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On admissible external loadings within the first and second strain gradient elasticity

Victor A. Eremeyev[1], Leonid P. Lebedev[2], and Michael J. Cloud[3]

Abstract

Recent extended applications of higher-order models of continuum mechanics as strain gradient elasticity and plasticity result in a certain extension of possible external loadings. In addition to known forces and couples we face double forces and hyper-stresses acting in the volume and on the external surface of an elastic solid body. Despite the fact that such higher-order loadings as double force or bi-moment are already known in structural mechanics, we still cannot say that these quantities are so obvious in continuum mechanics. Here we discuss the admissible external loadings including ones acting in a volume, on a surface, along a curve, and even at some points. To establish this possibility, we use the Virtual Work Principle (VWP) generalized to first and second strain gradient elasticity. The principle is a foundation of weak solutions of the boundary-value problems under consideration. From the mathematical point of view, external loadings form a linear functional in a corresponding functional energy space where a weak solution exists. By admissible loadings we mean those for which this functional is bounded, or, in other words, the work of external forces is meaningful and the total energy is finite. Then, using Sobolev's imbedding theorems we characterize a class of admissible external loadings in the VWP form and discuss their physical meaning.

Keywords

strain gradient elasticity, weak solutions, Sobolev spaces, linear continuous functional

1 Introduction

By the strain gradient elasticity we mean a model of an elastic continuum with a strain energy density dependent on deformation gradients of order higher than one. In particular, we call the first and second strain gradient elasticity the models where strain energy densities depend on the second and third gradients of placement vector. The idea of such continua appeared almost the same time with development classic elasticity, see [1–6] for historical developments in the field of strain gradient elasticity and other generalized continua. Within rational continuum mechanics the first and second strain gradient elasticity were introduced by Toupin [7,8] and Mindlin [9–11]. Nowadays these models found various applications for modelling of material behaviour at small scales [12–16] and for description of composite materials with essential difference in elastic stiffness such as beam lattices and fabrics [17–21], see also [22] and the reference therein. Let us note that considering such beam-lattice structures as pantographic metamaterials one faced with a certain modification of the Toupin–Mindlin strain gradient elasticity called gradient incomplete models where a strain energy density is degenerated in comparison with the full model [18,23–26].

Solution of the boundary-value problems within the strain gradient elasticity requires an efficient tool such as the finite element method (FEM) adopted for systems of partial differential equations (PDEs) of higher order. FEM as well as some other techniques is based on the Virtual Work Principle (the VWP) [27–30]. The corresponding solutions are called *weak* and in the case of classic problems were studied in many works, see, e.g. [31–36].

To introduce the weak statement of an equilibrium boundary value problems for hyperelastic materials, they use the VWP which for conservative loads is equivalent to the Lagrange variational principle. For linear problems the expression for the strain energy in terms of the components of the some strain tensors, is a finite quadratic non-negative

[1] DICAAR, University of Cagliari, Italy
[2] Universidad Nacional de Colombia, Colombia
[3] Lawrence Technological University, USA

Corresponding author:
Victor A. Eremeyev, DICAAR, University of Cagliari, via Marengo 2, 09-123, Cagliari, Italy
Email: victor.eremeyev@gmail.com

form W at each point, the work of the loads is presented by a linear functional denoted by F, which depends on the displacement vector $\mathbf{u}$. The Lagrange variational principle used for the weak formulation of the problem states that in some class of the displacements, the solution of a linear equilibrium problem attains on $\mathbf{u}$ at which the total energy functional $J(\mathbf{u}) \equiv E(\mathbf{u}) - F(\mathbf{u})$ admits minimum. Here $E(\mathbf{u}) = \int_V W(\mathbf{u})dV$ is an energy functional whereas W is a strain energy density. The equation of the weak (or energy) formulation of the problem is $\delta J = 0$, it coincides with the equation of the WVP principle for dead loads but the VWP can also be applied in the situations when the Lagrange principle is not applicable.

In linear elasticity the strain energy function W is non-negative but its kernel, the set of $\mathbf{u}$ where $W(\mathbf{u}) = 0$, normally is the set of all small rigid displacements $\mathbf{u} = \mathbf{a} + \mathbf{x} \times \mathbf{b}$ of the body as a rigid one. Here $\mathbf{x}$ is a position vector, $\mathbf{a}$, $\mathbf{b}$ are arbitrary but fixed constant vectors, and $\times$ stands for the cross product. For some models like for the plane membrane, the kernel is only a part of the set of rigid displacements, whereas for some other models the kernel may include additional deformations, see, e.g.,[37,38]. The class of the displacements where we should seek the weak solution is a set of functions equipped with some kinematic restrictions, which can be boundary conditions which eliminate free rigid motions of the body without deformations. This class is introduced usually as the completion of smooth functions satisfying kinematic restrictions, in the energy norm which is

$$\|\mathbf{u}\|_E^2 = 2 \int\limits_V W(\mathbf{u})dV.$$

This norm defines an energy inner product satisfying $(\mathbf{u}, \mathbf{u})_E = \|\mathbf{u}\|_E^2$.

For hyperelastic models it can be usually proved that such an energy space E is a subspace of a Sobolev's space or of combination of some Sobolev' spaces. Sobolev's imbedding theorems allow us to establish for which loads the work functional $F(\mathbf{u})$ is continuous in E. For the continuous work functional, by Riesz representation theorem, $F(\mathbf{v}) = (\mathbf{u}^*, \mathbf{v})_E$ for any $\mathbf{v} \in E$. So the WVP equation defining a weak solution reduces to a quite simple equation in space E:

$$(\mathbf{u}, \mathbf{v})_E - (\mathbf{u}^*, \mathbf{v})_E = 0,$$

which must be valid for any $\mathbf{v} \in E$. This equation has the only solution $\mathbf{u}^* \in E$ (up to free rigid motions) which is the needed weak solution to the problem under consideration.

Let us underline an essential difference between $W(\mathbf{u})$ and $F(\mathbf{u})$. A strain energy density W represents a constitutive equation, i.e. it describes the material behaviour, whereas F relates to external loads, which, at first glance, can be applied independently on material properties. However, external loads must be consistent with a material model. For example, it is obvious that an elastic string resists to an external transverse force and cannot resist to an external bending moment, whereas an elastic beam resists to both external forces and couples. In fact, this difference in resistance to external loadings constitute the difference between models of elastic strings and beams. Within the classic elasticity one can apply a mass force and surface traction whereas mass and surface couples are forbidden. The latter are admissible in the case of micropolar continua[39,40] and couple stress elasticity[7,8,41]. Within higher order models external loadings may include not only forces and couple but also "exotic" ones as generalized surface traction and hyper-stresses[42,43]. Let us note that extensions as bi-moments[44–46] or double forces[47] are already known in structural mechanics.

So the main step of the investigation of weak solutions is to establish the properties of the elements of space E and of the work functional F. In the following we call external loadings *admissible* if they satisfy

- $F(\mathbf{u})$ must be consistent with $\delta E(\mathbf{u})$, i.e. the work of external loadings must be consistent with the first variation of the energy functional;
- $F(\mathbf{u})$ must be a continuous linear functional in a corresponding energy space E, so a weak solution $\mathbf{u}_0$ is a minimizer of a total energy functional J: $J(\mathbf{u}_0) = \min\limits_{\mathbf{u}\in E} J(\mathbf{u})$.

In other words by admissible external loadings we mean those which can be introduced in a model under consideration and which result in the bounded total energy. For example, external couples are not admissible within elasticity but they are admissible for micropolar media. Similarly, a point force is not admissible for an elastic membrane as it results in unbounded energy, but it is admissible in the theory of Euler–Bernoulli beams or in the plate theory.

Let us illustrate the situation considering a well-known and relatively simple problem (theoretically, not practically) of finding of weak solutions for equilibrium problem of a plane membrane occupying a bounded domain $S \subset \mathbb{R}^2$. For simplicity, let the rigidity constant of the membrane be $a = 1$ and so the equation of the membrane is

$$\nabla^2 u + f = 0, \qquad (x, y) \in S,$$

where u is the normal deflection of the membrane and f is a normal load. Kinematic restrictions for the membrane can be a boundary condition $u = 0$ on some part ∂S_1 of the boundary ∂S or, if it is a free membrane without kinematic boundary conditions, then the restriction for u may be of the form $\int_S u dS = 0$ (in this way we delete the membrane rigid motions).

Here the strain energy functional is $E(u) \equiv \int_S W(u) dS$ with $W(u) = \frac{1}{2}(u_{,x}^2(x,y) + u_{,y}^2(x,y))$. For brevity we denote partial derivatives with respect to Cartesian coordinates x and y as $u_{,x}$ and $u_{,y}$, respectively. So the energy norm in E on the set of u with the above restrictions is

$$\|u\|_E^2 = \int_S (u_{,x}^2(x,y) + u_{,y}^2(x,y)) dS.$$

Its kernel, that is the set of u for which $\|u\|_E = 0$, is the set of all constant functions $u(x, y) = c$. As is known, under the above restrictions, E is a subspace of Sobolev's space $W^{1,2}(S)$ with the norm equivalent to the norm of $W^{1,2}(S)$ and so by the imbedding theorem, E is continuously imbedded to $L^p(S)$ for any finite $p > 1$ and to $L^q(C)$, $q > 1$, for any piecewise smooth curve C in S. It is also known that E is not imbedded continuously to the set of continuous functions on S.

A weak solution $u \in E$ of the equilibrium problem should satisfy the equation

$$\int\limits_S (u_{,x} v_{,x} + u_{,y} v_{,y}) dS - F(v) = 0$$

for any $v \in E$, where the work functional $F(v)$ can be of the form

$$F(v) = \int\limits_S f(x,y) v(x,y) dS + \int_C \phi(s) v(x(s), y(s)) ds.$$

Here ϕ is a normal force along curve C. Forces f, ϕ should belong to spaces $L^{p'}(S)$, $L^{q'}(C)$, where $p' = p/(p-1)$, $q' = q/(q-1)$ with which, by the imbedding theorem, F is a linear continuous functional in E as

$$\left| \int_S f v dS \right| \leq \|f\|_{L^{p'}}(S) \|v\|_{L^p}(S)$$

and

$$\left| \int_C \phi v ds \right| \leq \|\phi\|_{L^{q'}}(C) \|v\|_{L^q}(C).$$

The weak setup for the membrane problem does not allow us to introduce the work of a point force F_0 at point $(a, b) \in S$ into the work functional F as $F_0 v(a, b)$ is not a continuous functional in E. Of course this does not mean that for such forces there are no solutions to equilibrium membrane problem. Such solutions can exist but their strain energy is infinite and so they cannot be considered under the weak setup of the problem.

Here we should also note that for the free membrane, the solution is defined up to an indefinite constant c: it is $u^*(x, y) + c$ but to have a solution for a free membrane we should require the self-balance of the external forces which here is $\int_S f dS + \int_C \phi ds = 0$.

As a result of above-provided analysis for an elastic membrane, the *admissible* loads for the membrane are: a surface load $f \in L^{p'}(S), p' > 1$ distributed over the surface S or its part and a line force $\phi \in L^{q'}(C), q' > 1$ distributed along a curve. *Point-forces are not admissible.*

In what follows we will watch how to realize the scheme for more complicated problems of hyperelasticity.

The paper is organized as follows. First, we briefly introduce a strain energy density within the first and second strain gradient elasticity. For the first strain gradient elasticity a strain energy density depends on strains and the second gradients of displacements, whereas for the second strain gradient elasticity the strain energy depends also on the third gradients of displacements. Considering the first variation of the energy functionals we introduce the corresponding energy spaces and characterize them within Sobolev spaces $W^{2,2}$ and $W^{3,2}$. Finally, using Sobolev's imbedding theorems we discuss the possible forms of a linear continuous functional in these spaces and discuss the physical meaning of admissible external loadings. The latter includes various distributions of forces, couples and higher-order loadings. In the following we use index-free tensor calculus, so vectors and tensors of any order will be denoted by boldface symbols as in[48,49].

2 Strain energy density

Let V be a bounded domain in $\mathbb{R}^n$, $n = 2, 3$, which is occupied by an elastic body $\mathcal{B}$. Small deformations of $\mathcal{B}$ can be described though a displacement field $\mathbf{u}$

$$\mathbf{u} = \mathbf{u}(\mathbf{x}), \tag{1}$$

where $\mathbf{x}$ is a position vector in V with the Cartesian coordinates x_1,..., x_n.

In what follows we consider hyperelastic materials, i.e. the materials with given strain energy density. Within the strain gradient elasticity approach, a strain energy density depends on the first and higher-order gradients of $\mathbf{u}$. After Mindlin and Eshel[11] we assume the following form for a strain energy density (Mindlin's form I)

$$W = W_1(\boldsymbol{\varepsilon}(\mathbf{u}), \nabla\nabla\mathbf{u}), \tag{2}$$

where $\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}(\mathbf{u}) = \frac{1}{2}(\nabla\mathbf{u} + \nabla\mathbf{u}^T)$ is the linear strain tensor, "T" denotes the transpose of a second-order tensor, ∇ is the nabla-operator

$$\nabla = \mathbf{i}_1 \frac{\partial}{\partial x_1} + \ldots + \mathbf{i}_n \frac{\partial}{\partial x_n,}$$

$n = 2, 3$, and $\mathbf{i}_k$ are the Cartesian unit basis vectors, $k = 1, \ldots n$.

For a second strain gradient elastic material, the strain energy density takes the form as in Mindlin[10]

$$W = W_2(\boldsymbol{\varepsilon}(\mathbf{u}), \nabla\nabla\mathbf{u}, \nabla\nabla\nabla\mathbf{u}). \tag{3}$$

In the following we suppose W_1 and W_2 to be positive definite quadratic forms of their arguments, tensorial components in canonic basis. In other words we assume that

$$\begin{aligned} &C_1 \left(\|\boldsymbol{\varepsilon}\|^2 + \|\nabla\nabla\mathbf{u}\|^2\right) \\ &\qquad \geq W_1(\boldsymbol{\varepsilon}(\mathbf{u}), \nabla\nabla\mathbf{u}) \geq C_2 \left(\|\boldsymbol{\varepsilon}\|^2 + \|\nabla\nabla\mathbf{u}\|^2\right), \end{aligned} \tag{4}$$

$$\begin{aligned} &C_3 \left(\|\boldsymbol{\varepsilon}\|^2 + \|\nabla\nabla\mathbf{u}\|^2 + \|\nabla\nabla\nabla\mathbf{u}\|^2\right) \\ &\qquad \geq W_2(\boldsymbol{\varepsilon}(\mathbf{u}), \nabla\nabla\mathbf{u}, \nabla\nabla\nabla\mathbf{u}) \\ &\qquad\qquad \geq C_4 \left(\|\boldsymbol{\varepsilon}\|^2 + \|\nabla\nabla\mathbf{u}\|^2 + \|\nabla\nabla\nabla\mathbf{u}\|^2\right), \end{aligned} \tag{5}$$

where C_m here and in what follows are positive constants independent of $\mathbf{u}$. $\|\cdot\|$ denotes the Euclidian norms in spaces of second-, third-, and fourth-order tensors in canonic basis. Note that in the following we assume that all the quantities are dimensionless, otherwise we have to consider the constants with dimensional units.

For specific forms of (2) and (3) we refer to works[10,11,16,50–54], in which some examples for isotropic and anisotropic materials are given.

3 Virtual work principle for hyperelastic materials

As for others generalized continuum theories, for the derivation of corresponding boundary-value problems we can use the variational approach considering the stationarity of the total energy functional (Lagrange's variational principle). For classic problems, for the Lagrange principle we can refer to[27], whereas for the strain gradient elasticity and the other types of continua with microstructure we refer to[4,16,28–30,49,55–58].

The energy functionals take the form

$$E_1[\mathbf{u}] = \int\limits_V W_1\, dV, \quad E_2[\mathbf{u}] = \int\limits_V W_2\, dV. \tag{6}$$

The Lagrange variational principle states that the equilibrium solution of the problem under consideration attains at maximum value of the total energy functional and vice versa. For the stationarity, we get the following variational equations

$$\delta E_1 - \delta A_1 = 0, \quad \delta E_2 - \delta A_2 = 0, \tag{7}$$

where δ stands for the symbol of variation and δA_α, $\alpha = 1, 2$, is the work of external loadings which will be discussed in the following.

The first variations of E_1 and E_2 can be represented as follows[10]

$$\delta E_1 = B_1(\mathbf{u}, \mathbf{v}) \equiv \int\limits_V \left[\boldsymbol{\sigma}(\mathbf{u}) : \boldsymbol{\varepsilon}(\mathbf{v}) + \boldsymbol{\mu}_1(\mathbf{u}) \vdots \nabla\nabla\mathbf{v}\right] dV, \tag{8}$$

$$\begin{aligned}\delta E_2 = B_2(\mathbf{u}, \mathbf{v}) \equiv \int\limits_V \Big[&\boldsymbol{\sigma}(\mathbf{u}) : \boldsymbol{\varepsilon}(\mathbf{v}) + \boldsymbol{\mu}_1(\mathbf{u}) \vdots \nabla\nabla\mathbf{v} \\ &+ \boldsymbol{\mu}_2(\mathbf{u}) :: \nabla\nabla\nabla\mathbf{v}\Big] dV,\end{aligned} \tag{9}$$

where $\mathbf{v} = \delta\mathbf{u}$. We have introduced stress tensor $\boldsymbol{\sigma}$, hyper-stress tensors $\boldsymbol{\mu}_1$ and $\boldsymbol{\mu}_2$ by

$$\begin{aligned}\boldsymbol{\sigma} =& \frac{\partial W_\alpha}{\partial \boldsymbol{\varepsilon}}, \quad \boldsymbol{\mu}_1 = \frac{\partial W_\alpha}{\partial \nabla\nabla\mathbf{u}}, \quad \alpha = 1,2,\\ \boldsymbol{\mu}_2 =& \frac{\partial W_2}{\partial \nabla\nabla\nabla\mathbf{u}}.\end{aligned}$$

Obviously, $\boldsymbol{\sigma}$, $\boldsymbol{\mu}_1$, and $\boldsymbol{\mu}_2$ are linear tensor-valued functions of their arguments. In addition, $\boldsymbol{\sigma}$, $\boldsymbol{\mu}_1$, and $\boldsymbol{\mu}_2$ are tensors of second-, third-, and fourth-order, respectively, for which we use respective inner products ":", "$\vdots$", and "::". For dyads, triads and tetrads these operations are given by the formulae[10]

$$\begin{aligned}&(\mathbf{a}\otimes\mathbf{b}) : (\mathbf{c}\otimes\mathbf{d}) = (\mathbf{a}\cdot\mathbf{c})(\mathbf{b}\cdot\mathbf{d}),\\ &(\mathbf{a}\otimes\mathbf{b}\otimes\mathbf{c})\vdots(\mathbf{d}\otimes\mathbf{e}\otimes\mathbf{f}) = (\mathbf{a}\cdot\mathbf{d})(\mathbf{b}\cdot\mathbf{e})(\mathbf{c}\cdot\mathbf{f}),\\ &(\mathbf{a}\otimes\mathbf{b}\otimes\mathbf{c}\otimes\mathbf{d}) :: (\mathbf{e}\otimes\mathbf{f}\otimes\mathbf{g}\otimes\mathbf{h}) = (\mathbf{a}\cdot\mathbf{e})(\mathbf{b}\cdot\mathbf{f})(\mathbf{c}\cdot\mathbf{g})(\mathbf{d}\cdot\mathbf{h}).\end{aligned}$$

Here $\cdot$ and $\otimes$ stand for the dot and dyadic (tensorial) products, respectively.

By definition, $B_1(\mathbf{u},\mathbf{v})$ and $B_2(\mathbf{u},\mathbf{v})$ are symmetric bilinear forms in $\mathbf{u}$ and $\mathbf{v}$,

$$B_1(\mathbf{u},\mathbf{v}) = B_1(\mathbf{v},\mathbf{u}), \quad B_2(\mathbf{u},\mathbf{v}) = B_1(\mathbf{v},\mathbf{u}).$$

They have the properties of an inner product in a vector space. Moreover, for quadratic functionals E_1 and E_2 we have

$$B_1(\mathbf{u},\mathbf{u}) = 2E_1[\mathbf{u}], \quad B_2(\mathbf{u},\mathbf{u}) = 2E_2[\mathbf{u}]. \tag{10}$$

From (4) and (5) it follows that B_1 and B_2 are coercive and bounded

$$B_1(\mathbf{u},\mathbf{u}) \geq C_2 \int\limits_V \left(\|\boldsymbol{\varepsilon}\|^2 + \|\nabla\nabla\mathbf{u}\|^2\right) dv, \tag{11}$$

$$B_2(\mathbf{u},\mathbf{u}) \geq C_4 \int\limits_V \left(\|\boldsymbol{\varepsilon}\|^2 + \|\nabla\nabla\mathbf{u}\|^2 + \|\nabla\nabla\nabla\mathbf{u}\|^2\right) dv, \tag{12}$$

and

$$B_1(\mathbf{u},\mathbf{v}) \le C_5 \left[\int\limits_V \left(\|\boldsymbol{\varepsilon}(\mathbf{u})\|^2 + \|\nabla\nabla\mathbf{u}\|^2\right)\, dv\right]^{1/2} \left[\int\limits_V \left(\|\boldsymbol{\varepsilon}(\mathbf{v})\|^2 + \|\nabla\nabla\mathbf{v}\|^2\right)\, dv\right]^{1/2}, \tag{13}$$

$$B_2(\mathbf{u},\mathbf{v}) \le C_6 \left[\int\limits_V \left(\|\boldsymbol{\varepsilon}(\mathbf{u})\|^2 + \|\nabla\nabla\mathbf{u}\|^2 + \|\nabla\nabla\nabla\mathbf{u}\|^2\right)\, dv\right]^{1/2} \left[\int\limits_V \left(\|\boldsymbol{\varepsilon}(\mathbf{v})\|^2 + \|\nabla\nabla\mathbf{v}\|^2 + \|\nabla\nabla\nabla\mathbf{v}\|^2\right)\, dv\right]^{1/2}. \tag{14}$$

By standard techniques of calculus of variations, integrating by parts in (8) and (9), from (7), we can derive the equations for corresponding boundary-value problems and natural boundary conditions. Moreover, the form of work functionals δA_α should be compatible with the first variation of corresponding B_k. Instead of the deduction of these, here we use (7) with (8) and (9) as a basis to introduce the weak solutions to the problems.

4 Weak solutions and the corresponding energy spaces

4.1 Mathematical preliminaries

In order to introduce a weak solution to the problem under consideration we have to specify geometrical properties of domain V and of the functional spaces.

We will use the following functional spaces[59].

$C^m(V)$ is the space of functions continuous together with partial derivatives up to the order m in open domain V, $C^0(V) \equiv C(V)$ and $C^\infty(V) = \bigcap\limits_{m=0}^{\infty} C^m(V)$.

$C^m(\overline{V})$ is a Banach space of bounded and uniformly continuous functions together with their partial derivatives up to the order m in the closed domain $\overline{V}$, $C^0(\overline{V}) \equiv C(\overline{V})$. Space $C^m(\overline{V})$ is equipped with the norm

$$\|u\|_{C^m(\overline{V})} = \max_{0\le|k|\le m} \max_{\mathbf{x}\in\overline{V}} \left|D^k u\right|$$

and so

$$\|u\|_{C(\overline{V})} = \max_{\mathbf{x}\in\overline{V}} |u|.$$

Here $D^k u$ denotes a partial derivative of order k, where $k = (k_1, \ldots k_n)$ is a multiindex and $|k| = \sum_{i=1}^n |k_i|$. So

$$D^k u = \frac{\partial^{|k|} u}{\partial x_1^{k_1} \ldots \partial x_n^{k_n}}.$$

For $0 < \lambda \le 1$ $C^{m,\lambda}(\overline{V})$ is a Banach space of Hölder continuous functions with the norm

$$\|u\|_{C^{m,\lambda}(\overline{V})} = \|u\|_{C^m_B(\overline{V})} + \max_{0\le k\le m} \sup_{\mathbf{x},\mathbf{y}\in\overline{V},\mathbf{x}\neq\mathbf{y}} \frac{\left|D^k u(\mathbf{x}) - D^k u(\mathbf{y})\right|}{|\mathbf{x}-\mathbf{y}|^\lambda}.$$

By $C_0(V)$, $C_0^m(V)$, $C_0^\infty(V)$, etc., we mean the spaces of continuous functions with compact support in V.

$L^p(V)$, $p \ge 1$, is a Banach space of measurable functions equipped with the norm

$$\|u\|_{L^p(V)} = \left(\int_V |u|^p \, dV\right)^{1/p}.$$

Finally, we introduce Sobolev's spaces as follows

$$W^{m,p}(V) = \{u \in L^p(V), D^k u \in L^p(V)\}$$

for $0 \le |k| \le m$. $W^{m,p}(V)$ has the norm

$$\|u\|_{W^{m,p}(V)} = \left(\sum_{0\le|k|\le m} \|D^k u\|^p_{L^p(V)}\right)^{1/p}. \tag{15}$$

$W_0^{m,p}(V)$ is the closure (completion) of functions of $C_0^\infty(V)$ in the norm (15).

Considering vector-valued functions $\mathbf{u}$ we will use the notations like $\mathbf{u} \in \mathbf{L}^p(V)$, $\mathbf{u} \in \mathbf{C}^m(\overline{V})$ if any Cartesian component of $\mathbf{u}$ belongs to $L^p(V)$ or to $C^m(\overline{V})$, etc.

Let $V = \overline{V}$ be a bounded closed set in R^2 or R^3 with a piecewise smooth boundary such that for V Sobolev imbedding theorems are valid[59].

4.2 Definition of weak solutions

As δA_1 and δA_2 are linear with respect to $\mathbf{v}$, we treat them as linear functionals

$$\delta A_1 = F_1(\mathbf{v}), \quad \delta A_2 = F_2(\mathbf{v}).$$

Now variational equations (7) take the form

$$B_1(\mathbf{u}, \mathbf{v}) = F_1(\mathbf{v}), \quad B_2(\mathbf{u}, \mathbf{v}) = F_2(\mathbf{v}). \tag{16}$$

In the following we assume that a part S_u of the boundary ∂V is fixed and non-empty. So for the first strain gradient elasticity model we have two vectorial kinematic boundary conditions

$$\mathbf{u}\Big|_{S_u} = \mathbf{0}, \quad \frac{\partial \mathbf{u}}{\partial n}\Big|_{S_u} = \mathbf{0}, \tag{17}$$

where $\frac{\partial}{\partial n} = \mathbf{n} \cdot \nabla$ denotes the normal derivative and $\mathbf{n}$ is the unit outward normal to S_u. For the second strain gradient elasticity model we have three vectorial kinematic boundary conditions

$$\mathbf{u}\Big|_{S_u} = \mathbf{0}, \quad \frac{\partial \mathbf{u}}{\partial n}\Big|_{S_u} = \mathbf{0}, \quad \frac{\partial^2 \mathbf{u}}{\partial n^2}\Big|_{S_u} = \mathbf{0}. \tag{18}$$

As $\mathbf{v} = \delta \mathbf{u}$, it has the same properties as $\mathbf{u}$, so $\mathbf{v}$ also satisfies (17) or (18). In mathematical physics (17) (18) are called Dirichlet's conditions. On the other part $S_t = \partial V \backslash S_u$ we assume natural boundary conditions, which follow from $(16)_1$ or $(16)_2$. For brevity, we do not write out them here.

With kinematic boundary conditions (17) and (18) $B_1(\mathbf{u}, \mathbf{u})$ and $B_2(\mathbf{u}, \mathbf{u})$ respectively have all the properties of a squared norm. So we introduce the following energy spaces.

Definition 1. *Energy spaces E_1 and E_2 are respectively the completions of $\mathbf{C}^2(V)$ functions with boundary conditions (17) and (18) in the norms*

$$\|\mathbf{u}\|_1 = [B_1(\mathbf{u},\mathbf{u})]^{1/2}, \quad \|\mathbf{u}\|_2 = [B_2(\mathbf{u},\mathbf{u})]^{1/2}.$$

By Definition 1, E_1 and E_2 are Hilbert spaces with inner products defined as follows

$$(\mathbf{u},\mathbf{v})_1 = B_1(\mathbf{u},\mathbf{v}), \quad (\mathbf{u},\mathbf{v})_2 = B_2(\mathbf{u},\mathbf{u}).$$

It is easy to see that on E_1 and E_2 the introduced norms are equivalent to the norms in $\mathbf{W}^{2,2}(V)$ and $\mathbf{W}^{3,2}(V)$, respectively, as it follows from the inequalities for $\mathbf{u}$ in E_1 or in E_2

$$C_7\|\mathbf{u}\|_{\mathbf{W}^{2,2}(V)} \le \|\mathbf{u}\|_1 \le C_8\|\mathbf{u}\|_{\mathbf{W}^{2,2}(V)},$$
$$C_9\|\mathbf{u}\|_{\mathbf{W}^{3,2}(V)} \le \|\mathbf{u}\|_2 \le C_{10}\|\mathbf{u}\|_{\mathbf{W}^{3,2}(V)}$$

with constants C_7, C_8, C_9, and C_{10} independent on $\mathbf{u}$.

Using the energy spaces we introduce corresponding weak solutions as follows.

Definition 2. *For the first strain gradient elasticity, $\mathbf{u}_1 \in E_1$ is a weak solution of the problem under consideration if $\mathbf{u}_1$ satisfies the equation*

$$(\mathbf{u}_1,\mathbf{v})_1 = F_1(\mathbf{v}) \qquad \forall\ \mathbf{v} \in E_1. \tag{19}$$

Similarly, for the second strain gradient elasticity, $\mathbf{u}_2 \in E_2$ is a weak solution if $\mathbf{u}_2$ satisfies

$$(\mathbf{u}_2,\mathbf{v})_2 = F_2(\mathbf{v}) \qquad \forall\ \mathbf{v} \in E_2. \tag{20}$$

Now existence and uniqueness of weak solutions depends on the properties of $F_1(\mathbf{v})$ and $F_2(\mathbf{v})$. We formulate the basic theorem.

Theorem 1. *Let F_1 be a linear continuous functional in E_1 and F_2 be a linear continuous functional in E_2. Then there exist weak solutions $\mathbf{u}_1$ and $\mathbf{u}_2$ in the sense of Definition 2. These solutions are unique. Furthermore, $\mathbf{u}_1$ and $\mathbf{u}_2$ are minimizers of the total energy functionals $J_1(\mathbf{u})$ and $J_2(\mathbf{u})$,* *respectively.*

$$\begin{aligned}
\textit{For } \mathbf{u}_1: \quad & J_1(\mathbf{u}_1) = \min_{\mathbf{u}\in E_1} J_1(\mathbf{u}),\\
& J_1(\mathbf{u}) = \frac{1}{2}(\mathbf{u},\mathbf{u})_1 - F_1(\mathbf{u}).\\
\textit{For } \mathbf{u}_2: \quad & J_2(\mathbf{u}_2) = \min_{\mathbf{u}\in E_2} J_2(\mathbf{u}),\\
& J_2(\mathbf{u}) = \frac{1}{2}(\mathbf{u},\mathbf{u})_2 - F_2(\mathbf{u}).
\end{aligned}$$

Proof. By the Riesz representation theorem (see, e.g.[35]) there exist uniquely defined elements $\mathbf{u}_1^* \in E_1$ and $\mathbf{u}_2^* \in E_2$ such that

$$F_1(\mathbf{v}) = (\mathbf{u}_1^*, \mathbf{v})_1, \quad F_2(\mathbf{v}) = (\mathbf{u}_2^*, \mathbf{v})_2.$$

So (19) and (20) transform into

$$\begin{aligned}
(\mathbf{u},\mathbf{v})_1 &= (\mathbf{u}_1^*, \mathbf{v})_1 \qquad \forall\ \mathbf{v} \in E_1,\\
(\mathbf{u},\mathbf{v})_2 &= (\mathbf{u}_2^*, \mathbf{v})_2 \qquad \forall\ \mathbf{v} \in E_2.
\end{aligned}$$

Thus, $\mathbf{u}_1 = \mathbf{u}_1^*$ and $\mathbf{u}_2 = \mathbf{u}_2^*$.

The minimizing properties of weak solutions follow from the standard analysis of quadratic functionals, see, e.g.[35].

□

As we can see, the key of the proof is the continuity of linear functionals F_1 and F_2 in chosen energy spaces. To illustrate this let us consider a simple example. We return to to the equilibrium problem for a plane membrane with clamped boundary under transverse pressure f in more details. Its weak solution is given by the equation

$$(u,v)_{W^{1,2}(S)} = F(v), \quad F(v) = \int\limits_S fv\,dS.$$

Obviously, here the energy space is $W_0^{1,2}(S)$, $S \subset \mathbb{R}^2$. So if $F(v)$ is continuous in $W_0^{1,2}(S)$ there exists a unique weak solution satisfying the last equation for any $v \in W_0^{1,2}(S)$. Now we could give an exact characterization of f by saying that f belongs to the so-called negative Sobolev space, but this is no more than saying that the corresponding functional of work is continuous in the Sobolev space. This condition is difficult to verify. Therefore, we prefer to characterize the admissible f in terms of spaces L^p which can be verified. We will continue to do the same in what follows.

According Sobolev's imbedding theorem[59] we have continuous imbedding

$$W_0^{1,2}(S) \to L^q(S)$$

for any $1 < q < \infty$. Using Hölder's inequality

$$|F(v)| \le \left(\int\limits_S |f|^p \, dS \right)^{\frac{1}{p}} \left(\int\limits_S |u|^q \, dS \right)^{\frac{1}{q}}, \quad \frac{1}{p} + \frac{1}{q} = 1,$$

we can conclude that $F(v)$ is a linear continuous functional in $W^{1,2}(S)$ for $f \in L^p(S)$, $p > 1$.

Similarly, we can prove that

$$F_C(v) = \int\limits_C \phi v \, ds,$$

where $C \subset S$ is a smooth curve, is also linear continuous functional in $W^{1,2}(S)$ for $\phi \in L^p(L)$, $1 < p < \infty$. Physically, this means that the action of a line force is also admissible in this case.

By Riesz' representation theorem, a general form of a linear continuous functional in $W^{1,2}(S)$ or in $W_0^{1,2}(S)$ has the form[60]

$$F(v) = \int\limits_S (fv + \mathbf{f} \cdot \nabla v) \, dS$$

for $f \in L^p(S)$ with some $p > 1$, and $\mathbf{f} \in \mathbf{L}^2(S)$. This form of $F(v)$ changes the strong (classic) formulation of the membrane equilibrium problem. Now we get

$$-\nabla \cdot \nabla u = f - \nabla \cdot \mathbf{f}, \quad \mathbf{x} \in S \subset \mathbb{R}^2, \quad u\big|_{\partial S} = 0.$$

Such extension of the right side of a more general equations equation was discussed by Ladyzhenskaya[33]. From the physical point of view, here $\mathbf{f}$ is similar to surface couples distributed over V. Note that in the shell theory in the loads there are members of the nature similar to $\nabla \cdot \mathbf{f}$ derived from the work of external distributed couples on the surface. However they enter to the distributed forces on the shell surface and are not mentioned as the distributed couples in the final equations in displacements.

Note that we have no embedding into $C(S)$ or even into $L^\infty(S)$. So we cannot apply a point force to membrane as in this case the work of external forces is not bounded. To demonstrate this behavior let as consider an example. Let S be a circle $S = \{\mathbf{x} : \quad |\mathbf{x}| \le a\}$, where $a = (\mathrm{e}-1)^{-1} \approx 0.58$, and function

$$w = w(x, y) = \ln \ln \left(1 + \frac{1}{r}\right), \quad r = (x^2 + y^2)^{1/2}.$$

Obviously, $w(x, y) = 0$ if $x^2 + y^2 = a^2$. One can easily prove by straightforward calculations that $w \in W^{1,2}(S)$ but $w \notin L^\infty(S)$.

We introduce a sequence of external loads defined as

$$f_k(\mathbf{x}) = \begin{cases} \dfrac{f_0 k^2}{\pi} & \mathbf{x} \in S_k \subset S, \qquad S_k = \{\mathbf{x} : \quad |\mathbf{x}| \le \frac{1}{k}\} \\ 0 & \mathbf{x} \notin S_k \end{cases},$$

where $f_0 > 0$ is a constant. So for each k the resultant force acting on the membrane is f_0. In addition we also define a sequence of displacements

$$u_k(x, y) = \min\{k, w(x, y)\}.$$

Obviously, $\max\limits_{\mathbf{x} \in S} |u_k(x, y)| = k$. Since $|u_{k,x}| \le |w_{,x}|$ and $|u_{k,y}| \le |w_{,y}|$, we have

$$E(u_k) \le E(w) = \mathrm{Ei}(1, 1) + 1 - \mathrm{e}(-1) \approx 0.852,$$

where Ei is the exponential integral[61]. So $E(u_k)$ is uniformly bounded. Moreover, $E(u_k) \to E(w)$ at $k \to \infty$. On the other hand, considering $F(u_k)$ we get $F(u_k) = f_0 k$. Thus, the total energy functional J is unbounded from below as

$$J(u_k) = E(u_k) - F(u_k) = E(u_k) - f_0 k \to -\infty \quad \text{at} \;\; k \to \infty.$$

Note that assuming $f_0 < 0$ we get $J(u_k) \to +\infty$ at $k \to \infty$.

In other words, considering a point force applied at $r = 0$ we came to unbounded total energy functional. Moreover, replacing a point force by a sequence of approximations, we cannot observe any convergence to any energy solution as the limit of these “approximations” does not exist.

5 The functional of the work of external loadings

Now let us discuss the problem of admissible loads in the gradient elasticity from Section 4.2. From the physical point of view, F_1 and F_2 represent the work of external loadings, i.e. forces, couples, and hyper-stresses. First, let us consider loadings acting in V which are compatible with required properties of F_1 and F_2.

5.1 Dirichlet-type problem

Let the whole boundary of B be clamped. So we have (17) and (18) as boundary conditions. In this case elements of E_1 and E_2 belong to $W_0^{2,2}(V)$ and $W_0^{3,2}(V)$, respectively. The general representation of a linear continuous functional in $W_0^{m,2}(V)$, see[60]

$$F(\mathbf{v}) = \int\limits_V \left(\mathbf{f} \cdot \mathbf{v} + \sum_{1 \le |k| \le m} \mathbf{f}_k \cdot D^k \mathbf{v} \right) dV, \tag{21}$$

where it is sufficient to request that $\mathbf{f}_m \in L^2(V)$, $\mathbf{f} \in \mathbf{L}^p(V)$ and $\mathbf{f}_k \in \mathbf{L}^{p_k}(V)$ for some p, p_k which we specify later. From the physical point of view, $\mathbf{f}$ is a body force per unit volume, whereas $\mathbf{f}_k$ could be called generalized external loadings, similarly to[10]. Note that Mindlin[10] restricted himself to body forces but considered generalized surface tractions.

5.1.1 First strain gradient elasticity In this case the energy space coincides with $W_0^{2,2}(V)$. According to Sobolev's imbedding theorem we have the following imbedding:

$$V \subset \mathbb{R}^3 : \quad W_0^{2,2}(V) \to C^{0,\lambda}(V), \quad 0 < \lambda \le \frac{1}{2}, \tag{22}$$

$$V \subset \mathbb{R}^2 : \quad W_0^{2,2}(V) \to C^{0,\lambda}(V), \quad 0 < \lambda < 1. \tag{23}$$

Admissible loads: as in the both cases of E_1 the functions can be considered as continuous, this means that $\mathbf{f}$ can include few point forces. By Sobolev's imbedding theorem, the distributed part of the admissible load $\mathbf{f}$ can belong to $L^p(V)$ with $p \ge 1$ when $V \in \mathbb{R}^2$. In case $V \in \mathbb{R}^3$ the distributed part of the admissible load $\mathbf{f}_1$ can belong to $L^p(V)$ with $p \ge 6/5$. In addition to point forces, some surface and line forces, i.e. forces distributed along a smooth surface and curve, are also possible.

For example, in the case of a line force distributed along the equatorial line of an elastic sphere, we can obtain a smooth solution within strain gradient elasticity[62], whereas in classical elasticity such a problem leads to a singular solution[63].

Note that here and in what follows we present the data for the load as if they are done in Cartesian coordinates or in coordinates with singular points. So we exclude some vicinity of the origin in polar coordinates, for example. For the coordinate systems with singularities the conditions change at singular points.

5.1.2 Second strain gradient elasticity Here we have imbeddings

$$V \subset \mathbb{R}^3: \quad W_0^{3,2}(V) \to C^{1,\lambda}(V), \quad 0 < \lambda \leq \frac{1}{2}, \tag{24}$$

$$V \subset \mathbb{R}^2: \quad W_0^{3,2}(V) \to C^{1,\lambda}(V), \quad 0 < \lambda < 1. \tag{25}$$

Admissible loads: By these imbeddings, in the both cases of E_2 the functions can be considered as continuous together with the first derivatives. This means that the components of admissible loads $\mathbf{f}$ and $\mathbf{f}_1$ can contain few point members. As to the distributed parts of the load the components of $\mathbf{f}$ and $\mathbf{f}_1$ can belong to $L^p(V)$ with any $p \geq 1$ for the both $V \in \mathbb{R}^2$ and $V \in \mathbb{R}^3$. As to the components of $\mathbf{f}_2$, for $V \in \mathbb{R}^2$ they can belong to $L^p(V)$ with $p \geq 1$ and for $V \in \mathbb{R}^3$ the components can belong to $L^p(V)$ with $p \geq 6/5$.

From the physical point of view here we have a richer picture of external loadings. Point forces, point double forces (moments), as well surface distributions of forces and double forces are also possible within the second strain gradient elasticity.

5.2 Mixed boundary conditions

Now let us consider the problems with mixed boundary conditions. If the boundary restrictions eliminate the free rigid motions, the admissible loads inside the body obey the same restrictions as for the body with clamped boundary. So we need only to consider the restrictions for the loads on the body. Here we will still denote the energy space by E_1 and E_2 as in previous section but now the Dirichlet's boundary conditions are given only on some part S_u of the boundary of V as it is in formulas (17), (18).

The work of external loadings now takes the form

$$F(\mathbf{v}) = \int\limits_V \left(\mathbf{f} \cdot \mathbf{v} + \sum_{1 \le |k| \le m} \mathbf{f}_k \cdot D^k \mathbf{v} \right) dV,$$
$$+ \int\limits_{S_f} \left(\mathbf{t} \cdot \mathbf{v} + \sum_{1 \le |k| \le m} \mathbf{t}_k \cdot D_n^k \mathbf{v} \right) dS$$

where $\mathbf{t}$ is a traction, $\mathbf{t}_1$ and $\mathbf{t}_2$ are the generalized surface tractions[10], $S_f = \partial V \backslash S_u$, $D_n = \mathbf{n} \cdot \nabla \mathbf{u}$ is the normal derivative. Here $m = 1, 2$.

5.3 Free surface

6 Conclusions

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