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ABSTRACT

The Yang model is an example of noncommutative geometry on a background spacetime of constant curvature. We discuss the Hermitian realizations of its associated algebra on phase space in a perturbative expansion up to sixth order. We also discuss its realizations on extended phase spaces, that include additional tensorial and/or vectorial degrees or freedom.

I. INTRODUCTION

In recent years noncommutative models in curved spacetime have been extensively investigated, either from a formal perspective,[1–13] also in connection with quantum field theory,[14–16] or in view of their application in the study of phenomenological effects in cosmology.[17,18]

However, the first example of noncommutativity on a curved spacetime background was proposed by Yang[19] already in 1947, soon after Snyder had introduced the idea of a noncommutative spacetime.[20] Yang's proposal was based on an algebra which included phase space and Lorentz generators, where the commutation relations between the components of the position operators, as well as those of the momentum operators were not trivial, giving rise to a spacetime displaying both noncommutativity and curvature.

The noncommutative Yang algebra is a 15-parameter algebra, isomorphic to $so(1,5)$, defined by the relations

$$\begin{aligned}
&[\hat{x}_\mu, \hat{x}_\nu] = i\beta^2 M_{\mu\nu}, \qquad [\hat{p}_\mu, \hat{p}_\nu] = i\alpha^2 M_{\mu\nu}, \qquad [\hat{x}_\mu, \hat{p}_\nu] = i\eta_{\mu\nu} h,\\
&[h, \hat{x}_\mu] = i\beta^2 \hat{p}_\mu, \qquad [h, \hat{p}_\mu] = -i\alpha^2 \hat{x}_\mu, \qquad [M_{\mu\nu}, h] = 0,\\
&[M_{\mu\nu}, \hat{x}_\lambda] = i(\eta_{\mu\lambda}\hat{x}_\nu - \eta_{\nu\lambda}\hat{x}_\mu), \qquad [M_{\mu\nu}, \hat{p}_\lambda] = i(\eta_{\mu\lambda}\hat{p}_\nu - \eta_{\nu\lambda}\hat{p}_\mu),\\
&[M_{\mu\nu}, M_{\rho\sigma}] = i(\eta_{\mu\rho}M_{\nu\sigma} - \eta_{\mu\sigma}M_{\nu\rho} - \eta_{\nu\rho}M_{\mu\sigma} + \eta_{\nu\sigma}M_{\mu\rho}),
\end{aligned} \tag{1}$$

where α and β are real parameters and $\eta_{\mu\nu}$ the flat metric and we use natural units, $\hbar = c = 1$. The Yang algebra is a Lie algebra, and therefore automatically satisfies the Jacobi identities.

We interpret the operators $\hat{x}_\mu$ and $\hat{p}_\mu$ as coordinates of the quantum phase space, $M_{\mu\nu}$ as generators of the Lorentz transformations and h as a further scalar generator, necessary to close the algebra. The algebra (1) is invariant under Born duality,[21] $\alpha \leftrightarrow \beta$, $\hat{x}_\mu \to -\hat{p}_\mu$, $\hat{p}_\mu \to \hat{x}_\mu$, $M_{\mu\nu} \leftrightarrow M_{\mu\nu}$, $h \leftrightarrow h$. It contains as subalgebras both the de Sitter and the Snyder algebras, to which it reduces in the limit $\beta \to 0$ and $\alpha \to 0$, respectively.

We have investigated the Yang model in previous papers. In particular, in 11 and 12 we have considered noncommutative models in a spacetime of constant curvature and discussed their realizations on a quantum phase space. These models preserve the Lorentz invariance and, besides Yang proposal, include some generalizations.[1,2] Later, in 13, we have discussed the possibility of obtaining Yang model by symmetry breaking of an algebra defined in an extended quantum phase space that includes also tensorial generators, of the kind introduced in 22–26 for the Snyder model.

Following 12, in this paper we shall investigate the realizations of the Yang algebra in terms of a restricted number of operators of a Hilbert space, the simplest case being realizations in terms of phase space variables x_μ and p_μ.[11,12] However, we shall use a more efficient procedure than in 12 for going to higher orders. Moreover, several possibilities arise depending on how many operators are introduced to generate the Hilbert space, as we discuss in Secs. II–V. For example, one may consider the Lorentz generators as independent from the phase space ones, as proposed in 22–25 in the case of the Snyder model. Some choices may be useful to obtain Hopf algebra structures, which are not possible in a phase space realization. In general, we shall only obtain perturbative realizations of the algebra, since analytic results seem to be out of reach.

II. REALIZATIONS OF YANG MODEL ON QUANTUM PHASE SPACE

In this section, we look for Hermitian realizations in quantum phase space, with

$$\hat{x}_\mu^\dagger = \hat{x}_\mu, \qquad \hat{p}_\mu^\dagger = \hat{p}_\mu, \qquad M_{\mu\nu}^\dagger = M_{\mu\nu}, \qquad h^\dagger = h, \tag{2}$$

where $\hat{x}_\mu$, $\hat{p}_\mu$, $M_{\mu\nu}$ and h are functions of phase space operators x_μ and p_μ that satisfy the Heisenberg algebra

$$[x_\mu, x_\nu] = [p_\mu, p_\nu] = 0, \qquad [x_\mu, p_\nu] = i\eta_{\mu\nu}, \tag{3}$$

with

$$M_{\mu\nu} = x_\mu p_\nu - x_\nu p_\mu, \tag{4}$$

and $p_\mu \triangleright 1 = 0$, $M_{\mu\nu} \triangleright 1 = 0$. Of course the realizations so defined can be implemented in a Hilbert space by the standard substitution $x_\mu \to x_\mu$, $p_\mu \to -i\frac{\partial}{\partial x^\mu}$, or $x_\mu \to i\frac{\partial}{\partial p^\mu}$, $p_\mu \to p_\mu$.

In the limit $\alpha = 0$, a Hermitian realization of the algebra (1) is given by[12]

$$\hat{x}_\mu(\beta) = \frac{1}{2}\left(x_\mu\sqrt{1-\beta^2p^2} + \sqrt{1-\beta^2p^2}\ x_\mu\right), \qquad \hat{p}_\mu = p_\mu, \qquad h = \sqrt{1-\beta^2p^2}. \tag{5}$$

Analogously, when $\beta = 0$, a realization is

$$\hat{p}_\mu(\alpha) = \frac{1}{2}\left(p_\mu\sqrt{1-\alpha^2x^2} + \sqrt{1-\alpha^2x^2}\ p_\mu\right), \qquad \hat{x}_\mu = x_\mu, \qquad h = \sqrt{1-\alpha^2x^2}. \tag{6}$$

However, when both $\alpha \neq 0$ and $\beta \neq 0$, we get

$$\begin{aligned}
&[\hat{x}_\mu(\beta), \hat{p}_\nu(\alpha)] = \\
&\frac{i}{2}\ \eta_{\mu\nu}\left(\sqrt{(1-\alpha^2x^2)(1-\beta^2p^2)} + \sqrt{(1-\beta^2p^2)(1-\alpha^2x^2)}\ \right) + \frac{1}{4}x_\mu(p_\nu K + Kp_\nu) + \frac{1}{4}(p_\nu K + Kp_\nu)x_\mu = \\
&\frac{i}{2}\ \eta_{\mu\nu}\left(\sqrt{(1-\alpha^2x^2)(1-\beta^2p^2)} + \sqrt{(1-\beta^2p^2)(1-\alpha^2x^2)}\ \right) + \frac{1}{4}p_\nu(x_\mu K + Kx_\mu) + \frac{1}{4}(x_\mu K + Kx_\mu)p_\nu,
\end{aligned} \tag{7}$$

where

$$\begin{aligned}
K = \sum_{m,n=0}^{\infty}\binom{\frac{1}{2}}{m}\binom{\frac{1}{2}}{n}(-\beta^2)^m(-\alpha^2)^n[p^{2m}, x^{2n}] = \\
-i\left(\alpha^2\beta^2D + \frac{\alpha^2\beta^4}{4}(p^2D + Dp^2) + \frac{\alpha^4\beta^2}{4}(x^2D + Dx^2) + \cdots\right),
\end{aligned} \tag{8}$$

with $D = \frac{1}{2}(x\cdot p + p\cdot x)$.(We denote $x^2 = x_\alpha x_\alpha$, $x\cdot p = x_\alpha p_\alpha$, and so on.)

This result is different from $i\eta_{\mu\nu}h(x,p)$ and therefore $\hat{x}_\mu(\beta)$ and $\hat{p}_\mu(\alpha)$ are not a realization of the Yang algebra. In order to construct a true realization of the Yang algebra, we fix $\hat{p}_\mu = \hat{p}_\mu(\alpha)$ and define $\hat{x}_\mu = e^{iG}\hat{x}_\mu(\beta)e^{-iG}$, choosing G such that $[\hat{x}_\mu, \hat{p}_\nu] = i\eta_{\mu\nu}h$. In general, we can expand G as

$$G = \sum_{m,n=1}^{\infty}\alpha^{2m}\beta^{2n}g_{2m,2n}, \tag{9}$$

where $g_{2m,2n}$ are functions of x^2, p^2 and D. From $\hat{x}_\mu = e^{iG}\hat{x}_\mu(\beta)e^{-iG}$, it follows

$$\hat{x}_\mu = \hat{x}_\mu(\beta) + i[G, \hat{x}_\mu(\beta)] + \frac{i^2}{2!}[G,[G,\hat{x}_\mu(\beta)]] + \cdots. \tag{10}$$

Then, up to sixth order in α and β, we get

$$[G, \hat{x}_\mu(\beta)] = \alpha^2\beta^2[g_{22}, x_\mu] - \frac{\alpha^2\beta^4}{4}[g_{22}, x_\mu p^2 + p^2 x_\mu] + \alpha^2\beta^4[g_{24}, x_\mu] + \alpha^4\beta^2[g_{42}, x_\mu]. \tag{11}$$

Hence,

$$\begin{aligned}[\hat{x}_\mu, \hat{p}_\nu] &= [\hat{x}_\mu(\beta), \hat{p}_\nu(\alpha)] + i\alpha^2\beta^2[[g_{22}, x_\mu], p_\nu] - i\frac{\alpha^2\beta^4}{4}[[g_{22}, x_\mu p^2 + p^2 x_\mu], p_\nu] \\ &\quad + i\alpha^2\beta^4[[g_{24}, x_\mu], p_\nu] + i\alpha^4\beta^2[[g_{42}, x_\mu], p_\nu] - \frac{i}{4}\alpha^4\beta^2[[g_{22}, x_\mu], p_\nu x^2 + x^2 p_\nu].\end{aligned} \tag{12}$$

Substituting in (7), it follows

$$\begin{aligned}&[\hat{x}_\mu, \hat{p}_\nu] = \\ &\frac{i}{2}\eta_{\mu\nu}\left(\sqrt{(1-\alpha^2x^2)(1-\beta^2p^2)} + \sqrt{(1-\beta^2p^2)(1-\alpha^2x^2)}\right) - \frac{i\alpha^2\beta^2}{4}(x_\mu(p_\nu D + Dp_\nu) + (p_\nu D + Dp_\nu)x_\mu) \\ &- \frac{i\alpha^2\beta^4}{16}\left(x_\mu\left(p_\nu(p^2D + Dp^2) + (p^2D + Dp^2)p_\nu\right) + \left(p_\nu(p^2D + Dp^2) + (p^2D + Dp^2)p_\nu\right)x_\mu\right) \\ &- \frac{i\alpha^4\beta^2}{16}\left(x_\mu\left(p_\nu(x^2D + Dx^2) + (x^2D + Dx^2)p_\nu\right) + \left(p_\nu(x^2D + Dx^2) + (x^2D + Dx^2)p_\nu\right)x_\mu\right) \\ &+ i\alpha^2\beta^2[[g_{22}, x_\mu], p_\nu] - i\frac{\alpha^2\beta^4}{4}\left[[g_{22}, x_\mu p^2 + p^2 x_\mu], p_\nu\right] + i\alpha^2\beta^4[[g_{24}, x_\mu], p_\nu] + i\alpha^4\beta^2[[g_{42}, x_\mu], p_\nu] \\ &- \frac{i}{4}\alpha^4\beta^2\left[[g_{22}, x_\mu], p_\nu x^2 + x^2 p_\nu\right].\end{aligned} \tag{13}$$

Requiring that only terms proportional to $\eta_{\mu\nu}$ survive, one obtains (see the Appendix)

$$g_{22} = \frac{1}{6}\left(D^3 - \frac{1}{2}D\right), \qquad g_{24} = -\frac{1}{16}(Dp^2 + p^2D), \qquad g_{42} = -\frac{1}{16}(Dx^2 + x^2D). \tag{14}$$

Hence, at this order,

$$G = \frac{\alpha^2\beta^2}{6}\left(D^3 - \frac{1}{2}D\right) - \frac{\alpha^2\beta^2}{16}\left(D(\alpha^2x^2 + \beta^2p^2) + (\alpha^2x^2 + \beta^2p^2)D\right), \tag{15}$$

and then

$$\begin{aligned}\hat{x}_\mu &= \frac{1}{2}\left(x_\mu\sqrt{(1-\beta^2p^2)} + \sqrt{(1-\beta^2p^2)}\ x_\mu\right) + \frac{\alpha^2\beta^2}{4}\left(x_\mu D^2 + D^2 x_\mu\right) \\ &\quad - \frac{\alpha^2\beta^2}{16}\left[x_\mu(\alpha^2x^2 + \beta^2p^2) + (\alpha^2x^2 + \beta^2p^2)x_\mu + 2\beta^2(Dp_\mu + p_\mu D)\right. \\ &\quad \left. - \beta^2\left((x_\mu p^2 + p^2 x_\mu)D^2 + D^2(x_\mu p^2 + p^2 x_\mu)\right)\right], \\ \hat{p}_\mu &= \frac{1}{2}\left(p_\mu\sqrt{1-\alpha^2x^2} + \sqrt{1-\alpha^2x^2}\ p_\mu\right),\end{aligned} \tag{16}$$

and $[\hat{x}_\mu, \hat{p}_\nu] = i\eta_{\mu\nu}h$, with

$$\begin{aligned}h &= \frac{1}{2}\left(\sqrt{(1-\alpha^2x^2)(1-\beta^2p^2)} + \sqrt{(1-\beta^2p^2)(1-\alpha^2x^2)} + \alpha^2\beta^2D^2\right) \\ &\quad + \frac{\alpha^2\beta^2}{8}\left(\beta^2p^2 - \alpha^2x^2 + \beta^2(p^2D^2 + D^2p^2) - \alpha^2(x^2D^2 + D^2x^2)\right).\end{aligned} \tag{17}$$

We point out that infinitely many realizations can be obtained from $\hat{x}_\mu$ and $\hat{p}_\mu$ in (16) by similarity transformations, defined by acting simultaneously with $e^{iG(D,x^2,p^2)}$ on all generators $\hat{x}_\mu$, $\hat{p}_\mu$, $M_{\mu\nu}$ and h obtained above. Note that $M_{\mu\nu}$ is invariant under these transformations because G is a function of Lorentz-invariant operators, but h is not invariant since $[G,h] \neq 0$.

Special classes of realizations are

$$\begin{aligned}\hat{x}_\mu(c_1) &= e^{-ic_2G}\hat{x}_\mu e^{ic_2G} = e^{ic_1G}\hat{x}_\mu(\beta)\ e^{-ic_1G},\\ \hat{p}_\mu(c_2) &= e^{-ic_2G}\hat{p}_\mu e^{ic_2G} = e^{-ic_2G}\hat{p}_\mu(\alpha)\ e^{ic_2G},\\ h(c_1,c_2) &= e^{-ic_2G}h\ e^{ic_2G},\end{aligned} \tag{18}$$

with G given in (15) and $c_1 + c_2 = 1$.

At sixth order in α and β we have

$$\begin{aligned}\hat{x}_\mu(c_1) = \frac{1}{2}\Big(x_\mu\sqrt{(1-\beta^2p^2)} + \sqrt{(1-\beta^2p^2)}\ x_\mu\Big) + c_1\frac{\alpha^2\beta^2}{4}\big(x_\mu D^2 + D^2x_\mu\big)\\ - c_1\frac{\alpha^2\beta^2}{16}\big[x_\mu(\alpha^2x^2+\beta^2p^2) + (\alpha^2x^2+\beta^2p^2)x_\mu + 2\beta^2(Dp_\mu + p_\mu D)\\ - \beta^2\big((x_\mu p^2 + p^2x_\mu)D^2 + D^2(x_\mu p^2 + p^2x_\mu)\big)\big],\end{aligned}$$

$$\begin{aligned}\hat{p}_\mu(c_2) = \frac{1}{2}\Big(p_\mu\sqrt{(1-\alpha^2x^2)} + \sqrt{(1-\alpha^2x^2)}\ p_\mu\Big) + c_2\frac{\alpha^2\beta^2}{4}\big(p_\mu D^2 + D^2p_\mu\big)\\ - c_2\frac{\alpha^2\beta^2}{16}\big[p_\mu(\alpha^2x^2+\beta^2p^2) + (\alpha^2x^2+\beta^2p^2)p_\mu + 2\alpha^2(Dx_\mu + x_\mu D)\\ - \alpha^2\big((p_\mu x^2 + x^2p_\mu)D^2 + D^2(p_\mu x^2 + x^2p_\mu)\big)\big],\end{aligned}$$

$$\begin{aligned}h(c_1,c_2) = \frac{1}{2}\Big(\sqrt{(1-\alpha^2x^2)(1-\beta^2p^2)} + \sqrt{(1-\beta^2p^2)(1-\alpha^2x^2)} + \alpha^2\beta^2D^2\Big)\\ + (c_1-c_2)\frac{\alpha^2\beta^2}{8}\big(\beta^2p^2 - \alpha^2x^2 + \beta^2(p^2D^2 + D^2p^2) - \alpha^2(x^2D^2 + D^2x^2)\big).\end{aligned} \tag{19}$$

In particular, for $c_1 = c_2 = \frac{1}{2}$,

$$h\Big(\frac{1}{2},\frac{1}{2}\Big) = \frac{1}{2}\Big(\sqrt{(1-\alpha^2x^2)(1-\beta^2p^2)} + \sqrt{(1-\beta^2p^2)(1-\alpha^2x^2)} + \alpha^2\beta^2D^2\Big) \tag{20}$$

and

$$[h(c_1,c_2),\hat{x}_\mu(c_1)] = i\beta^2\hat{p}_\mu(c_2), \qquad [h(c_1,c_2),\hat{p}_\mu(c_2)] = -i\alpha^2\hat{x}_\mu(c_1). \tag{21}$$

III. REALIZATIONS OF EXTENDED YANG MODEL ON QUANTUM PHASE SPACE

A different realization of the Yang algebra can be obtained introducing additional tensorial generators $\hat{x}_{\mu\nu} = -\hat{x}_{\nu\mu}$, similarly to what has been done in 22–26 for the Snyder model or in 27 for a more general setting. They are assumed to satisfy

$$\begin{aligned}&[\hat{x}_{\mu\nu},\hat{x}_{\rho\sigma}] = i(\eta_{\mu\rho}\hat{x}_{\nu\sigma} - \eta_{\mu\sigma}\hat{x}_{\nu\rho} - \eta_{\nu\rho}\hat{x}_{\mu\sigma} + \eta_{\nu\sigma}\hat{x}_{\mu\rho}),\\ &[\hat{x}_{\mu\nu}, x_\lambda] = 0, \qquad [\hat{x}_{\mu\nu}, p_\lambda] = 0.\end{aligned} \tag{22}$$

In this case, we consider realizations of Lorentz generators of the form

$$M_{\mu\nu} = \hat{x}_{\mu\nu} + x_\mu p_\nu - x_\nu p_\mu, \tag{23}$$

and $M_{\mu\nu} \triangleright 1 = \hat{x}_{\mu\nu} \triangleright 1 = x_{\mu\nu}$, where $x_{\mu\nu}$ are commuting variables.

In the limit $\alpha = 0$, a realization of the Yang algebra is given by

$$\hat{x}_\mu(\beta) = \frac{1}{2}\Big(x_\mu\sqrt{1-\beta^2p^2} + \sqrt{1-\beta^2p^2}x_\mu\Big) - \beta^2\hat{x}_{\mu\alpha}\frac{p_\alpha}{1+\sqrt{1-\beta^2p^2}}, \qquad \hat{p}_\mu = p_\mu, \qquad h = \sqrt{1-\beta^2p^2}. \tag{24}$$

Analogously, when $\beta = 0$,

$$\hat{p}_\mu(\alpha) = \frac{1}{2}\Big(p_\mu\sqrt{1-\alpha^2x^2} + \sqrt{1-\alpha^2x^2}p_\mu\Big) + \alpha^2\hat{x}_{\mu\alpha}\frac{x_\alpha}{1+\sqrt{1-\alpha^2x^2}}, \qquad \hat{x}_\mu = x_\mu, \qquad h = \sqrt{1-\alpha^2x^2}. \tag{25}$$

Also in this case, if both $\alpha \neq 0$ and $\beta \neq 0$, $\hat{x}_\mu(\beta)$ and $\hat{p}_\mu(\alpha)$ do not constitute a realization of the Yang algebra, since $[\hat{x}_\mu(\beta), \hat{p}_\nu(\alpha)] \neq i\eta_{\mu\nu}h$. Therefore, as in the previous section, in order to construct a realization in terms of the extended algebra (22), we fix $\hat{p}_\mu = \hat{p}_\mu(\alpha)$ and define $\hat{x}_\mu = e^{iG}\hat{x}_\mu(\beta)e^{-iG}$, constructing the operator G in such a way that $[\hat{x}_\mu, \hat{p}_\nu] = i\eta_{\mu\nu}h$.

From the expansion (9), we get at fourth order in α, β,

$$G = \alpha^2\beta^2\left[\frac{1}{6}\left(D^3 - \frac{1}{2}D\right) - \frac{1}{8}\hat{x}_{\alpha\beta}(x_\alpha p_\beta + p_\beta x_\alpha)D - \frac{1}{8}\hat{x}_{\alpha\gamma}\hat{x}_{\beta\gamma}(x_\alpha p_\beta + p_\alpha x_\beta)\right]. \tag{26}$$

Hence,

$$\begin{aligned}\hat{x}_\mu = \hat{x}_\mu(\beta) &+ \frac{\alpha^2\beta^2}{4}\left(x_\mu D^2 + D^2 x_\mu\right) + \frac{\alpha^2\beta^2}{4}\hat{x}_{\mu\alpha}x_\alpha D - \frac{\alpha^2\beta^2}{8}\left(\hat{x}_{\alpha\beta}(x_\alpha p_\beta + p_\beta x_\alpha)x_\mu\right)\\ &- \frac{\alpha^2\beta^2}{8}(\hat{x}_{\alpha\gamma}\hat{x}_{\mu\gamma} + \hat{x}_{\mu\gamma}\hat{x}_{\alpha\gamma})x_\alpha\end{aligned} \tag{27}$$

and

$$h = \frac{1}{2}\left(\sqrt{(1-\alpha^2x^2)(1-\beta^2p^2)} + \sqrt{(1-\beta^2p^2)(1-\alpha^2x^2)} + \alpha^2\beta^2D^2\right). \tag{28}$$

There are infinitely many realizations obtained from $\hat{x}_\mu$ and $\hat{p}_\mu$ with arbitrary similarity transformations that are invariant under Lorentz transformations and act on all generators $\hat{x}_\mu$, $\hat{p}_\mu$ and h simultaneously. Note that $M_{\mu\nu}$ is invariant under these transformation but h is not.

IV. REALIZATIONS OF YANG MODEL ON DOUBLE QUANTUM PHASE SPACE

A different class of realizations can be obtained by adding to the generators x_μ, p_ν of the Heisenberg algebra new generators q_μ and k_μ satisfying a second Heisenberg algebra,

$$[q_\mu, q_\nu] = 0, \qquad [k_\mu, k_\nu] = 0, \qquad [q_\mu, k_\nu] = i\eta_{\mu\nu}, \tag{29}$$

with

$$[x_\mu, q_\nu] = [x_\mu, k_\nu] = 0, \qquad [p_\mu, q_\nu] = [p_\mu, k_\nu] = 0, \tag{30}$$

and

$$p_\mu \triangleright 1 = 0, \qquad k_\mu \triangleright 1 = 0, \qquad x_\mu \triangleright 1 = x_\mu, \qquad q_\mu \triangleright 1 = q_\mu. \tag{31}$$

These realizations are more symmetric in the phase space variables and might permit the definition of a Hopf structure. We shall call the phase space obtained by the addition of q_μ and k_μ double quantum phase space.

In the limit $\alpha \to 0$, a realization of the Yang model in this space is given by

$$\begin{aligned}&\hat{x}_\mu(\beta) = \frac{1}{2}\left(x_\mu\sqrt{1-\beta^2p^2} + \sqrt{1-\beta^2p^2}\;x_\mu\right) + \frac{b}{2}\left(k_\mu\sqrt{1-\frac{\beta^2q^2}{b^2}} + \sqrt{1-\frac{\beta^2q^2}{b^2}}\;k_\mu\right),\\ &\hat{p}_\mu = q_\mu + \tilde{b}\,p_\mu, \qquad h = \tilde{b}\sqrt{1-\beta^2p^2} - b\sqrt{1-\frac{\beta^2q^2}{b^2}}, \qquad M_{\mu\nu} = x_\mu p_\nu - x_\nu p_\mu + q_\mu k_\nu - q_\nu k_\mu,\end{aligned} \tag{32}$$

with nonvanishing parameters b and $\tilde{b}$, with $\tilde{b} - b = 1$. Analogously, when $\beta = 0$,

$$\begin{aligned}&\hat{p}_\mu(\alpha) = \frac{1}{2}\left(q_\mu\sqrt{1-\alpha^2k^2} + \sqrt{1-\alpha^2k^2}\;q_\mu\right) + \frac{\tilde{b}}{2}\left(p_\mu\sqrt{1-\frac{\alpha^2x^2}{\tilde{b}^2}} + \sqrt{1-\frac{\alpha^2x^2}{\tilde{b}^2}}\;p_\mu\right),\\ &\hat{x}_\mu = x_\mu + b\,k_\mu, \qquad h = -b\sqrt{1-\alpha^2k^2} + \tilde{b}\sqrt{1-\frac{\alpha^2x^2}{\tilde{b}^2}}, \qquad M_{\mu\nu} = x_\mu p_\nu - x_\nu p_\mu + q_\mu k_\nu - q_\nu k_\mu.\end{aligned} \tag{33}$$

As usual, if both $\alpha \neq 0$ and $\beta \neq 0$, $\hat{x}_\mu(\beta)$ and $\hat{p}_\mu(\alpha)$ are not a realization of the Yang algebra.

In order to construct realizations of the Yang model in this space, as in Secs. II and III, we set $\hat{p}_\mu = \hat{p}_\mu(\alpha)$ and $\hat{x}_\mu = e^{iG}\hat{x}_\mu(\beta)e^{-iG}$, and construct the operator G such that $[\hat{x}_\mu, \hat{p}_\nu] = i\eta_{\mu\nu}h$. In general, G can be expanded as in (9).

Proceeding as usual, we get at fourth order in α and β,

$$G = \frac{\alpha^2\beta^2}{6}\left[\frac{1}{\tilde{b}^2}\left(D^3 - \frac{1}{2}D\right) - \frac{1}{b^2}\left(\tilde{D}^3 - \frac{1}{2}\tilde{D}\right)\right], \tag{34}$$

where $\tilde{D} = \frac{1}{2}(k \cdot q + q \cdot k)$. Hence,

$$\hat{x}_\mu = \hat{x}_\mu(\beta) + \frac{\alpha^2\beta^2}{4}\left[\frac{1}{\tilde{b}^2}(x_\mu D^2 + D^2 x_\mu) + \frac{1}{b}(k_\mu \tilde{D}^2 + \tilde{D}^2 k_\mu)\right] \tag{35}$$

and

$$\begin{aligned} h = {} & \frac{\tilde{b}}{2}\left(\sqrt{(1-\beta^2p^2)\left(1-\frac{\alpha^2x^2}{\tilde{b}^2}\right)} + \sqrt{\left(1-\frac{\alpha^2x^2}{\tilde{b}^2}\right)(1-\beta^2p^2)}\right) \\ & + \frac{b}{2}\left(\sqrt{(1-\alpha^2k^2)\left(1-\frac{\beta^2q^2}{b}\right)} + \sqrt{\left(1-\frac{\beta^2q^2}{b}\right)(1-\alpha^2k^2)}\right) + \frac{\alpha^2\beta^2}{2}\left(\frac{1}{\tilde{b}}D^2 - \frac{1}{b}\tilde{D}^2\right). \end{aligned} \tag{36}$$

Again, infinitely many realizations can be obtained by acting on (35) and (36) with similarity transformations.

V. REALIZATIONS OF EXTENDED YANG MODEL ON DOUBLE QUANTUM PHASE SPACE

Let us finally consider the Yang model with both additional phase space generators and additional Lorentz generators. Realizations of this kind have been considered in 13 in a slightly different formalism, see Sec. VI. The additional Lorentz generators $\hat{x}_{\mu\nu}$ are introduced as in Sec. III, such that

$$M_{\mu\nu} = \hat{x}_{\mu\nu} + x_\mu p_\nu - x_\nu p_\mu + q_\mu k_\nu - q_\nu k_\mu, \tag{37}$$

and $M_{\mu\nu} \triangleright 1 = \hat{x}_{\mu\nu} \triangleright 1 = x_{\mu\nu}$, where $x_{\mu\nu}$ are commutative parameters.

Proceeding as usual, one can show that realizations up to second order in α^2, β^2 are in this case

$$\begin{aligned} \hat{x}_\mu &= x_\mu - \frac{\beta^2}{4}(x_\mu p^2 + p^2 x_\mu) + b\ k_\mu - \frac{\beta^2}{4b}(k_\mu q^2 + q^2 k_\mu) - \frac{\beta^2}{2\tilde{b}}(x_{\mu\alpha}p_\alpha - x_{\mu\alpha}q_\alpha), \\ \hat{p}_\mu &= q_\mu - \frac{\alpha^2}{4}(q_\mu k^2 + k^2 q_\mu) + \tilde{b}\ p_\mu - \frac{\alpha^2}{4\tilde{b}}(p_\mu x^2 + x^2 p_\mu) + \frac{\alpha^2}{2b}(x_{\mu\alpha}k_\alpha + x_{\mu\alpha}x_\alpha), \end{aligned} \tag{38}$$

with

$$h = 1 - \frac{1}{2}\left(\frac{\alpha^2}{\tilde{b}}x^2 + \beta^2\tilde{b}\ p^2 - \frac{\beta^2}{b}q^2 - \alpha^2 b\ k^2\right). \tag{39}$$

Also in the present case infinitely many realizations can be obtained by similarity transformations.

VI. CONCLUDING REMARKS

In this paper we have assumed that in the limit $\alpha = 0, \beta = 0$, the Yang algebra (1) reduces to the ordinary Heisenberg algebra with Lorentz algebra action and $h \to 1$. Realizations are obtained in terms of quantum phase space and double quantum phase space with or without tensorial coordinates.

An approach to the Yang algebra alternative to the one we have considered here is to view it as a Lie algebra with 15 generators $\hat{x}_\mu, \hat{p}_\mu, M_{\mu\nu}$ and $\hat{h}$. When all structure constants go to zero, it reduces to a commutative space with coordinates x_μ, q_μ, $x_{\mu\nu}$ and h with relations $\hat{x}_\mu \triangleright 1 = x_\mu$, $\hat{p}_\mu \triangleright 1 = q_\mu$, $M_{\mu\nu} \triangleright 1 = x_{\mu\nu}$ and $\hbar = 0$. Realizations of this Yang algebra can be found using the method of realizations of Lie algebras described in 10 and 27. These realizations are linear in the position coordinates, but are given by power series in the momenta. Such approach was used in 23–25 for the extended Snyder model and in 13 for the Yang model.

We also notice that the Yang model can be obtained from the $so(1,5)$ algebra with 15 generators M_{AB} $(A, B = 1, \ldots, 5)$, through the relations $\hat{x}_\mu = \beta M_{\mu 4}$, $\hat{p}_\mu = \alpha M_{\mu 5}$ and $\hat{h} = \alpha\beta M_{45}$. A realization of $so(1,5)$ in symmetric ordering has been presented in 27 and can be used for the Yang model as well.

As future prospects of our investigations we may envisage the possibility of constructing a star product and a twist using the double quantum phase space. Also the definition of a field theory on a spacetime based on the Yang model can be pursued from the present results and would be of great interest.

Finally, we remark that at the classical level, the symplectic structure of the Snyder-like models can be obtained from dynamical models, see for example.[5,7,28,29] In a similar way one can write down an action for the Yang model, for example using the general procedure illustrated in 30, (see also Ref. 31). However, the results are cumbersome and we shall not report them here.

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AUTHOR DECLARATIONS

Conflict of Interest

The authors have no conflicts to disclose.

Author Contributions

Tea Martinić-Bilać: Conceptualization (equal); Formal analysis (equal); Investigation (equal); Validation (equal); Writing – review & editing (equal). **Stjepan Meljanac**: Conceptualization (equal); Formal analysis (equal); Investigation (equal); Methodology (equal); Supervision (equal); Validation (equal); Writing – original draft (equal). **Salvatore Mignemi**: Conceptualization (equal); Formal analysis (equal); Investigation (equal); Methodology (equal); Validation (equal); Writing – original draft (equal).

DATA AVAILABILITY

Data sharing is not applicable to this article as no new data were created or analyzed in this study.

APPENDIX: PROOF OF RELATIONS (14)

In this the appendix we give some details on the calculations leading to (14).

Starting from (13), we can compute the terms proportional to $\alpha^2\beta^2$. These are given by

$$i[[g_{22},x_\mu],p_\nu]-\frac{i}{4}(x_\mu(p_\nu D+Dp_\nu)+(p_\nu D+Dp_\nu)x_\mu)=\Big[i[g_{22},x_\mu]-\frac{1}{4}(x_\mu D^2+D^2x_\mu),p_\nu\Big]+\frac{i}{2}\eta_{\mu\nu}D^2. \tag{A1}$$

Requiring that only terms proportional to $\eta_{\mu\nu}$ survive in (A1), one gets

$$i[g_{22},x_\mu]=\frac{1}{4}(x_\mu D^2+D^2x_\mu), \tag{A2}$$

which is solved by

$$g_{22}=\frac{1}{6}\Big(D^3-\frac{1}{2}D\Big). \tag{A3}$$

Then,

$$i\Big[\Big[g_{22},\frac{1}{2}(x_\mu p^2+p^2x_\mu)\Big],p_\nu\Big]\approx-\frac{1}{8}\big((x_\mu p^2+p^2x_\mu)A_\nu+A_\nu(x_\mu p^2+p^2x_\mu)\big) \tag{A4}$$

and

$$i\Big[[g_{22},x_\mu],\frac{1}{2}(p_\nu x^2+x^2p_\nu)\Big]\approx-\frac{1}{8}\big((p_\nu x^2+x^2p_\nu)B_\mu+B_\mu(p_\nu x^2+x^2p_\nu)\big), \tag{A5}$$

with $A_\mu=Dp_\mu+p_\mu D$ and $B_\mu=Dx_\mu+x_\mu D$ and the $\approx$ symbol means that we are discarding the terms proportional to $\eta_{\mu\nu}$.

Substituting in (13) gives at order $\alpha^2\beta^4$,

$$i[[g_{24},x_\mu],p_\nu]+\frac{i}{16}\big((x_\mu p^2+p^2x_\mu)A_\nu+A_\nu(x_\mu p^2+p^2x_\mu)\big)-\frac{i}{16}\big(x_\mu p^2A_\nu+x_\mu A_\nu p^2+p^2A_\nu x_\mu+A_\nu p^2x_\mu\big)\approx i[[g_{24},x_\mu],p_\nu]+\frac{i}{4}p_\mu p_\nu. \tag{A6}$$

The last expression vanishes if

$$[g_{24},x_\mu]=-\frac{i}{8}(Dp_\mu+p_\mu D), \tag{A7}$$

and then, up to terms that give contributions proportional to $\eta_{\mu\nu}$,

$$g_{24}=-\frac{1}{16}(Dp^2+p^2D). \tag{A8}$$

At order $\alpha^4\beta^2$, one gets instead

$$i[[g_{42},x_\mu],p_\nu] + \frac{i}{16}\big((p_\nu x^2 + x^2 p_\nu)B_\mu + B_\mu(p_\nu x^2 + x^2 p_\nu)\big) - \frac{i}{16}\big(p_\nu x^2 B_\mu + p_\nu B_\mu x^2 + x^2 B_\mu p_\nu + B_\mu x^2 p_\nu\big) \approx i[[g_{42},x_\mu],p_\nu] + \frac{i}{4}x_\mu x_\nu. \tag{A9}$$

The last expression vanishes up to terms proportional to $\eta_{\mu\nu}$ if

$$[g_{42},x_\mu] = \frac{i}{8}x^2 x_\mu, \tag{A10}$$

and then

$$g_{42} = -\frac{1}{16}(Dx^2 + x^2 D). \tag{A11}$$

More generally, if we define

$$\hat{p}_\mu(\alpha) = \sum_{m=0}^{\infty}\alpha^{2m}p_\mu^{(2m)}, \qquad \hat{x}_\mu(\beta) = \sum_{n=0}^{\infty}\beta^{2n}x_\mu^{(2n)}, \tag{A12}$$

and

$$\hat{x}_\mu = e^{iG}\hat{x}_\mu(\beta)e^{-iG} = \hat{x}_\mu(\beta) + \sum_{n=1}^{\infty}\frac{1}{n!}(\mathrm{ad}_{\ iG})^n\hat{x}_\mu(\beta), \tag{A13}$$

with

$$G = \sum_{m,n=1}^{\infty}\alpha^{2m}\beta^{2n}g_{2m,2n}, \qquad h = \sum_{m,n=0}^{\infty}\alpha^{2m}\beta^{2n}h_{2m,2n}, \tag{A14}$$

and $h_{0,0} = 1$, then from $[\hat{x}_\mu,\hat{p}_\nu(\alpha)] = i\eta_{\mu\nu}h$, we get at order $\alpha^{2m}\beta^{2n}$,

$$[x_\mu^{(2n)},p_\nu^{(2m)}] + \sum_{\substack{m_1+m_2=m\\ n_1+n_2=n}}\Big[\Big[ig_{2m_1,2n_1},x_\mu^{(2n_2)}\Big],p_\nu^{(2m_2)}\Big] + \frac{1}{2!}\sum_{\substack{m_1+m_2+m_3=m\\ n_1+n_2+n_3=n}}\Big[\Big[ig_{2m_1,2n_1},\Big[ig_{2m_2,2n_2},x_\mu^{(2n_3)}\Big]\Big],p_\nu^{(2m_3)}\Big] + \cdots = i\eta_{\mu\nu}h_{2m,2n}. \tag{A15}$$

The last term on the left hand side has the form $\frac{1}{k!}\Big[(\mathrm{ad}_{\ iG_{2,2}})^k(x_\mu),p_\nu\Big]$, where $k = \min(m,n)$. These relations can be solved recursively to compute $g_{2m,2n}$ and $h_{2m,2n}$ using the results for g at lower orders.

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