_3}$ represents the 3-dimensional traceless part of the viscous stress tensor. The notation $t^{<\alpha\beta>_3}$ highlights the three-dimensional traceless component, defined as $t^{<\alpha\beta>_3} = \left(h^\alpha_\mu h^\beta_\nu - \frac{1}{3}h^{\alpha\beta}h_{\mu\nu}\right)t^{\alpha\beta}$. The subscript 3 is included to distinguish it from the fully traceless part $t^{<\alpha\beta>} = \left(g^\alpha_\mu g^\beta_\nu - \frac{1}{4}g^{\alpha\beta}g_{\mu\nu}\right)t^{\alpha\beta}$.

These variables satisfy the following constraints:

$$U^\alpha U_\alpha = c^2\,, \quad q^\alpha U_\alpha = 0\,, \quad t^{<\alpha\beta>_3}\,U\alpha = 0\,, \quad t^{<\alpha\beta>_3}\,g\alpha\beta = 0\,. \tag{4}$$

In Eckart's formulation, the four-velocity is collinear with V^α as evident from Eq. $(3)_1$. Furthermore, this approach ensures that no dissipative contributions appear in either the rest-mass density current or the energy density.

In the second perspective, which corresponds to the Landau–Lifshitz one [11], the observer choose to focus on the energy. In this framework, no net energy flux is present but it is possible to observe a dissipative contribution for mass particles J^α. A distinct four-velocity U_L^α is introduced which is related to the Eckart one by the following equation:

$$U_L^\alpha = U^\alpha + J^\alpha \,. \tag{5}$$

The vector J^α is orthogonal to U^α and, in a first-order approximation, it can be expressed as

$$J^\alpha = \frac{q^\alpha}{e+p} \,. \tag{6}$$

For further details, see [22,25]. Substituting Eq. (6) into Eq. (5), we find:

$$U_L^\alpha = U^\alpha + \frac{q^\alpha}{e+p} \,. \tag{7}$$

In the Landau–Lifshitz frame, the field variables are related to physical quantities as follows:

$$\begin{aligned} V_L^\alpha &= m\,n_L\,U_L^\alpha - \frac{m\,n_L}{e_L+p_L}\,q_L^\alpha \,, \\ T_L^{\alpha\beta} &= \frac{e_L}{c^2}\,U_L^\alpha U_L^\beta - (p_L + \Pi_L)\,\Delta^{\alpha\beta} + P^{<\alpha\beta>_3} \,. \end{aligned} \tag{8}$$

Here, the subscript L distinguishes the Landau–Lifshitz frame variables from those in the Eckart frame. Additionally, $\Delta^{\alpha\beta} = g^{\alpha\beta} - \frac{1}{c^2}U_L^\alpha U_L^\beta$ is the projector onto the subspace orthogonal to U_L^α and $P^{<\alpha\beta>_3}$ represents the pressure deviator. The subscript 3 refers to the traceless part of the tensor, defined similarly as before but using $-\Delta^{\alpha\beta}$ instead of $h^{\alpha\beta}$. The variables are constrained by the following relations:

$$U_L^\alpha\,U_{L\alpha} = c^2 \,, \quad q_L^\alpha\,U_{L\alpha} = 0 \,, \quad P^{<\alpha\beta>_3}\,U_{L\alpha} = 0 \,, \quad P^{<\alpha\beta>_3}\,g_{\alpha\beta} = 0 \,. \tag{9}$$

This approach ensures that the four-velocity U_L^α is an eigenvector of the stress–energy tensor. In fact, thanks to the constraints in Eq. (9), it can be shown that $T_{\alpha\beta\,L} U_L^\alpha = e_L U_L^\beta$.

In absence of dissipative phenomena (perfect fluids or non-perfect fluids at equilibrium) the thermodynamic fluxes Π, q^α and $t^{<\alpha\beta>_3}$ are null. As a consequence the two four-velocities match (see Eq. (7)) and the Eckart four-velocity, which is aligned with V^α, also becomes an eigenvector of the stress–energy tensor. This also leads to an identical interpretation of the physical quantities in both frames (see Eq. (7) and compare Eqs. (3) and (8)). However, when non-ideal fluids are out of equilibrium, the two frames diverge, resulting in different definitions for the tensors in the system (1) and discrepancies in the way variables change with respect to their equilibrium values.

In the Eckart approach is assumed that the quantities n, e and U^α remain unchanged from their equilibrium values, and we have the following relations:

$$V^\alpha - V_E^\alpha = 0 \,, \quad T^{\alpha\beta} - T_E^{\alpha\beta} = \Pi\,h^{\alpha\beta} + \frac{2}{c^2}\,q^{(\alpha}U^{\beta)} + t^{<\alpha\beta>_3} \,.$$

The subscript E represents the equilibrium value of the quantity. This leads to the absence of dissipative contributions to the rest-mass density current and the energy density:

$$V^\alpha = V_E^\alpha \,, \qquad \left(T^{\alpha\beta} - T_E^{\alpha\beta}\right) U_\alpha U_\beta = 0 \,.$$

In contrast, in the Landau–Lifshitz frame, the quantities n_L, e_L and U_L^α remain unchanged, and we have:

$$V_L^\alpha - (V_L^\alpha)_E = -\frac{m\,n_L}{e_L+p_L}\,q_L^\alpha \neq 0 \,, \qquad T_L^{\alpha\beta} - (T_L^{\alpha\beta})_E = -\Pi_L\,\Delta_L^{\alpha\beta} + P_3^{<\alpha\beta>} \,. \tag{10}$$

This leads to the following relations:

$$\left[V_L^\alpha - (V_L^\alpha)_E\right] U_{L\alpha} = 0 \,, \quad \left[T_L^{\alpha\beta} - (T_L^{\alpha\beta})_E\right] U_{L\beta} = 0 \,,$$

which ensures there is no net energy flux.

Remark 1. In both approaches, the particle number density is often assumed to be conserved. This does not mean that n is identical in the two frames; rather, it means that its evolution follows the same conservation equation in both cases. Strictly speaking, they are not exactly the same when considering higher-order terms or non-equilibrium situations where diffusion effects are strong. However, at first order (which is often the level considered in practical applications), the difference is negligible, and it is convenient to assume that n remains unchanged in both frames. Moreover, the four-velocity remains the same as in equilibrium but the four-velocity of the other approach changes. In the Eckart frame, we have $U^\alpha - U_E^\alpha = 0$ and $U_L^\alpha - U_{L\,E}^\alpha \neq 0$ while in the Landau–Lifshitz frame, $U^\alpha - U_E^\alpha \neq 0$ and $U_L^\alpha - U_{L\,E}^\alpha = 0$.

The two approaches differ also for the expression traditionally used for the collision term Q, that allows to define the production terms $I^{\alpha_1\cdots\alpha_n}$. In Landau–Lifshitz frame the most diffused expression of Q, is

$$Q = -\frac{U_L^\alpha\,p_\alpha}{c^2\,\tau}\,\left(f - f_E\right) \,, \tag{11}$$

proposed by Anderson and Witting [26]. It ensures zero production terms for mass and energy–momentum. However, if applied within the Eckart framework by simply replacing U_α with $U_{L\alpha}$, it does not yield zero production terms for mass and energy–momentum. For this reason, the following alternative expression for Q has been proposed in papers [27–30], as a variant of the Anderson–Witting model in the Eckart frame:

$$Q = \frac{U^\alpha\,p_\alpha}{c^2\,\tau}\left(f_E - f - f_E p^\mu q_\mu \frac{1+\frac{\mathcal{I}}{mc^2}}{bmc^2}\right) . \tag{12}$$

In [29] a similar result has been found also for the Marle model [31]. The extra term $f_E p^\mu q_\mu \frac{1+\frac{\mathcal{I}}{mc^2}}{bmc^2}$ in Eq. (12) ensures zero production terms for mass and momentum–energy in the Eckart frame, both for monatomic and polyatomic gases.

4. A 15 moments model for polyatomic gas in the Landau–Lifshitz description

In this section, the simplest but significant model of RET for polyatomic gases, aside from Euler, within the Landau–Lifshitz framework is proposed and analysed. In papers [17,32], it has been demonstrated that this model corresponds to system (1) with $N = 2$:

$$\partial_\alpha A^\alpha = 0, \quad \partial_\alpha A^{\alpha\beta} = 0, \quad \partial_\alpha A^{\alpha\beta\gamma} = I^{\beta\gamma}, \tag{13}$$

The tensors are still defined as moments of the distribution functions, as in Eq. (2), but their macroscopic representations are evaluated in the Landau–Lifshitz frame. For instance, the quantities $A^\alpha = V_L^\alpha$ and $A^{\alpha\beta} = T_L^{\alpha\beta}$ are defined as in Eq. (10).

System (13) consists of 15 independent equations but involves more than 15 variables. To solve the system is first necessary to close it. For this purpose, we adopt the Maximum Entropy Principle (MEP) (see, for example, [7,33]). This approach allows to obtain a more explicit closure compared to the classical symmetrization procedure by Ruggeri–Strumia [10], which introduces undetermined constants of integration. For instance, in [16], it was shown that the Ruggeri–Strumia procedure yields the closure except for an arbitrary function of a single scalar, whereas the MEP provides an explicit expression for this function.

4.1. Distribution function and moments at equilibrium

First of all we determine the equilibrium distribution function. We consider an Eulerian fluid governed by the 5 balance Eqs. $(13)_{1,2}$, with the equilibrium part of (8).

$$A_E^\alpha \equiv V_E^\alpha = m\,n_L U_L^\alpha, \qquad A_E^{\alpha\beta} \equiv T_E^{\alpha\beta} = -p_L\Delta^{\alpha\beta} + \frac{e_L}{c^2}\,U_L^\alpha U_L^\beta. \tag{14}$$

MEP requires that the appropriate distribution function $f \equiv f(x^\alpha, p^\alpha, \mathcal{I})$ is the one which maximizes the entropy density at equilibrium:

$$h_E = -k_B\, c\, U_{L\alpha} \int_{\mathbb{R}^3} \int_0^{+\infty} f \cdot (\ln f)\, p^\alpha\, \phi(\mathcal{I})\, d\mathcal{I}\, d\boldsymbol{P},$$

under the constraints that the temporal part $V_L^\alpha U_{L\alpha}$ and $T_L^{\alpha\beta} U_{L\beta}$ are prescribed. Here k_B is the Boltzmann constant.

The balance equation of the entropy is a constrain for our system that can be imposed by using properly the Lagrange method. We can define the Lagrangian

$$\begin{aligned}\mathcal{L} = U_{L\alpha} \Bigg\{ & -k_B\, c\, U_{L\alpha} \int_{\mathbb{R}^3} \int_0^{+\infty} f \cdot (\ln f)\, p^\alpha\, \phi(\mathcal{I})\, d\mathcal{I}\, d\boldsymbol{P} \\ & + \lambda \left[V_L^\alpha - mc \int_{\mathbb{R}^3} \int_0^{+\infty} f \cdot (\ln f)\, p^\alpha\, \phi(\mathcal{I})\, d\mathcal{I}\, d\boldsymbol{P} \right] \\ & + \lambda_\beta \left[T_L^{\alpha\beta} - \frac{1}{mc} \int_{\mathbb{R}^3} \int_0^{+\infty} f \cdot (\ln f)\, p^\alpha\, \phi(\mathcal{I})\, d\mathcal{I}\, d\boldsymbol{P} \right] \Bigg\},\end{aligned}$$

where λ and λ_β are the Lagrange multipliers. The Lagrangian $\mathcal{L}$ have to be maximized by the distribution function and this implies that $\frac{\partial \mathcal{L}}{\partial f} = 0$, i.e.

$$\int_{\mathbb{R}^3} \int_0^{+\infty} \left[-k_B \left(ln f_L + 1 \right) - \lambda m - \frac{1}{mc^2} \lambda_\beta p^\beta \left(mc^2 + \mathcal{I} \right)^n \phi(\mathcal{I}) \right] U_{L\alpha} p^\alpha\, \phi(\mathcal{I})\, d\mathcal{I}\, d\boldsymbol{P} = 0$$

and

$$f_E = e^{-1-\frac{\chi}{k_B}}, \quad \text{with} \quad \chi = m\,\lambda + \lambda_\mu\, p^\mu \left(1 + \frac{\mathcal{I}}{m\,c^2} \right). \tag{15}$$

The Lagrange multipliers λ, λ_μ have the following expression

$$\lambda_E = -\frac{g}{T_L}, \quad \lambda_E^\mu = \frac{U_L^\mu}{T_L}. \tag{16}$$

The equilibrium chemical potential $g = \varepsilon + p/\rho - T\,S$ can be determined by following the procedure dictated in [8], where $\rho = nm$ is the mass density, ε is the internal energy, T is the absolute temperature, and has to be evaluated in the Landau–Lifshitz frame.

The thermal and caloric equation of state allow to determine explicit expressions for the number particle and for pressure and energy. We can multiply Eq. (14)$_1$ times $\frac{1}{c^2} U_L^\alpha$ and Eq. (14)$_2$ times $\frac{1}{c^2} U_L^\alpha U_L^\alpha$ and $\Delta^{\alpha\beta}$. After inserting the distribution function (15) into these expressions, and apply the change of integration variables

$$\begin{aligned} p^0 &= mc \cosh s \\ p^1 &= mc \sinh s \sin\vartheta \cos\varphi \\ p^2 &= mc \sinh s \sin\vartheta \sin\varphi \\ p^3 &= mc \sinh s \cos\vartheta \qquad \text{with } s \in [0, +\infty[,\ \vartheta \in [0, \Pi[,\ \varphi \in [0, 2\Pi[\end{aligned} \tag{17}$$

with Jacobian $m^3 c^3 \sinh^2 s \cosh s \sin\vartheta$, we obtain

$$n_L = 4\pi m^3 c^3 e^{-1-\frac{m}{k_B}\lambda_E} \int_0^{+\infty} J^*_{2,1}\, \phi(\mathcal{I})\, d\mathcal{I}, \tag{18}$$

$$p_L = \frac{n_L mc^2}{\gamma}, \qquad e_L = n_L mc^2 \frac{\int_0^{+\infty} J^*_{2,2} \left(1 + \frac{\mathcal{I}}{mc^2}\right) \phi(\mathcal{I})\, d\mathcal{I}}{\int_0^{+\infty} J^*_{2,1}\, \phi(\mathcal{I})\, d\mathcal{I}}. \tag{19}$$

As in [12], the functions J^* are defined as

$$J^*_{m,n}(\gamma) = \int_0^{+\infty} e^{-\gamma^* \cosh s} \sinh^m s \cosh^n s\, d\, s,$$

with $\gamma^* = \gamma \left(1 + \frac{\mathcal{I}}{mc^2}\right)$ and $\gamma = \frac{mc^2}{k_B T}$.

By using Eqs. (16) and (18), the equilibrium distribution function for a rarefied polyatomic gas that maximizes the entropy can be also written as

$$f_E = \frac{n_L}{A(\gamma)} \frac{1}{4\pi m^3 c^3} e^{-\frac{1}{k_B T}\left[\left(1+\frac{\mathcal{I}}{mc^2}\right) U_{L\beta} p^\beta\right]}, \quad A(\gamma) = \int_0^{+\infty} J^*_{2,1}\, \phi(\mathcal{I})\, d\mathcal{I}, \tag{20}$$

that is a generalization of the Jüttner distribution function.

The moments in equilibrium state $A_E^{\alpha\alpha_1\cdots\alpha_j}$ for $j \geq 2$ can be obtained by substituting the expression (20) of the equilibrium distribution function into Eq. (2)$_1$. Otherwise, they can be represented in a macroscopic way by using the representation theorems:

$$A_E^{\alpha_1\cdots\alpha_{j+1}} = \sum_{k=0}^{\left[\frac{j+1}{2}\right]} \rho c^{2k} \theta_{k,j} (-1)^k \Delta^{(\alpha_1\alpha_2} \cdots \Delta^{\alpha_{2k-1}\alpha_{2k}} U_L^{\alpha_{2k+1}} \cdots U_L^{\alpha_{j+1})}, \tag{21}$$

where

$$\theta_{k,j} = \frac{1}{2k+1} \binom{j+1}{2k} \frac{\int_0^{+\infty} J^*_{2k+2,j+1-2k} \left(1 + \frac{\mathcal{I}}{mc^2}\right)^j \phi(\mathcal{I})\, d\mathcal{I}}{\int_0^{+\infty} J^*_{2,1}\, \phi(\mathcal{I})\, d\mathcal{I}} \tag{22}$$

are dimensionless functions depending only on γ. The above expression (22) can be obtained by comparing Eq. (21) with the corresponding expression obtained by putting (15) in (2)$_1$, contracting with $\Delta^{(\alpha_1\alpha_2} \cdots \Delta^{\alpha_{2h-1}\alpha_{2h}} U_L^{\alpha_{2h+1}} \cdots U_L^{\alpha_{j+1})}$ and by using the change of variables (17).

The functions $\theta_{k,j}$ have the same expression of the corresponding Eckart ones so they satisfy the same conditions reported in paper [19]:

$$\begin{aligned} &\theta_{0,0} = 1, \\ &\theta_{0,1} = \omega(\gamma), \\ &\theta_{0,j+1} = \omega(\gamma)\,\theta_{0,j} - \theta'_{0,j} && \text{with} \quad ' = \frac{d}{d\gamma}, \\ &\theta_{h,j+1} = \frac{j+2}{\gamma}\left(\theta_{h,j} + \frac{j+3-2h}{2h}\theta_{h-1,j}\right) && \text{for } h = 1, \ldots, \left[\frac{j+1}{2}\right], \\ &\theta_{\frac{j+2}{2},j+1} = \frac{1}{\gamma}\theta_{\frac{j}{2},j} && \text{for } j \text{ even}, \end{aligned} \tag{23}$$

where

$$\omega(\gamma) = \frac{\int_0^{+\infty} J^*_{2,2} \left(1 + \frac{\mathcal{I}}{mc^2}\right) \phi(\mathcal{I})\, d\mathcal{I}}{\int_0^{+\infty} J^*_{2,1}\, \phi(\mathcal{I})\, d\mathcal{I}}.$$

Moreover, from Eq. (23)$_5$ with $j = 0$ we have

$$\theta_{1,1} = \frac{1}{\gamma}. \tag{24}$$

The above equations allows to rewrite the expression of p_L and e_L in terms of the functions $\theta_{h,j}$ as follows:

$$p_L = n_L\, m\, c^2 \theta_{1,1} \qquad e_L = n_L\, m\, c^2 \theta_{0,1}. \tag{25}$$

Remark 2. It is noteworthy that, as in the Eckart approach [19] and in the non-relativistic case [32], all the scalar coefficients can be expressed in terms of the function $\omega(\gamma)$ and its derivatives with respect to γ (or with respect to the temperature T). Moreover, ω is closely related to the internal energy ε. In fact, since $e = \rho c^2 + \rho\varepsilon$, we deduce from (19)$_2$ that $\varepsilon = c^2(\omega - 1)$.

4.2. Non equilibrium distribution function and closure of the 15 moments model

Application of the MEP to the full system, under the constraints that the temporal part $V_L^\alpha U_{L\alpha}$, $T_L^{\alpha\beta} U_{L\alpha}$ and $A_L^{\alpha\beta\gamma} U_{L\alpha}$ are prescribed, with the same procedure described in Section 4.1, gives:

$$f_{15} = e^{-1-\frac{\chi}{k_B}}, \quad \text{with} \quad \chi = m\,\lambda + \lambda_\mu\, p^\mu \left(1 + \frac{\mathcal{I}}{m\,c^2}\right) + \frac{1}{m} \lambda_{\mu\nu}\, p^\mu p^\nu \left(1 + \frac{\mathcal{I}}{m\,c^2}\right)^2, \tag{26}$$

where $\lambda, \lambda_\mu, \lambda_{\mu\nu}$ are the Lagrange multipliers. According with Müller and Ruggeri [7] equilibrium in RET is the state in which the entropy production vanishes and hence attains its minimum value. Using this definition, it is possible to prove that the Lagrange multipliers relative to the balance laws of nonequilibrium variables vanish, and only the five Lagrange multipliers corresponding to the equilibrium conservation laws remain, i.e. that referred to the Euler system analysed in Section 4.1. In the present case, we have Eq. (16) plus $\lambda_{\mu\nu_E}=0$.

We emphasize that Eq. (26) represents the distribution function that maximizes the entropy for processes near thermodynamic equilibrium and is referred exclusively to the present 15 moments model.

We can choose as independent variables, the standard 14 physical variables: ρ_L, T_L, U_L^α, Π_L, q_L^α, $P^{<\alpha\beta>_3}$, where $q_L^\alpha=-\frac{e_L+p_L}{\rho_L}\Delta^{\alpha\beta}V_L^\beta$ and $P^{<\alpha\beta>_3}=T_L^{\mu\nu}\left(\Delta_\mu^\alpha\Delta_\nu^\beta-\frac{1}{3}\Delta^{\alpha\beta}\Delta_{\mu\nu}\right)$ plus

$$\Omega=\frac{4}{c^2}U_{L\alpha}U_{L\beta}U_{L\gamma}\left(A_L^{\alpha\beta\gamma}-(A_L^{\alpha\beta\gamma})_E\right) \tag{27}$$

as the 15th variable.

The particle number vector and the energy–momentum tensor are expressed by Eqs. (8) where the pressure and the energy as function of mass density and temperature are given in (19). So we have to determine now, the expression of the triple tensor $A_L^{\alpha\beta\gamma}$ and of the production term $I_L^{\beta\gamma}$.

Inserting the distribution function (26) into the moments (2), we obtain the following system:

$$\begin{aligned}
&\frac{k_B n_L}{e_L+p_L}q_L^\alpha=V_L^\alpha-V_{LE}^\alpha=V_{LE}^\alpha(\lambda-\lambda_E)+T_{LE}^{\alpha\mu}\left(\lambda_\mu-\lambda_{\mu_E}\right)+A_{LE}^{\alpha\mu\nu}\lambda_{\mu\nu}\,,\\
&-\frac{k_B}{m}\left(P^{<\alpha\beta>_3}-\Pi_L\Delta^{\alpha\beta}\right)=T_{LE}^{\alpha\beta}(\lambda-\lambda_E)+A_{LE}^{\alpha\beta\mu}\left(\lambda_\mu-\lambda_{\mu_E}\right)+A_{LE}^{\alpha\beta\mu\nu}\lambda_{\mu\nu}\,,\\
&-\frac{k_B}{m}\left(A^{\alpha\beta\gamma}-A_{LE}^{\alpha\beta\gamma}\right)=A_{LE}^{\alpha\beta\gamma}(\lambda-\lambda_E)+A_{LE}^{\alpha\beta\gamma\mu}\left(\lambda_\mu-\lambda_{\mu_E}\right)+A_{LE}^{\alpha\beta\gamma\mu\nu}\lambda_{\mu\nu}\,,
\end{aligned} \tag{28}$$

where the equilibrium values of the tensors $A_{LE}^{\alpha\beta\mu}$, $A_{LE}^{\alpha\beta\mu\nu}$ and $A_{LE}^{\alpha\beta\mu\nu\gamma}$ can be obtained from Eq. (21) and are

$$\begin{aligned}
&A_{LE}^{\alpha\beta\gamma}=\rho\,\theta_{0,2}\,U_L^\alpha U_L^\beta U_L^\gamma-\rho c^2\,\theta_{1,2}\,\Delta^{(\alpha\beta}U_L^{\gamma)},\\
&A_{LE}^{\alpha\beta\mu\nu}=\rho\,\theta_{0,3}\,U_L^\alpha U_L^\beta U_L^\mu U_L^\nu-\rho c^2\,\theta_{1,3}\,\Delta^{(\alpha\beta}U_L^\mu U_L^{\nu)}+\rho c^4\,\theta_{2,3}\,\Delta^{(\alpha\beta}\Delta^{\mu\nu)},\\
&A_{LE}^{\alpha\beta\gamma\mu\nu}=\rho\,\theta_{0,4}\,U_L^\alpha U_L^\beta U_L^\gamma U_L^\mu U_L^\nu-\rho c^2\theta_{1,4}\,\Delta^{(\alpha\beta}U_L^\gamma U_L^\mu U_L^{\nu)}+\rho c^4\theta_{2,4}\,\Delta^{(\alpha\beta}\Delta^{\gamma\mu}U_L^{\nu)}\,,
\end{aligned} \tag{29}$$

with the θ's given in (22).

The system (28) permits to deduce the 15 Lagrange multipliers in terms of the 15 field variables, including Ω given in (27), and then we can obtain the remaining part of the tensor $A_L^{\alpha\beta\gamma}$.

To solve this system, we consider first Eq. $(28)_1$ contracted with $U_{L\alpha}$, Eq. $(28)_2$ contracted with $U_{L\alpha}U_{L\beta}$, Eq. $(28)_3$ contracted with $U_{L\alpha}U_{L\beta}U_{L\gamma}/c^3$, Eq. $(28)_2$ contracted with $\Delta_{\alpha\beta}/3$ and $(28)_3$ contracted with $U_{L\alpha}\Delta_{\beta\gamma}/(3\,c^2)$, obtaining the system

$$\begin{aligned}
&\theta_{0,0}\,(\lambda-\lambda_E)+\theta_{0,1}\,U_L^\mu\left(\lambda_\mu-\frac{U_{L\mu}}{T}\right)+\theta_{0,2}\,U_L^\mu U_L^\nu\lambda_{\mu\nu}-\frac{c^2}{3}\theta_{1,2}\,\Delta^{\mu\nu}\lambda_{\mu\nu}=0\,,\\
&\theta_{0,1}\,(\lambda-\lambda_E)+\theta_{0,2}\,U_L^\mu\left(\lambda_\mu-\frac{U_{L\mu}}{T}\right)+\theta_{0,3}\,U_L^\mu U_L^\nu\lambda_{\mu\nu}-\frac{c^2}{6}\,\theta_{1,3}\,\Delta^{\mu\nu}\lambda_{\mu\nu}=0\,,\\
&\theta_{0,2}\,(\lambda-\lambda_E)+\theta_{0,3}\,U_L^\mu\left(\lambda_\mu-\frac{U_{L\mu}}{T}\right)+\theta_{0,4}\,U_L^\mu U_L^\nu\lambda_{\mu\nu}\\
&-\frac{c^2}{10}\,\theta_{1,4}\,\Delta^{\mu\nu}\lambda_{\mu\nu}=-\frac{k_B}{4\,m^2\,n_L\,c^4}\,\Omega\,,\\
&\theta_{1,1}\,(\lambda-\lambda_E)+\frac{1}{3}\theta_{1,2}\,U_L^\mu\left(\lambda_\mu-\frac{U_{L\mu}}{T}\right)+\frac{1}{6}\theta_{1,3}\,U_L^\mu U_L^\nu\lambda_{\mu\nu}\\
&-\frac{5}{9}c^2\theta_{2,3}\,\Delta^{\mu\nu}\lambda_{\mu\nu}=-\frac{k_B}{m^2n_Lc^2}\,\Pi_L\,,\\
&\frac{1}{3}\theta_{1,2}\,(\lambda-\lambda_E)+\frac{1}{6}\,\theta_{1,3}\,U_L^\mu\left(\lambda_\mu-\frac{U_{L\mu}}{T}\right)+\frac{1}{10}\,\theta_{1,4}\,U_L^\mu U_L^\nu\lambda_{\mu\nu}-\frac{c^2}{9}\,\theta_{2,4}\,\Delta^{\mu\nu}\lambda_{\mu\nu}=\\
&\qquad=+\frac{k_B}{3m^2c^4n_L}\left(A^{\alpha\beta\gamma}-A_{LE}^{\alpha\beta\gamma}\right)U_{L\alpha}\Delta_{\beta\gamma}\,.
\end{aligned}$$

This is a system of 5 equations in the 4 unknowns $\lambda-\lambda_E$, $U_L^\mu\left(\lambda_\mu-\frac{U_{L\mu}}{T}\right)$, $U_L^\mu U_L^\nu\lambda_{\mu\nu}$, $\Delta^{\mu\nu}\lambda_{\mu\nu}$; in order to have solutions, the determinant of the complete matrix must be zero, i.e.,

$$0=\begin{vmatrix}
\theta_{0,0} & \theta_{0,1} & \theta_{0,2} & \frac{1}{3}\theta_{1,2} & 0\\
\theta_{0,1} & \theta_{0,2} & \theta_{0,3} & \frac{1}{6}\,\theta_{1,3} & 0\\
\theta_{0,2} & \theta_{0,3} & \theta_{0,4} & \frac{1}{10}\,\theta_{1,4} & -\frac{k_B}{4mc^4}\,\Omega\\
\theta_{1,1} & \frac{1}{3}\,\theta_{1,2} & \frac{1}{6}\,\theta_{1,3} & \frac{5}{9}\,\theta_{2,3} & -\frac{k_B}{mc^2}\,\Pi_L\\
\frac{1}{3}\,\theta_{1,2} & \frac{1}{6}\,\theta_{1,3} & \frac{1}{10}\,\theta_{1,4} & \frac{1}{9}\,\theta_{2,4} & \frac{k_B}{3mc^4}\left(A^{\alpha\beta\gamma}-A_{LE}^{\alpha\beta\gamma}\right)U_{L\alpha}\Delta_{\beta\gamma}
\end{vmatrix}. \tag{30}$$

By defining

$$D_4=\begin{vmatrix}
\theta_{0,0} & \theta_{0,1} & \theta_{0,2} & \frac{1}{3}\theta_{1,2}\\
\theta_{0,1} & \theta_{0,2} & \theta_{0,3} & \frac{1}{6}\,\theta_{1,3}\\
\theta_{0,2} & \theta_{0,3} & \theta_{0,4} & \frac{1}{10}\,\theta_{1,4}\\
\theta_{1,1} & \frac{1}{3}\,\theta_{1,2} & \frac{1}{6}\,\theta_{1,3} & \frac{5}{9}\,\theta_{2,3}
\end{vmatrix},$$

$$N^\Pi=-\begin{vmatrix}
\theta_{0,0} & \theta_{0,1} & \theta_{0,2} & \frac{1}{3}\theta_{1,2}\\
\theta_{0,1} & \theta_{0,2} & \theta_{0,3} & \frac{1}{6}\,\theta_{1,3}\\
\theta_{0,2} & \theta_{0,3} & \theta_{0,4} & \frac{1}{10}\,\theta_{1,4}\\
\frac{1}{3}\,\theta_{1,2} & \frac{1}{6}\,\theta_{1,3} & \frac{1}{10}\,\theta_{1,4} & \frac{1}{9}\,\theta_{2,4}
\end{vmatrix},\quad
N^\Omega=\begin{vmatrix}
\theta_{0,0} & \theta_{0,1} & \theta_{0,2} & \frac{1}{3}\theta_{1,2}\\
\theta_{0,1} & \theta_{0,2} & \theta_{0,3} & \frac{1}{6}\,\theta_{1,3}\\
\theta_{1,1} & \frac{1}{3}\,\theta_{1,2} & \frac{1}{6}\,\theta_{1,3} & \frac{5}{9}\,\theta_{2,3}\\
\frac{1}{3}\,\theta_{1,2} & \frac{1}{6}\,\theta_{1,3} & \frac{1}{10}\,\theta_{1,4} & \frac{1}{9}\,\theta_{2,4}
\end{vmatrix},$$

the (30) gives:

$$\frac{1}{3\,c^2}\left(A_L^{\alpha\beta\gamma}-A_{LE}^{\alpha\beta\gamma}\right)U_{L\alpha}\Delta_{\beta\gamma}=-\frac{N^\Pi}{D_4}\,\Pi-\frac{N^\Omega}{D_4}\frac{1}{4c^2}\,\Omega\,. \tag{31}$$

We contract now Eq. $(28)_1$ with Δ_α^δ, Eq. $(28)_2$ with $U_{L\alpha}\,\Delta_\beta^\delta$, Eq. $(28)_3$ with $U_{L\alpha}U_{L\beta}\Delta_\gamma^\delta$ and again $(28)_3$ with $\Delta_\alpha^\delta\Delta_{\beta\gamma}$, obtaining the system

$$\begin{aligned}
&\theta_{1,1}\,\Delta^{\delta\mu}(\lambda_\mu-\lambda_{\mu_E})+\frac{2}{3}\theta_{1,2}\,U_L^\mu\Delta^{\delta\nu}\lambda_{\mu\nu}=-\frac{k_B}{m\,c^2(e_L+p_L)}\,q_L^\delta\,,\\
&\theta_{1,2}\,\Delta^{\delta\mu}(\lambda_\mu-\lambda_{\mu_E})+\theta_{1,3}\,U_L^\mu\Delta^{\delta\nu}\lambda_{\mu\nu}=0\,,\\
&\theta_{1,3}\,\Delta^{\delta\mu}(\lambda_\mu-\lambda_{\mu_E})+\frac{6}{5}\theta_{1,4}\,U_L^\mu\Delta^{\delta\nu}\lambda_{\mu\nu}=\frac{6\,k_B}{m^2c^6n_L}\left(A_E^{\alpha\beta\gamma}-A_{LE}^{\alpha\beta\gamma}\right)U_{L\alpha}U_{L\beta}\Delta_\gamma^\delta\,,\\
&\frac{5}{3}\theta_{2,3}\,\Delta^{\delta\mu}(\lambda_\mu-\lambda_{\mu_E})+\frac{2}{3}\theta_{2,4}\,U_L^\mu\Delta^{\delta\nu}\lambda_{\mu\nu}=-\frac{k_B}{m^2n_Lc^4}\left(A_E^{\alpha\beta\gamma}-A_{LE}^{\alpha\beta\gamma}\right)\Delta_{\alpha\beta}\Delta_\gamma^\delta\,.
\end{aligned}$$

By eliminating the parameters $\Delta^{\delta\mu}(\lambda_\mu-\lambda_{\mu_E})$ and $U_L^\mu\Delta^{\delta\nu}\lambda_{\mu\nu}$ from these equations, we obtain

$$\begin{aligned}
&\left(A_E^{\alpha\beta\gamma}-A_{LE}^{\alpha\beta\gamma}\right)U_{L\alpha}U_{L\beta}\Delta_\gamma^\delta=\frac{m\,n_L\,c^4}{6(e_L+p_L)}\frac{N_3}{D_3}\,q^\delta\,,\\
&\left(A_E^{\alpha\beta\gamma}-A_{LE}^{\alpha\beta\gamma}\right)\Delta_{\alpha\beta}\Delta_\gamma^\delta=-\frac{m\,n_L\,c^2}{e_L+p_L}\frac{N_{31}}{D_3}\,q^\delta\,,
\end{aligned} \tag{32}$$

with

$$D_3^L=\begin{vmatrix}\theta_{1,1} & \frac{2}{3}\theta_{1,2}\\ \theta_{1,2} & \theta_{1,3}\end{vmatrix},\quad
N_3^L=\begin{vmatrix}\theta_{1,2} & \theta_{1,3}\\ \theta_{1,3} & \frac{6}{5}\,\theta_{1,4}\end{vmatrix},\quad
N_{31}^L=\begin{vmatrix}\theta_{1,2} & \theta_{1,3}\\ \frac{5}{3}\,\theta_{2,3} & \frac{2}{3}\,\theta_{2,4}\end{vmatrix}.$$

We contract now Eq. $(28)_2$ with $\Delta_\alpha^{<\delta}\,\Delta_\beta^{\theta>_3}$ and $(28)_3$ with $\Delta_\alpha^{<\delta}\,\Delta_\beta^{\theta>_3}U_{L\gamma}$, obtaining

$$\begin{aligned}
&-\frac{k_B}{m}\,P^{<\delta\theta>_3}=\frac{2}{3}\,mn_Lc^4\theta_{2,3}\,\Delta^{\mu<\delta}\Delta^{\theta>_3\nu}\lambda_{\mu\nu}\,,\\
&\left(A_E^{\alpha\beta\gamma}-A_{LE}^{\alpha\beta\gamma}\right)\Delta_\alpha^{<\delta}\,\Delta_\beta^{\theta>_3}U_{L\gamma}=-\frac{2}{15}\,\frac{m^2n_L\,c^6}{k_B}\,\theta_{2,4}\,\Delta^{\mu<\delta}\Delta^{\theta>_3\nu}\lambda_{\mu\nu}\,,
\end{aligned}$$

from which it follows

$$\left(A_E^{\alpha\beta\gamma}-A_{LE}^{\alpha\beta\gamma}\right)\Delta_\alpha^{<\delta}\,\Delta_\beta^{\theta>_3}U_{L\gamma}=C_5\,c^2\,P^{<\delta\theta>_3}\qquad\text{with}\qquad C_5=\frac{1}{5}\,\frac{\theta_{2,4}}{\theta_{2,3}}\,. \tag{33}$$

Finally, $(28)_3$ contracted with $\Delta_\alpha^{<\delta}\,\Delta_\beta^\theta\,\Delta_\gamma^{\psi>_3}$ gives

$$mn\,c^6\Delta_\alpha^{<\delta}\,\Delta_\beta^\theta\,\Delta_\gamma^{\psi>_3}=0\,.$$

This result, jointly with (31), (32), (33), gives the decomposition of the triple tensor $A_L^{\alpha\beta\gamma}$:

$$A_L^{\alpha\beta\gamma} - A_{LE}^{\alpha\beta\gamma} = \frac{1}{4c^4}\,\Omega\, U_L^\alpha U_L^\beta U_L^\gamma + \frac{3}{4c^2}\frac{N^\Omega}{D_4}\,\Omega\,\Delta^{(\alpha\beta}U_L^{\gamma)} + 3\frac{N^\Pi}{D_4}\,\Pi\,\Delta^{(\alpha\beta}U_L^{\gamma)} + \frac{n_L\, m}{2(e_L+p_L)}\frac{N_3^L}{D_3^L}\, q_L^{(\alpha}U_L^\beta U_L^{\gamma)} - \frac{3}{5}\frac{n_L\, mc^2}{e_L+p_L}\frac{N_{31}^L}{D_3^L}\,\Delta^{(\alpha\beta}q_L^{\gamma)} + 3C_5\, P^{(<\alpha\beta>_3}U_L^{\gamma)}\,.$$

Thanks to Eq. $(29)_1$, we have the closure of the triple tensor in terms of the physical variables:

$$\begin{aligned} A_L^{\alpha\beta\gamma} &= \left(\rho_L\,\theta_{0,2} + \frac{1}{4c^4}\,\Omega\right) U_L^\alpha U_L^\beta U_L^\gamma \\ &- \left(\rho_L\, c^2\,\theta_{1,2} - \frac{3}{4c^2}\frac{N^\Omega}{D_4}\,\Omega - 3\frac{N^\Pi}{D_4}\,\Pi\right)\Delta^{(\alpha\beta}U_L^{\gamma)} \\ &+ \frac{\rho_L}{2(e_L+p_L)}\frac{N_3^L}{D_3^L}\, q_L^{(\alpha}U_L^\beta U_L^{\gamma)} - \frac{3}{5}\frac{\rho_L c^2}{e_L+p_L}\frac{N_{31}^L}{D_3^L}\,\Delta^{(\alpha\beta}q_L^{\gamma)} \\ &+ 3C_5\, P^{(<\alpha\beta>_3}U_L^{\gamma)}\,. \end{aligned} \tag{34}$$

To conclude the determination of the closure it is necessary to determine the explicit expression of the production terms. By substituting the expression (11) of the collision term Q into Eq. $(2)_2$ with $n = 0, 1, 2$ and applying the constrains (9), we have

$$I = -\frac{U_{L\alpha}}{c^2\tau}\left(V_L^\alpha - V_{LE}^\alpha\right) = \frac{U_{L\alpha}}{c^2\tau}\frac{\rho_L\, q_L^\alpha}{e_L+p_L} = 0 \quad \text{(Mass production)},$$

$$I^\beta = -\frac{U_{L\alpha}}{c^2\tau}\left(T_L^{\alpha\beta} - T_{LE}^{\alpha\beta}\right) = -\frac{U_{L\alpha}}{c^2\tau}\left[\Pi_L\,\Delta^{\alpha\beta} + P^{<\alpha\beta>_3}\right] = 0\,\text{(Momentum–energy production)},$$

$$I^{\beta\gamma} = -\frac{U_{L\alpha}}{c^2\tau}\left(A_L^{\alpha\beta\gamma} - A_{LE}^{\alpha\beta\gamma}\right) = -\frac{1}{\tau}\left[\frac{1}{4c^4}\Omega\, U_L^\beta U_L^\gamma + \frac{1}{4c^2}\frac{N^\Delta}{D_4}\,\Omega\Delta^{\beta\gamma} + \frac{N^\Pi}{D_4}\,\Pi_L\,\Delta^{\beta\gamma} + \frac{\rho_L}{3(e_L+p_L)}\frac{N_3^L}{D_3^L}\, U^{L(\beta}\, q_L^{\gamma)} + C_5\, P^{<\beta\gamma>_3}\right].$$

5. Comparison of the closures obtained with the two approaches

In the following we compare the closure obtained in the Eckart and Landau–Lifshitz frames. For this aim it is first necessary to find the mapping between the variables of the two approaches and show that the transformation is invertible.

5.1. The transformation law between the sets of variables in the two descriptions

We start by obtaining the explicit transformation law of the Landau–Lifshitz 4-velocity into Eckart variables. From Eq. (7) we have the transformation law of the two four-velocities. Moreover,

$$\Delta^{\alpha\beta} = -h^{\alpha\beta} - \frac{2}{(e+p)\,c^2}\,U^{(\alpha}\,q^{\beta)}\,, \tag{35}$$

where the second order term has been omitted. This is not a restriction, as also Eq. (7) is a first order approximation.

From $(16)_2$, compared with the corresponding of Eckart, we have

$$\frac{U^\alpha}{T} = \frac{U_L^\alpha}{T_L}.$$

By using the definition (7) we obtain

$$\frac{U^\alpha}{T} = \frac{1}{T_L}U^\alpha + \frac{q^\alpha}{e+p},$$

that, contracted with U^α and using $(4)_2$, gives $T = T_L$. From the definition of γ and θs in Section 4.1 it follows that they are also equal in the two frames.

- We consider now the transformation that expresses the variables of the Landau–Lifshitz description in terms of the Eckart ones.

We can compare the expression of the mass density current in the two frames, i.e. Eqs. $(3)_1$ and $(8)_1$, obtaining

$$m\,n\,U^\alpha = m\,n_L\,U_L^\alpha - \frac{m\,n_L}{e_L+p_L}\,q_L^\alpha. \tag{36}$$

We can contract the above equation with $U_{L\alpha}$ obtaining $n_L = \frac{n}{c^2}\,U^{L\alpha}\,U_\alpha$. If we use Eq. (7), we obtain

$$n_L = n \qquad \text{and} \qquad p_L = p, \ \text{ as p=n } k_B \text{ T}. \tag{37}$$

Moreover, by using Eq. $(37)_1$ into (36) and using Eq. (7) we have

$$\frac{q_L^\alpha}{e_L+p_L} = U_L^\alpha - U^\alpha = \frac{q^\alpha}{e+p}.$$

We can also compare the expression of the energy–momentum tensor in the two frames, i.e. Eqs. $(3)_2$ and $(8)_2$, obtaining

$$\begin{aligned} &\frac{e}{c^2}\,U^\alpha U^\beta + (p+\Pi)\,h^{\alpha\beta} + \frac{2}{c^2}\,U^{(\alpha}q^{\beta)} + t^{<\alpha\beta>_3} \\ &\quad = \frac{e_L}{c^2}\,U_L^\alpha U_L^\beta - (p_L+\Pi_L)\,\Delta^{\alpha\beta} + P^{<\alpha\beta>_3}\,. \end{aligned} \tag{38}$$

The above equation contracted with $U_{L\alpha}U_{L\beta}$ and using Eq. (7) gives

$$e_L = e$$

Eq. (38) contracted with $\Delta^{\alpha\beta}$ and using Eqs. (35) and $(37)_2$ gives

$$\Pi_L = \Pi.$$

Eq. (38) contracted with $U_{L\alpha}\Delta_\beta^\gamma$ and $\Delta_\alpha^{<\gamma}\,\Delta_\beta^{\delta>_3}$ gives

$$q_L^\gamma = q^\gamma\,, \qquad\qquad P^{<\gamma\delta>_3} = t^{<\gamma\delta>_3}.$$

Finally, we can compare Eq. (34) with the corresponding Eq. (35) of [19] and contract with $U_{L\alpha}U_{L\beta}U_{L\gamma}$ obtaining

$$\Omega = \Delta.$$

So we have obtained that, up to first order

$$n_L = n\,,\, p_L = p\,,\, \Pi_L = \Pi\,,\, U^{L\alpha} = U^\alpha + \frac{q^\alpha}{e+p}\,,\, e_L = e\,,\, q^{CK\,\alpha} = q^\alpha\,,\, P^{<\delta\gamma>_3(1)} = t^{<\delta\gamma>_3}\,,\, \Omega = \Delta\,. \tag{39}$$

- Now we express the Eckart variables in terms of the Landau–Lifshitz's one.
Eq. (36), contracted with itself, gives

$$n = n_L\left(1 + \frac{q_L^\alpha q_{L\,\alpha}}{c^2\left(e_L+p_L\right)^2}\right)^{\frac{1}{2}}. \tag{40}$$

Eq. (36) with (40) gives

$$U^\alpha = \left(1 + \frac{q_L^\alpha q_{L\,\alpha}}{c^2\left(e_L+p_L\right)^2}\right)^{-\frac{1}{2}}\left(U_L^\alpha - \frac{q_L{}^\alpha}{e_L+p_L}\right), \tag{41}$$

that is useful also to find the expression of the projector $h^{\alpha\beta}$. Moreover, as $p = n\,k_B\,T$, we have

$$p = n_L\,k_B\,T\left(1 + \frac{q_L^\alpha q_{L\,\alpha}}{c^2\left(e_L+p_L\right)^2}\right)^{\frac{1}{2}}, \tag{42}$$

From (40), (41) and (42) we see that the transformation law between the sets of fundamental variables ρ, u^α and p, in the two descriptions is not linear. As the definition of the Landau–Lifshitz in terms of the Eckart variables, widely used in literature, is a first order approximation we consider appropriate to take a linear approximation of the transformation laws we find above. In such way we have

$$n = n_L\,, \qquad p = p_L\,, \qquad U^\alpha = U_L^\alpha - \frac{q_L^\alpha}{e_L+p_L}\,.$$

Eq. (38) contracted with $\frac{U_\alpha U_\beta}{c^2}$ and $\frac{h_{\alpha\beta}}{3}$ gives

$$e = e_L\,, \qquad \Pi = \Pi_L\,.$$

Eq. (38) contracted with $U_\alpha h^\gamma_\beta$ and $h^{<\gamma}_\alpha h^{\delta>_3}_\beta$ gives

$$q^\gamma = q^\gamma_L\,, \qquad t^{<\gamma\delta>_3} = P^{<\gamma\delta>_3}.$$

Finally, we can compare Eq. (34) with the corresponding Eq. (35) of [19] and contract with $U_\alpha U_\beta U_\gamma$ obtaining

$$\Delta = \Omega.$$

It is remarkable that these eqs. are the inverse of Eqs. (39).

We emphasize that in both cases the above expressions are linear approximations of the transformation laws between the variables of the two frames.

5.2. Mapping the tensors between the two approaches

In this section we compare the expressions of the tensors present in system (13) in the two frames.

We have

$$V^\alpha_L = m\,n_L\,U^\alpha_L - \frac{m\,n_L}{e_L+p_L}\,q^\alpha_L = m\,n\left(U^\alpha + \frac{q^\alpha}{e+p}\right) - \frac{m\,n}{e+p}\,q^\alpha = m\,n\,U^\alpha = V^\alpha.$$

where Eqs. (39) have been used, in order to express the quantities in the Eckart frame. Moreover we have

$$\begin{aligned} T^{\alpha\beta}_L &= \frac{e}{c^2}\,U^\alpha_L U^\beta_L - (p_L+\Pi_L)\,\Delta^{\alpha\beta} + P^{<\alpha\beta>_3} = \\ &= \frac{e}{c^2}\,U^\alpha U^\beta + \frac{e}{c^2}\frac{2}{e+p}\,U^{(\alpha}q^{\beta)} + (p+\Pi)\,h^{\alpha\beta} + \frac{2}{c^2}\frac{p}{e+p}\,U^{(\alpha}q^{\beta)} + t^{<\alpha\beta>_3} \\ &= \frac{e}{c^2}\,U^\alpha U^\beta + \frac{2}{c^2}U^{(\alpha}q^{\beta)} + (p+\Pi)\,h^{\alpha\beta} + t^{<\alpha\beta>_3} = T^{\alpha\beta}, \end{aligned}$$

where Eqs. (39) have been used, in order to express the quantities in the Eckart frame and the second order terms have been omitted. We also have

$$\begin{aligned} A^{\alpha\beta\gamma}_E &= \rho\,\theta_{0,2}\,U^\alpha U^\beta U^\gamma - \rho c^2\,\theta_{1,2}\,\Delta^{(\alpha\beta}U^{\gamma)} \\ &= \rho\,\theta_{0,2}\,U^\alpha U^\beta U^\gamma + \rho c^2\,\theta_{1,2}\,h^{(\alpha\beta}U^{\gamma)} \\ &+ \frac{3\rho\,\theta_{0,2}+2\rho\theta_{1,2}}{(e+p)}U^{(\alpha}U^\beta q^{\gamma)} + \frac{\rho c^2\,\theta_{1,2}}{e+p}\,h^{(\alpha\beta}q^{\gamma)}\,, \end{aligned}$$

where the higher order terms have been omitted. This results shows that also the equilibrium part of $A^{\alpha\beta\gamma}$, that is of order 0 in the Landau–Lifshitz description, produces a first order term when it is converted in the Eckart variables.

Now we analyse the complete expression of $A^{\alpha\beta\gamma}$, in order to compare it with the corresponding of Eckart (Eq. (35) of paper [19])

$$\begin{aligned} A^{\alpha\beta\gamma} =& \rho\,\theta_{0,2}U^\alpha U^\beta U^\gamma + \frac{1}{4c^4}\,\Delta\,U^\alpha U^\beta U^\gamma + \rho\,c^2\,\theta_{1,2}h^{(\alpha\beta}U^{\gamma)} \\ &- \frac{3}{4c^2}\frac{N^\Delta}{D_4}\,\Delta h^{(\alpha\beta}U^{\gamma)} - 3\frac{N^\Pi}{D_4}\,\Pi\,h^{(\alpha\beta}U^{\gamma)} + \\ &\frac{3}{c^2}\frac{N_3}{D_3}\,q^{(\alpha}U^\beta_L U^{\gamma)}_L + \frac{3}{5}\frac{N_{31}}{D_3}\,h^{(\alpha\beta}q^{\gamma)} + 3C_5 t^{(<\alpha\beta>_3}U^{\gamma)}\,, \end{aligned} \tag{43}$$

where N^Δ, N^Π, D_4, N_3, N_{31} and D_3 are defined at pag. 8 of paper [3].

We notice the presence of different matrices in the two expressions. As they are composed by θs, that are equal in the two frames, we can see that N^Π and D_4 are exactly the same and for this reason we left the same symbol, while

$$N^\Delta = N^\Omega\,, \qquad D^L_3 = \frac{2}{3}D_3. \tag{44}$$

By using Eqs. (39) and (44) in the expression (34) of $A^{\alpha\beta\gamma}_L$ and eliminating the second order terms we have

$$\begin{aligned} A^{\alpha\beta\gamma}_L =& \rho\,\theta_{0,2}U^\alpha U^\beta U^\gamma + \frac{3\rho\theta_{0,2}}{e+p}U^{(\alpha}U^\beta q^{\gamma)} + \rho\,c^2\,\theta_{1,2}h^{(\alpha\beta}U^{\gamma)} + \frac{\rho\,c^2\,\theta_{1,2}}{e+p}h^{(\alpha\beta}q^{\gamma)} + \\ &\frac{2\rho\,\theta_{1,2}}{e+p}U^{(\alpha}U^\beta q^{\gamma)} + \frac{1}{4c^4}\,\Delta\,U^\alpha U^\beta U^\gamma - \frac{3}{4c^2}\,\frac{N^\Delta}{D_4}\,\Delta h^{(\alpha\beta}U^{\gamma)} - 3\frac{N^\Pi}{D_4}\,\Pi\,h^{(\alpha\beta}U^{\gamma)} \\ &+ \frac{\rho}{2(e+p)}\frac{N^L_3}{D^L_3}\,q^{(\alpha}U^\beta_L U^{\gamma)}_L + \frac{3}{5}\frac{\rho c^2}{e+p}\frac{N^L_{31}}{D^L_3}\,h^{(\alpha\beta}q^{\gamma)} + 3C_5 t^{(<\alpha\beta>_3}U^{\gamma)}\,. \end{aligned}$$

Comparing the above equation with (43) we see that the majority of terms simplify while there remains the terms in $U^{(\alpha}U^\beta q^{\gamma)}$ and those in $h^{(\alpha\beta}q^{\gamma)}$, so they coincide if and only if the following conditions are satisfied:

$$\begin{aligned} &(3\rho\theta_{0,2}+2\rho\,\theta_{1,2})D_3 + \frac{3\rho}{4}N^L_3 = \frac{3}{c^2}N_3(e+p) \\ &\rho\,c^2\,\theta_{1,2}D_3 + \frac{9}{10}\rho c^2 N^L_{31} = \frac{3}{5}N_{31}(e+p). \end{aligned} \tag{45}$$

The first one of these can be written as

$$\begin{vmatrix} \frac{3\rho}{4} & \frac{3}{4}\theta_{1,1} & \frac{1}{2}\theta_{1,2} \\ \frac{3}{c^2}\,(e+p) & \theta_{1,2} & \theta_{1,3} \\ 2(3\rho\,\theta_{0,2}+2\rho\theta_{1,2}) & \theta_{1,3} & \frac{6}{5}\theta_{1,4} \end{vmatrix} = 0\,,$$

and this is an identity because the second column is equal to the first one multiplied by $\frac{1}{\gamma\rho}$. This is easy to verify for the first element by recalling that $\frac{1}{\gamma} = \theta_{1,1}$ (see Eq. (24)). For the second element we have to use Eqs. (25) for energy and pressure the recursive relation $(23)_4$ with $h = j = 1$. According to the third element we have simply to use Eq. $(23)_4$ with $h = 1$ and $j = 2$.

Condition $(45)_2$ can be written as

$$\frac{3}{5}\begin{vmatrix} \frac{3}{2}\rho c^2 & \frac{3}{2}\theta_{1,1} & \theta_{1,2} \\ 3(e+p) & \theta_{1,2} & \theta_{1,3} \\ \frac{5}{3}\rho c^2\theta_{1,2} & \frac{5}{3}\theta_{2,3} & \frac{2}{3}\theta_{2,4} \end{vmatrix} = 0\,,$$

and this is an identity because the second column is equal to the first one multiplied by $\frac{1}{\gamma\rho c^2}$. For the first two elements it is possible to follow the same steps used in the previous matrix, while for the third term we have to use the recursive relation Eq. $(23)_5$ with $j = 2$.

There remains the analysis of the right hand sides of (13). Since Eq. (11) produces $I = I^\alpha = 0$ its linearization will also satisfy this requirement (the linearized expression of zero is zero). Therefore it is necessary only to compare the production term $I^{\beta\gamma}$ in $(31)_3$ with the corresponding expression (45) of [19]. Thanks to Eqs. (39) and (44) it becomes

$$I^{\beta\gamma} = -\frac{1}{\tau}\left[\frac{1}{4c^4}\,\Delta\,U^\beta U^\gamma - \frac{1}{4c^2}\frac{N^\Delta}{D_4}\,\Delta h^{\beta\gamma} - \frac{N^\Pi}{D_4}\,\Pi\,h^{\beta\gamma} + \frac{m\,n}{2(e+p)}\frac{N^L_3}{D_3}\,U^{(\beta}q^{\gamma)} + C_5\,t^{<\beta\gamma>_3}\right].$$

The above is equal to the corresponding of paper [19] iff

$$\frac{\rho}{2(e+p)}\frac{N^L_3}{D_3} = \frac{1}{c^2}\left(2\frac{N_3}{D_3} - \frac{\theta_{1,3}}{\theta_{1,2}}\right).$$

By substituting the expression of the matrices involved in the above expression, by using Eqs. (25) for e and p, the recursive formula $(16)_4$ with $h = j = 1$ and (24), it is easy to prove that the above is an identity.

6. Conclusions

In this paper we presented a relativistic moments model for the description of dissipative polyatomic gases in the Landau–Lifshitz description. Such frame is fundamental in describing, for example, high energy phenomena in which the mass of the involved particles is almost null. The proposed model presents 15 independent equations, as prescribed by recent results in Rational Extended Thermodynamics [17] proving that the choice of the moments number cannot be arbitrary if we want the model to be consistent from a mathematical and physical point of view. We found a closure for this model and make a comparison with results already present in literature in the Eckart description, finding that they coincide, at least up to first order with respect to equilibrium. Finally we show that the production terms obtained in the Landau–Lifshitz description and based on the Anderson–Witting collision term coincide, within the approximation we talked about earlier, with that obtained in the Eckart approach by using the collision term proposed by Pennisi–Ruggeri in [16] and containing an additional term proportional to the heat flux.

By incorporating dissipative dynamics into relativistic gas models, researchers can achieve a more accurate description of transport phenomena and the non-equilibrium behaviour of gases in extreme relativistic conditions.

In the future we aim to show that the 14 moments model for relativistic monatomic gases proposed by Cercignani and Kremer in [15] is contained in the present one as a singular limit. Moreover we aim to find the transport coefficients predicted by the present model and make a comparison with that already present in the literature and obtained in the Eckart frame in order to evaluate possible difference.

Funding

This research was supported by GNFM/INdAM and National Recovery and Resilience Plan (NRRP)– Project Title eINS Ecosystem of Innovation for Next Generation Sardinia– CUP F53C22000430001 funded by the European Union–NextGenerationEU, Project Code ECS0000038.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

No data was used for the research described in the article.

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